Reply to "Comment on 'Scrutinizing $\pi\pi$ scattering in light of recent lattice phase shifts'"

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The commenting manuscript by Beveren and Rupp (referred to as BR's comment [1]) raises some doubts about our published paper [2] (referred to as GGXZ paper in the following) on the reliability and the novelty of the results.

Before we reply to the comments, we need to highlight some essential points of GGXZ paper. In GGXZ paper, we have used the Peking University (PKU) representation to extract the information of the S-matrix poles from the lattice phase shifts of the $\pi\pi$ scattering with $m_\pi = 391$, 236 MeV by Hadron Spectrum Collaboration (HSC) [3–6]. The starting point is that a reliable analysis should respect the unitarity, crossing symmetry and analyticity of the scattering amplitude. In our analysis, the PKU representation respects the analyticity and unitary, and the Balanchandran-Nuyts-Roskies (BNR) relation which relates all the three channels with $IJ = 00$, 11, 20 in the unphysical region between $s = 0$ and $s = 4m_\pi^2$ is also imposed to include the constraints from the crossing symmetry. We stress that the BNR relation plays a much more relevant role in the present discussion when the σ becomes a bound state below the two-pion threshold than the situation with a physical pion mass when the σ is a broad resonance above the two-pion threshold. By carefully comparing the different scenarios, with and without the BNR relation constraints, we concluded that the scenario with a virtual-state (VS) pole and a bound-state (BS) pole in $IJ = 00$ is better in describing the lattice data and fulfilling the BNR relations simultaneously than the one with only a $IJ = 00$ BS pole. The scenario with only one BS pole can also reasonably describe the HSC phase-shift data. However, in this case the crossing symmetry is badly violated. Therefore, we would like to emphasize again that the crossing symmetry plays an important role in reaching our conclusion.

Their main criticism to our paper is that we omit the contribution of inelastic channels in our analysis and they argue that this is important in affecting the existence of the virtual state. They use their own coupled channel model as the main tool to support their argument. We would like to point out the differences between their approaches and ours to clarify this issue.

Although the inclusion of the dynamics above the inelastic channels may have some effect on the determination of the σ pole contents, our study reveals that the crossing symmetry implemented via the BNR relation in the unphysical energy region from $s = 0$ to $s = 4m_\pi^2$ is more relevant for investigation of the bound-state σ below threshold than the distant effects above the inelastic channels. The reason is simple: The bound-state σ is located inside the BNR integration ranges and the inelastic channels are quite distant from such ranges. Furthermore, we need to point out that a sophisticated model respecting the crossing symmetry should at least have a left-hand cut in the partial wave amplitude from the crossed t- and u-channel dynamics. As far as we see, neither the effect of left-hand cut nor the fulfillment of crossing symmetry for a large pion mass when σ is turning into a bound state, are explicitly addressed in the BR's commenting paper. In our opinion, the argument of s-t channel duality of the dual model in their comment does not mean their model includes the crossing symmetry and there is no analytical proof that their model satisfies the s-t duality. Including infinite s-channel poles also does not mean that the model satisfies crossing symmetry. A simple counterexample is that in the nonrelativistic potential model, there could be infinite poles in the s channel, but we know that it does not have crossing symmetry. Also, in our opinion, being consistent with the phase shift data in the physical region and σ pole position at physical pion mass is not enough to demonstrate the consistency with the crossing symmetry and the S-matrix in the unphysical region when pion mass is 391 MeV.

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Furthermore, it is well-known that the negative phase shift of $IJ = 20$ $\pi\pi$ channel is mainly contributed to by the left-hand cut from the crossed channels, whereas a model without the left-hand cut and crossing symmetry could hardly describe the $IJ = 20$ phase shift. Thus, it is unclear to us for the large pion mass, when the σ becomes a bound state below two-pion threshold, whether their model could correctly describe the S-matrix in the unphysical region between $s = 0$ and $s = 4m_\pi^2$, which is exactly where the BNR relations are evaluated and the bound-state and virtual-state poles are located. Regarding their statement that "*However, for* $m_\pi = 391$ MeV *and with the fitted value of the overall coupling, we find a bound state at 760 MeV, compatible with the lattice result of 758 MeV in [2], but no nearby VS pole. This is to be contrasted to the dispersive analysis in [1] for the same* $m_\pi = 391$ MeV, *which extracted BS and VS poles at 1 MeV and 73 MeV below threshold, respectively*," we have not seen any such documented analysis at $m_\pi = 391$ MeV in their published papers. From the BR's comment paper, we also have not seen any numerical discussion about the left-hand cut and crossing symmetry in their model for the situation when σ becomes a bound state.

