Generalized Unruh effect: A potential resolution to the black hole information paradox

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We generalize the vacuum-Unruh effect to arbitrary excited states in the Fock space and find that the Unruh mode at the horizon induces coherent excitation on the canonical background ensemble measured by an accelerated observer. When there is only one type of Unruh mode in the system, for example, the ones outgoing from a black hole horizon, the mapping from an arbitrary density matrix on the maximal foliation to a vector space spanned by the pseudothermal density matrix on the partitioned spacetime wedge is one-to-one. Hence, we propose that the information of the particles that is inside a collapsing shell; thus, inside the asymptotic black hole horizon is at least partially retrievable by measuring the deviation of the Hawking radiation from the blackbody radiation spectrum. This work shows that the long-standing black hole information confusion might come from overlooking the possibility that the information could be preserved much better than we have expected in a seemingly nonunitary process when the partitions of the system are strongly entangled.

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I. INTRODUCTION

In Ref. [1], W. G. Unruh first demonstrated a mechanism now called the Unruh effect, which states that a uniformly accelerating observer would detect a thermal bath from expressing the stationary vacuum state in terms of a different set of generator/annihilation basis defined along the timelike killing vector in their relatively accelerating coordinate system. The Unruh effect is a direct result of the nonunique canonical quantization of a field living in a Riemannian spacetime [2]. Following the original derivation on a vacuum state, the Unruh effect has been mainly quoted and discussed in the originally proposed scenario of a vacuum state, in the past decades [3–5]. Most famously, the Unruh effect has helped understanding the Hawking radiation [6,7] and is sometimes quoted as an alternative way for interpreting the black hole radiation/evaporation phenomenon. In recent years, the field has seen a growing number of works utilizing the Unruh effect to understand the entanglement generation and degradation in curved spacetime [8–14].

Schematically, the Unruh effect prescribes that under a Bougoliubov basis transformation, the Minkowski vacuum is transferred into a thermal ensemble on left for a right Rindler wedge; likewise, the Kruskal vacuum is transferred into a thermal ensemble living outside or inside the horizon of a Schwartzschild black hole metric. However, when we scrutinize what lies in the core of the derivation of the Unruh effect, we can easily see how it can be applied to much more general scenarios. First of all, the mixture of positive and negative frequency modes is not only happening in Minkowski-Rindler observer pairs and Kruskal-Schwarzschild observer pairs. Although a unified theory has not been developed, in recent decades, we have seen many examples of positive and negative frequency modes mixing in noninertial frames, usually investigated by a tool called Bogoliubov transformation [15]. A common feature those scenarios share is the local partition of the spacetime manifold by a particle horizon, which could be induced by acceleration, gravitation, or inflation [16–18]. The second direction of generalizing the Unruh effect is that Bogoliubov basis transformation is essentially a basis transformation that can be applied to arbitrary states in the Hilbert space, not just the vacuum state. In fact, given that in quantum field theory in curved spacetime framework the vacuum is defined based on the local timelike Killing vector, it is not that special from the point of view of quantum field theory in the curved spacetime. Vacuum is just one among infinitely many other states in the Fock states. The calculation in this paper is mainly dedicated to the generalization of the Unruh effect onto nonvacuum states. We start from Minkowski metric, then further into asymptotic Schwarzschild black hole scenarios depicted by collapsing shell metric. By the end of this paper, as we go into a discussion of the wider application of the generalized Unruh effect from the perspective of view of the first point made in this paragraph, we will see that a new definition of

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vacuum modulo any canonical ensemble might be in call for further development of quantum field theory in curved spacetime, and possibly for an inclusive description of quantum gravity.

Unruh effect calculations are usually done in these three steps:

(i) Firstly, we secure two complete sets of orthonormal modes for the solution of classical field theory on a four-dimensional Lorentzian manifold. They each correspond to a maximal Cauchy surface foliation of the complete Lorentzian manifold. Each one of the orthonormal basis modes sets gives a complete expression of the solutions to the classical field equation of motion. Their positive and negative frequency modes would be mixed, in the mutual transformation due to different time foliations that only coincide on one slice.

(ii) Secondly, we canonically quantize the field and obtain the creation and annihilation operators corresponding to the two sets of modes, which are the equation of motion solutions secured in the previous step. For a given state in the Fock space of one type of the quantized equation of motion eigenmodes, we carry out Bougoliubov transformation to map the state onto another set of the basis of the quantum field.

(iii) Lastly, due to the existence of a horizon, for an observer living on one of the partial foliations of the spacetime manifold, we contract out the states living on the other partition of the spacetime manifold.

Notice that the last step by its nature breaks the unitarity: we partition an entangled quantum system into two parts separated by a horizon then trace out one of them. However, information is not necessarily lost, at least might not be at an as severe extent as thought before, in this nonunitary process, thanks to the strong entanglement across the horizon. This point has been made previously in the series of works by Lochan and Padmanabhan [19,20]. We formulate this idea in a more explicitly quantum mechanical way by adopting the density matrix representation of countably infinite dimensional Hilbert space of the scalar field eigenmodes. We also would like to point out a more radical implication of this phenomenon on the approach to quantum gravity/unification of general relativity and quantum field theory.

The above summary might seem too abstract as a starting point. In the main body of this paper, we actually begin with this rather concrete question: given a pure state in the Fock space of Minkowski metric, what an observer accelerating along a parabolic worldline on a Rindler wedge would see. The answer is seemingly simple and has been treated straightforwardly so far, that they will see particles accelerating in the opposite direction, sitting in a thermal bath. Such approximation is adopted in, for example, [21] in relative scenarios. This straightforward picture works well in most cases, but this paper may reveal more of the story through a deeper contemplation and more detailed calculation.

One of the highly intriguing interpretations of the generalized Unruh effect is its application to the "Black Hole Information Paradox" [22–24]. By scrutinizing the evolution of plane wave modes inside the shell of a collapsing shell metric, we found a nontrivial mapping between arbitrary density matrix generated by the in shell plane wave Fock states and the pseudothermal density matrix generated by the positive frequency modes living at the asymptotically flat region outside the Schwarzschild black hole horizon. We thus draw the following implication from our calculations: we may restore the information of the amplitudes and frequencies of the infalling particles collapsed into a black hole event horizon, that exists only for a finite time due to Hawking radiation, by measuring the deviation of the Hawking radiation spectrum from the perfect blackbody radiation spectrum. Here, we continue to refer to the generally featured black hole emission from the horizon as Hawking radiation. The concept of "stimulated Hawking radiation" has been studied for a long time [25–29]. Just like the original Unruh effect can be regarded as an alternative interpretation of the Hawking radiation without localized wave packet assumption, the stimulated Hawking radiation could be seen as a similar precursor of the generalized Unruh effect in this work.

This paper is organized in the following way. In Sec. II, we go through the well-established prerequisite for this work, formatting the definitions and conventions. They include the specification of spacetime metric, equation of motion of the field, and canonical quantizations. In Sec. III, the calculation of a single Unruh particle state contracting $\mathrm{R}-(R+)$ states out to an $\mathrm{R}+(R-)$ density matrix is presented. In Sec. IV, we generalize the results from the previous section to an arbitrary eigenstate with two types of Unruh modes. Up to this point, we work in Minkowski $\leftrightarrow$ Rindler scenario. Starting from Sec. V, we explore the Kruskal $\leftrightarrow$ Schwarzschild scenario. We investigate the evolution of an in-going mode living on a collapsing shell metric, its behavior at the horizon, and its density matrix measured by a future observer outside. In Sec. VI, we calculate the energy spectrum corresponding to the pseudothermal density matrices we obtained from Secs. III and IV. In Sec. VII, we discuss the assumptions, caveats, and further implications of this work. The conclusion is in Sec. VIII.

II. SPECIFICATION OF THE SYSTEM

We start from Minkowski spacetime with $(+---)$ signature,

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{1}$$

After coordinate transformation,

$$t = 2\sqrt{\rho}\sinh\frac{1}{2}\tau, \qquad z = 2\sqrt{\rho}\cosh\frac{1}{2}\tau \quad (R+) \tag{2}$$

$$t = -2\sqrt{\rho}\sinh\frac{1}{2}\tau, \qquad z = -2\sqrt{\rho}\cosh\frac{1}{2}\tau \quad (R-) \tag{3}$$

$$t = 2\sqrt{\rho}\cosh\frac{1}{2}\tau, \qquad z = 2\sqrt{\rho}\sinh\frac{1}{2}\tau \quad (F) \tag{4}$$

$$t = -2\sqrt{\rho}\cosh\frac{1}{2}\tau, \qquad z = -2\sqrt{\rho}\sinh\frac{1}{2}\tau \quad (P). \tag{5}$$

We get the Rindler metric,

$$ds^2 = \rho d\tau^2 - \frac{d\rho^2}{\rho} - dx^2 - dy^2 \quad (R\pm) \tag{6}$$

$$ds^2 = -\rho d\tau^2 + \frac{d\rho^2}{\rho} - dx^2 - dy^2 \quad (F, P). \tag{7}$$

The four wedges above cover the full Minkowski spacetime.

The constant proper acceleration worldlines in Minkowski spacetime are expressed by the hyperbola,

$$z^2 - t^2 = 4\rho = \frac{1}{\alpha^2}, \tag{8}$$

where α is the constant proper acceleration. Hence, for any modes expressed in terms of τ, we can get the corresponding mode for $d\tau' = \sqrt{\rho}d\tau = \frac{1}{2\alpha}d\tau$ by multiplying the frequency with 2α and deform the ρ dependent part of the solution accordingly.

Next, we set up a free scalar field obeying the equation of motion,

$$(\nabla^2 - \mu^2)\phi = 0, \tag{9}$$

where ∇ is the covariant derivative. The Hamiltonian of this scalar field only has the dynamical and the mass terms, with no interactions of any sort. Hence, thermalization does not happen in this system except through gravitation. This massive real scalar field could serve as a minimal toy model for the investigation of the collapsing shell and asymptotically black hole metric later in Sec. V.

Now we quantize the field Φ defined on the full manifold that can be charted by the Minkowski metric or the Rindler metric. First, we solve the equation of motion (9) for the classical field ϕ, and find two sets of positive frequency basis for the solutions [1],

$$\phi^M_{\omega,\vec{k}}(x) = \frac{e^{-i\omega t}}{[(2\pi)^3 2\omega]^{1/2}} e^{i\vec{k}\cdot\vec{x}} \tag{10}$$

$$\phi^R_{\tilde{\omega},\vec{q}}(\tilde{x}) = \frac{e^{-i\tilde{\omega}\tau}}{[(2\pi)^3 2\tilde{\omega}]^{1/2}} g(\rho) e^{i(q_x x + q_y y)}, \tag{11}$$

where $g(\rho)$ satisfies

$$\left[\rho\frac{d}{d\rho}\rho\frac{d}{d\rho} + \tilde{\omega}^2 - (\mu^2 + q_x^2 + q_y^2)\rho\right] g(\rho) = 0 \tag{12}$$

and $\omega = \sqrt{\mu^2 + \vec{k}^2}$.

Notice that, by its physical definition ρ is positive, and we can duplicate $\phi^R_{\tilde{\omega},\vec{q}}(\tilde{x})$ to cover the modes on R^+ and R^-. On future and past wedges, things are more complicated. Because $\frac{d}{d\rho}$ is the timelike direction there, and it is not a killing vector, we do not have solution modes that can be expressed as constant frequency waves in terms of the time (ρ). However, we can analytically continue the solutions $\phi^{R+}_{\tilde{\omega},\vec{q}}(\tilde{x})$ to a future wedge, and $\phi^{R-}_{\tilde{\omega},\vec{q}}(\tilde{x})$ to past wedges. This way of continuation can label the equation of motion solutions on F, P with $\tilde{\omega}, \vec{q}$, as a complete set. Because the combination of $\phi^{R+}_{\tilde{\omega},\vec{q}}(\tilde{x})$ and $\phi^{R-}_{\tilde{\omega},\vec{q}}(\tilde{x})$ on the spacelike surface $t = 0$ can fully represent $\phi^M_{\omega,\vec{k}}(x)$, and as long as a field is fully determined by the initial condition on certain spacelike slice, the decomposition of that field on this specific slice can be applied to the full foliated spacetime even if the explicit analytical form of the solution remains unknown, as the time evolution of the field is completely unitary.

In the following text, $\phi^{R+}_{\tilde{\omega},\vec{q}}(\tilde{x})$ is by definition nonzero on $R+$, F, and $\phi^{R-}_{\tilde{\omega},\vec{q}}(\tilde{x})$ is by definition nonzero on $R-$, P (even though we have saved exploring the analytical expression of them on P, F). Combining the two, we can get a full piece of $\phi^M(t_0, \vec{x})$ on any t_0 slice. Thus we define the creation and annihilation operators through the integral of each solution mode with the quantum field Φ, both living on the spacetime manifold charted by certain coordinate systems,

$$a_{\omega,\vec{k}} = \left\langle \phi^M_{\omega,\vec{k}}(x), \Phi(x) \right\rangle \tag{13}$$

$$a^\dagger_{\omega,\vec{k}} = \left\langle \phi^{*M}_{\omega,\vec{k}}(x), \Phi(x) \right\rangle \tag{14}$$

$$b_{\tilde{\omega},\vec{q}} = \left\langle \phi^{R+}_{\tilde{\omega},\vec{q}}(\tilde{x}), \Phi(\tilde{x}) \right\rangle \tag{15}$$

$$b^\dagger_{\tilde{\omega},\vec{q}} = \left\langle \phi^{*R+}_{\tilde{\omega},\vec{q}}(\tilde{x}), \Phi(\tilde{x}) \right\rangle \tag{16}$$

$$d_{\tilde{\omega},\vec{q}} = \left\langle \phi^{R-}_{\tilde{\omega},\vec{q}}(\tilde{x}), \Phi(\tilde{x}) \right\rangle \tag{17}$$

$$d^\dagger_{\tilde{\omega},\vec{q}} = \left\langle \phi^{*R-}_{\tilde{\omega},\vec{q}}(\tilde{x}), \Phi(\tilde{x}) \right\rangle. \tag{18}$$

Remark.—Quantum field $\Phi(x)$, or $\Phi(\tilde{x})$, is a scalar field defined on the full spacetime manifold $\mathcal{M}$ that is an identical collection of Lorentzian manifold points for

Minkowski or Rindler coordinate system. When there is a boundary of $\mathcal{M}$, $\partial\mathcal{M}$ is transferred accordingly under two coordinate systems. We assume in our setup the boundary terms are vanishing.

Minkowski and Rindler foliation structures have a shared Cauchy surface $t = \tau = 0$. The canonical quantization should be done on this specific shared time slice because only this hypersurface sets the initial condition along every time-flow (forward or backward) direction. So the bracket in the above creator/annihilator definitions should be an integral either on the shared Cauchy surface $t = \tau = 0$, or by default over the full chart of 4D spacetime manifold if the shared time slice was not identified. Doing otherwise will obtain an incomplete set of creator/annihilator to represent the quantum field states living on the entire manifold, under one of the reference frames in the pair that we are concern about for the problem.

The quantum field can thus be decomposed in two ways,

$$\Phi(x) = \int dk^3 \left[a_{\omega,\vec{k}} \phi^M_{\omega,\vec{k}}(x) + a^\dagger_{\omega,\vec{k}} \phi^{*M}_{\omega,\vec{k}}(x) \right] \tag{19}$$

$$= \int dq^3 \left[b_{\tilde{\omega},\vec{q}} \phi^{R+}_{\tilde{\omega},\vec{q}}(\tilde{x}) + d^\dagger_{\tilde{\omega},\vec{q}} \phi^{R-}_{\tilde{\omega},\vec{q}}(\tilde{x}) \right. \tag{20}$$

$$\left. + d_{\tilde{\omega},\vec{q}} \phi^{*R-}_{\tilde{\omega},\vec{q}}(\tilde{x}) + b^\dagger_{\tilde{\omega},\vec{q}} \phi^{*R+}_{\tilde{\omega},\vec{q}}(\tilde{x}) \right] \tag{21}$$

with canonical quantization [30],

$$\left[a_{\omega,\vec{k}}, a^\dagger_{\omega',\vec{k}'} \right] = \delta^3(\vec{k} - \vec{k}') \tag{22}$$

$$\left[b_{\omega,\vec{q}}, b^\dagger_{\omega',\vec{q}'} \right] = \delta^3(\vec{q} - \vec{q}') \tag{23}$$

$$\left[d_{\omega,\vec{q}}, d^\dagger_{\omega',\vec{q}'} \right] = \delta^3(\vec{q} - \vec{q}'). \tag{24}$$

All other a or b, d commutators are zero. $a, a^\dagger$ does not necessarily commute with the Rindler creating and annihilating operators.[1]

In the following sections, we will first calculate how a single frequency mixed Unruh mode is observed by an R+ or R− observer, then proceed to the multiparticle case.

