Search for an ultraviolet zero in the seven-loop beta function of the $\lambda\phi_4^4$ theory

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We investigate whether the seven-loop beta function of the $\lambda\phi_4^4$ theory exhibits evidence for an ultraviolet zero. In addition to a direct analysis of the beta function, we calculate and study Padé approximants and discuss effects of scheme transformations on the results. Confirming and extending our earlier studies of the five-loop and six-loop beta functions, we find that in the range of λ where the perturbative calculation of the seven-loop beta function is reliable, the theory does not exhibit evidence for an ultraviolet zero.

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I. INTRODUCTION

In this paper we consider the renormalization-group (RG) behavior of the $\lambda\phi^4$ field theory in $d = 4$ spacetime dimensions, where ϕ is a real scalar field. This theory, commonly denoted ϕ_4^4, is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\nu\phi)(\partial^\nu\phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.1}$$

The Lagrangian (1.1) is invariant under the global discrete $\mathbb{Z}_2$ symmetry $\phi \to -\phi$. Quantum loop corrections lead to a dependence of the physical quartic coupling $\lambda = \lambda(\mu)$ on the Euclidean energy/momentum scale μ at which this coupling is measured. The dependence of $\lambda(\mu)$ on μ is described by the RG beta function of the theory, $\beta_\lambda = d\lambda/dt$, or equivalently, $\beta_a = da/dt$, where $dt = d\ln\mu$ [1] and

$$a \equiv \frac{\lambda}{(4\pi)^2}. \tag{1.2}$$

(The argument μ will often be suppressed in the notation.) Since we will investigate the properties of the theory for large μ in the ultraviolet (UV), the value of m^2 will not play an important role in our analysis. For technical convenience, we assume that m^2 is positive. At a reference scale μ_0, the quartic coupling $\lambda(\mu_0)$ is taken to be positive for the stability of the theory. The one-loop term in this beta function has a positive coefficient, so that for small λ, $\beta_\lambda > 0$, and hence as $\mu \to 0$, the coupling $\lambda(\mu) \to 0$, i.e., the theory is infrared (IR)-free. This perturbative result is in agreement with nonperturbative approaches [2]; some reviews include [3,4].

The beta function β_a has the series expansion

$$\beta_a = a\sum_{\ell=1}^{\infty} b_\ell a^\ell. \tag{1.3}$$

The n-loop ($n\ell$) beta function, denoted $\beta_{a,n\ell}$, is given by Eq. (1.3) with the upper limit of the loop summation index $\ell = n$ instead of $\ell = \infty$. The one-loop and two-loop terms in β_a are independent of the scheme used for regularization and renormalization, while terms of loop order $\ell \geq 3$ are scheme-dependent [5,6]. For the $\mathrm{O}(N)$ $\lambda|\vec{\phi}|^4$ theory with an N-component field, $\vec{\phi} = (\phi_1, \ldots, \phi_N)$, the coefficients b_1, b_2, and b_3 were calculated in [5]. Higher-loop coefficients b_ℓ with $\ell \geq 3$ have been computed using the $\overline{\mathrm{MS}}$ minimal subtraction scheme [7,8]. A calculation of b_5 and discussion of earlier computations of b_4 and b_5 (e.g., [9–11]) were given in [4,12]. The coefficient b_6 was calculated for $N = 1$ in [13] and for general N in [14]. Most recently, the seven-loop coefficient b_7 was calculated in [15]. In analyzing the series expansion (1.3), one recalls that it is an asymptotic expansion, and the large-order behavior has been the subject of extensive study [16], including [17] and references therein.

