Electromagnetic energy loss of axion stars

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The rate of energy exchange of spatially localized and gravitationally bound axion configurations with electromagnetic radiation is investigated in the presence of strong static magnetic fields. A fully analytic treatment is achieved based on variationally optimized separable spatiotemporal clump profiles. For dilute axion stars the equation of the energy variation is reinterpreted as a rate equation of the axion number. Its solutions hints at an asymptotic $\sim t^{1/5}$ increase of the clump size.

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I. INTRODUCTION

A boson star is expected to form from free massive scalar particles merely due to their gravitational interaction. Such objects were first studied numerically by Kaup [1] and Ruffini and Bonnazola [2]. In the context of axions, a leading dark matter candidate, this phenomenon was first investigated by Tkachev [3]. The characteristics of axionic scalar stars have been extensively explored in the literature [4–8]. Numerical studies by Chavanis and Delfini [9] including quartic self-interactions of axions pointed out the existence of a stability edge in the mass-radius relation for this so-called *dilute* branch. Semianalytical solutions in the gravity-dominated branch were constructed by Eby *et al.* [10]. Another branch where higher n-point axion interactions can stabilize even high-mass localized configurations was discovered by Braaten *et al.* [11] and called *dense* axion stars.

In the equations of gravitationally bound axion stars the energy is dominated by the rest mass of axions and one can apply the nonrelativistic approximation. Then, the axion number content of a field configuration is an approximately conserved quantity. When one includes via higher-order couplings the generation of more energetic (relativistic) particles, some particles get the chance to escape from the Bose-Einstein condensate of the star. This effect can be accounted for by introducing a non-Hermitian term into the axion potential [12] which arises from scattering processes of the full theory involving the high-momentum tail of the axion field [13,14]. In the case of the dilute branch the main mechanism leading to the depletion of the axionic medium is the two-photon decay of axions [15]. A detailed discussion of the lifetime of axion stars can be found in Ref. [16].

Strong external magnetic fields are present around neutron stars (10^4–10^{11} T), in particular around magnetars (10^9–10^{11} T). Electromagnetic radiation from axion stars embedded in strong magnetic fields was studied in Refs. [17,18]. A generic dimensionless combination of some axion data and the magnetic field strength was suggested to determine the order of magnitude of the electromagnetic decay time in Ref. [19]. A computational algorithm to include the radiation backreaction effect quantitatively in the evolution of dense branch stars was put forward and implemented very recently [20].

The aim of the present short paper is to outline a fully analytic computation of the time evolution of the particle number of a gravitationally bound (dilute) axion clump in a strong magnetic field. The treatment relies on the discussion of the energy-momentum transfer between axions and the electromagnetic field, analyzed in our previous publication [21]. The rate equation for the change of the energy in the axion sector will be analyzed, taking into account fully the backreaction of the emerging electromagnetic radiation (Sec. II). In the course of the calculation a spherically symmetric axion configuration is assumed. Its separable spatiotemporal profile function is given in terms of variationally optimized trial functions. Their variational treatment is reviewed in Sec. III. Based on such profile functions, a rather compact analytic formula is given in Sec. IV for the right-hand side of the energy rate equation. It is translated into the rate equation of the axion number in Sec. V, which is solved explicitly for two trial spatial profiles used already in the literature. The asymptotic solution of this equation leads to conjecturing *powerlike* time dependence for the particle depletion of axion stars due to electromagnetic radiation.

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II. ELECTROMAGNETIC ENERGY BALANCE OF AN AXION CLUMP

Temporal and spatial variations of an axion field $a(\mathbf{x},t)$ in the presence of a static magnetic background field $B_0(\mathbf{x})$ represent effectively electromagnetic source densities, the strength of which is determined by the axion–two-photon coupling $g_{a\gamma\gamma}$ [15,22]:

$$\begin{aligned}\mathbf{j}_a(\mathbf{x},t) &= -g_{a\gamma\gamma}\dot{a}(\mathbf{x},t)\mathbf{B}_0(\mathbf{x}),\\ \rho_a(\mathbf{x},t) &= g_{a\gamma\gamma}\mathbf{B}_0(\mathbf{x})\cdot\nabla a(\mathbf{x},t).\end{aligned} \tag{1}$$