III. UNRUH EFFECT GENERALIZED TO SINGLE POSITIVE MINKOWSKI FREQUENCY STATE

We now carry out a quantum mechanical way of calculation from a single particle state generated by Unruh mode to a density matrix tracing out one of the Rindler wedges. This process has been done in [1] for a Minkowski vacuum, and we do it for an Unruh wave packet excited state $|\tilde{\omega}(\vec{q})\rangle_U$.

A. An Unruh wave packet excited on Minkowski vacuum

Skipping the standard Bogoliubov transformation of the classical field solutions in Minkowski/Rindler metrics, we start from Eq. (2.19a) in [1]. It gives the relationship between creation and annihilation operators as the result of the Bogoliubov transformation of the classical field modes,

$$\left(e^{\pi\tilde{\omega}} b_{\tilde{\omega},\vec{q}} - e^{-\pi\tilde{\omega}} d^\dagger_{\tilde{\omega},\vec{q}} \right) |0\rangle_M = 0 \tag{25}$$

$$\left(e^{-\pi\tilde{\omega}} b^\dagger_{\tilde{\omega},\vec{q}} - e^{\pi\tilde{\omega}} d_{\tilde{\omega},\vec{q}} \right) |0\rangle_M = 0. \tag{26}$$

Let us define the following annihilators for Minkowski vacuum:

$$u_{\tilde{\omega},\vec{q}} = \frac{1}{\sqrt{2\sinh 2\pi\tilde{\omega}}} \left(e^{\pi\tilde{\omega}} b_{\tilde{\omega},\vec{q}} - e^{-\pi\tilde{\omega}} d^\dagger_{\tilde{\omega},\vec{q}} \right) \tag{27}$$

$$v_{\tilde{\omega},\vec{q}} = \frac{1}{\sqrt{2\sinh 2\pi\tilde{\omega}}} \left(e^{-\pi\tilde{\omega}} b^\dagger_{\tilde{\omega},\vec{q}} - e^{\pi\tilde{\omega}} d_{\tilde{\omega},\vec{q}} \right) \tag{28}$$

that satisfies the normal commutating relationships,

$$\left[u_{\omega,\vec{q}}, u^\dagger_{\omega',\vec{q}'} \right] = \delta^3(\vec{q} - \vec{q}'), \tag{29}$$

$$\left[v_{\omega,\vec{q}}, v^\dagger_{\omega',\vec{q}'} \right] = \delta^3(\vec{q} - \vec{q}'), \tag{30}$$

with all other commutators in the above operators set equal to zero. We call these operators Unruh creators and annihilators type I and II, and call the mode they generate Unruh mode/excitation.

B. A single Unruh excitation viewed by a Rindler observer, on R+ and R−

Given the transformation between creators and annihilators, in principle, we have obtained a transformation between the Fock states under two basis. To investigate how a single Unruh excitation on a Minkowski vacuum is viewed by a Rindler observer on R+ or R−, we will act $u^\dagger_{\tilde{\omega},\vec{q}}$ on both sides of the equation connecting Minkowski and

[1] Here we referred to Srednicki equations (3.19) and (3.29). The normalization of $\Phi(x)$ and Minknowski modes here should give the same result as (3.19) in Srednicki, just moving the ω normalizing factor into $a_{\omega,\vec{k}}$ and $\phi^M_{\omega,\vec{k}}(x)$. As for the normalization of the Rindler modes, we swipe them under the definition of $g(\rho)$.

Fulling-Rindler vacuum, then trace out the Hilbert space for the states confined on $R-$ or $R+$. We are expected to obtain a highly stochastic density matrix due to the entanglement between two wedges, and it is distinguishable from the canonical thermal density matrix that a Minkowski vacuum would resolve into. By measuring the observables of this density matrix, for example, the energy spectrum, an observer on either $R+$ or $R-$ could resume the Unruh mode generated on the Minkowski vacuum that we started from. This result applies for both type I and type II Unruh modes, and they each are detectable on either $R+$ or $R-$.

The traditional Unruh effect for Minkowski vacuum is formally expressed by Eq. (2.19b) in [1],

$$|0\rangle_M = Z\left[\prod_q \exp\left(e^{-2\pi\tilde{\omega}} b^\dagger_{\tilde{\omega},\vec{q}} d^\dagger_{\tilde{\omega},\vec{q}}\right)\right]|0\rangle_F, \tag{31}$$

where $|0\rangle_F$ is the Fulling-Rindler vacuum vanished by b, d annihilators, and the normalization constant $Z^{-2} = \sum_{\mathcal{N}} e^{-4\pi E_{\rm tot}}$ is the canonical ensemble partition function. $\mathcal{N}$ are all the possible configurations of the eigenmodes, which will be explained further in Eq. (41), and $E_{\rm tot} = \sum_i n_i\tilde{\omega}_i$ is the total energy corresponding to the eigenmode.

Acting a type-I and type-II combined Unruh creator $A_I u^\dagger_{\tilde{\omega},\vec{q}} + A_{II} v^\dagger_{\tilde{\omega},\vec{q}}$, where $A_I^2 + A_{II}^2 = 1$, on both side of Eq. (31),

$$|\tilde{\omega}(\vec{q})\rangle_U = ZA_I\left(\frac{e^{\pi\tilde{\omega}}}{\sqrt{2\sinh 2\pi\tilde{\omega}}} b^\dagger_{\tilde{\omega},\vec{q}}\hat{S} - \frac{e^{-\pi\tilde{\omega}}}{\sqrt{2\sinh 2\pi\tilde{\omega}}} d_{\tilde{\omega},\vec{q}}\hat{S}\right) \tag{32}$$

$$+ZA_{II}\left(\frac{e^{\pi\tilde{\omega}}}{\sqrt{2\sinh 2\pi\tilde{\omega}}} d^\dagger_{\tilde{\omega},\vec{q}}\hat{S} - \frac{e^{-\pi\tilde{\omega}}}{\sqrt{2\sinh 2\pi\tilde{\omega}}} b_{\tilde{\omega},\vec{q}}\hat{S}\right), \tag{33}$$

where

$$\hat{S} = \left[\prod_q \exp\left(e^{-2\pi\tilde{\omega}} b^\dagger_{\tilde{\omega},\vec{q}} d^\dagger_{\tilde{\omega},\vec{q}}\right)\right]. \tag{34}$$

Using

$$[b^\dagger_{\tilde{\omega},\vec{q}}, \hat{S}] = [d^\dagger_{\tilde{\omega},\vec{q}}, \hat{S}] = 0, \tag{35}$$

$$[d_{\tilde{\omega},\vec{q}}, \hat{S}] = e^{-2\pi\tilde{\omega}}\hat{S} b^\dagger_{\tilde{\omega},\vec{q}}, \tag{36}$$

$$[b_{\tilde{\omega},\vec{q}}, \hat{S}] = e^{-2\pi\tilde{\omega}}\hat{S} d^\dagger_{\tilde{\omega},\vec{q}}, \tag{37}$$

we get

$$|\tilde{\omega}(\vec{q})\rangle_U = Ze^{-\pi\tilde{\omega}}\sqrt{2\sinh 2\pi\tilde{\omega}} \tag{38}$$

$$\times\left(A_I b^\dagger_{\tilde{\omega},\vec{q}} + A_{II} d^\dagger_{\tilde{\omega},\vec{q}}\right)\hat{S}|0\rangle_F. \tag{39}$$

Next, we trace out the states generated by the $R-$ creators, $d^\dagger$, acting on $|0\rangle_F$, to get the density matrix on $R+$, $\rho_U^{R+}(q)$. The result consists of four terms,

$$\rho_U^{R+}(q) = |A_I|^2\hat{\rho}^{b^\dagger b} + |A_{II}|^2\hat{\rho}^{d^\dagger d} + A_I A_{II}^*\hat{\rho}^{b^\dagger d} + A_I^* A_{II}\hat{\rho}^{d^\dagger b}, \tag{40}$$

where q has energy component $\tilde{\omega}$, and $\hat{\rho}^{d^\dagger b} = \hat{\rho}^{b^\dagger d^\dagger}$. All four terms have q dependence through b/d creator/annihilators.

Before we demonstrate the detailed expressions of $\hat{\rho}$ in the above equation, let us introduce some convenient notations. We denote the configuration of momentum q of an eigenstate in the following way:

$$|\mathcal{N}\rangle = \{q_1, n_1, \ldots q_i, n_i\}, \qquad n_i = 0, 1, 2, \ldots, \tag{41}$$

where q is the four momentum, including $\tilde{\omega}$ as a component. With respect to the configuration $|\mathcal{N}\rangle$, a state missing one particle of momentum q is denoted by

$$|\mathcal{N}, n_q - 1\rangle, \tag{42}$$

and similarly, for other number modifications of n_q. The states in this notation are normalized, in the sense that

$$|\mathcal{N}, n_q - 1\rangle = \frac{a_{\omega,q}}{\sqrt{n_q}}|\mathcal{N}\rangle, \tag{43}$$

$$|\mathcal{N}, n_q + 1\rangle = \frac{a^\dagger_{\omega,q}}{\sqrt{n_q + 1}}|\mathcal{N}\rangle. \tag{44}$$

This notation can be applied to any set of eigenstates and corresponding creators and annihilators. The vacuum that $|\mathcal{N}\rangle$ corresponds to will be labeled in the subscripts later on.

With $|\mathcal{N}\rangle$ notation, we can denote

$$\hat{S}|0\rangle_F = \left[\prod_q \exp\left(e^{-2\pi\tilde{\omega}} b^\dagger_{\tilde{\omega},\vec{q}} d^\dagger_{\tilde{\omega},\vec{q}}\right)\right]|0\rangle_F \tag{45}$$

$$= \sum_{\mathcal{N}} e^{-2\pi E_{\rm tot}}|\mathcal{N}\rangle_{R+}|\mathcal{N}\rangle_{R-}, \tag{46}$$

where $E_{\rm tot} = \sum_i \tilde{\omega}_i n_i$, and $\mathcal{N}$ runs over all the possible configurations of the eigenstates of Φ generated by the quantization of Eq. (11) as indicated by $R+/R-$ subscripts.

We start from the calculation of $\hat{\rho}^{b^\dagger b}$,

$$\hat{\rho}^{b^\dagger b}(q) = Z^2 e^{-2\pi\tilde{\omega}} 2\sinh 2\pi\tilde{\omega} \tag{47}$$

$$\times\sum_{\mathcal{N}}\sum_{\mathcal{N}'} e^{-2\pi E'_{\rm tot}}{}_{R-}\langle\mathcal{N}|\mathcal{N}'\rangle_{R-}b_q^\dagger|\mathcal{N}'\rangle_{R+} \tag{48}$$

$$\otimes\sum_{\mathcal{N}''} e^{-2\pi E''_{\rm tot}}{}_{R+}\langle\mathcal{N}''|b_{q\,R-}\langle\mathcal{N}''|\mathcal{N}\rangle_{R-} \tag{49}$$

$$= Z^2 e^{-2\pi\tilde{\omega}}2\sinh 2\pi\tilde{\omega} \tag{50}$$

$$\times\sum_{\mathcal{N}} e^{-4\pi E_{\rm tot}}(n_q+1)|\mathcal{N},n_q+1\rangle_{R+R+}\langle\mathcal{N},n_q+1|. \tag{51}$$

Next, the result for $\hat{\rho}^{d^\dagger d}$ is

$$\hat{\rho}^{d^\dagger d}(q) = Z^2 e^{-2\pi\tilde{\omega}}2\sinh 2\pi\tilde{\omega} \tag{52}$$

$$\times\sum_{\mathcal{N}}\sum_{\mathcal{N}'} e^{-2\pi E'_{\rm tot}}{}_{R-}\langle\mathcal{N}|d_q^\dagger|\mathcal{N}'\rangle_{R-}|\mathcal{N}'\rangle_{R+} \tag{53}$$

$$\otimes\sum_{\mathcal{N}''} e^{-2\pi E''_{\rm tot}}{}_{R+}\langle\mathcal{N}''|_{R-}\langle\mathcal{N}''|d_q|\mathcal{N}\rangle_{R-} \tag{54}$$

$$= Z^2 e^{-2\pi\tilde{\omega}}2\sinh 2\pi\tilde{\omega} \tag{55}$$

$$\times\sum_{\mathcal{N}} e^{-4\pi E_{\rm tot}}(n_q+1)|\mathcal{N}\rangle_{R+\,R+}\langle\mathcal{N}|. \tag{56}$$

Lastly, the result for $\hat{\rho}^{b^\dagger d}$ is

$$\hat{\rho}^{b^\dagger d}(q) = Z^2 e^{-2\pi\tilde{\omega}}2\sinh 2\pi\tilde{\omega} \tag{57}$$

$$\times\sum_{\mathcal{N}}\sum_{\mathcal{N}'} e^{-2\pi E'_{\rm tot}}{}_{R-}\langle\mathcal{N}|\mathcal{N}'\rangle_{R-}b_q^\dagger|\mathcal{N}'\rangle_{R+} \tag{58}$$

$$\otimes\sum_{\mathcal{N}''} e^{-2\pi E''_{\rm tot}}{}_{R+}\langle\mathcal{N}''|_{R-}\langle\mathcal{N}''|d_q|\mathcal{N}\rangle_{R-} \tag{59}$$

$$= Z^2 e^{-2\pi\tilde{\omega}}2\sinh 2\pi\tilde{\omega} \tag{60}$$

$$\times\sum_{\mathcal{N}} e^{-2\pi\tilde{\omega}}e^{-4\pi E_{\rm tot}}\sqrt{(n_q+2)(n_q+1)} \tag{61}$$

$$\times|\mathcal{N},n_q+2\rangle_{R+\,R+}\langle\mathcal{N}|. \tag{62}$$

Substituting the above terms back to Eq. (40), we get

$$\rho_U^{R+}(q) = Z^2 e^{-2\pi\tilde{\omega}}2\sinh 2\pi\tilde{\omega}\bigg[|A_I|^2\sum_{\mathcal{N}} e^{-4\pi E_{\rm tot}}(n_q+1)|\mathcal{N},n_q+1\rangle_{R+\,R+}\langle\mathcal{N},n_q+1| \tag{63}$$

$$+|A_{II}|^2\sum_{\mathcal{N}} e^{-4\pi E_{\rm tot}}(n_q+1)|\mathcal{N}\rangle_{R+\,R+}\langle\mathcal{N}| \tag{64}$$

$$+\left(A_I A_{II}^*\sum_{\mathcal{N}} e^{-2\pi\tilde{\omega}}e^{-4\pi E_{\rm tot}}\sqrt{(n_q+2)(n_q+1)}|\mathcal{N},n_q+2\rangle_{R+\,R+}\langle\mathcal{N}| + \text{H.c.}\right)\bigg] \tag{65}$$

For an R− observer, $\rho_U^{R-}(q)$ switches A_I and A_{II}, and changes the subscript $R+$ to $R-$ for the states.

The first thing we would notice is that $\hat{\rho}^{b^\dagger b}$ and $\hat{\rho}^{d^\dagger d}$ only involves diagonal terms, while $\hat{\rho}^{b^\dagger d}$ only has nondiagonal terms. Remembering that Z^{-2} is partition function, we find that $\mathrm{Tr}(\hat{\rho}^{b^\dagger b}) = 1$ and $\hat{\rho}^{b^\dagger b} \geq 0$. Namely, both diagonal matrices $\hat{\rho}^{b^\dagger b}$ and $\hat{\rho}^{d^\dagger d}$ could be normalized density matrices, assuming such notation can be applied to infinite dimensional Hilbert space. As a result, the trace of $\rho_U^{R+}(q)$ is also safely equal to 1.

When the Unruh mode is purely type I, $\rho_U^{R+}(q)$ only have $\hat{\rho}^{b^\dagger b}$ term, and $\rho_U^{R-}(q)$ only have $\hat{\rho}^{d^\dagger d}$ term. Comparing $\hat{\rho}^{b^\dagger b}$ and $\hat{\rho}^{d^\dagger d}$, we notice that although the normalized states have different labels, the matrix components are equal to each other term by term. Hence, the Von-Neumann entropy $S = -\mathrm{tr}\rho\log\rho$ of $\rho_U^{R+}(q)$ and $\rho_U^{R-}(q)$ for type I Unruh mode must be the same. It is expected because $|\tilde{\omega}(\vec{q})\rangle_U$ is a pure state. The bipartite of a pure state must have equal entropy [31].