An interesting question is whether, for the region of λ where a perturbative calculation of β_λ is reliable, this beta function exhibits evidence for a zero at some (positive) value of the quartic coupling. This would be an ultraviolet fixed point (UVFP) of the renormalization group; i.e., as $\mu \to \infty$, $\lambda(\mu)$ would approach this value (from below). In previous work we have investigated this question up to the five-loop order for the $\mathrm{O}(N)$ $\lambda|\vec{\phi}|^4$ theory in [18] and up to the six-loop order for the real $\lambda\phi^4$ theory in [19] and the $O(N)$ $\lambda|\vec{\phi}|^4$ theory in [20], finding evidence against such a UVFP. In the present paper, using the results of [15], we extend our analysis to the seven-loop level. Our analysis in [20] covered a large range of specific N values and also included an argument for the absence of a UV zero in the (rescaled) n-loop beta function at large N [see Eqs. (3.12) and (3.13) in [20]]. Indeed, in the $N \to \infty$ limit, the (rescaled) beta function of this $O(N)$ $\lambda|\vec{\phi}|_4^4$ theory is one-loop exact [3], and hence obviously has no UV zero. We will focus on the $N = 1$ theory here.

In view of this previous evidence against a UV zero in β_λ and associated UVFP in the $O(N)$ $\lambda|\vec{\phi}|^4$ theory, it is worthwhile to mention one case where an IR-free quantum field theory is known to have a UVFP, namely, the nonlinear $O(N)$ σ model in $d = 2 + \epsilon$ spacetime dimensions. In this theory, an exact solution was obtained in the limit $N \to \infty$ with $\lambda(\mu)N = x(\mu)$ a fixed function of μ and yielded the beta function

$$\beta_x = \frac{dx}{dt} = \epsilon x\left(1 - \frac{x}{x_{\rm UV}}\right) \qquad (1.4)$$

for small ϵ, where $x_{\rm UV} = 2\pi\epsilon$ is a UV fixed point of the renormalization group [21]. Since the leading term in β_x is positive for $\epsilon > 0$, this theory is IR-free. Thus, in this nonlinear $O(N)$ σ model in $d = 2 + \epsilon$ dimensions, the coupling $x(\mu)$ flows (monotonically) from $x = 0$ at $\mu = 0$ to $x = x_{\rm UV}$ as $\mu \to \infty$. Note that by making $\epsilon \ll 1$ one can arrange that the UVFP at $x_{\rm UV} = 2\pi\epsilon$ occurs at an arbitrarily small value of the scaled coupling x.

This paper is organized as follows. In Sec. II we review some relevant background. In Sec. III we present the results of our analysis of the seven-loop beta function. Section IV contains a further analysis of this question of a UV zero using Padé approximants, while Sec. V discusses effects of scheme transformations. Our conclusions are given in Sec. VI.

II. BETA FUNCTION

The n-loop truncation of (1.3), denoted $\beta_{a,n\ell}$, is a polynomial in a of degree $n+1$ having an overall factor of a^2. We may extract this factor and define a reduced beta function

$$\beta_{a,r} = \frac{\beta_a}{\beta_{a,1\ell}} = \frac{\beta_a}{b_1 a^2} = 1 + \frac{1}{b_1}\sum_{\ell=2}^{\infty} b_\ell a^{\ell-1}. \qquad (2.1)$$

The n-loop truncation of $\beta_{a,r}$, denoted $\beta_{a,r,n\ell} \equiv R_n$, is defined by taking the upper limit of the sum in (2.1) to be $\ell = n$ rather than $\ell = \infty$.

The first two coefficients in the beta function of this theory are $b_1 = 3$ and $b_2 = -17/3$ [5]. The coefficients b_ℓ with $3 \le \ell \le 7$ and the resultant higher-loop beta function discussed below are calculated in the $\overline{\rm MS}$ scheme. The coefficients up to the five-loop level are [4,5,9,12]

$$b_3 = \frac{145}{8} + 12\zeta_3 = 32.5497, \qquad (2.2)$$

$$b_4 = -\frac{3499}{48} - 78\zeta_3 + 18\zeta_4 - 120\zeta_5 = -271.606, \qquad (2.3)$$