This means that in Lorentz gauge one finds the following retarded scalar and vector potentials generated by the axions, using natural units ($\hbar = c = 1$):

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \int d^3x' \frac{\mathbf{j}_a(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|},\\ A_0(\mathbf{x},t) &= \int d^3x' \frac{\rho_a(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}.\end{aligned} \tag{2}$$

It is worthwhile to emphasize that $|\mathbf{B}_0(\mathbf{x})| \gg |\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{x},t)|$, by assumption, throughout the investigation below. The rate of the energy exchange of any axion configuration with electromagnetic fields formally coincides with the expression of the work power of electrically charged currents, as was emphasized recently by various authors [20,21] (see also the Appendix of Ref. [23]):

$$\begin{aligned}\frac{dE_a}{dt} &= \int d^3x\,\mathbf{j}_a(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\\ &= \int d^3x\,g_{a\gamma\gamma}\dot{a}(\mathbf{x},t)\mathbf{B}_0(\mathbf{x})\cdot(\dot{\mathbf{A}}(\mathbf{x},t)+\nabla_{\mathbf{x}}A_0(\mathbf{x},t)).\end{aligned} \tag{3}$$

One might also note that the rate of momentum change of the axion clump due to the presence of a topologically nontrivial electromagnetic field density is given as

$$\begin{aligned}\frac{d\mathbf{P}_a}{dt} &= \int d^3x\,g_{a\gamma\gamma}\nabla a(\mathbf{x},t)\\ &\quad\times[\mathbf{B}_0(\mathbf{x})\cdot(-\dot{\mathbf{A}}(\mathbf{x},t)-\nabla_{\mathbf{x}}A_0(\mathbf{x},t))].\end{aligned} \tag{4}$$

One substitutes the expressions (2) of the potentials into Eq. (3) and performs in the second term (involving the scalar potential) a partial integration. Then one can exploit the continuity equation $\partial_t\rho_a = -\nabla\mathbf{j}_a$, which is also valid for the axionic "charge" and "current" densities to arrive at

$$\begin{aligned}\frac{dE_a}{dt} &= \int d^3x \int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\\ &\quad\times\left[\mathbf{j}_a(\mathbf{x},t)\cdot\frac{\partial}{\partial t}\mathbf{j}_a(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|)\right.\\ &\quad\left.+\rho_a(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|)\frac{\partial}{\partial t}\rho_a(\mathbf{x},t)\right].\end{aligned} \tag{5}$$

The clump oscillates with some average frequency ω_a. The retardation dependence of ρ_a, $\mathbf{j}_a$ can be expanded into Taylor series in the near zone, where $\omega_a|\mathbf{x}-\mathbf{y}| \ll 1$. Separating the first term of the expansions from the rest, one finds the time derivative of an "electromagnetic" contribution to the energy of the axion configuration, ΔE_a. It has the obvious interpretation of being the electrostatic and magnetostatic energy of the near zone promptly following the oscillation of the corresponding source densities. The rest ($W_{\mathrm{rad-loss}}$) can be associated with the energy lost by the clump via electromagnetic radiation,

$$\begin{aligned}\frac{dE_a}{dt} &= -\frac{d\Delta E_a}{dt} - W_{\mathrm{rad-loss}}\\ \Delta E_a &= \frac{1}{2}g_{a\gamma\gamma}^2\int d^3x\int d^3x' \frac{B_{0i}(\mathbf{x})B_{0j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\\ &\quad\times[\delta_{ij}\dot{a}(\mathbf{x},t)\dot{a}(\mathbf{x}',t)+\nabla_{x_i}a(\mathbf{x},t)\nabla_{x'_j}a(\mathbf{x}',t)].\end{aligned} \tag{6}$$

One might attempt to represent the energy loss with just the next term of the Taylor expansion. Also one notes (exploiting the continuity equation) that there is no contribution from the second ("electric") term proportional to $(\int d^3x\dot{\rho}(\mathbf{x},t))^2$. The "magnetic" piece suggests an analogy with the so-called "three-dot" force exerted by the emitted electromagnetic radiation on the motion of its pointlike electric source:

$$\begin{aligned}&W_{\mathrm{rad-loss}}\\ &\quad= -g_{a\gamma\gamma}^2\int d^3x\dot{a}(\mathbf{x},t)\mathbf{B}_0(\mathbf{x})\cdot\int d^3x'\dddot{a}(\mathbf{x}',t)\mathbf{B}_0(\mathbf{x}').\end{aligned} \tag{7}$$

