Internal force distributions in the ’t Hooft-Polyakov monopole and Julia-Zee dyon

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(Received 18 January 2023; accepted 22 February 2023; published 13 March 2023)

The energy-momentum tensor (EMT) of the ’t Hooft-Polyakov monopole and the Julia-Zee dyon are studied. This tensor contains important information about the pressure and the shear force distributions which define mechanical properties of systems. Obtaining the violation of the local stability criterion for the magnetic monopole and dyon we decompose the EMTs into the long- and short-range parts. This decomposition depends on the Abelian field strength tensor which cannot be uniquely defined. We suggest to use the modified ’t Hooft definition for the tensor. Finally, the long- and short-range parts of the EMTs are computed and new equilibrium equations are obtained. Numerical values for masses, D-terms, and various mean square radii for the monopole and the dyon are also computed.

DOI: 10.1103/PhysRevD.107.055015

I. INTRODUCTION

A magnetic monopole is a hypothetical particle carrying a magnetic charge like an electron carrying an electric charge. Maxwell formulated his elegant equations in 1865 without a monopole because there was no evidence for it. Despite the fact that the monopole is still experimentally not detected there is also no proof of its nonexistence. In 1974 Polyakov and ’t Hooft showed in Refs. [1,2] that a magnetic monopole exists in all grand unified theories of elementary particles as a static soliton solution of the classical equations of motion. In 1975 Zee and Julia generalized the ’t Hooft-Polyakov monopole by including the electric charge; they called such particle a dyon in Ref. [3].

Usually the mass and related properties of a monopole are studied. In this work we are more interested in mechanical properties of a monopole and a dyon such as pressure, shear force distributions, and the mechanical stability condition which can be obtained from the corresponding energy-momentum tensor (EMT). Although the main purpose of the paper is to study the mechanical properties of the monopole and dyon, we are also interested in studying the local stability criterium of an arbitrary system with finite energy obtained in 2016 in Ref. [4]. The force distributions obtained from the EMT could be useful in experimental searches of monopoles and studying the monopole stability can be helpful for theoretical insights into the stability of classical solutions in gauge theories.

The organization of the paper is as follows. In Sec. II we introduce main definitions corresponding to an EMT and discuss the stability conditions for a static and nonstatic EMT. In Sec. III we discuss the EMT of the ’t Hooft-Polyakov monopole and its consequences. We also separate the EMT into the long- and short-range parts in this section introducing the Abelian strength tensor. In Sec. IV we discuss the properties of a Julia-Zee dyon. We conclude in the last section.

II. STABILITY CONDITIONS

A. EMT densities

The energy-momentum tensor $T_{\mu\nu}(x)$ can be defined as a functional variation of the matter part of the action in curved space-time with respect to metric $g^{\mu\nu}(x)$ as follows:

$$T_{\mu\nu}(x) = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}(x)}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}, \tag{1}$$

where $\eta_{\mu\nu}$ is a Minkowski tensor with the metric signature $(1,-1,-1,-1)$.

The EMT encodes fundamental properties of a system. The T_{00} component of the EMT defines the energy density in the studied system corresponding to the mass distribution in the static approximation. The integral of the $T_{00}(r)$ over the full space defines the full energy of a system or its mass in the rest frame of the system

$$M = \int d^3 r T_{00}(r). \tag{2}$$

The most interesting part of the EMT for our purpose is the ij components, which define the stress tensor. The T_{ij} components can be associated, according to Ref. [5], with the distribution of the shear forces $s(r)$ and the elastic pressure $p(r)$ inside the system. For a spherically symmetric system in static approximation in three dimensions the stress tensor is decomposed in $s(r)$ and $p(r)$ as follows:

$$T_{ij}(\vec{r}) = \left(\frac{r_i r_j}{r^2} - \frac{1}{3}\delta_{ij}\right)s(r) + \delta_{ij}p(r). \tag{3}$$