and

$$b_5 = \frac{764621}{2304} + \frac{7965}{16}\zeta_3 - \frac{1189}{8}\zeta_4 + 987\zeta_5 + 45\zeta_3^2 - \frac{675}{2}\zeta_6 + 1323\zeta_7 = 2848.57, \qquad (2.4)$$

where the floating-point values are given to the indicated accuracy and

$$\zeta_s = \sum_{n=1}^{\infty}\frac{1}{n^s} \qquad (2.5)$$

is the Riemann zeta function. If $s = 2r$ is even, then ζ_s can be expressed as a rational number times π^{2r}, namely $\zeta_{2r} = (-1)^{r+1}B_{2r}(2\pi)^{2r}/[2(2r)!]$, where B_n are the Bernoulli numbers; however, we leave these ζ_{2r} in their generic form here and below. The six-loop coefficient is [13,14]

$$b_6 = -\frac{18841427}{11520} - \frac{779603}{240}\zeta_3 + \frac{16989}{16}\zeta_4 - \frac{63723}{10}\zeta_5 - \frac{8678}{5}\zeta_3^2 + \frac{6691}{2}\zeta_6 + 162\zeta_3\zeta_4 - \frac{63627}{5}\zeta_7 - 4704\zeta_3\zeta_5 + \frac{264543}{25}\zeta_8 - \frac{51984}{25}\zeta_{3,5} - 768\zeta_3^3 - \frac{46112}{3}\zeta_9 = -34776.13, \qquad (2.6)$$

where [22]

$$\zeta_{3,5} = \sum_{m>n\ge 1}\frac{1}{n^3 m^5}. \qquad (2.7)$$

The seven-loop coefficient is considerably more complicated than b_6, and we refer the reader to [15] for the analytic expression. The numerical value is

$$b_7 = 474651.0. \qquad (2.8)$$

Thus, in summary, the seven-loop beta function of the $\lambda\phi^4$ theory (calculated in the $\overline{\rm MS}$ scheme) is

$$\beta_{a,7\ell} = a^2\left(3 - \frac{17}{3}a + 32.5497a^2 - 271.606a^3 + 2848.57a^4 - 34776.1a^5 + 474651a^6\right). \qquad (2.9)$$

III. ZEROS OF THE n-LOOP BETA FUNCTION UP TO LOOP ORDER $n = 7$

In this section we investigate a possible UV zero, denoted $a_{\mathrm{UV},n\ell}$, of the n-loop beta function, $\beta_{a,n\ell}$. The double zero of $\beta_{a,n\ell}$ at $a = 0$ is always present (independent of n); this is an infrared zero and hence will not be of interest here.

A necessary condition for there to be robust evidence for a UV zero in the beta function of an IR-free theory is that the values calculated at successive loop orders should be close to each other. Although the two-loop beta function $\beta_{a,2\ell}$ does have a UV zero, at $a_{\mathrm{UV},2\ell} = 9/17 = 0.52941$, we found that the three-loop beta function $\beta_{a,3\ell}$ has no UV zero and, while a UV zero is present in $\beta_{a,4\ell}$, it occurs at a considerably smaller value, namely $a_{\mathrm{UV},4\ell} = 0.23332$. At the five-loop level, $\beta_{a,5\ell}$ has no UV zero, while at the six-loop level, although $\beta_{a,6\ell}$ has a UV zero, it occurs at a still smaller value, $a_{\mathrm{UV},6\ell} = 0.16041$ [18,19]. Thus, the results of this analysis show that the necessary condition that the beta function calculated to successively higher loop order should exhibit values of $a_{\mathrm{UV},n\ell}$ that are close to each other is not satisfied by this theory. At seven-loop order, using $\beta_{a,7\ell}$ from [15], we find that this function has no physical UV zero. Instead, the zeros are comprised of three complex-conjugate pairs, $-0.102135 \pm 0.079848i$, $0.0142348 \pm 0.136854i$, and $0.124533 \pm 0.0659940i$. Summarizing,

$$a_{\mathrm{UV},2\ell} = 0.52941, \qquad a_{\mathrm{UV},4\ell} = 0.23332,$$
$$a_{\mathrm{UV},6\ell} = 0.16041 \qquad \text{no } a_{\mathrm{UV},n\ell} \quad \text{for } n = 3,5,7. \tag{3.1}$$