Our notation in this section has already implicitly assumed the quantum mechanical linear algebraic representation of the infinite-dimensional Hilbert space in QFT. Tracing back to the origin where this assumption is introduced, the notation $|\mathcal{N}\rangle$ in Eq. (41) is actually not as innocent as it seems—at this step, we implicitly assumed countability of the momentum defined with the eigenmodes of the field equation of motion solutions. It means that throughout this paper, the maximal spacetime manifold we consider should be bounded, or equivalently, we should have a lower cutoff on the acceleration α (minimum deviation from the inertial frame) so that the energy levels of the scalar field are discrete. Without losing generality, we assume the minimal interval between energy levels to be a constant h_0. The set of Fock states spanning the Hilbert space of this QFT system thus could be countably infinite. One of the strategies to count the states is to continuously increase $E_{\rm tot}$ from minimum possible value μ and to attribute a natural number to each eigenstate in ascending $E_{\rm tot}$ sequence.

IV. BEYOND SINGLE UNRUH EXCITATION

Now we generalize the calculation in the above section into the full Fock space generated by the Unruh creators $\{u^\dagger, v^\dagger\}$. This Unruh Fock space should be a basis transformation with respect to the one generated by the Minkowski Klein-Gordon wave creators $a^\dagger$ since they share the same vacuum. Actually, the classical field solutions have shown that the Unruh modes are plane wave decomposition of the field in $\log U$ space, instead of in the natural light cone coordinate, $U = t - x$ space. Without proof here, we quote the results from [1,3] that the transformation between plane wave modes and the Unruh modes is positive frequency to positive frequency. Thus, our conclusion in this section for the Unruh Fock space is in general applicable to the original Minkowski plane wave Fock space, with a basis transformation from the plane waves to the plane waves in $\log x$ space.

A general Fock state generated by the Unruh modes above Minkowski vacuum can be expressed by

$$|\mathcal{N}_I \mathcal{N}_{II}\rangle_U = \frac{1}{\sqrt{\alpha_1!\ldots\alpha_N!\beta_1!\ldots\beta_N!}} \times u_{q_1}^{\dagger,\alpha_1}\ldots u_{q_N}^{\dagger,\alpha_N} v_{q_1}^{\dagger,\beta_1}\ldots v_{q_N}^{\dagger,\beta_N}\hat{S}|0\rangle_F, \tag{66}$$

where α_i, β_i are the number counts for certain momentum modes, and they can be zero.

Utilizing the commutators between $b, d, b^\dagger, d^\dagger$, and $\hat{S}$ in Eqs. (23), (24), and (35)–(37), we notice that

$$|\mathcal{N}_I \mathcal{N}_{II}\rangle_U = \prod_{q_i}\sum_m B(\tilde{\omega}_i|m+\alpha_i-\beta_i, m) \times b_{q_i}^{\dagger,m+\alpha_i-\beta_i} d_{q_i}^{\dagger,m}\hat{S}|0\rangle_F. \tag{67}$$

Here, assuming $\alpha_i > \beta_i$, $m = 0, \ldots\beta_i$, and $B(\tilde{\omega}_i|m + \alpha_i - \beta_i, m)$ are positive coefficients that can be analytically calculated from the commutators between $\hat{S}, b, d, b^\dagger, d^\dagger$. Thus, for an arbitrary eigenstate in the Minkowski vacuum Unruh mode Fock space, the building block for the density matrix on ρ^{R+} or ρ^{R-} is given by

$$\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q) = B(\tilde{\omega}|l,m)B^*(\tilde{\omega}|n,s)\sum_{\mathcal{N}}\sum_{\mathcal{N}'} e^{-2\pi E'_{\rm tot}}{}_{R-}\langle\mathcal{N}|d^{\dagger,m}|\mathcal{N}'\rangle_{R-} b_q^{\dagger,l}|\mathcal{N}'\rangle_{R+}\sum_{\mathcal{N}''} e^{-2\pi E''_{\rm tot}}{}_{R+}\langle\mathcal{N}''|b^n_q{}_{R-}\langle\mathcal{N}''|d^s|\mathcal{N}\rangle_{R-} \tag{68}$$

$$= B(\tilde{\omega}|l,m)B^*(\tilde{\omega}|n,s)\sum_{\mathcal{N},n_q^{\rm min}} e^{-4\pi E_{\rm tot}} e^{2\pi(m+s)\tilde{\omega}}\frac{n_q!}{(n_q-m)!(n_q-s)!}\sqrt{(n_q-m+l)!(n_q-s+n)!} \tag{69}$$

$$\times|\mathcal{N}, n_q - m + l\rangle_{R+\ R+}\langle\mathcal{N}, n_q - s + n|, \tag{70}$$

where the configuration runs over $\mathcal{N}$, with the minimum number of q momentum modes $n_q^{\rm min} = \max[m,s]$. l, m, n, s runs over non-negative integers. The normalization factor $B(\tilde{\omega}_i|l,m)$ ensures the traces of diagonal matrices $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^l d^m}(q)$ equal one. An arbitrary $\rho_{R+}(\Omega)$ tracing out R− states can be expressed by a linear combination of $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}$, with the modulation of diagonal components $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^l d^m}(q)$ being unity.

The expression above is only for a single Unruh mode frequency; because the creators and annihilators of different momentum commute with each other, the generalization to multiple momenta is straightforward. We just need to manipulate n_{q_i} with respect to the mode number eigenvalue configuration $\mathcal{N}$, for each momentum q_i. Notice that the choice of $\mathcal{N}$ is meaningful and nontrivial under the relabeling of n_q, because the Boltzmann factor $e^{-4\pi E_{\rm tot}}$ nontrivially weighs a specific block $|\mathcal{N}, n_q - m + l\rangle_{R+\ R+}\langle\mathcal{N}, n_q - s + n|$ in the density matrix.

Equation (70) is all one needs to calculate the contracted density matrix on a partitioned spacetime wedge from an original density matrix living on the fully accessible spacetime. Thus, we have a nontrivial mapping from the Minkowski-Unruh Fock states[2] to the pseudothermal density matrix on R+ and R−. We give the name "pseudothermal" density matrix to those infinite term density matrices like $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q)$, that slightly deviates from

[2]They stand for the Fock states of the Unruh wave modes number eigenstates excited on Minkowski vacuum. Similarly, we would have Minkowski-Plane wave Fock states, etc.

the canonical ensemble. Also, notice that the pseudothermal density matrix cannot be constructed as the linear combination of a canonical ensemble and the excitation of the required number of certain momentum. The physical meaning of this fact is that the pseudothermal density matrix originated from the entanglement of excited states across the event horizon is distinguishable from the excitation on one of the Rindler wedges backlighted by the thermal bath generated by the vacuum at the horizon.

In Sec. III B, we calculated the special cases for $lmns = 1010$, 0101, and 1001. They each agree with the general form Eq. (70).

It is obvious that the matrices $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q)$ are linearly independent in terms of the matrix summation algebra.What is more, it is impossible for different Fock states, i.e., the eigenstates corresponding to Klein-Gordon equation solutions, $|\mathcal{N}_I\mathcal{N}_{II}\rangle_U$ to be mapped into the same $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q)$ linear combination because the $lmns$ maximum coefficient term for certain configuration $|\mathcal{N}_I\mathcal{N}_{II}\rangle_U$ is given by $l = n = \alpha, m = s = \beta$. With the always positive coefficient $B(\tilde{\omega}|l, m)$, the $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q)$ linear combination obtained from a distinct $|\mathcal{N}_I\mathcal{N}_{II}\rangle_U$ state is guaranteed to be different from the pseudothermal density matrices linear combination corresponding to other Unruh mode eigenstates.

However, it needs further discussion to clarify whether each unique density matrix of the form $\sum_{i,j} A_{ij}|\mathcal{N}_I^i\mathcal{N}_{II}^i\rangle_U\ {}_U\langle\mathcal{N}_I^j\mathcal{N}_{II}^j|$ would be contracted into a unique $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q)$ linear combination. The answer is likely yes, but we leave it as an open question without carrying out a concrete mathematical proof in the scope of this paper. The complication comes from the contribution to the same $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q)$ term from different $|\mathcal{N}_I^i\mathcal{N}_{II}^i\rangle_U\ {}_U\langle\mathcal{N}_I^j\mathcal{N}_{II}^j|$ terms, when two types of Unruh modes exist simultaneously.

The toy model of our massive scalar experiences no interaction other than gravity. Hence, in principle, any slight deviation from the canonical ensemble in the density matrix would be preserved during its propagation on a Rindler wedge, without further thermalization from collisions. An observer sitting away from the horizon on a Rindler wedge can then measure the observables of this density matrix, then infer the original state living across the horizon on the Minkowski vacuum. When the acceleration is small, we asymptotically go back to the inertial frame case, where the pseudothermal density matrix is dominated by a peak corresponding to the Minkowski vacuum Unruh mode.

We are skipping the formal setups of the detectors and the formal definitions of the observable operators here.[3] We believe the measurement can be done in a fairly standard way in the asymptotically flat region far from the Rindler horizons. The measurement in the asymptotically flat region is sufficient for learning the property of the ensemble on any equal-time slice, because in our noninteracting massive scalar field toy model, the density matrix evolves trivially in energy-momentum eigenstates.

Remark.—The Boltzmann factor $e^{-4\pi E_{\rm tot}}$ should be treated exactly, carefully, without any approximation in the rest of this paper, where $E_{\rm tot}$ is the total energy for the configuration $\mathcal{N}$, not the physical total energy of the energy eigenstate. Relabeling n_q will only change the reference zero energy point for every component in the density matrix $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q)$. In some previous work, for example, the Horowitz-Maldacena conjecture [32] paper, people regarded this seemingly boring factor as an insignificant algebraic label without tracking the details in it. This might exactly be the approximation erasing all the information that a contracted Rindler density matrix carries. $E_{\rm tot}$ depends on everything in the number eigenvalue configuration $\mathcal{N}$. The coherent mismatch between the Boltzmann factor of $\mathcal{N}$ and the physical state $|\mathcal{N}, n_q - 1\rangle\langle\mathcal{N}, n_q - 1|$, for example, is telling the stories about the Minkowski pure state before contracting a Rindler wedge out.

For an arbitrary Unruh mode eigenstate simultaneously generated by both types of the creators, $u^\dagger$ and $v^\dagger$, those three observations hold:

(i) The relationship between R− and R+ observer density matrix is

$$\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q) = \hat{\rho}_{R-}^{b^{\dagger,m}d^{\dagger,l}b^s d^n}(q). \tag{71}$$

(ii) All the $\hat{\rho}_{R+}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q)$ components that appear in the representation of a Rindler wedge contraction for Unruh mode eigenstate obey the identity $l - m = n - s \equiv \delta$. Thus, $\rho_{R\pm}$ is always diagonal for Unruh mode eigenstates.

(iii) Regardless of the Hilbert space on which they are built, pseudothermal matrices have component-level symmetry,

$$\hat{\rho}_{ii}^{b^{\dagger,l}d^{\dagger,m}b^n d^s}(q) = \hat{\rho}_{i+\delta,i+\delta}^{b^{\dagger,m}d^{\dagger,l}b^s d^n}(q), \tag{72}$$

when the identity $l - m = n - s$ holds. Without losing generality i starts from 0, and the components with indices smaller than δ of the matrix on the right-hand side vanish. An illustration of this symmetry is $\hat{\rho}^{b^\dagger b}(q)$ and $\hat{\rho}^{d^\dagger d}(q)$ in the single particle case in Sec. III.

The above features of the pseudothermal density matrices grant us the same conclusion as the single particle case in Sec. III, for an arbitrary Unruh mode eigenstate: The Von-Neumann entropy $S = -\mathrm{tr}(\rho \log \rho)$ on R+ or R− wedge partition of an Unruh mode eigenstate has equal value.

However, the true physical implication of pseudothermal density matrix features on the amount of entanglement and

[3]Something way more complicated than Unruh-DeWitt detector is required to measure the frequency space details of the density matrix.

TABLE I. Creators and their visibility under different bases of the quantum field modes. Their definitions are specified in Eqs. (13)–(18). Minkowski plane waves and Minkowski Unruh modes are generated on the same vacuum with different classical waveform decompositions, while Rindler observers' creator and annihilation operators are mixed in the Bogoliubov transformations from the former two. "Not visible" in the table means that a generator or annihilator on one subspace of the manifold is not affecting the states excited on the other subspace of the manifold.

Creators	Minkowski plane waves	Minkowski Unruh modes	R+	R−
$u^\dagger$, $v^\dagger$	$a^\dagger$	$\cdots$	$(b^\dagger, d)$ or $(d^\dagger, b)$	
$a^\dagger$	$\cdots$	$u^\dagger$, $v^\dagger$	$(b^\dagger, d)$ or $(d^\dagger, b)$	
$b^\dagger$	$(a^\dagger, a)$	$(u^\dagger, v)$	$\cdots$	Not visible
$d^\dagger$	$(a^\dagger, a)$	$(u, v^\dagger)$	Not visible	$\cdots$

information might need more careful calculation with well-defined metrics like negativity, relative entropy defined for QFT, etc. [33,34], applied to the countably infinite dimensional density matrix representation here. We leave it to future works due to a lack of expertise on these topics.

Given the preemptive statements about the information and entanglement metrics above, it is still useful to remind ourselves that metrics are only handy algebraic compression of the information carried by the specific quantum state or ensemble. The symmetry between partially traced ρ_{R+} and ρ_{R-} of Unruh eigenstates, at least, strongly implies the equal measurability of a mode excited on maximal foliation on both partitions separated by an event horizon.

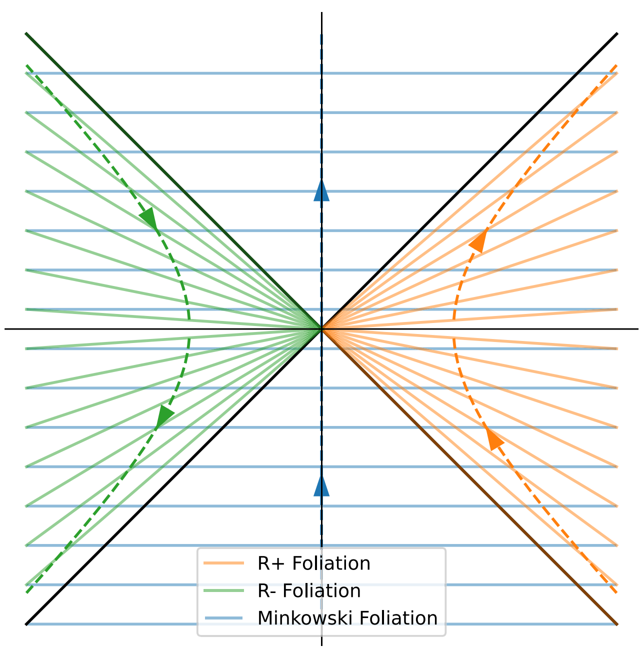

FIG. 1. The foliation structure of Minkowski metric and Rindler metric. Blue horizontal lines are the equal time slices under Minkowski foliation, and the inclined straight lines going through the origin from green to orange are the equal time slices under Rindler foliation. The flow of time is forward for $R+$ and backward for $R-$ observer, with respect to the time flow for a Minkowski observer.

Only the mode generated on a Rindler vacuum, namely the particle defined with respect to the proper time of the accelerated observer, is absolutely not detectable on the complementary Rindler wedge. The representation of a mode generated by different types of creators is summarized in Table I. The foliation structures of the Minkowski $\leftrightarrow$ R+, R− spacetime are illustrated in Fig. 1.

V. A POTENTIAL SOLUTION TO THE BLACK HOLE INFORMATION PARADOX WITHOUT QUANTUM GRAVITY

In this section, we will explore the applicability of the results in Secs. III and IV on the Kruskal-Schwarzschild relative noninertial frames pair. To analogously find the mapping like (Minkowski-Unruh Fock space density matrix $\leftrightarrow$ R$\pm$ pseudothermal density matrix) in the previous section for the (infalling, black-hole-forming star density matrix $\leftrightarrow$ Schwarzschild outside/inside pseudothermal density matrix) duality, we need two conditions to be satisfied:

(i) *Condition 1*: A positive frequency mode under Kruskal foliation consists of a mixture of positive and negative frequency modes under Schwarzschild inner and outer foliation.

(ii) *Condition 2*: The infalling positive frequency plane waves inside the collapsing shells are imprinted as Kruskal positive frequency boundary conditions on the past event horizon, which is an effective extrapolation of the collapsing shell metric toward infinite past.

Why do we need (but possibly, not only need) these two conditions for the purpose of extracting information from a black hole using the generalized Unruh effect? The importance of the first condition is plain to see, as it is the trigger of any generalized Unruh effect discussed in Sec. I. The mixture of positive and negative frequency modes results in the mixture of creation and annihilation operators thus leading to the informative entanglement between two partitions of the manifold, R+/R− or outside/inside the event horizon.