One can average this expression over the average period $T = 2\pi/\omega_a$ and obtain with partial time integration the compact formula

$$\overline{W_{\mathrm{rad-loss}}}^T = g_{a\gamma\gamma}^2\overline{\left(\int d^3x\ddot{a}(\mathbf{x},t)\mathbf{B}_0(\mathbf{x})\right)^2}^T. \tag{8}$$

In the next section we summarize some well-established facts about axion stars of spherically symmetric shape while also assuming separable time and space dependences of their profile. A simple variational approximation to its binding energy allows one to determine its size and establish a functional relation with the number of its constituent axions. It becomes clear that for gravitationally bound axion clumps one cannot justify the above Taylor expansion; rather, one has to find an exact treatment for retardation effects.

III. PROFILE OF AN AXION STAR

Here we outline the construction steps and the main features characterizing the spatial profile of scalar stars emerging from an equilibrium between gravitational attraction and kinetic pressure. For this a reduction to a nonrelativistic approximation of the full theory is performed. We mostly follow the treatments of Refs. [14,24].

The Hamiltonian governing the dynamics of the axion condensate in its own gravitational field reads

$$H = \int d^3x \left[\frac{1}{2}(\dot{a}(\mathbf{x},t))^2 + \frac{1}{2}(\nabla a(\mathbf{x},t))^2 + \frac{1}{2}m_a^2 a(\mathbf{x},t)^2\right] + U_{\mathrm{grav}}. \tag{9}$$

The gravitational energy is determined by the mass-density distribution $\rho_{\mathrm{mass}}(\mathbf{x},t)$ of axions:

$$U_{\mathrm{grav}} = -\frac{G_N}{2}\int d^3x \int d^3x' \frac{\rho_{\mathrm{mass}}(\mathbf{x},t)\rho_{\mathrm{mass}}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}. \tag{10}$$

In the condensate all particles have nearly the rest-mass energy m_a, and therefore the field $a(\mathbf{x},t)$ is parametrized as the product of the corresponding harmonic oscillation with a slowly varying amplitude,

$$a(\mathbf{x},t) = \frac{1}{\sqrt{2m_a}}(e^{-im_a t}\psi(\mathbf{x},t) + e^{im_a t}\psi^*(\mathbf{x},t)),$$
$$\psi(\mathbf{x},t) = e^{-i\mu_g t}\tilde{\psi}(\mathbf{x}). \tag{11}$$

The time dependence of the slowly varying term is approximated by a small frequency shift $\mu_g \ll m_a$. When substituting this ansatz into the canonical equations one keeps only the first time derivative of $\psi(\mathbf{x},t)$ in view of its assumed slow variation. This results in the following equation:

$$i\dot{\psi} = -\frac{1}{2m_a}\triangle\psi + \frac{\delta U_{\mathrm{grav}}}{\delta(\psi^*\psi)}\psi. \tag{12}$$

As a consequence one can define a density $\psi^*\psi$, which is conserved in the present approximation. Its integral is identified with the axion number N of the clump,

$$\frac{d}{dt}(\psi^*\psi) = 0, \rightarrow N_a = \int d^3x|\tilde{\psi}(\mathbf{x})|^2 = \mathrm{const}. \tag{13}$$

The corresponding mass density $\rho_{\mathrm{mass}} = m_a\tilde{\psi}^*\tilde{\psi}$ can be used in the expression of the gravitational energy:

$$U_{\mathrm{grav}} = -\frac{G_N}{2}\int d^3x \int d^3x' \frac{(m_a|\tilde{\psi}(\mathbf{x})|^2)(m_a|\tilde{\psi}(\mathbf{x}')|^2)}{|\mathbf{x}-\mathbf{x}'|}. \tag{14}$$