The calculations up to seven loops show a pattern, namely that for even $n = 2, 4, 6$, $\beta_{a,n\ell}$ has a zero, $a_{\mathrm{UV},n\ell}$, but the values for different n are not close to each other, while for odd $n = 1, 3, 5, 7$, $\beta_{a,n\ell}$ has no UV zero.

In Fig. 1 we plot the n-loop beta functions for $2 \le n \le 7$ loops. Another way to show this information is via the n-loop reduced beta function, $\beta_{a,r,n\ell} = R_n$. We plot R_n in Fig. 2 for $2 \le n \le 7$. The results discussed above are evident in these figures. First, one may inquire how large the interval is in a over which the calculations of $\beta_{a,n\ell}$ to the respective n-loop orders are in mutual agreement. As one can see from Figs. 1 and 2, the n-loop beta functions $\beta_{a,n\ell}$ with $2 \le n \le 7$ only agree with each other well over the small interval of couplings $0 \le a \lesssim 0.05$. As shown in Fig. 1, the $\beta_{a,n\ell}$ with even $n = 2, 4, 6$ reach maxima and then decrease, crossing the (positive) real axis at different values listed in Eq. (3.1), while the $\beta_{a,n\ell}$ with odd n increase monotonically with a. This seven-loop analysis confirms and extends our conclusions in [19,20] at the six-loop level that the zero in the two-loop beta function of the $\lambda\phi^4$ theory occurs at too large a value of a for the perturbative calculation to be reliable.

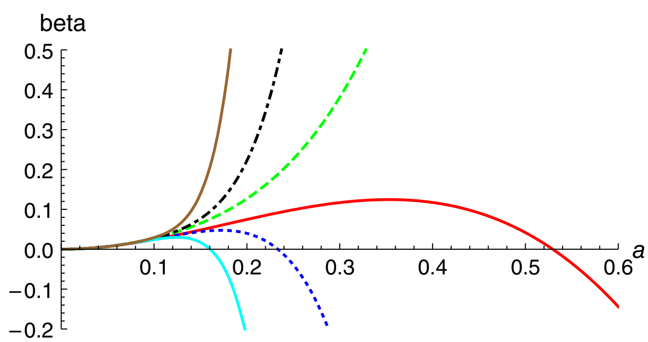

FIG. 1. Plot of the n-loop β function $\beta_{a,n\ell}$ as a function of a for (i) $n = 2$ (red, solid), (ii) $n = 3$ (green, dashed), (iii) $n = 4$ (blue, dotted), (iv) $n = 5$ (black, dot-dashed), (v) $n = 6$ (cyan, solid), and (vi) $n = 7$ (brown, solid). At $a = 0.16$, going from bottom to top, the curves are for $n = 6$, $n = 4$, $n = 2$, $n = 3$, $n = 5$, and $n = 7$.

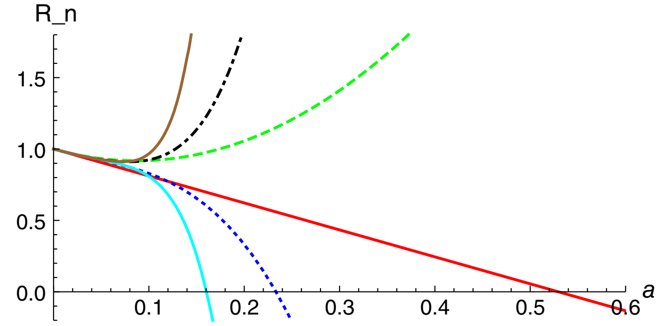

FIG. 2. Plot of the ratio R_n of the n-loop beta function $\beta_{a,n\ell}$ divided by $\beta_{a,1\ell}$, as a function of a for (i) $n = 2$ (red, solid), (ii) $n = 3$ (green, dashed), (iii) $n = 4$ (blue, dotted), (iv) $n = 5$ (black, dot-dashed), (v) $n = 6$ (cyan, solid), and (vi) $n = 7$ (brown, solid). At $a = 0.16$, going from bottom to top, the curves are for $n = 6$, $n = 4$, $n = 2$, $n = 3$, $n = 5$, and $n = 7$.