The second condition echoes an existing type of view that the information of a collapsed black hole is imprinted on its event horizon, or some deformation/stretching of that surface [35,36]. They are largely stimulated by the holography, AdS/CFT line of thoughts [37,38]. This work shares many common grounds with them, with one obvious difference. We take a look at the postulates of Susskind *et al.*'s work [36] as an example. Their postulate 2 and 3, the semiclassical field equations and discrete energy level of the field living on black hole metric, are adopted in this work as well. However, we deliberately break the unitarity from their postulate 1 in the process of formation and evaporation of the black hole. The mapping we investigate is between the density matrix generated by maximal foliation eigenstates and the pseudothermal density matrix subset in the full density matrix set generated by the partitioned spacetime manifold eigenstates. A pure state in the former is mapped into a highly stochastic pseudothermal ensemble in the latter, but we will see that the information imprinted on the past horizon should still be retrievable to a certain extent due to the coherent excitation on the stochastic background.

A chronicle way to understand the relationship between these two conditions is that condition 2 resolves an ingoing field mode living on an evolving astrophysical black hole metric into an approximated boundary condition, at the past event horizon, which is an extrapolation of collapsing shell metric to the infinite past. Condition 1 secures the solution of outgoing ensemble under the boundary condition granted by condition 2.

Remark.—One potential strategy to connect our result to those previous works postulating unitarity is to define the thermal ensemble modulo the temperature as the vacuum state, and any pseudothermal ensemble that deviates from the thermal ensemble as excited states. Such a new formalism should redefine the unitarity on the basis of a thermal and pseudothermal ensemble instead of the old-fashioned pure states, which could be regarded as a special case of $T = 0$. More discussion towards the end of Sec. V B.

Since the two conditions are fairly standard in previous literature on the black hole information problem topic, it is an option for experienced readers to treat them as standard postulates and jump ahead to Sec. V B. The following Secs. V A and V A 1 are dedicated to arguing the validity of the two conditions by painting more details on the original arguments made by [1], which demonstrated the feasibility of approximating the collapsing shell information by the positive Kruskal frequency boundary conditions living on the past horizon of the Schwarzschild black hole. The improvements here are that our calculations will be in 4D spacetime and that we will try to amend some minor errors in [1] along the road.

In any case, once the two conditions above hold, the calculations and arguments for the generalized Unruh effect in Minkowski/Rindler case after Eq. (25) naturally follow. Because the explicit expression of the metric or the Klein-Gordon equation solutions is not used anywhere in the derivation about the quantum mechanical aspect of the problem, after canonical quantization.

Now let us start from the Schwarzschild and Kruskal metrics, using the conventions in [1] Eqs. (2.20) and (2.25).

A. Mixture of positive and negative frequency modes across the asymptotic event horizon

The Schwarzschild metric,

$$ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{73}$$

Under a coordinate transformation, it can be written as Kruskal metric [39],

$$ds^2 = 2M\frac{e^{-r/2M}}{r}dUdV - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{74}$$

$$U = -4Me^{-\frac{1}{4M}(t-r-2M\ln(\frac{r}{2M}-1))}, \quad \text{for } r \geq 2M \tag{75}$$

$$U = 4Me^{-\frac{1}{4M}(t-r-2M\ln(1-\frac{r}{2M}))}, \quad \text{for } r < 2M \tag{76}$$

$$V = 4Me^{\frac{1}{4M}(t+r+2M\ln(\frac{r}{2M}-1))}, \quad \text{for } r \geq 2M \tag{77}$$

$$V = 4Me^{\frac{1}{4M}(t+r+2M\ln(1-\frac{r}{2M}))}, \quad \text{for } r < 2M. \tag{78}$$

There are multiple ways to write down the Kruskal coordinates, and our expression here is consistent with the Kruskal diagram with Schwarzschild chart as shown in Fig. 2.

The relationship,

$$U = T - X \tag{79}$$

$$V = T + X \tag{80}$$

is always satisfied, for region I,

$$T = 4M\sqrt{\frac{r}{2M} - 1}e^{r/4M}\sinh t/4M \tag{81}$$

$$X = 4M\sqrt{\frac{r}{2M} - 1}e^{r/4M}\cosh t/4M, \tag{82}$$

and region II,

$$T = 4M\sqrt{1 - \frac{r}{2M}}e^{r/4M}\cosh t/4M \tag{83}$$

$$X = 4M\sqrt{1 - \frac{r}{2M}}e^{r/4M}\sinh t/4M. \tag{84}$$

Here, we do not extend the original Schwarzschild spacetime; instead, we glue together the past outgoing

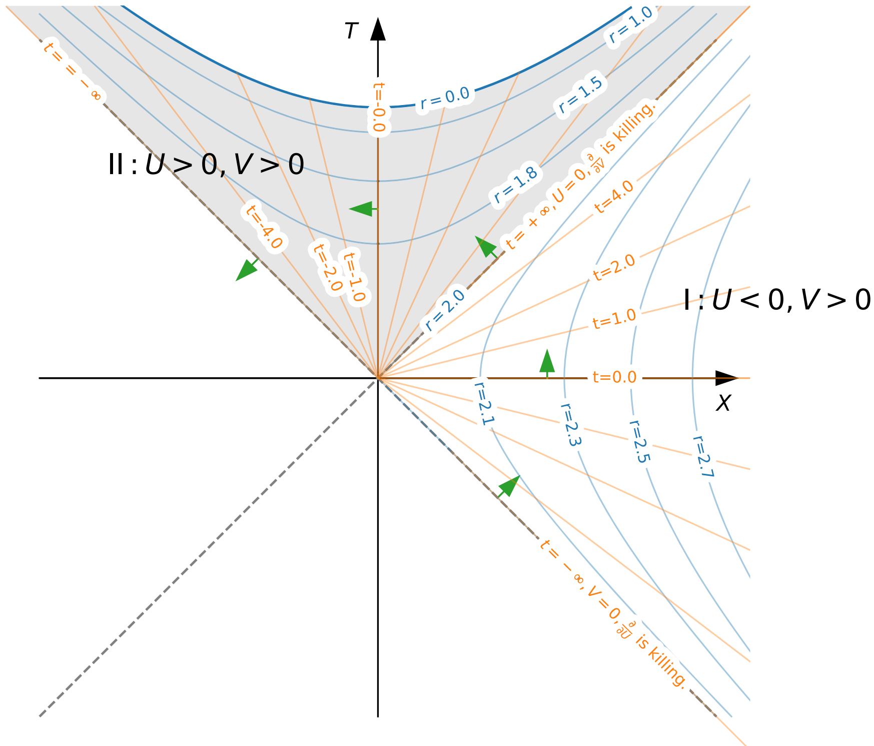

FIG. 2. Kruskal metric with Schwarzschild coordinates charted. Green arrows are local timelike killing directions.

horizon and the future outgoing horizon so that the two regions are connected through the surface $t \to -\infty$, $r = 2M$. Without losing generality, it can be done by enforcing boundary condition $\phi(U \to +\infty) = \phi(U \to -\infty)$. For an astrophysical black hole, regions I and II are sufficient to describe the physics we care about. Such an eternally existing Kruskal manifold exactly expressed by Eq. (74) is only a far field ideal approximation of the collapsing shell metric that we will consider in the next subsection, so we do not need to worry too much about the singularity at $r = 0$ and the artificially exerted periodicity at infinitely far past and future outgoing event horizons.

Remark.—Considering the bounded distance in the light cone coordinate where asymptotic Schwarzschild approximation is valid, as an evaporating black hole only exists in finite time, the outgoing light cone coordinate boundary conditions at far past and future outgoing horizon should be at finite extremals $\phi(U = U_{\max}) = \phi(U = U_{\min})$. The finiteness of the astrophysical black hole naturally discretizes the energy levels ω of the system. The argument here does not remove the assumption of discrete energy levels; instead, it is just a self-consistency check.

Since confining to the regions I and II of the Kruskal metric grants us only one outgoing/past horizon and ingoing/future horizon, we will use each pair of the words interchangeably in the following text.

In Kruskal metric, the covariant vector $\frac{\partial}{\partial U}$ is Killing on the past horizon $V = 0$, and $\frac{\partial}{\partial V}$ is Killing on the future horizon $U = 0$. The proof is briefly shown as follows. Labeling the coordinates (U, V, θ, ψ), we can check the Killing vector conditions for covariant vectors $\xi^U_\mu = (1, 0, 0, 0)$ and $\xi^V_\mu = (0, 1, 0, 0)$,

$$\nabla_\mu \xi^U_\nu + \nabla_\nu \xi^U_\mu = -2\Gamma^0_{\mu\nu} = \delta_{\mu 0}\delta_{\nu 0} 2 g^{01} g_{10,r} \frac{\partial r}{\partial U} = 0. \tag{85}$$

Substituting Eqs. (75)–(78) in, we get [40]

$$r = 2M\left(1 + W_0\left(\frac{\mathrm{UV}}{16M^2 e}\right)\right), \tag{86}$$

where $W_0(z)$ is the positive branch of Lambert W function [40]. On both future and past horizons, $UV = 0$, and the derivative of Lambert W function $W'_0(0) = 1$. On past horizon, $V = 0$, hence $\frac{\partial r}{\partial U} = 0$ and $\frac{\partial}{\partial U}$ is killing, vice versa.

Hence, on the past horizon $\mathcal{H}^-$, $V = 0$, the solutions to the Klein-Gordon equation for a scalar field in Kruskal coordinates are featured by the modes $e^{-i\omega U}$. The positive frequency modes are those analytic and bound in the lower half complex plane of $\mathrm{Im}(U) < 0$. Similarly, the decomposition of Kruskal modes at the future horizon is represented by $e^{-i\omega V}$ due to the killing of the timelike vector field $\frac{\partial}{\partial V}$ in that region.

After discussing the eigenmodes on the maximal foliation in the Kruskal metric, we now look into the field equation solutions in the Schwarzschild metric. [1] showed that the semiclassical field solutions in Schwarzschild metric near the horizon are representable by a set of eigenmodes $e^{\pm i\omega t}e^{\pm i\omega r_*}$, where $r_* = r + 2M\ln(\frac{r}{2M} - 1)$ outside the horizon and $r_* = r + 2M\ln(1 - \frac{r}{2M})$ inside the horizon. We can denote the inner and outer near-horizon solutions by

$$\phi_\omega^{\rm in}(r^*, t) = \begin{cases} e^{2\pi M\omega} e^{\pm i\omega t} e^{\pm i\omega r_*}, & \text{for } r \leq 2M \\ 0, & \text{for } r > 2M, \end{cases} \quad (87)$$

$$\phi_\omega^{\rm out}(t, r^*) = \begin{cases} e^{-2\pi M\omega} e^{\pm i\omega t} e^{\pm i\omega r_*}, & \text{for } r \geq 2M \\ 0, & \text{for } r < 2M, \end{cases} \quad (88)$$

with the same mathematical expression modulo a normalization factor, but vanish on the complementary side. We also assume a normalization factor difference $e^{\pm 2\pi M\omega}$ between inside and outside modes possibly due to the volume difference between two partitions of the 4D subspacetime.

By observing Eqs. (75)–(78), we notice that

$$\left(\frac{U}{4M}\right)^{i4M\omega} = e^{-i\omega(t - r_*)}, \quad U > 0, \quad (89)$$

$$\left(-\frac{U}{4M}\right)^{i4M\omega} = e^{-i\omega(t - r_*)}, \quad U < 0, \quad (90)$$

on the real axis of U. Inside the horizon, r_* is the timelike direction; hence, the expression above implies the mixture of the positive frequency t modes outside the horizon and negative frequency r_* modes inside the horizon, with the same frequency amplitude.

The left-hand side expression in Eqs. (89) and (90) satisfies the condition of being bound in the lower half complex plane for U. Next, we combine them in a way that secures analyticity along the full U real axis. Notice that the Kruskal-Unruh mode $\phi_\omega(U)$ should be continuous across $U = 0$, and the first derivative is nonzero (those Kruskal-Unruh modes are supposed to be the free-falling fields that actually travel across the horizon, i.e., nonzero flux there). A relatively opposite sign between $U > 0$ and $U < 0$ regimes in the above expressions is then necessary. Otherwise, they are out of phase across $U = 0$.

Combining our knowledge about the positive frequency Kruskal U modes and their relation with Schwarzschild eigenmodes, we can write down the decomposition,

$$\phi_\omega(U) \propto (e^{2\pi M\omega}\phi_\omega^{\rm out}(t, r^*) - e^{-2\pi M\omega}\phi_\omega^{in,*}(r^*, t)) \propto \begin{cases} \left(-\frac{U}{4M}\right)^{i4M\omega}, & U < 0 \\ -\left(\frac{U}{4M}\right)^{i4M\omega}, & U > 0 \end{cases}, \quad (91)$$

when constrained to the real axis of U, as illustrated in Fig. 3. $\phi_\omega^{out/\rm in}(t, r^*)$ are the positive frequency modes corresponding to each of their own timelike coordinates. $\phi_\omega^{\rm out}(t, r^*)$ is outgoing from the horizon, and $\phi_\omega^{\rm in}(r^*, t)$ is also departing from the horizon on the other side.[4]

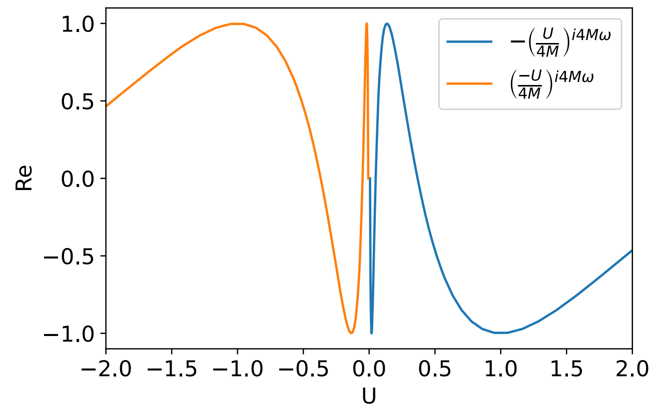

FIG. 3. A Kruskal Unruh mode expressed in Eq. (91) modulo normalization factor. $\omega = \frac{5\pi}{4M\Delta\log(|U|)}$.

By observing the mathematical expression of the Unruh modes in Eq. (91), we notice that the Unruh modes are basically the plane waves in $-\log U$ space, instead of the normal plane waves in U space ($e^{-i\omega U}$). And it is these Unruh mode frequencies in $-\log U$ space, not the plane waves with respect to the Kruskal lightcone coordinate U, that determines the Schwarzschild plane wave frequencies corresponding to the t foliation infinitely far way from the horizon. This point can be clearly seen from the Eq. (90), where the equality is connecting a Kruskal-Unruh mode on the left-hand side and a Schwarzschild plane wave mode outside/inside the horizon on the right-hand side.

Before we depart from this section, we would like to stress again that the neat eternal black hole considered here is only an asymptotic approximation of the collapsing shell metric representing an astrophysical black hole, in its most condensed limit. It is well known that an observer outside a Schwarzschild radius will never witness the exact accretion of an ingoing particle onto the event horizon, in the sense that such an event is not on any of the equal-time Cauchy surfaces of the outside observer. On the other hand, Schwarzschild metric is always a valid local approximation of the metric near an observer, outside a sphere enclosing a bulk of mass. From this perspective of view, all massive objects are gravitationally thermalized and evaporating. The formation of an event horizon exactly at r_s corresponding to the enclosed mass is never accomplished before the evaporation of the asymptotic black hole, from the perspective of view of any observer outside the sphere. In other words, quite intuitively, singularity does not

[4]Falling inward might be a confusing way to put it, although people are used to saying so. Let us not be fooled by the letter label, and remember that r is timelike inside the horizon. t increase with increasing r^* (along the time flow) is the most accurate description for mode $\phi_\omega^{\rm in}(r^*, t)$.

emerge in the collapse of an astrophysical black hole, no matter how seemingly close it is to an exact event horizon; the existence of a global event horizon is equivalent to the existence of a singularity, or a defect of the spacetime manifold from the first place.

On this note, the question "what if a particle falls into the black hole event horizon" is actually a problematic, even if not completely wrong question to be asked by an observer outside the horizon, because it is incompatible with the locality of physics rules. In general relativity, the physics rules, represented by some equations of motion for the fields, are the same in different reference frames. Such consistency is known to be broken when the measurements from different reference frames are used in a single set of equations.[5] We often confusingly find ourselves in the paradox of infinite redshift or diverging energy-momentum density at the black hole horizon, or information loss after reaching the exact event horizon. It is likely only because we were carrying out illegal maths that simultaneously admit the measurements by different observers. An observer that has appeared outside the horizon far past, at the exact event horizon, and outside the horizon far future does not exist.

A reasonable speculation is that an eternal black hole with an event horizon can only exist as a conceptual approximation, not in the real physical world. It is in the stance of an infinitely extending metal plate in the electrodynamic problems when we concern about physics instead of maths.