Substituting Eq. (11) into the equation of ψ one arrives at an eigenvalue equation for μ_g,

$$\mu_g\tilde{\psi}(\mathbf{x}) = -\frac{1}{2m_a}\triangle\tilde{\psi}(\mathbf{x}) + \tilde{\psi}(\mathbf{x})G_N\int d^3x' \frac{m_a^2|\tilde{\psi}(\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|}. \tag{15}$$

The corresponding nonrelativistic Hamilton operator is readily identified and its minimum in the space of $\tilde{\psi}$ functions determines the binding energy of the clump:

$$H_{\mathrm{non-rel}} = \int d^3x \frac{1}{2m_a}|\nabla\tilde{\psi}|^2 + U_{\mathrm{grav}},$$
$$E_{\mathrm{ground}} = \min[H_{\mathrm{non-rel}}(\tilde{\psi})]. \tag{16}$$

The ground-state energy and the gravitational energy of the ground-state profile determine the frequency shift due to the binding:

$$N\mu_g = \int d^3x \frac{1}{2m_a}|\nabla\tilde{\psi}|^2 + 2U_{\mathrm{grav}} = E_{\mathrm{ground}} + U_{\mathrm{grav}}. \tag{17}$$

In the present paper we will be satisfied with a variational estimate after choosing a spherically symmetric ansatz for $\tilde{\psi}$:

$$\tilde{\psi}(\mathbf{x}) = wF(\xi), \qquad \xi = \frac{|\mathbf{x}|}{R}, \qquad w^2 = \frac{N}{C_2R^3}. \tag{18}$$

Here we have introduced the parameter R to be determined variationally, which characterizes the spatial extension of the clump. The quantity w is determined by the normalization condition (13). The energy function in units of m_a depends on the dimensionless parameters (m_aR, N) and also on the combination $G_Nm_a^2$ of the physical constants:

$$E(R,N) = m_a\left(\frac{D_2}{2C_2}\frac{1}{(m_aR)^2} - \frac{B_4}{2C_2^2}\frac{1}{m_aR}G_Nm_a^2N\right), \tag{19}$$

where specific integrals over the profile function $F(\xi)$ are introduced,

$$C_2 = 4\pi\int_0^\infty d\xi\xi^2F^2(\xi), \qquad D_2 = 4\pi\int_0^\infty d\xi\xi^2F'^2(\xi),$$
$$B_4 = 32\pi^2\int_0^\infty d\xi\xi F^2(\xi)\int_0^\xi d\eta\eta^2F'(\eta). \tag{20}$$

One minimizes $E(R,N)$ at a fixed value of N with respect to m_aR which yields

$$X \equiv (m_a R)_{\rm opt} = \frac{2C_2 D_2}{B_4}\frac{1}{G_N m_a^2 N},$$
$$E_a = N m_a + E_{\rm ground}, \qquad N\mu_g = \frac{3}{2}E_{\rm ground},$$
$$E_{\rm ground} = -m_a \frac{B_4^2}{8C_2^3 D_2}(G_N m_a^2 N)^2 \equiv E_g(X(NG_n m_a^2)). \quad (21)$$

The order of magnitude of the coefficient built from integrals over the clump profile is $\leq \mathcal{O}(10^2)$. Choosing for the axion mass $m_a = 10^{-14}$ GeV, one finds $G_N m_a^2 \approx 5\times 10^{-66}$. Guth *et al.* [24] argued that during the formation of axion stars (near the temperature of the QCD transition) $N \approx 10^{61}$. Using these values one finds that $X \gg 1$ and $\mu_g \ll m_a$. Retardation effects cannot be treated perturbatively for this solution.

The full energy of the axionic clump is the sum of the rest masses of the free axions and the binding energy. Both X and E_g depend parametrically on $G_N m_a^2 N \sim 5\times 10^{-66} N$.