IV. ANALYSIS WITH PADÉ APPROXIMANTS

One can gain further insight into the behavior of the beta function by the use of Padé approximants (PAs). We carried out this analysis up to the six-loop level in [19,20], finding no indication of a physical UV zero, and here we extend it to the seven-loop level. Since the double zero in $\beta_{a,n\ell}$ at $a = 0$ is not relevant to the question of a UV zero, we use the reduced beta function $\beta_{a,r,n\ell}$ for this Padé analysis. The $[p,q]$ Padé approximant to $\beta_{a,r,n\ell}$ is the rational function [23]

$$[p,q]_{\beta_{a,r,n\ell}} = \frac{1 + \sum_{j=1}^{p} r_j a^j}{1 + \sum_{k=1}^{q} s_k a^k} \tag{4.1}$$

with $p + q = n - 1$, where the coefficients r_j and s_j are independent of a. At seven-loop order, we can calculate the Padé approximants $[p,q]_{\beta_{a,r,7\ell}}$ with $[p,q]$ taking on the values [6,0], [5,1], [4,2], [3,3], [2,4], [1,5], and [0,6]. Since the loop order is understood, we write $[p,q]_{\beta_{a,r,7\ell}} \equiv [p,q]$ for brevity of notation. The PA [6,0] is equivalent to $\beta_{a,r,7\ell}$ itself, which we have already analyzed, and the PA [0,6] has no zeros; thus, we focus here on the remaining five Padé approximants.

We list our results for these Padé approximants to $\beta_{a,r,7\ell}$ below:

$$[5,1] = \frac{1 + 11.760a - 14.931a^2 + 57.552a^3 - 286.17a^4 + 1367.8a^5}{1 + 13.649a}, \tag{4.2}$$

$$[4,2] = \frac{1 + 20.541a + 75.687a^2 - 49.670a^3 + 81.973a^4}{1 + 22.430a + 107.21a^2}, \tag{4.3}$$

$$[3,3] = \frac{1 + 25.073a + 152.81a^2 + 155.99a^3}{1 + 26.962a + 192.89a^2 + 318.33a^3}, \tag{4.4}$$

$$[2,4] = \frac{1 + 22.314a + 103.55a^2}{1 + 24.203a + 138.42a^2 + 89.390a^3 - 91.252a^4}, \tag{4.5}$$

$$[1,5] = \frac{1 + 14.023a}{1 + 15.912a + 19.205a^2 - 45.828a^3 + 196.10a^4 - 910.03a^5}. \tag{4.6}$$