Remark.—In two special cases, (Minkowski $\leftrightarrow$ Rindler) and (Schwarzschild $\leftrightarrow$ Kruskal) noninertial frame pairs, we found the opposite time-flow phenomena on a shared Cauchy surface in those diffeomorphism connected metric pairs of the same $(+,-,-,-)$ pseudo-Riemannian manifold. Inertial frame pairs, connected by special diffeomorphisms generated by Poincaré groups, do not have opposite time flows. To be more specific, in the (Minkowski $\leftrightarrow$ Rindler) case, for a flat manifold with no singularity, as shown in Fig. 1, on $t=\tau=0$ Cauchy surface $(\frac{\partial}{\partial t},\frac{\partial}{\partial \tau})|_{R+}>0$ while $(\frac{\partial}{\partial t},\frac{\partial}{\partial \tau})|_{R-}<0$. Similarly, in the (Schwarzschild $\leftrightarrow$ Kruskal) case for a flat manifold with a singularity, on the Cauchy surface, the past horizon $\mathcal{H}^-$, that is shared by both foliation structures, $(\frac{\partial}{\partial U},\frac{\partial}{\partial t})|_{\rm out}>0$, while $(\frac{\partial}{\partial U},\frac{\partial}{\partial r_*})|_{\rm in}<0$. Notice that the time-flow direction is not specified by being labeled by some letter related to "t," but is determined as the killing direction with non-negative signature. We suspect that this kind of spacetime foliation structure is what truly underlies the Unruh effects; yet, the differential manifold knowledge required to unveil the fundamentals of such phenomena in general noninertial frame pairs is beyond our reach in this paper.

The outcome of the above opposite foliation structure after quantization manifests as follows. In noninertial frame problems, the Hilbert spaces of advanced (one-way) time-evolving states do not coincide entirely in two ways of foliation. When we write down the exact equality between the states in two relatively accelerating frames, formally it is done in a more inclusive Hilbert space that incorporates both advanced and retarded field solutions, or states, after quantization. Those retarded states vanish soon enough into the investigation under the lower-bounded algebra we defined, $a|0\rangle=0$.

[5] For example, Einstein equations are bound to be broken if we use the Ricci curvature expressed Newtonian gauge and energy-momentum tensor expressed in synchronous gauge.

1. Ingoing modes inside the shell of a collapsing shell metric

In this section, we will move on to the discussion of astrophysical black holes that form and evaporate in a finite lifetime. The goal is to verify the condition 2, which helps argue that the infalling positive frequency modes can be represented by the Kruskal outgoing horizon positive frequency modes. The Penrose diagram of such a finite-lifetime black hole is illustrated in Fig. 4.

The scalar field modes that start evolving from the past Cauchy surface i^- for $\mu=0$ or $\mathcal{I}^-$ for $\mu\neq 0$ can be classified into two types: one never travels near the black hole horizon during the finite lifetime of an astrophysical black hole, and the other has substantial amplitude near the horizon $[\phi(r_s<r<r_s+\delta)>\epsilon$, for t during $\tau_{\rm BH}]$. We call them transmitted and reflected modes, respectively. The transmitted modes are those matters floating around the black hole and never get close enough to them, thus are not plagued by the black hole information problem; they unitarily evolve from the past before black hole formation to the future after black hole evaporation. The reflected modes travel through the horizon, and they are the main investigation objects in any black hole information problems.

The four-dimensional collapsing shell metric, generalized from the 2D collapse from [1], can be written as

$$ds^2=\begin{cases} d\tau^2-dr^2, & r<\hat{R}(\tau)\\ \left(1-\frac{2M}{r}\right)dt^2-\frac{dr^2}{1-2M/r}, & r>\hat{R}(\tau). \end{cases} \tag{92}$$

The shell radius is given by

$$\hat{R}(\tau)=\begin{cases} R_0, & \tau<0\\ R_0-\nu\tau, & \tau>0. \end{cases} \tag{93}$$

The collapsing shell approximately describes such a physical system, where a shell of matter shrinks its size at velocity ν. The collapsing of a bulky distribution of the matter, or the additional matter accreting onto an existing

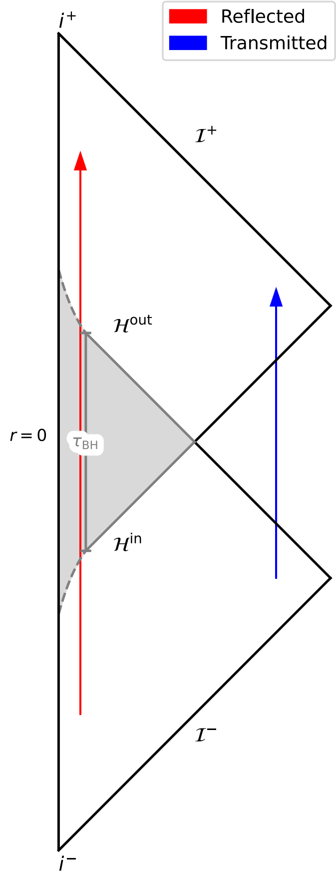

FIG. 4. A finite-lifetime black hole. The blue arrow denotes the transmitted scalar field modes that have negligible amplitude near the horizon; They are the matters floating around and never get absorbed into the black hole. The red arrow denotes the modes reflected by the event horizon.

black hole could be decomposed by layers of collapsing shells with the in shell metric modified from the completely flat Minkowski one. We do not consider the backreaction of the field on the metric. Hence, strictly speaking, the retrieval of the information calculation in the following subsection based on the current subsection is only for the perturbative part of the whole chunk of collapsed mass. A complete, nonlinear level analysis is not available until the puzzle of curved spacetime and quantum field theory unification is fully resolved, i.e., the inclusive formulation of gravitational and particle interactions.

It is assumed that ν is fast enough $1-\nu < \frac{4M}{R_0+2M} \ll 1$ for simplicity in the later calculations. Using light cone coordinates,

$$\bar{U} = \tau - r + R_0, \qquad \bar{V} = \tau + r - R_0 \tag{94}$$

$$\bar{u} = t - r_* + R_{0*}, \qquad \bar{v} = t + r_* - R_{0*}, \tag{95}$$

we can rewrite the collapsing shell metric as

$$ds^2 = \begin{cases} d\bar{U}d\bar{V} - r^2 d\Omega^2, & \text{inside the shell} \\ (1-2M/r)d\bar{u}d\bar{v} - r^2 d\Omega^2 & \text{outside the shell} \end{cases}. \tag{96}$$

$\bar{u}, \bar{v}, \bar{U}, \bar{V}$ are related by the condition that the ds on the shell match each other expressed by in shell or out shell coordinates [1],

$$\frac{du}{dU} = \begin{cases} \left(1-\frac{2M}{R_0}\right)^{-\frac{1}{2}}, & \bar{u}, \bar{U} < 0 \\ \frac{\hat{R}(\frac{\bar{U}}{1+\nu})}{(1+\nu)\left(\hat{R}\left(\frac{\bar{U}}{1+\nu}\right)-2M\right)} & \left[\nu + \left(1-\frac{2M(1-\nu^2)}{\hat{R}\left(\frac{\bar{U}}{1+\nu}\right)}\right)^{1/2}\right], \\ & \bar{u}, \bar{U} > 0 \end{cases} \tag{97}$$

$$\frac{dv}{dV} = \begin{cases} \left(1-\frac{2M}{R_0}\right)^{-\frac{1}{2}}, & \bar{v}, \bar{V} < 0 \\ \frac{\hat{R}\left(\frac{\bar{V}}{1+\nu}\right)}{(1+\nu)\left(\hat{R}\left(\frac{\bar{V}}{1+\nu}\right)-2M\right)} & \left[\left[1-\frac{2M(1-\nu^2)}{\hat{R}\left(\frac{\bar{V}}{1+\nu}\right)}\right]^{1/2} - \nu\right]. \\ & \bar{v}, \bar{V} > 0 \end{cases} \tag{98}$$

We put a bar on these light cone coordinates to distinguish them from the Kruskal metric ones in the previous subsection.

The Klein-Gordon equation for the massive scalar in this collapsing shell coordinate becomes

$$r^2 \frac{\partial}{\partial \bar{U}} \frac{\partial}{\partial \bar{V}} \phi + r\left(\frac{\partial r}{\partial \bar{U}} \frac{\partial \phi}{\partial \bar{V}} + \frac{\partial r}{\partial \bar{V}} \frac{\partial \phi}{\partial \bar{U}}\right) - \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\phi\right) - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial^2 \psi}\phi - \mu^2 r^2 \phi = 0 \quad \text{in shell} \tag{99}$$

$$r^2\left(1-\frac{2M}{r}\right)^{-1} \frac{\partial}{\partial \bar{u}} \frac{\partial}{\partial \bar{v}} \phi + r\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{\partial r}{\partial \bar{u}} \frac{\partial \phi}{\partial \bar{v}} + \frac{\partial r}{\partial \bar{v}} \frac{\partial \phi}{\partial \bar{u}}\right) - \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\phi\right) - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial^2 \psi}\phi - \mu^2 r^2 \phi = 0 \quad \text{out shell.} \tag{100}$$

The angular part of the scalar field can be solved by spherical harmonics. The remaining light cone coordinate dependent part of the scalar field $\phi(U, V)$ resolves into

$$r^2 \frac{\partial}{\partial \bar{U}} \frac{\partial}{\partial \bar{V}} \phi + r \frac{\partial \phi}{\partial r} + \ell(\ell+1)\phi - \mu^2 r^2 \phi = 0 \quad \text{in shell} \tag{101}$$

$$r^2 \left(1 - \frac{2M}{r}\right)^{-1} \frac{\partial}{\partial \bar{u}} \frac{\partial}{\partial \bar{v}} \phi + r \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{\partial r}{\partial \bar{u}} \frac{\partial \phi}{\partial \bar{v}} + \frac{\partial r}{\partial \bar{v}} \frac{\partial \phi}{\partial \bar{u}}\right) + \ell(\ell+1)\phi - \mu^2 r^2 \phi = 0 \quad \text{out shell.} \tag{102}$$

Even for a nonspinning Schwarzschild black hole, the scalar field solution in general does not have l exactly equal to zero. The non-exact-zero of the angular momentum for most of the scalar field modes forms a centrifugal barrier that leads to the $\phi(r=0)=0$ solution for the in shell equation of motion (101).

We are interested in the ϕ modes behavior near the horizon when the shell approaches the horizon. This condition specifies the final stage of the collapse $\tau \to \frac{R_0 - 2M}{\nu}$, $t \to +\infty$. The approaching to infinity of the out-shell time t when $\hat{R}(\tau)$ indefinitely approaches $2M$ demonstrates that for an observer outside the shell ($\frac{dt}{d\tau}$ diverges $\sim \frac{1}{\hat{R}-2M}$), the black hole event horizon only forms asymptotically. Outside the shell, near the horizon, as $(1 - 2M/r)^{-1}$ diverges and $\frac{\partial r}{\partial \bar{u}} \sim \frac{\partial r}{\partial \bar{v}} \sim 0$, the first term of the equation of motion dominates, and the equation of motion simplifies to

$$\frac{\partial}{\partial \bar{u}} \frac{\partial}{\partial \bar{v}} \phi = 0, \quad \text{Out shell}, t \to +\infty, r \to 2M. \tag{103}$$

This equation is solved by

$$\phi(\bar{u}, \bar{v}) = f(\bar{v}) + g(\bar{u}), \qquad \text{Out shell}, t \to +\infty, r \to 2M. \tag{104}$$

We assume the continuity of the scalar field across the shell, and we know that according to Eqs. (97) and (98), $\bar{u} = \hat{U}(\bar{U})$ and $\bar{v} = \hat{V}(\bar{V})$, where ˆ denotes a single variable function. Namely, out/ingoing light cone coordinates do not mix from the ds matching at the shell. Thus, we can adopt the ansatz for in shell equation of motion solution near the horizon with the same form,

$$\phi(\bar{U}, \bar{V}) = F(\bar{V}) + G(\bar{U}), \quad \text{In shell}, \tau \to \frac{\mathrm{R}_0 - 2\mathrm{M}}{\nu}, \mathrm{r} \to 2\mathrm{M}. \tag{105}$$

The continuum of ϕ across the shell near the horizon further implies

$$f(\bar{v}) = F(\hat{V}(\bar{v})), g(\bar{u}) = G(\hat{U}(\bar{u})). \tag{106}$$

The $\bar{u}(\bar{U})$ and $\bar{v}(\bar{V})$ dependent functions are outgoing and ingoing modes, respectively.

The centrifugal barrier effect discussed in the previous paragraphs tells us $\phi(r=0) = F(\bar{V}) + G(\bar{U}) = 0$. Strictly speaking, the separable ansatz is only justified near $r = 2M$; thus, the possible oscillatory motion between $0 < r < 2M$ might modify this condition by a phase shift, but it is unimportant for the following derivations. Inside the shell, along $r = 0$ worldline, we have the relationship $U - V = 2R_0$. Thus, the 1D functions $F(x)$ and $G(x)$ has the following relationship:

$$G(x) = -F(x - 2R_0). \tag{107}$$

Combining Eqs. (106) and (107), we get

$$g(\bar{u}) = -F(\hat{U}(\bar{u}) - 2R_0). \tag{108}$$

Remember that the metric, thus the equation of motion inside the shell, is simply of flat spacetime. Hence, the collapsing modes whose wavefronts evolve with decreasing r as τ increases could be represented by the positive frequency modes $F(\bar{V}) \sim e^{-i\Omega\bar{V}}$ just inside the shell.[6] These modes, when leaving the past horizon of Kruskal spacetime, are reflected into $g(\bar{u})$ in the form,

$$g(\bar{u}) \sim -e^{-i\Omega(\hat{U}(\bar{u}) - 2R_0)}. \tag{109}$$

A physics interpretation is that the infalling mode is bounced back by the centrifugal barrier at $r = 0$ (possibly with a phase shift).

We use $\sim$ symbol to waive the careful treatment of normalization factors, only focusing on the spacetime coordinates dependence of a mode.

By integrating Eq. (97), at $O(1-\nu)$ order, we have [1]

$$\hat{u}(\bar{U}) = \begin{cases} (1 - 2M/R_0)^{-1/2}\bar{U}, & \bar{U} < 0 \\ -4M \ln(1 - \nu\bar{U}/[(1+\nu)(R_0 - 2M)]) & . \\ +\bar{U} + O(1-\nu), & \bar{U} > 0 \end{cases} \tag{110}$$

[6]Whether $\mu = 0$ or not is unimportant here; the positive frequency modes solution for massive or massless particles have the same form.

Here, ν is the collapsing rate, which we assumed to be close to 1, thus a rapid collapse. In the final stage of collapse near horizon, we have $\tau \to \frac{R_0-2M}{\nu}$, $r \to 2M$; hence, $\bar{U} \to (R_0-2M)(\nu+1)/\nu$. In this region, $\bar{U}$ is positive, and the log term dominates over the linear term. Thus, at the past horizon of Kruskal metric, we have

$$\hat{U}(\bar{u}) \approx \frac{1+\nu}{\nu}(R_0-2M)[1-e^{-\bar{u}/(4M)}]. \tag{111}$$

Now we proceed to connect the collapsing shell picture with the black hole picture discussed in the previous subsection. Imagine that we have an observer sitting at $r_o \gg R_0$. The scalar field perturbative part has negligible backreaction on the metric, so this observer does not need to know the exact dynamics inside the shell to figure out the evolution of the scalar field local at r_o. An eternal black hole and a collapsing shell have the same effective metric locally near r_o. Thus, the evolution of the scalar field near the region of a distant observer in a collapsing shell metric is representable by the evolution of the scalar field in an eternal black hole metric. The dynamics of the scalar field happening inside the shell are imprinted as the boundary conditions at the asymptotic black hole event horizon extrapolated to $t \to -\infty$.

Matching the r and t coordinates of the collapsing shell metric and the imaginary eternal Schwarzschild black hole metric, the eternal black hole Kruskal metric light cone coordinate U defined in Eq. (78) is related to the collapsing shell light cone coordinate outside the shell by $U = -e^{-(\bar{u}-R_0-2M\ln(R_0-2M))/4M}$. Such matching should be asymptotically exact out shell in the far future, towards the final stage of the collapse.

Thus, we see an in shell ingoing mode $\phi(\bar{V}) \sim e^{-i\Omega\bar{V}}$ eventually evolves into

$$\phi(U) \sim e^{-i\Omega\xi U}, \tag{112}$$

on the asymptotic event horizon, which is effectively a boundary condition for far observers. ξ is a redshift factor $\xi = \frac{1+\nu}{\nu}\sqrt{R_0-2M}e^{-\frac{R_0}{4M}}$, always positive. Thus, each positive frequency $e^{-i\Omega\bar{V}}$ mode is mapped into a positive frequency $e^{-i\Omega' U}$ mode after a long time of evolution, and they each could be decomposed into the Schwarzschild positive and negative frequency modes mixture outside and inside the asymptotic event horizon.