Since here one can neglect the frequency shift μ_g relative to m_a, we use below the following trial function with separable time and spatial dependencies, and with real function $F(\xi)$:

$$a_{\rm dilute} = \sqrt{\frac{2}{m_a}}\cos(m_a t) w F(\xi). \quad (22)$$

IV. ELECTROMAGNETIC ENERGY LOSS INCLUDING RETARDATION EFFECTS

We return to the evaluation of the rate of energy change of the axion clump (5) after substituting Eq. (22) into Eq. (1). In the "magnetic" and "electric" parts of the integrand the following expressions multiply the squared photon-axion coupling:

$$\dot a(\mathbf{x},t)\ddot a(\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|)$$
$$= 2m_a^2 \sin(m_a t)\cos(m_a(t-|\mathbf{x}-\mathbf{x}'|)) w^2 F(\xi_x)F(\xi_{x'}),$$
$$(\mathbf{B}_0\cdot\nabla_x \dot a(\mathbf{x},t))(\mathbf{B}_0\cdot\nabla_{x'} a(\mathbf{x}', t-|\mathbf{x}-\mathbf{x}'|))$$
$$= 2B_0^2\left(\mathbf{n}_B\cdot\frac{\hat{\mathbf{x}}}{R}\right)\left(\mathbf{n}_B\cdot\frac{\hat{\mathbf{x}}'}{R}\right)$$
$$\times \sin(m_a t)\cos(m_a(t-|\mathbf{x}-\mathbf{x}'|)) w^2 F'(\xi_x)F'(\xi_{x'}). \quad (23)$$

In the third line the gradient operator applies only to the first arguments. The time average over the period $T = 2\pi/m_a$ is easily performed with the help of simple trigonometric identities leading to

$$\overline{\frac{dE_a}{dt}}^T = -\int d^3x \int d^3x' \frac{g_{a\gamma\gamma}^2 B_0^2 w^2}{|\mathbf{x}-\mathbf{x}'|}\sin(m_a|\mathbf{x}-\mathbf{x}'|) \times\left[m_a^2 F(\xi_x)F(\xi_{x'}) + \left(\mathbf{n}_B\cdot\frac{\hat{\mathbf{x}}}{R}\right)\left(\mathbf{n}_B\cdot\frac{\hat{\mathbf{x}}'}{R}\right) \times F'(\xi_x)F'(\xi_{x'})\right]. \quad (24)$$

One rediscovers the "three-dot" result for the energy loss (8) when one keeps only the first term of the Taylor series of $\sin(m_a|\mathbf{x}-\mathbf{x}'|)$. The near-zone electromagnetic contribution to the energy of the axion clump (6) oscillates periodically, and therefore its average vanishes. Now, one proceeds with the evaluation of the angular parts of the space integrations by exploiting the following factorization of the $|\mathbf{x}-\mathbf{x}'|$ dependence of the integrand:

$$\frac{e^{im_a|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = \left[im_a\sum_{l=0}^{\infty} j_l(m_a r_<)h_l^{(1)}(m_a r_>)\sum_{m=-l}^{l} Y_{lm}^*(\hat{\mathbf{x}})Y_{lm}(\hat{\mathbf{x}}')\right], \quad (25)$$

where $r_< = \min(|\mathbf{x}|, |\mathbf{x}|')$, $r_> = \max(|\mathbf{x}|, |\mathbf{x}|')$. In the actual expression of the integrand of the time-averaged energy loss one has to take the imaginary part of both sides. The angular integrations of the "magnetic" term only receive contributions from the $l = 0$ spherical harmonics, while in the "electric" term only Y_{10} contributes when the z axis is chosen along $\hat{\mathbf{n}}_B$. The relevant imaginary parts turn out to be invariant under the exchange of $r_<$ and $r_>$:

$$\mathrm{Im}[ij_0(m_a r_<)h_0^{(1)}(m_a r_>)] = \frac{\sin(m_a r_<)}{m_a r_<}\cdot\frac{\sin(m_a r_>)}{m_a r_>}, \quad (26)$$

$$\mathrm{Im}[ij_1(m_a r_<)h_1^{(1)}(m_a r_>)] = \left(\frac{\sin(m_a r_<)}{(m_a r_<)^2} - \frac{\cos(m_a r_<)}{m_a r_<}\right) \times\left(\frac{\sin(m_a r_>)}{(m_a r_>)^2} - \frac{\cos(m_a r_>)}{m_a r_>}\right). \quad (27)$$