To improve the GGXZ analysis for the purpose of including the $K\bar{K}$ contributions, one needs to parametrize the contributions of right-hand cut integral and the higher poles including $f_0(980)$ pole, as we did in Ref. [7]. One needs to keep in mind that here we have $m_\pi = 391$ MeV, and in this case, according to the lattice result, $m_K = 549$ MeV and the $f_0(980)$ is found to be at $1.166(45) - \frac{i}{2} \times 0.181(68)$ GeV [8]. Now that our S-matrix is parametrized with respect to s, the analytic structure of the S-matrix in the s plane is shown in Fig. 1. We can see that the $K\bar{K}$ threshold ~ 1.21 GeV^2 and f_0 pole at $s_{f_0} = 1.35 - 0.21i$ GeV^2 are far from the $\pi\pi$ threshold ~ 0.61 GeV^2. The $K\bar{K}$ cut and the $f_0(980)$ is farther away from the unphysical region than the left-hand cut, so they are not expected to affect much of the physics below the $\pi\pi$ threshold. As we have shown in the paper, since the BNR relation imposes constraints in the whole unphysical region below the $\pi\pi$ threshold, if only one σ bound state is included then the χ^2 contribution from BNR is large, whereas after the virtual state is included the χ^2 from BNR is significantly improved. This is the essential reason why we include the virtual state. In order to quantitatively address the concern of the BR's comment about the effects above the inelastic $K\bar{K}$ channel, we have tried to explicitly include the $f_0(980)$ pole fixed by the lattice study [8] in the PKU parametrization. Within our expectation, the $f_0(980)$ can not replace the virtual state to improve the χ^2 significantly, and including both virtual state and the $f_0(980)$ does not change the previous result much, since the $f_0(980)$ is far away from the unphysical region between $s = 0$ and $s = 4m_\pi^2$ where the BNR relations are evaluated.

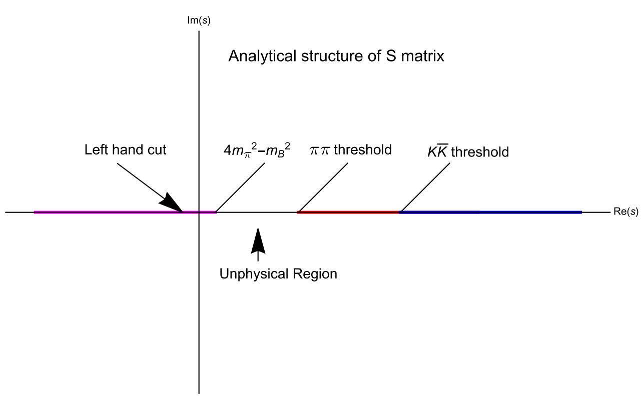

FIG. 1. The cut structure of the $IJ = 00$ $\pi\pi$ scattering amplitude when one of the σ pole becomes a bound state pole with $m_\pi = 391$ MeV and $m_K = 549$ MeV. There appears another branch point at $4m_\pi^2 - m_B^2$, which contribute another left-hand cut.

Besides, in the original papers by the HSC group [3–6], though the $s\bar{s}$ operator is included at $m_\pi = 391$ MeV, the authors also stated that the $K\bar{K}$ operators are not important in determining the σ pole below the $\pi\pi$ threshold, which can be seen from their remark "*We also include several $K\bar{K}$-like operators, of analogous construction to the $\pi\pi$ operators, although they are not vital in the determination of the spectrum below the $K\bar{K}$ threshold*." In the updated paper of HSC group [8], in order to include the analysis of $f_0(980)$, they include the whole $s\bar{s}$, $K\bar{K}$, $\eta\eta$ operators, etc. The σ pole position only moves from 758(4) MeV to 745(5) MeV, which also demonstrates that including the inelastic degrees of freedom does not affect very much the physics below the $\pi\pi$ threshold.