Remark.—The approach in this note could be regarded as a realization of the Horowitz-Maldacena conjecture [32] in some sense. There, they assumed the infalling state is entangled with the vacuum (perfectly thermal after tracing out the external part) Unruh state inside the horizon. This note shows that this extra entanglement is redundant—the Unruh state itself can be richer than a pure vacuum and sets up the boundary conditions.

B. The one-to-one mapping between infalling physical states and outside-horizon pseudothermal physical states

We have seen in Sec. IV that, when there are two types of Unruh modes generated simultaneously in a spacetime, it was quite difficult to identify the invertible mapping between the density matrix space generated by the Unruh Fock space and the $\hat{\rho}^{lmns}$ vectors. Because when we have both the type I and the type II Unruh modes, even for a basis Fock state, there are multiple $\hat{\rho}^{lmns}$ terms due to the necessity to commute $b^\dagger$ and b coming from $u^\dagger$ and $v^\dagger$. Things are more approachable in the Kruskal-Schwarzschild duality where we only have one type of Unruh mode moving outwards. In this section, we will try to identify the invertible mapping from an arbitrary density matrix in the Kruskal-Unruh eigenmodes Fock space to the $\hat{\rho}^{lmns}$-basis-spanned subspace of the density matrices generated by the Schwarzschild outside-horizon eigenmodes Fock space. We will be focusing on the states for a single momentum q in the rest of this section, as the proof of the invertible mapping in this section can be trivially generalized given the commutativeness of the operators with different momenta.

Canonically quantizing the Eq. (91), we get the expression of the Unruh creator in terms of the outside creator and the inside annihilator,

$$u_q^\dagger = \frac{1}{\sqrt{2\sinh 4\pi M\omega}}(e^{2\pi M\omega}b_q^\dagger - e^{-2\pi M\omega}d_q), \tag{113}$$

where $u^\dagger$ is the Unruh creator which generates a positive frequency mode on the past horizon of Kruskal spacetime. $b^\dagger$, $d^\dagger$ are the Schwarzschild creators outside and inside the horizon. As usual, they each do nothing on the complementary side.

A most general Hermitian density matrix built on the Unruh Fock space with a fixed momentum $\vec{q}_0, \omega_0$ is given by

$$\rho^K = \sum_{\alpha_1,\alpha_2} A_{\alpha_1\alpha_2}|\alpha_1\rangle\langle\alpha_2|, \tag{114}$$

where $A^*_{\alpha_1\alpha_2} = A_{\alpha_2\alpha_1}$, $\mathrm{tr}(A_{\alpha_1\alpha_2}) = 1$ and the matrix is positive semidefinite. α_1, α_2 are the non-negative integers, and $|\alpha\rangle$ is the normalized state with α Unruh particles of momentum q_0,

$$|\alpha\rangle = \frac{1}{\sqrt{\alpha!}}u^{\dagger,\alpha}|0\rangle_K = \frac{1}{\sqrt{\alpha!}}e^{-2\pi M\alpha\omega}(2\sinh 4\pi M\omega)^{\alpha/2}b^{\dagger,\alpha}\hat{S}|0\rangle_S, \tag{115}$$

where we used the commutation relations $[b^\dagger, d] = 0$, $[d, \hat{S}] = e^{-4\pi M\omega}\hat{S}b^\dagger$. The index K means Kruskal, S Schwarzschild, O outside the Schwarzschild horizon, and I inside the Schwarzschild horizon.

The general ρ^K above should be able to describe any possible infalling states/ensembles at the perturbative level. We contract out the states inside the Schwarzschild horizon to find the density matrix outside the horizon,

$$\rho^O(A_{\alpha_1\alpha_2}) = \sum_{\mathcal{N}}{}_I\langle\mathcal{N}|\rho^K|\mathcal{N}\rangle_I \tag{116}$$

$$= \sum_{\alpha_1\alpha_2} A_{\alpha_1\alpha_2}\hat{\rho}^{b^{\dagger,\alpha_1}b^{\alpha_2}}, \tag{117}$$

where the basis matrices $\hat{\rho}^{b^{\dagger,\alpha_1}b^{\alpha_2}}$ are the special cases of Eq. (70) with $l = \alpha_1, m = 0, n = \alpha_2, s = 0$,

$$\hat{\rho}^{b^{\dagger,\alpha_1}b^{\alpha_2}} = Z^2 e^{-2\pi M(\alpha_1+\alpha_2)\omega}(2\sinh 4\pi M\omega)^{\frac{\alpha_1+\alpha_2}{2}} \sum_{\mathcal{N},n_q=0}^{\infty} e^{-8\pi M E_{\rm tot}}\sqrt{C(n_q+\alpha_1,\alpha_1)C(n_q+\alpha_2,\alpha_2)}|\mathcal{N}, n_q+\alpha_1\rangle_O\, {}_O\langle\mathcal{N}, n_q+\alpha_2| \tag{118}$$

$C(n_q+\alpha_1,\alpha_1) = \frac{(n_q+\alpha_1)!}{n_q!\alpha_1!}$ is the binomial coefficient, and Z^{-2} is the partition function.

As discussed in Sec. IV, we call $\hat{\rho}^{b^{\dagger,\alpha_1}b^{\alpha_2}}$ a set of basis matrices for the outside-horizon physics, just like $\hat{\rho}^K = |\alpha_1\rangle\langle\alpha_2|$ on the Kruskal side, because they are linearly independent of each other in terms of matrix summation algebra. The diagonal ones $\hat{\rho}^{b^{\dagger,\alpha}b^{\alpha}}$ are also properly normalized; hence, a normalized, positive semidefinite, and Hermitian density matrix $\hat{\rho}^{b^{\dagger,\alpha_1}b^{\alpha_2}}$ is safely traced into a normalized, positive semidefinite, and Hermitian density matrix ρ^O. The expression of $\rho^O = \sum_{\alpha_1\alpha_2} A_{\alpha_1\alpha_2}\hat{\rho}^{b^{\dagger,\alpha_1}b^{\alpha_2}}$ is unique, and we have built a one-to-one mapping between the set of arbitrary Kruskal positive frequency modes density matrices and a subset of the outside-horizon density matrices spanned by the normalized linear combination of pseudothermal density matrices $\hat{\rho}^{b^{\dagger,\alpha_1}b^{\alpha_2}}$.

The space of $\rho^O(A_{\alpha_1\alpha_2})$, or the subspace of the density matrices being a normalized linear combination of the pseudothermal density matrices, apparently is not the full density matrix space for the physical ensembles living outside the horizon, and it should not be. For example, $\rho^O = |\alpha\rangle_O\, {}_O\langle\alpha|$, a pure state living outside the horizon, cannot be decomposed into $\hat{\rho}^{b^{\dagger\alpha_1}b^{\alpha_2}}$ basis at nonzero temperature T. One might wonder how we mapped a full density matrix space into a subspace of density matrices, but it is not mysterious at all for infinite element groups—consider how one-to-one mapping can be easily built between integers and positive integers. The surface gravity with respect to a specific observer raises the ground state of that observer by a thermal ensemble at the temperature corresponding to the surface gravity. Such a gravitational thermalization could have memorizing feature; i.e., the original Kruskal density matrix ρ^K could be a thermal/pseudothermal density matrix itself, at a different temperature. We will encounter such a scenario when approximating the collapsing astrophysical black hole by layers of collapsing shell metrics, which we leave for future works.

A relativistic perspective of view is helpful to understand such a phenomenon. The stance of inertial and noninertial frames are always interchangeable. The dimensionality of their Hilbert space and density matrix space are infinity; their density matrices have this one-to-one mapping that could preserve the information from one space in the subspace at a certain temperature of the other; their Hilbert spaces are different spaces by definition, only with isomorphic representation structure.

Given the particle state/density matrix describing the collapsing history, one could analytically calculate the Page curve; i.e., the information $I_r = S_{\rm therm} + {\rm Tr}(\rho_r \log\rho_r)$ from the density matrix at each stage of the evaporation, using the pseudothermal density matrices in Eq. (118). On a rather loose end, the one-to-one mapping between density matrices itself might be sufficient for us to use the degrees of freedom counting argument in the original proof of Page curve [41]. Without validation on the details here, we notice that a recent work [42] starting from the same point as our paper; i.e., the Unruh effect on excited states has already presented a calculation of the Page curve in their framework.

Remark.—As first mentioned in the remark block in Sec. V, although the mapping from a pure state density matrix to a mixed one is not unitary from a traditional quantum mechanics point of view, the unitarity could be "resumed" by redefining the states basis matrices labeled by $\hat{\rho}^{b^{\dagger,\alpha_1}b^{\alpha_2}}$ quantum numbers modulo temperature in this supposedly nonunitary contracting operation. As long as the one-to-one mapping holds, such formalism should be feasible. The speculation is that we should probably change the definition of the vacuum in our relativistic QFT theory. We suspect that in an advanced representation of the physical states living on certain curved spacetime manifold, the vacuum should be defined as degenerating with any thermal ensemble, and by acting creators on the traditional vacuum or a thermal ensemble density matrix, instead of state vector representation, we get a spectrum of excitations. Actually, simple algebra shows that the hierarchy of

$\hat{\rho}^{b^{\dagger,\alpha_1}b^{\alpha_2}}$ is connected by acting corresponding numbers of creators and annihilators on both sides of a canonical density matrix, multiplied by a normalization factor, in the high-frequency/low-temperature limit. What is more, by acting annihilator b on the ground state in this formalism, the canonical ensemble, we obtain the first excitation on the opposite side $\hat{\rho}^{d^\dagger d}$.

VI. CASE STUDY: SPECTRUM UNDER DIFFERENT BASIS

We intuitively have a rough picture of the physics that the pseudothermal density matrix represents. As briefly mentioned in Sec. IV, $\hat{\rho}$ has a bump around specific frequency sitting on top of a thermal ensemble. However, it is still very elucidating to see the detailed number counting and energy spectrum that an observer would measure in the asymptotically flat regime. We will go through the detailed calculation in this section.

Before we start, we would like to first specify several presumptions and approximations. The first thing to stress is again, that we are going to apply some quantum information and computation techniques well-established for finite-dimensional Hilbert space on the infinite-dimensional Hilbert space. Specifically, in this section, we need to use the concept of positive operator-valued measures (POVM) for number/energy measurements. Secondly, we ignore the gray-body factor, which causes a frequency dependent < 1 transmission rate across the potential barrier extended in the intermediate region away from the black hole horizon. The gray-body factor is well-studied in the particle packet perspective of view for Hawking flux [43], and in our scalar field eigenmode scenario, this effect originates from the complicated form of the eigenmodes in the transition region between the near horizon and asymptotically flat regions. Recall that (only) in those two extreme regimes the Klein-Gordon equation solutions have simple plane-wave-like expressions in terms of t, r or r_*, and they have ≤ 1 transmission rates in between captured by the gray-body factor. We argue that since the gray-body factor is related to the physics not in the immediate vicinity of the horizon, it could be mounted separately later in the standard way from previous literature. The result we present in this section are not considering the gray-body factor and assume the transmission rate $T = 1$.

To find the energy spectrum of a density matrix, we need to calculate the expectation value $\langle E(\omega)\rangle$ as a function of the frequency ω. In quantum mechanics, the expectation value of a positive operator-valued measure (POVM) for a density matrix can be calculated as [44]

$$\langle \mathbf{E}\rangle = \mathrm{Tr}(\rho\mathbf{E}) = \mathrm{Tr}\left(\sum_\alpha A_\alpha\rho_\alpha\mathbf{E}\right) = \sum_\alpha A_\alpha\langle\mathbf{E}\rangle_\alpha, \quad (119)$$

where $\sum_\alpha A_\alpha = 1$ and $\mathrm{Tr}(\rho_\alpha) = 1$. The energy in a certain frequency $\mathbf{E}_\omega$ is a POVM of our system, and we calculate the expectation value using the above equation, as a function of frequency, to obtain the energy spectrum.

We start with a wave packet of Unruh modes living on the outgoing horizon of the Schwarzschild black hole. We express this Unruh wave packet of the quantized real scalar field as follows:

$$|\Gamma\rangle_K = \int_0^\infty d\omega g(\omega) u_\omega^\dagger|0\rangle_K, \quad (120)$$

where $g(\omega)$ is the shape of an arbitrary wave packet, which is normalized to unity,

$$\int_0^\infty d\omega |g(\omega)|^2 = 1. \quad (121)$$

The density matrix living outside the Schwarzschild black hole horizon contracting out the inner physical states is given by

$$\rho^O = \sum_{\mathcal{N}} {}_I\langle\mathcal{N}|\Gamma\rangle_K\, {}_K\langle\Gamma|\mathcal{N}\rangle_I \quad (122)$$

$$= \int_0^\infty\int_0^\infty d\omega_1 d\omega_2 g(\omega_1) g^*(\omega_2)\hat{\rho}^{11}(\omega_1,\omega_2), \quad (123)$$

where $\hat{\rho}^{11}(\omega_1,\omega_2)$ is as in the Eq. (118).

Among all the $\hat{\rho}^{11}(\omega_1,\omega_2)$ terms, only the terms with equal frequency $\omega_1 = \omega_2 = \omega$ are contributing to the diagonal terms in density matrix ρ—“diagonal” means that a component is expressed as the direct product of two identical states $|\mathcal{N}\rangle\langle\mathcal{N}|$. With $\omega_1 \neq \omega_2$, the two states of a density matrix component, $|\mathcal{N}, n_{\omega_1}+1\rangle$ and $|\mathcal{N}, n_{\omega_2}+1\rangle$, are never identical throughout all the configurations $\mathcal{N}$. For the purpose of calculating the expectation value of a POVM, we focus on the diagonal terms in the density matrix from now on,

$$\mathrm{diag}(\rho^O) = \int_0^\infty d\omega |g(\omega)|^2 \hat{\rho}^{11}(\omega). \quad (124)$$

$\hat{\rho}^{11}(\omega)$ are the normalized density matrix components, with $\mathrm{Tr}(\hat{\rho}^{11}(\omega)) = 1$. Thus, for the measurement of energy distributed to a specific frequency,

$$\mathbf{E}_\omega|\mathcal{N}\rangle = n_\omega\omega, \quad (125)$$

the energy expectation value over the full density ρ^O in $d\omega$ band can be decomposed into

$$\langle\mathbf{E}_\omega\rangle = \mathrm{Tr}(\mathbf{E}_\omega\rho^O) \quad (126)$$

$$= \int_0^\infty d\omega' |g(\omega')|^2 \langle\mathbf{E}_\omega(\omega')\rangle, \quad (127)$$

where

$$\langle \mathbf{E}_\omega(\omega') \rangle = \mathrm{Tr}(\mathbf{E}_\omega \hat{\rho}^{11}(\omega')) \tag{128}$$

is the energy spectrum for a specific single Unruh mode contracted normalized pseudothermal density matrix $\hat{\rho}^{11}(\omega')$.

Next, we refer to the traditional derivation of the Planck formula to calculate $\langle \mathbf{E}_\omega(\omega') \rangle$ [45,46]. Because $\hat{\rho}^{11}(\omega')$ is normalized and the distribution of the occupation number of a frequency is uncorrelated with other frequencies (multiplicative coefficients for each frequency in all the density matrix elements), we can calculate the expectation of the energy of a frequency ω as

$$\langle \mathbf{E}_\omega \rangle_{\omega'} = \frac{\sum_{n_\omega} p_{\omega'}(\omega, n_\omega) \epsilon_\omega}{\sum_{n_\omega} p_{\omega'}(\omega, n_\omega)}, \tag{129}$$

where $p_{\omega'}(\omega, n_\omega)$ is the probability of we finding n_ω of a frequency in the configuration $\mathcal{N}$, in a system prepared with density matrix $\hat{\rho}^{11}(\omega')$. n_ω always run from 0 to ∞.