The $r_< \leftrightarrow r_>$ invariance has the consequence that the radial ξ_x and $\xi_{x'}$ integrations are independent and lead to the same result. By this observation one finds a rather compact expression for the averaged electromagnetic energy loss of an axion star:

$$\overline{\frac{dE_a}{dt}}^T = -m_a\cdot m_a N\cdot\frac{g_{a\gamma\gamma}^2 B_0^2}{m_a^2}\frac{X^3}{C_2}\left(I_{\rm mag}^2 + \frac{1}{3X^2}I_{\rm el}^2\right)$$
$$\equiv -m_a\cdot m_a N\cdot\frac{g_{a\gamma\gamma}^2 B_0^2}{m_a^2}\mathcal{F}(X(G_N m_a^2 N)),$$
$$I_{\rm mag} = \int d^3\xi\frac{\sin(X\xi)}{X\xi}F(\xi),$$
$$I_{\rm el} = \int d^3\xi\left(\frac{\sin(X\xi)}{(X\xi)^2} - \frac{\cos(X\xi)}{X\xi}\right)F'(\xi). \quad (28)$$

The first three terms on the right-hand side follow the parametric form denoted as τ_γ in Ref. [19]. The temporal dependence of the decay however is essentially influenced by the X-dependent (in the final count N-dependent) factor $\mathcal{F}(X)$. This effect was observed in Refs. [17,18]. In the case of clumps stabilized by the self-interaction of axions Refs. [17,20] numerically found $\mathcal{F} \approx 2$. These investigations all arrived at the conclusion that axionic clumps decay exponentially. The only difference is in the scaled value of the decay constant $m_a \tau_\gamma$. With the help of specific choices of $F(\xi)$ we shall evaluate this factor and establish its limiting behavior in the extreme dilute ($X \gg 1$) case, relevant to the above chosen values of the physically free parameters. It will be shown that this leads to a qualitatively different conclusion concerning the temporal dependence of the number of particles in an axion star, $N(t)$.

It is notable that a completely analogous analysis of the rate of momentum transfer given in Eq. (4) arising from the magnetically induced electromagnetic radiation to the spherically symmetric axion clump leads to the conclusion that the resulting force is zero.

V. TIME EVOLUTION OF AXION PARTICLE NUMBER FOR GENERIC CLUMP PROFILES

Several specific trial profile functions were used for estimating the energy of axionic clumps. Guth *et al.* [24] chose a simple exponential taking the example of the ground-state wave function of the hydrogen atom, although it does not satisfy the prescribed boundary condition $\lim_{r\to 0} \partial_r \tilde{\psi}(|\mathbf{x}|) = 0$. In Ref. [14] a Gaussian was used for the variational estimation. For our analytic calculation an alternative cosine profile is more convenient, which was shown [14] to reproduce the Gaussian very well in the range $|\mathbf{x}| < R$. We have computed the profile integrals (20) and the factor $\mathcal{F}$ [Eq. (28)] with the following trial profiles:

$$F^{\exp} = e^{-\xi}, \qquad X^{\exp} \approx 20(G_N m_a^2 N)^{-1},$$
$$E_g^{\exp} \approx -0.31(G_N m_a^2 N)^2 m_a, \tag{29}$$

and

$$F^{\cos} = \cos^2\left(\frac{\pi\xi}{2}\right), \qquad \xi < 1, \qquad X^{\cos} \approx 11.2(G_N m_a^2 N)^{-1},$$
$$E_g^{\cos} \approx -0.25(G_N m_a^2 N)^2 m_a. \tag{30}$$

The elementary integrals which determine $I_{\mathrm{mag}}, I_{\mathrm{el}}$ in Eq. (28) lead to apparently quite different expressions for the average energy loss per particle mass:

$$\frac{1}{m_a N}\frac{\overline{dE_a^{\exp T}}}{dt} = -m_a\left(\frac{g_{a\gamma\gamma}B_0}{m_a}\right)^2 \frac{256\pi}{3}\frac{X^3}{(1+X^2)^4}, \tag{31}$$

$$\frac{1}{m_a N}\frac{\overline{dE_a^{\cos T}}}{dt} = -m_a\left(\frac{g_{a\gamma\gamma}B_0}{m_a}\right)^2 \frac{4\pi^2}{C_2} \times \left\{X\left[\frac{\cos X}{X}\frac{\pi^2}{X^2-\pi^2} + \sin X\left(\frac{1}{X^2} - \frac{X^2+\pi^2}{(X^2-\pi^2)^2}\right)\right]^2 + \frac{\pi^4}{3X(X^2-\pi^2)^2}\left[\cos X - \frac{\sin X}{X}\frac{3X^2-\pi^2}{X^2-\pi^2}\right]^2\right\}. \tag{32}$$