We recall some necessary requirements for a zero of a $[p,q]$ Padé approximant to be physically relevant. These include the requirement that this zero should occur on the positive real axis in the complex a plane and the requirement that this zero of the PA should be closer to the origin $a = 0$ than any pole on the real positive a axis since otherwise the pole would dominate the IR to UV flow starting at the origin. If a Padé approximant were to exhibit such a zero, then one would proceed to inquire how close it is to any of the $a_{\mathrm{UV},n\ell}$ in Eq. (3.1). However, we find that none of these Padé approximants (4.2)–(4.6) has a zero on the positive real a axis. Explicitly, the [5,1] PA has two complex-conjugate pairs of zeros at $a = -0.12719 \pm 0.26046i$ and $a = 0.26922 \pm 0.20930i$, together with a real zero at $a = -0.074837$. This real zero is part of a nearly coincident pole-zero pair, with the pole of the [5,1] PA being located at $a = -0.073267$. The appearance of a nearly coincident pole-zero pair close to a point a_0 in a $[p,q]$ Padé approximant is typically an indication that the function that the PA is fitting has neither a pole nor a zero in the local neighborhood of a_0 since, as the locations of the nearly coincident pole-zero pair approach each other, they simply divide out in the ratio (4.1). Each of the Padé approximants that we calculate here has a pole-zero pair. The [4,2] PA has zeros at the complex-conjugate pair $a = 0.42009 \pm 0.96575i$, together with the real values $a = \{-0.16929, -0.064970\}$ and poles at $a = \{-0.14481, -0.064414\}$. The [3,3] PA has zeros at $a = \{-0.78531, -0.13282, -0.061458\}$ and poles at $a = \{-0.42342, -0.12140, -0.061112\}$. The [2,4] PA has zeros at $a = \{-0.15193, -0.063563\}$ and poles at $a = \{-0.69186, -0.13432, -0.063100, 1.8689\}$. Finally, the [1,5] PA has a zero at $a = -0.071313$ and poles at $a = \{-0.22780, -0.070185, 0.44160, 0.035937 \pm 0.39287i\}$. Thus, our analysis with Padé approximants of the seven-loop beta function yields the same conclusion as our analysis of the beta function itself, namely that there is no evidence for a stable, reliably perturbatively calculable UV zero up to this seven-loop level.

V. EFFECTS OF SCHEME TRANSFORMATIONS

Since the terms in the beta function at loop order $n \geq 3$ are scheme-dependent, it is necessary to assess the effect of scheme transformations in an analysis of zeros of a higher-loop beta function. A scheme transformation can be expressed as a mapping between a and a transformed coupling a',

$$a = a' f(a'), \tag{5.1}$$

where $f(a')$ is the scheme transformation function. Since this transformation has no effect in the free theory, one has $f(0) = 1$. We consider $f(a')$ functions that are analytic about $a = a' = 0$ and hence can be expanded in the form

$$f(a') = 1 + \sum_{s=1}^{s_{\max}} k_s (a')^s, \tag{5.2}$$

where the k_s are constants and $s_{\max}$ may be finite or infinite. The beta function in the transformed scheme, $\beta_{a'} = da'/d\ln\mu$, has the expansion

$$\beta_{a'} = a' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell. \tag{5.3}$$

In [24], formulas were derived for the b'_ℓ in terms of b_ℓ and the k_s. In addition to $b'_1 = b_1$ and $b'_2 = b_2$, these are

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1, \tag{5.4}$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1, \tag{5.5}$$

and so forth for higher ℓ. These results are applicable to the study of both an IR zero in the beta function of an asymptotically free theory and a possible UV zero in the beta function of an IR-free theory. They were extensively applied to assess scheme dependence in higher-loop studies of an IR fixed point in asymptotically free non-Abelian gauge theories [24–28].

For the present $\lambda\phi^4$ theory, a study of scheme dependence was carried out in [18]. It was shown that even when one shifts to a scheme different from the usual $\overline{\text{MS}}$ scheme, the beta function still does not satisfy a requisite condition for a physical UV zero, namely that the value of this zero (in a given scheme) should not change strongly when it is calculated to successive loop orders. This result from [18] also holds in the same way in the present seven-loop context.

VI. CONCLUSIONS

In this paper we have investigated whether the real scalar field theory with a $\lambda\phi^4$ interaction exhibits evidence of an ultraviolet zero in the beta function. Using the seven-loop coefficient b_7 from [15], our present study extends our previous six-loop study in [19,20] to the seven-loop level. Our work includes a study of the seven-loop beta function itself, together with an analysis of Padé approximants. We conclude that, for the range of couplings where the perturbative calculation of this beta function may be reliable, it does not exhibit robust evidence for an ultraviolet zero.

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