When $\omega \neq \omega'$, the distribution of $\langle \mathbf{E}_\omega(\omega') \rangle$ is plainly Bose-Einstein for our massive scalar field, as $p_{\omega'}(\omega, n_\omega) \propto e^{-n_\omega \omega/T}$, $\epsilon_\omega = n_\omega \omega$, where $T = \frac{1}{8\pi M}$.

$$\langle \mathbf{E}_\omega(\omega' \neq \omega) \rangle = \frac{\sum_{n=0}^{\infty} n\omega e^{-n\omega/T}}{\sum_{n=0}^{\infty} e^{-n\omega/T}} \tag{130}$$

$$= \frac{\omega \sum_{n=0}^{\infty} \frac{de^{-n\omega/T}}{d(-\omega/T)}}{\sum_{n=0}^{\infty} e^{-n\omega/T}} \tag{131}$$

$$= \frac{\omega \frac{d}{d(-\omega/T)} \left(1 - e^{-\omega/T}\right)^{-1}}{\left(1 - e^{-\omega/T}\right)^{-1}} \tag{132}$$

$$= \frac{\omega}{e^{\omega/T} - 1} \tag{133}$$

For $\omega = \omega'$, according to Eq. (118) with $\alpha_1 = \alpha_2 = 1$, $p_{\omega'}(\omega', n_{\omega'}) \propto (n_{\omega'} + 1) e^{-n_{\omega'}\omega'/T}$, $\epsilon_{\omega'} = (n_{\omega'} + 1)\omega'$. This distribution is manifestly different from the canonical one for other frequencies, especially with the number counting in frequency $\omega = \omega'$ starting from 1, instead of 0. It is this relative difference, of $\omega = \omega'$ compared with other canonically occupied frequencies, recording the information of ω' corresponding to the original Unruh mode. Such difference is nontrivial, i.e., cannot be eliminated by some sort of relabeling of the number counting. The energy expectation value of $\omega = \omega'$ frequency is given by

$$\langle \mathbf{E}_\omega(\omega' = \omega) \rangle = \frac{\sum_{n=0}^{\infty} \omega (n+1)^2 e^{-n\omega/T}}{\sum_{n=0}^{\infty} (n+1) e^{-n\omega/T}} \tag{134}$$

$$= \omega \frac{\sum_{n=0}^{\infty} (n+1) n e^{-n\omega/T} + \sum_{n=0}^{\infty} (n+1) e^{-n\omega/T}}{\sum_{n=0}^{\infty} \frac{d}{d(e^{-\omega/T})} e^{-(n+1)\omega/T}} \tag{135}$$

$$= \omega \frac{e^{-\omega/T} \sum_{n=0}^{\infty} \frac{d^2}{d(e^{-\omega/T})^2} e^{-(n+2)\omega/T} + \frac{d}{d(e^{-\omega/T})} ((1 - e^{-\omega/T})^{-1} - 1)}{\frac{d}{d(e^{-\omega/T})} ((1 - e^{-\omega/T})^{-1} - 1)} \tag{136}$$

$$= \omega \frac{e^{-\omega/T} \frac{d^2}{d(e^{-\omega/T})^2} ((1 - e^{-\omega/T})^{-1} - 1 - e^{-\omega/T}) + (1 - e^{-\omega/T})^{-2}}{(1 - e^{-\omega/T})^{-2}} \tag{137}$$

$$= \omega \frac{2e^{-\omega/T} (1 - e^{-\omega/T})^{-3} + (1 - e^{-\omega/T})^{-2}}{(1 - e^{-\omega/T})^{-2}} \tag{138}$$

$$= \omega \frac{e^{\omega/T} + 1}{e^{\omega/T} - 1}. \tag{139}$$

Supposing that the discrete energy levels labeled by ω and ω' are distinguishable at discretization unit h_0, we can combine the expression of $\langle \mathbf{E}_\omega(\omega' \neq \omega) \rangle$ and $\langle \mathbf{E}_\omega(\omega' = \omega) \rangle$ into a delta function,

$$\langle \mathbf{E}_\omega(\omega') \rangle = \frac{\omega}{e^{\omega/T} - 1} + \delta(\omega' - \omega) h_0 \omega' \frac{e^{\omega'/T}}{e^{\omega'/T} - 1}. \tag{140}$$

Similarly, the number counting spectrum,

$$\langle \mathbf{n}_\omega(\omega')\rangle = \frac{1}{e^{\omega/T}-1} + \delta(\omega'-\omega)h_0\frac{e^{\omega'/T}}{e^{\omega'/T}-1}. \quad (141)$$

Substituting $\langle \mathbf{E}_\omega(\omega')\rangle$ back to Eq. (127), the integration gives us,

$$\langle \mathbf{E}_\omega\rangle = \frac{\omega}{e^{\omega/T}-1} + h_0|g(\omega)|^2\frac{\omega e^{\omega/T}}{e^{\omega/T}-1}. \quad (142)$$

The delta function $\delta(\omega'-\omega)$ for each individual $\hat{\rho}^{11}_{\omega'}$ ensures us the simple convolution with the Unruh wave

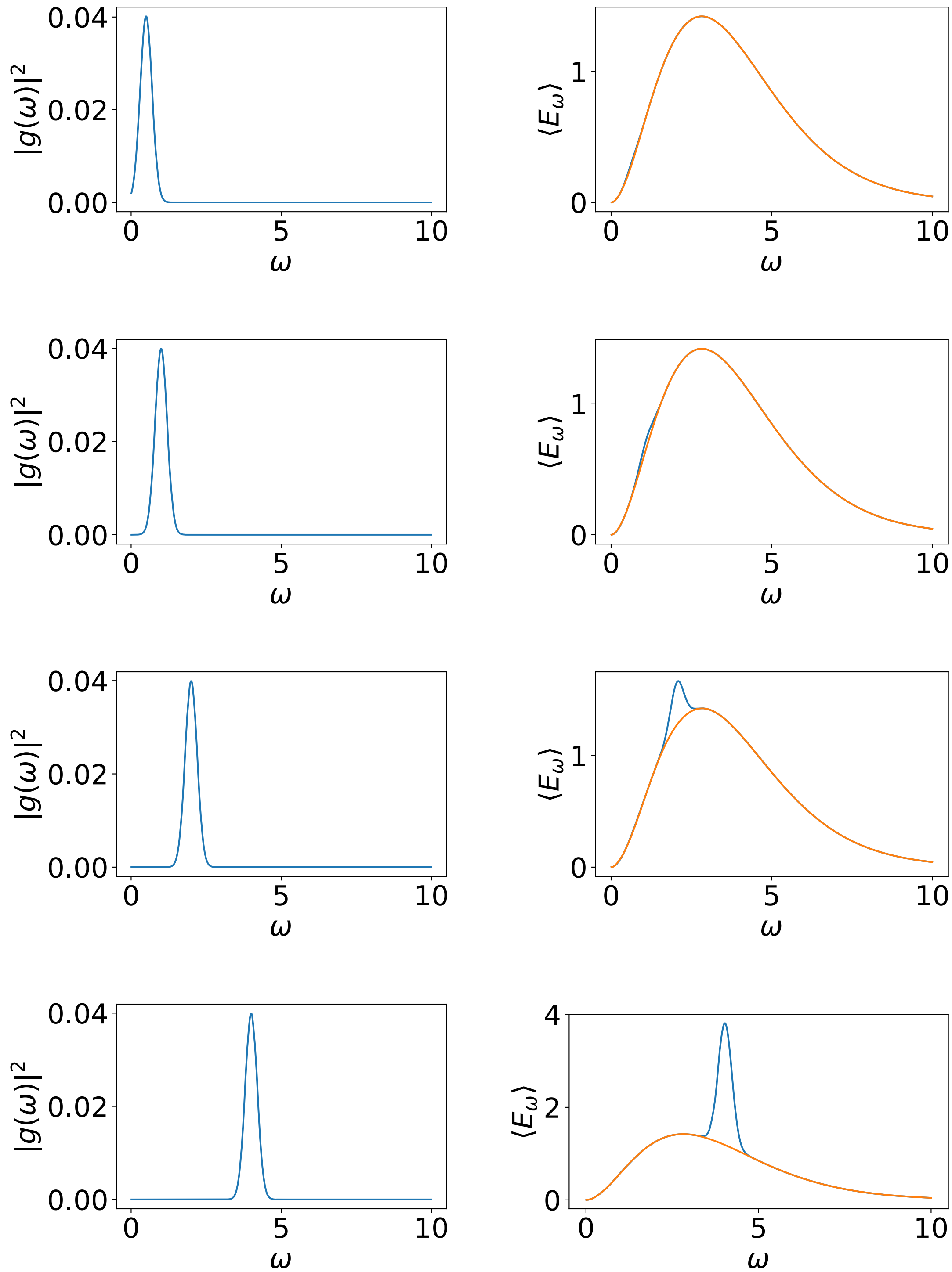

FIG. 5. Left panels: the Gaussian wave packets of type-I Unruh modes. Right panels: the energy spectrum on R+ (or outside the horizon), after tracing out the states on R− (or inside the horizon).

packet in the pseudothermal spectrum. The first term gives us a blackbody radiation spectrum, while the second term enhances the power by a factor of $h_0|g(\omega)|^2 e^{\omega/T}$. This result echoes the stimulated Hawking radiation by Jürgen Audretsch and Rainer Müller in 1992 [27], with a focus on showing how the deviation of the Hawking radiation from a perfect blackbody spectrum incorporates the information of the waveform of the specific in-going excitation. Notice that the low-temperature limit $T \to 0$ of Eq. (142) goes back to the inertial frame result,

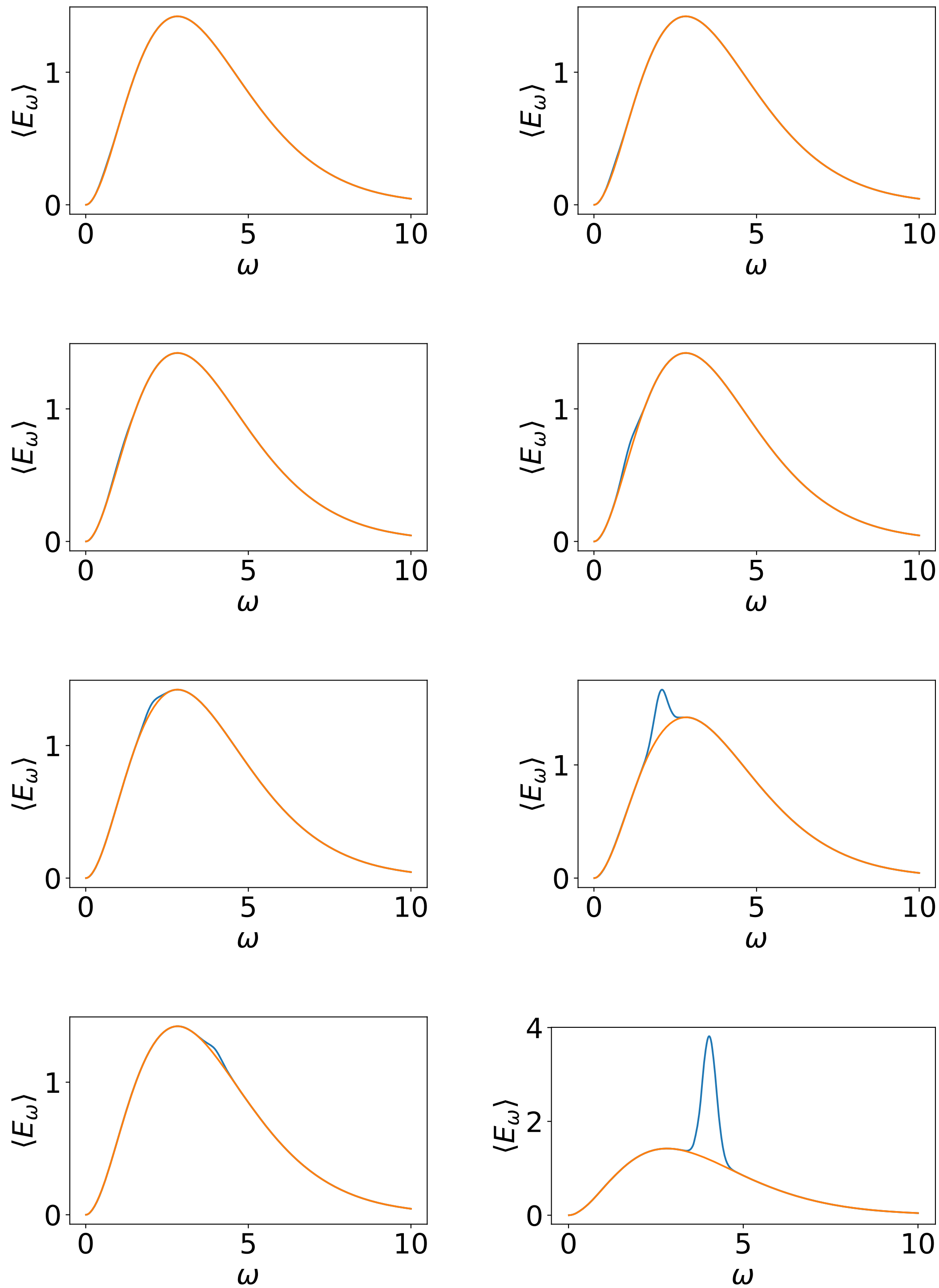

FIG. 6. Left panels: the energy spectrum on R− (or inside the horizon), after tracing out the states on R+ (or outside the horizon). Right panels: the energy spectrum on R+ (or outside the horizon), after tracing out the states on R− (or inside the horizon). The parity is broken by the asymptotic direction of the initial Unruh mode.

$$\langle \mathbf{E}_\omega \rangle|_{T\to 0} = |g(\omega)|^2 h_0 \omega. \tag{143}$$

In Fig. 5, we illustrate several Gaussian $g(\omega) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(\omega-\omega_0)^2}{4\sigma^2}}$ wave packets and their corresponding energy spectrum $\langle \mathbf{E}_\omega \rangle$ measured by an observer infinitely far away outside the horizon. We multiply Eq. (142) by ω^2, which is achievable by assuming a uniform angular modes distribution, to compare with the blackbody radiation spectrum depicted by the orange curves. We set the frequency discretization constant $h_0 = 1$ for illustration purposes. Actually, changing the excitation number in Eq. (120) to α multiplies the second term in Eq. (142) by α, so Fig. 5 could be physically achieved by a large number of excitations setting $\alpha h_0 = 1$. The subtlety is that by contributing a nontrivial amount of energy into the energy spectrum, the Unruh modes at the horizon presumed to be perturbative might break the assumption of no backreaction. Such complexity can be ignored for now as the illustration here is only dedicated to showing that there will be a featured deviation from the blackbody spectrum in Hawking radiation, considering the generalized Unruh effect.

The applicability of the above calculation of the pseudothermal ensemble spectrum is wide, for example, we could calculate the spectrum for higher number excitation, realistic Unruh wave packets at the horizon, angular distribution, fermion generalized Unruh effect, etc. The technique in this section can be fairly straightforwardly applied those further investigations into the collapsing history of a black hole. In any case, hopefully, the amazing bottom line has been illustrated by those plots in Fig. 5 clearly, that by observing the offset of the Hawking radiation spectrum from the blackbody spectrum, we could infer the information of the particles that have fallen into the black hole. Again, our result is only valid at the perturbative level.

Although we assumed a uniform angular distribution of the infalling modes in our illustration, it is useful to notice that the angular distribution of the infalling modes will be preserved when they propagate out as Hawking radiations for nonrotating Schwarzschild black holes. Because the scalar field eigenmodes in Kruskal and Schwarzschild metrics have the same angular dependent part. In reality, the Hawking radiation for cosmological black holes is at temperatures so low that it has never been observed. The results obtained here might need to be tested in some other laboratories like asteroidal mass primordial black holes, which are still in the open window to take all the dark matter [47], or at the particle horizons that could be produced by accelerations or simulated systems in the lab. The inflation and reheating research can also potentially investigate the effect of the particle horizon formed from the rapid expansion of the Universe using the generalized Unruh effect as a tool.

Remark.—Back to the Rindler-Minkowski case. The results shown by Fig. 5 are also representable for the $\hat{\rho}^{bb}$ on $R+$ wedge starting from an excitation on the Minkowski vacuum. The energy-frequency spectrum is very intuitive here: The observer on R+ measures a peak on top of the thermal bath whose temperature is proportional to the acceleration; when the particle has very high energy compared to the background temperature, the observation goes back to Minkowski case; And when the acceleration is high, the low-energy particle is buried under the hot thermal bath. This serves as a decent sanity check of our result. The implication for an R− observer coming after that is rather counterintuitive. The spectrum of $\hat{\rho}^{dd}$ calculated in the similar way as $\hat{\rho}^{bb}$ is shown in Fig. 6. Although at a much lower amplitude, the same frequency bump caused by $C(n+1,1) = n+1$ factor is always present in the spectrum observed by an $R-$ observer, just as implied by the equal Von-Neumann entropy of ρ^{bb} and ρ^{dd}. Remember that the Unruh mode we start from is a purely type-I mode in this case, whose propagation asymptotically align with the trajectory of an R+ observer. At the theoretical level, it seems that the information of this oppositely traveling particle is not completely lost on the $R-$ wedge. The engraving of a piece of information on a horizon seems to be equally observable by both sides no matter which direction it is traveling.