However, the asymptotic behaviors for both small and very large values of X turn out to be the same:

$$\mathcal{F}(X(G_N m_a^2 N)) \sim X^3, \quad X \ll 1, \quad \sim X^{-5}, \quad X \gg 1. \tag{33}$$

The coinciding asymptotic behaviors prompt the conjecture that they reflect the nature of the exact solution.

This conjecture can actually be proven for profile functions which are nonzero in the interval $\xi \in (0, \Lambda)$ and fulfill at the upper end the boundary conditions

$$F(\Lambda) = F'(\Lambda) = 0. \tag{34}$$

The asymptotic behavior can be constructed in the same steps for both I_{magn} and I_{el}. Here we give some details for the first one. The limiting behavior when $X \to 0$ is easily established by performing the limit directly in the integrand:

$$\lim_{X\to 0} I_{\mathrm{magn}} = \int d^3\xi F(\xi). \tag{35}$$

For the large-X asymptotics it is convenient to introduce the integration variable $u = X\xi$ and perform three (!) partial u integrations, taking into account the boundary conditions (34) in the intermediate steps. This results in

$$\lim_{X\to\infty} I_{\mathrm{magn}} = -\frac{8\pi}{X^4}F'(0) + \frac{4\pi}{X^5}(\Lambda X\cos(\Lambda X) + 3\sin(\Lambda X))F''(\Lambda) + \frac{4\pi}{X^6}\int_0^{\Lambda X} du(3\sin u - u\cos u)F'''(u/X). \tag{36}$$

One promptly recognizes that the contribution from the last integral is at most $\mathcal{O}(X^{-5})$, and therefore the leading asymptotics reads as

$$\lim_{X\to\infty} I_{\mathrm{magn}} = -\frac{8\pi}{X^4}F'(0) + \frac{4\pi}{X^4}\Lambda\cos(\Lambda X)F''(\Lambda) \tag{37}$$

In the case of the exponential ansatz one has $F^{\exp,\prime}(0) = -1, F^{\exp,\prime\prime}(\Lambda) = 0$, for the cosine ansatz $F^{\cos,\prime}(0) = 0, F^{\cos,\prime\prime}(\Lambda) = \pi^2/2$ and the above formula reproduces the asymptotics directly obtained from Eqs. (31) and (32).

The same steps lead for I_{el}/X^2 to the same $\sim X^{-4}$ behavior in the above class of axion clump profiles. The class is generic enough (although not the most general) to support the above conjecture for the asymptotics of the exact solution. Now we turn to the rate equation of the axion number.

In regions where the gravitational binding energy is small, one can use the approximate relation $E_a \approx N m_a$ which allows to reinterpret the rate equation of the energy as an equation for the rate of change of the axion number of the clump:

$$\frac{1}{m_a}\frac{1}{m_a N}\overline{\frac{dE_a}{dt}}^T \approx \frac{1}{N}\frac{dN}{d(m_a t)} = -\frac{g_{a\gamma\gamma}^2 B_0^2}{m_a^2}\mathcal{F}(X(G_N m_a^2 N)). \qquad (38)$$

This can certainly be applied for $X \gg 1$ and eventually one finds a slow algebraic blowing up of the clump which is accompanied (in view of the inverse relation of X and N) by the diminishing of the number of axions due to electromagnetic radiation:

$$X(t) \sim (G_N g_{a\gamma\gamma}^2 B_0^2 m_a t)^{1/5}, \qquad X \gg 1. \qquad (39)$$

VI. SUMMARY

Starting from the equation governing the energy balance of the axion sector of axion electrodynamics a dynamical rate equation has been derived for the particle number content of a gravitationally bound axion clump. Our fully analytic treatment was based on assuming a separable single-frequency, spherically symmetric ansatz for the spatiotemporal profile of the axion star. Variational treatments of two rather different looking explicit trial spatial profile functions led to the same behavior of the relative axion number rate in both the small-size and large-size regimes. This experience led us to conjecture a universal asymptotic form of the rate equation for large axion numbers. In this region the characteristic size parameter of the object slowly increases as $\sim t^{1/5}$.