Hereby, we clarify the author's confusion about our statement that "*The description of the σ at $m_\pi = 391$ MeV as a pair of bound and virtual poles in our study is a novel finding in the analysis of the lattice data, and to our knowledge the role of the virtual state pole has not been explicitly reported in other analyses of lattice data as in Refs. [41,42]*." Obviously, this statement has two special premises; the first one is the special condition where $m_\pi = 391$ MeV, and the second refers to analyzing the specific lattice data set from the HSC. To the best of our knowledge, we do not find such a conclusion in the previous publications when analyzing the HSC lattice data at $m_\pi = 391$ MeV. In such a special context, no one could assert that it is a general result that there should be a virtual state there without a careful study like we did. The commenting authors also maintain that there is no virtual state in their analysis, this also demonstrates that our result is different from others, thus is novel in this special context. Figure 1 in BR's comment is done by varying the coupling constant λ. However, Fig. 8 in GGXZ paper is given in a totally different context, i.e., with varying pion mass. Though the figure in GGXZ looks similar to the one in theirs, one can not expect that it can be deduced from the behavior of varying coupling constant, since the mass describes the intrinsic property of the propagation of pion and has no direct relation with the coupling constant. In our paper, we have also explicitly mentioned that some other theoretical approaches also predicted the σ pole trajectory with respect to the change of pion masses: "*Such a pole behavior of the σ resonance has also been noticed in the calculations by unitarizing χPT amplitude with a varying quark mass m_q [9,44,45].*"

Finally, we take this opportunity to add a supplementary discussion of the left-hand cut in our GGXZ paper. After the publication, we realized that the left-hand cut discussed in our paper is still not complete. In fact, after the σ resonance becomes the bound state at $s = m_B^2$ below the $\pi\pi$ threshold, one more left-hand cut appears with a branching point at $4m_\pi^2 - m_B^2$, which should also be included in the BNR-relation, as shown in Fig. 1. The omitted left-hand cut contributed by the bound-state σ of the crossed channel could be sizable since this cut overlaps with the energy region where BNR relation is evaluated. Combining with the BNR relation, this extra left-hand cut effect could be important in determining the fate of the virtual state revealed in our paper. No matter what direction the virtual-state pole finally goes to after the inclusion of the extra left-hand cut, it is mainly influenced by the crossing-symmetry requirement via the BNR relations, since the explicit inclusion of higher excited state $f_0(980)$ is verified to play little role in the BNR relation. We will leave the detailed discussion about the effect of this extra left-hand cut to a future work.

To conclude, the statement of the novelty of our result is clear: Our conclusion about the σ pole contents is given for a specific value of pion mass at $m_\pi = 391$ MeV by simultaneously including the constraints from the lattice phase shifts and the crossing symmetry imposed by the BNR relations in the energy region between 0 and $2m_\pi$. Our study reveals that the left-hand cut and crossing symmetry play more important roles in determining the poles in unphysical region, instead of the distant effects lying above the inelastic channels, although the latter can affect the phase shifts to some extent in the elastic energy region. We have not seen a numerical discussion of the crossing symmetry in the commenting paper at large pion masses when σ becomes a bound state below $\pi\pi$ threshold. The insubstantial argument of s-t duality in the commenting paper is not enough to prove the crossing symmetry in their model. As shown in Fig. 1, the left-hand cut is closer to the unphysical region where the BNR relations take effect and the σ pole locates than the $K\bar{K}$ threshold and the $f_0(980)$ pole. To obtain reliable results concerning the σ state below $\pi\pi$ threshold for large pion masses, the left-hand cuts and constraints of crossing symmetry below the $\pi\pi$ threshold are important theoretical ingredients, which BR's comment paper completely lacks. A numerical discussion of left-hand cut contribution and crossing symmetry in their model is necessary if they want to compare their result with our result of the dispersive approach.

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