VII. DISCUSSIONS

A. Assumptions and caveats

We would like to preemptively summarize the price for the neat implication of this work. We could not exhaust the potential problems, but we try our best to list the assumptions and caveats that come in at different steps of our derivation in this paper here.

(1) The infinite dimensionality of the Hilbert space of a quantum field theory system. By postulating the scalar field energy levels (equation of motion solution eigenvalues) to be discrete, we obtain an infinite but countable dimension of the Hilbert space, thus the density matrices. However, as far as we know, many of the quantum information techniques under the density matrix formalism used in this paper have only been well-established for finite-dimensional quantum systems. Since most of the pseudothermal density matrices are well-behaved (diagonalizable), the main concern is the normalizability of the density matrix. We verify the normalization of Eq. (118) with $\alpha_1 = \alpha_2$ by explicitly calculating the series summation backstage, and speculate that the consistent physics interpretation should not be spoiled for no reason. But truth be told, strictly speaking, the normalization of pseudothermal density matrices $\hat{\rho}$ is speculation at this point. Even if the normalization has no problem, the direct applicability of quantum information concepts like Von-Neumann entropy and

POVM on infinite dimensional Hilbert space is still an open question.

(2) The transformation between plane wave modes and Unruh modes. As mentioned before, it takes a transformation from $e^{-i\omega x}$ and $x^{-i\omega}$ basis to go from one way of quantization to another. The exponential relationships between the Fourier transform variable here could cause severe blue- and redshifts. When considering the discrete eigenvalues, such log-space transformation poses a question of the energy level structures that can truly span mutually complete basis modes.

(3) The practical measurability of the pseudothermal spectrum. The one-to-one mapping between an arbitrary Unruh mode density matrix and the pseudothermal density matrix vector only ensures the conceptual invertibility of the operation. Due to the high stochasticity of the pseudothermal ensemble, the story could be stated in another way that the information will indeed be buried under the thermal bath. Even though there is coherent excitation on the whole ensemble, it might be too weak to be detected.

(4) Enhanced emission from the black holes. An obvious risk presents for the generally enhanced Hawking radiation. In extreme cases, excitation peaks could even potentially make a black hole no longer black, when $g(\omega)$ spectrum deposits its energy in high frequencies. However, there is a possibility that high-frequency emission spectra are indeed allowed by the realistic black holes depending on their collapsing history: We could have already observed them but could not distinguish them from the bright accretion disc emissions, or have attributed them to the unexplained diffuse cosmic ray excess [48,49]. The enhanced Hawking radiation will also accelerate the evaporation, which brings in problems or opportunities depending on the masses of the primordial black holes.

B. Generalized Unruh effect on other particle horizons

The general Unurh effect technique is in principle applicable to any particle horizon, regardless of the cause of it being the acceleration, massive objects, or inflation.

The Bogoliubov transformation approach has been explored for particle production during inflation for a long time. A recent paper by Kaneta *et al.* [50] have carried out an instance of comparing Boltzmann and Bogoliubov ways of dark matter production during inflation, and they reached a great agreement. It is an encouraging work unveiling the alternativeness of the gravitational thermalization picture and the direct quantization of the graviton picture. One major merit of the Bogoliubov over the Boltzmann approach though, is that it can make analytical predictions at very low-k regimes, which is not something that can be done with the Boltzmann approach by carrying out conventional particle theory calculation at the perturbative level for the quantized gravitons. The reason is that at the low-k regime, the scalar field stops being localized enough and loses its particle property. It is in analogy to the stretch of the waveform near the asymptotic Schwarzschild black hole in our work.

Despite the advantage at the low-k regime, the Bogoliubov approach has an obvious disadvantage in the traditional vacuum initial state treatment. The method to incorporate other nongravitational particle productions has been unknown in the Bogoliubov approach. The generalized Unruh effect working with the excitation at the horizon as initial conditions could fill this gap and give an inclusive description of different particle production mechanisms through a differentiated Bogoliubov approach.

Asymptotic horizons are always formed when an observer accelerates with respect to the surrounding system. We could break down a smooth geodesic into a series of short hyperbolas, then use the Minkowski $\leftrightarrow$ Rindler generalized Unruh effect to investigate how a bunch of information restored in local field operators $\phi(O)$ in an element volume O is passed from upstream to downstream of a bundle of geodesics. Of course, there is arbitrariness in the definition of upstream/downstream, i.e., the direction of the time flow. Suppose at a certain world point P_i the geodesic is represented by the local hyperbola segment with acceleration α_i between two successive moments. The local field operators $\phi(P_i)$ passed its information to the local field operators at the next moment $\phi(P_{i+1})$, with a slight thermalization due to α_i, according to generalized Unruh effect.

We remember that there is arbitrariness in the definition of the time arrow in the first place. However, going in any arbitrary direction of time flow always add a thermal ensemble to the initial state, thus presumably increasing the entropy. We do not know which one leads to the other, but from the argument above it seems that there is a connection between the second law of thermodynamics and the simple law of causality, which forbids looping geodesics.

C. Emergent gravity from the stochasticity of quantum fields

The ubiquitousness of the local particle horizon and thus the local Unruh effects as described in the previous subsection intrigues us with this conceptual speculation:

$$\begin{aligned} \text{gravity} &\equiv \text{spacetime curvature} \\ &\equiv \text{temperature of the quantum fields} \end{aligned}$$

Each of these equivalences is supposedly revertible, and those quantities are completely local. By this proposed relationship we could derive the former from the latter or the other way around. Namely, it falls into the emergent gravity category of ideas. To the best of our knowledge, the thermodynamics interpretation of general relativity was first discussed in [51], proposing the connections like $T \leftrightarrow \kappa$, $dt \leftrightarrow dS$ that have been repetitively appearing throughout our paper. People have been reluctant to take it further than an interpretation since then. We think the way to go a step forward along their path could be to come up with a formal way to replace Ricci curvature in the Einstein-Hilbert action with a heat reservoir of the fields. By solving for the temperature of the fields from this refined action with a heat reservoir, if feasible, we can accordingly solve the metric of the spacetime.

We are going to briefly comment on these thought experiments that are dedicated to proving the quantum nature of gravity as a closing remark to the far-fetched discussion in this section. To the best of our knowledge, most, if not all, of those quantum gravity experiments' setups, are indistinguishable between a quantized gravitational field (metric) and the local application of the Unruh effect, namely the gravity emergent from the stochasticity of quantum fields—they both render the gravitational phenomena quantum natures. Usually, such experiments intend to prove the quantum nature of gravity by measuring the entanglement introduced or mediated by gravitational interaction. For example, the Gedankenexperiment, which is essentially a gravitational version of the Stern-Gerlach experiment [52] and Marletto and Vedral's proposal of double mass interferometer experiment [53]. What lies as the fundamental to these experiments is usually a perturbed gravitational field whose propagation is semiclassically describable, and the experiments are designed to show that such perturbed gravitational fields serve as the quantum entanglement messenger between two already well-established quantum systems. However, in a stochastic quantum field description of gravity, we do not need to introduce this additional quantum of graviton to mediate the entanglement—the ubiquitous quantum field of the two presumed quantum systems spread out in spacetime is already a mediator of the entanglement. The entanglement is passed on by local Unruh effects. In principle, no experiment can distinguish between the picture of "the graviton $h_{\mu\nu}$ mediated the entanglement between Q_1 and Q_2" and "Q_1 interact with Q_i distributed ubiquitously in the spacetime spanned until Q_2; thus, the two eventually brought into entanglement." They are just two alternative ways of telling the same story.

For the reasons presented in this section, we figure that emergent gravity from the stochasticity of the quantum fields might be a competitive way toward the incorporation of gravity into the quantum picture. This approach, as illustrated in the calculation in this paper, has the potential to produce calculable and testable predictions.

VIII. CONCLUSIONS

We practice a straightforward idea, that generalizes the Unruh effect applied usually on the vacuum state to arbitrary excited states in this paper. The result shows that the positive frequency excitation at the horizon induces a coherent excitation on each of the configurations of a canonical ensemble measured by an accelerating observer, thus illustrating itself as a featured peak on top of the featureless blackbody radiation spectrum. We call such an ensemble pseudothermal, which in principle (in terms of density matrix representation) is distinguishable from the linear combination of a canonical ensemble and a pure excitation state.

We apply our generalized Unruh effect result on a system with massive real scalar field in the collapsing shell metric, asymptotically a Schwarzschild black hole. We find that the particles inside the shell could be represented by the boundary condition of the Kruskal positive frequency modes living on the outgoing horizon, thus we could retrieve that information, at least partially, through the strong entanglement across the horizon. The Hawking radiation has a featured enhancement based on the collapsing history and the initial excitation inside the shell of the black hole.

There are numerous open questions extending both bottom-ward and up-ward from this work, as discussed in Sec. VII. Both theoretically and observationally, the generalized Unruh effect implies intriguing conclusions and speculations to be confronted in the future.

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[1] W. G. Unruh, Notes on black hole evaporation, Phys. Rev. D **14**, 870 (1976).
[2] S. A. Fulling, Nonuniqueness of canonical field quantization in Riemannian space-time, Phys. Rev. D **7**, 2850 (1973).
[3] L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, The Unruh effect and its applications, Rev. Mod. Phys. **80**, 787 (2008).
[4] C. Krishnan, Quantum field theory, black holes and holography, arXiv:1011.5875.
[5] W. G. Brenna, E. G. Brown, R. B. Mann, and E. Martin-Martinez, Universality and thermalization in the Unruh effect, Phys. Rev. D **88**, 064031 (2013).
[6] S. W. Hawking, Black hole explosions, Nature (London) **248**, 30 (1974).
[7] G. W. Gibbons and S. W. Hawking, Cosmological event horizons, thermodynamics, and particle creation, Phys. Rev. D **15**, 2738 (1977).
[8] I. Fuentes-Schuller and R. B. Mann, Alice Falls into a Black Hole: Entanglement in Non-Inertial Frames, Phys. Rev. Lett. **95**, 120404 (2005).
[9] G. Adesso and I. Fuentes-Schuller, Correlation loss and multipartite entanglement across a black hole horizon (2007), https://arxiv.org/pdf/quant-ph/0702001.pdf.
[10] E. Martin-Martinez and J. León, Fermionic entanglement that survives a black hole, Phys. Rev. A **80**, 042318 (2009).
[11] A. G. S. Landulfo and G. E. A. Matsas, Sudden death of entanglement and teleportation fidelity loss via the Unruh effect, Phys. Rev. A **80**, 032315 (2009).
[12] M. Montero and E. Martin-Martinez, The entangling side of the Unruh-Hawking effect, J. High Energy Phys. 07 (2011) 006.
[13] E. Martin-Martinez, L. J. Garay, and J. Leon, Unveiling quantum entanglement degradation near a Schwarzschild black hole, Phys. Rev. D **82**, 064006 (2010).
[14] M. Ramzan and M. Khan, Decoherence and entanglement degradation of a Qubit-Qutrit system in non-inertial frames, Quantum Inf. Process. **11**, 443 (2012).
[15] N. N. Bogolyubov, On a new method in the theory of superconductivity, Nuovo Cimento **7**, 794 (1958).
[16] F. A. T. Pannia, S. E. P. Bergliaffa, and N. Pinto-Neto, Particle production in accelerated thin bubbles, J. Cosmol. Astropart. Phys. 04 (2022) 015.
[17] S. Hemming and E. Keski-Vakkuri, Hawking radiation from AdS black holes, Phys. Rev. D **64**, 044006 (2001).
[18] D. J. H. Chung, Classical inflation field induced creation of superheavy dark matter, Phys. Rev. D **67**, 083514 (2003).
[19] K. Lochan and T. Padmanabhan, Inertial nonvacuum states viewed from the Rindler frame, Phys. Rev. D **91**, 044002 (2015).
[20] K. Lochan and T. Padmanabhan, Extracting Information about the Initial State from the Black Hole Radiation, Phys. Rev. Lett. **116**, 051301 (2016).
[21] D. A. T. Vanzella and G. E. A. Matsas, Decay of Accelerated Protons and the Existence of the Fulling-Davies-Unruh Effect, Phys. Rev. Lett. **87**, 151301 (2001).
[22] S. W. Hawking, Information loss in black holes, Phys. Rev. D **72**, 084013 (2005).
[23] S. B. Giddings, Black holes, quantum information, and unitary evolution, Phys. Rev. D **85**, 124063 (2012).
[24] X. Calmet and S. D. H. Hsu, A brief history of Hawking's information paradox, Europhys. Lett. **139**, 49001 (2022).
[25] R. M. Wald, Stimulated emission effects in particle creation near black holes, Phys. Rev. D **13**, 3176 (1976).
[26] J. D. Bekenstein and A. Meisels, Einstein A and B coefficients for a black hole, Phys. Rev. D **15**, 2775 (1977).
[27] J. Audretsch and R. Müller, Amplification of the black-hole Hawking radiation by stimulated emission, Phys. Rev. D **45**, 513 (1992).
[28] S. Weinfurtner, E. W. Tedford, M. C. J. Penrice, W. G. Unruh, and G. A. Lawrence, Measurement of Stimulated Hawking Emission in an Analogue System, Phys. Rev. Lett. **106**, 021302 (2011).
[29] I. Agullo, A. J. Brady, and D. Kranas, Quantum Aspects of Stimulated Hawking Radiation in an Optical Analog White-Black Hole Pair, Phys. Rev. Lett. **128**, 091301 (2022).
[30] M. Srednicki, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 2007).
[31] M. A. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2002).
[32] G. T. Horowitz and J. Maldacena, The black hole final state, J. High Energy Phys. 02 (2004) 008.
[33] M. B. Plenio and S. Virmani, An introduction to entanglement measures, Quantum Inf. Comput. **7**, 1 (2007).
[34] E. Witten, APS medal for exceptional achievement in research: Invited article on entanglement properties of quantum field theory, Rev. Mod. Phys. **90**, 045003 (2018).
[35] S. D. Mathur, The information paradox: A pedagogical introduction, Classical Quantum Gravity **26**, 224001 (2009).
[36] L. Susskind, L. Thorlacius, and J. Uglum, The stretched horizon and black hole complementarity, Phys. Rev. D **48**, 3743 (1993).
[37] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998).
[38] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, The entropy of Hawking radiation, Rev. Mod. Phys. **93**, 035002 (2021).
[39] G. 't Hooft, The quantum black hole as a theoretical lab, a pedagogical treatment of a new approach, in *56th International School of Subnuclear Physics: From Gravitational Waves to QED, QFD and QCD,* (2018), arXiv: 1902.10469.
[40] Wikipedia Contributors, Kruskal-Szekeres coordinates—Wikipedia, The free encyclopedia (2022), [Online; accessed 2-May-2022].
[41] D. N. Page, Information in Black Hole Radiation, Phys. Rev. Lett. **71**, 3743 (1993).
[42] J. Wang, Unruh effect of multiparticle states and black hole radiation, JUSTC **52**, 4 (2022).
[43] F. Gray and M. Visser, Greybody factors for Schwarzschild black holes: Path-ordered exponentials and product integrals, Universe **4**, 93 (2018).
[44] A. Peres, *Quantum Theory: Concepts and Methods* (Springer, New York, 2002).

[45] Unknown, The derivation of the Planck formula, 2022, https://edisciplinas.usp.br/pluginfile.php/48089/course/section/16461/qsp_chapter10-plank.pdf.

[46] R. K. Pathria, *Statistical Mechanics* (Elsevier, New York, 2016).

[47] B. Carr, K. Kohri, Y. Sendouda, and J. Yokoyama, Constraints on primordial black holes, Rep. Prog. Phys. **84**, 116902 (2021).

[48] J. Knodlseder *et al.*, The all-sky distribution of 511 keV electron-positron annihilation emission, Astron. Astrophys. **441**, 513 (2005).

[49] M. Ajello *et al.*, Fermi-LAT observations of high-energy γ-ray emission toward the galactic center, Astrophys. J. **819**, 44 (2016).

[50] K. Kaneta, S. Mook Lee, and K.-Y. Oda, Boltzmann or Bogoliubov? Approaches compared in gravitational particle production, arXiv:2206:2022.

[51] T. Jacobson, Thermodynamics of Space-Time: The Einstein Equation of State, Phys. Rev. Lett. **75**, 1260 (1995).

[52] A. Belenchia, R. M. Wald, F. Giacomini, E. Castro-Ruiz, Č. Brukner, and M. Aspelmeyer, Quantum superposition of massive objects and the quantization of gravity, Phys. Rev. D **98**, 126009 (2018).

[53] C. Marletto and V. Vedral, Gravitationally-Induced Entanglement between Two Massive Particles is Sufficient Evidence of Quantum Effects in Gravity, Phys. Rev. Lett. **119**, 240402 (2017).