[1] D. J. Kaup, Klein-Gordon geon, Phys. Rev. **172**, 1331 (1968).

[2] R. Ruffini and S. Bonnazola, Systems of self-gravitating particles in general relativity and the concept of an equation of state, Phys. Rev. **187**, 1767 (1969).

[3] I. I. Tkachev, Coherent scalar-field oscillations forming compact astrophysical object, Sov. Astron. Lett. **12**, 305 (1986).

[4] I. I. Tkachev, On the possibility of Bose-star formation, Phys. Lett. B **261**, 281 (1991).

[5] E. W. Kolb and I. I. Tkachev, Axion Miniclusters and Bose Stars, Phys. Rev. Lett. **71**, 3051 (1993).

[6] E. W. Kolb and I. I. Tkachev, Nonlinear axion dynamics and formation of cosmological pseudosolitons, Phys. Rev. D **49**, 5040 (1994).

[7] P. H. Chavanis, Mass-radius relation of Newtonian self-gravitating Bose-Einstein condensates via short range interactions: I. Analytical results, Phys. Rev. D **84**, 043531 (2011).

[8] P. H. Chavanis, Mass-radius relation of Newtonian self-gravitating Bose-Einstein condensates via short range interactions: II. Numerical results, Phys. Rev. D **84**, 043532 (2011).

[9] P. H. Chavanis and L. Delfini, Mass-radius relation of Newtonian self-gravitating Bose-Einstein condensates with short-range interactions: II. Numerical results, Phys. Rev. D **84**, 043532 (2011).

[10] J. Eby, P. Suranyi, C. Vaz, and L. Wijewardhana, Axion stars in the infrared limit, J. High Energy Phys. 03 (2015) 080.

[11] E. Braaten, A. Mohapatra, and H. Zhang, Dense Axion Stars, Phys. Rev. Lett. **117**, 121801 (2016).

[12] E. Braaten, A. Mohapatra, and H. Zhang, Emission of photons and relativistic axions from axion stars, Phys. Rev. D **96**, 031901 (2017).

[13] J. Eby, P. Suranyi, and L. Wijewardhana, The lifetime of axion stars, Mod. Phys. Lett. A **31**, 1650090 (2016).

[14] J. Eby, M. Leembruggen, P. Suranyi, and L. C. R. Wijewardhana, Collapse of axion stars, J. High Energy Phys. 12 (2016) 066.

[15] P. Sikivie, Experimental Tests of the "Invisible" Axion, Phys. Rev. Lett. **51**, 1415 (1983).

[16] E. Braaten and H. Zhang, Colloquium: The physics of axion stars, Rev. Mod. Phys. **91**, 041002 (2019).

[17] M. A. Amin, A. J. Long, Z.-G. Mou, and P. M. Saffin, Dipole radiation and beyond from axion stars in electromagnetic fields, J. High Energy Phys. 06 (2021) 182.

[18] S. Sen and L. Sivertsen, Electromagnetic radiation from axion condensates in a time dependent magnetic field, J. High Energy Phys. 05 (**2022**) 192.

[19] A. Arvinataki, S. Dimopoulos, S. Dubovsky, N. Kaloper, and J. March-Russell, String axiverse, Phys. Rev. D **81**, 123530 (2010).

[20] S. Sen and L. Sivertsen, Energy conservation and axion back reaction in a magnetic field, arXiv:2210.01149.

[21] A. Patkós, Radiation backreaction in axion electrodynamics, Symmetry **14**, 1113 (2022).

[22] P. Sikivie, Invisible axion search methods, Rev. Mod. Phys. **93**, 015004 (2021).

[23] J. Paixao, L. Ospedal, M. Neves, and J. Helayël-Neto, The axion-photon mixing in non-linear electrodynamic scenarios, J. High Energy Phys. 10 (2022) 160.

[24] A. H. Guth, M. P. Hertzberg, and C. Prescod-Weinstein, Do dark matter axion form a condensate with long range correlation?, Phys. Rev. D **92**, 103513 (2015).