The pressure and shear force distributions can be connected with the gravitational $D(t)$-form factor, which is involved in the parametrization of the matrix element of the EMT operator; see e.g. Ref. [5]. As it follows from Ref. [5], the D term (or the Druck term) $D \equiv D(0)$ can be obtained as

$$D = -\frac{4M}{15}\int d^3r r^2 s(r) = M\int d^3r r^2 p(r), \tag{4}$$

where M is the mass of the system. The value of the D term for a particle can be measured experimentally, like its mass; however, the value is difficult to extract from experimental data. For example, the experimental value of the D term for the proton is $D = -1.47 \pm 0.06 \pm 0.14$, where the first error is the statistical uncertainty, and the second error is due to the systematic uncertainties; see Ref. [6]. The value of the D term for the nucleon was first obtained theoretically in the Skyrme model in Ref. [7]; the authors obtained the value $D = -3.6$ and in the bag model the value of the D term for the nucleon is $D = -1.1$ [8], other values of the D term together with the corresponding references can be found in the Table 2 of Ref. [9]. The values of the D term for the monopole and the dyon are given in Secs. III and IV, respectively.

B. Static EMT

The EMT is the conserved Noether current associated with space-time translations, i.e. it satisfies the following condition:

$$\partial_\mu T^{\mu\nu} = 0. \tag{5}$$

For the static EMT Eq. (5) turns to $\partial_i T^{i\nu} = 0$. For the parametrization of Eq. (3) it implies that the pressure and shear force distributions satisfy the following equation which is also called the equilibrium equation:

$$\frac{\partial p(r)}{\partial r} + \frac{2}{3}\frac{\partial s(r)}{\partial r} + \frac{2}{r}s(r) = 0. \tag{6}$$

Assuming that the pressure and shear force densities decay at large distances faster than $\sim\frac{1}{r^3}$, multiplying on r^3 and integrating over r one obtains the von Laue stability condition given in Ref. [10],

$$\int d^3r p(r) = 0, \tag{7}$$

which is necessary for stability, but not sufficient, since it is also satisfied for unstable systems; see discussion in Ref. [4]. This condition is satisfied for any system, whose EMT is conserved and implies that the pressure distribution has at least one mode. Integrating the differential equation (6) from positive value r to infinity and requiring that $s(r)$ and $p(r)$ vanish at the infinity, one obtains

$$\frac{2}{3}s(r) + p(r) = 2\int_r^\infty dx \frac{s(x)}{x}. \tag{8}$$

This equation describes the equilibrium of the internal forces inside a system. The combination $\frac{2}{3}s(r) + p(r)$ describes the normal component of the total force exhibited by the system on an infinitesimal piece of area dS_i, which is denoted as $p_\mathrm{r}(r)$:

$$F^i(r) = T^{ij}(r)dS_j = \left(\frac{2}{3}s(r) + p(r)\right)dS^i = p_\mathrm{r}(r)dS^i \tag{9}$$

where $dS^i = dS r^i/r$. From the equilibrium equation (8) follows that for the positive shear force distribution $s(r)$ the normal force is always positive and for the negative $s(r)$ the normal force is always negative. In Ref. [4] it was argued that the normal force has to be directed outwards otherwise the system would collapse, this means that it has to be non-negative, i.e.

$$\frac{2}{3}s(r) + p(r) \geq 0. \tag{10}$$

Thus, according to Eq. (8) for negative shear force distribution the system is definitely unstable. The local stability condition is also necessary, but not sufficient for stability analogously to the von Laue condition. However, due to its local character, it is stronger than the von Laue condition.

C. EMT and external forces

The EMT conservation taking the form $\partial_i T^{ij}(r) = f^j(r)$ can be interpreted according to Ref. [11] as the equilibrium equation for internal stress and external force f_j (per unit of the volume). According to the parametrization in Eq. (3) the equilibrium equation gets the following form

$$\frac{d}{dr}\left(\frac{2}{3}s(r) + p(r)\right) + 2\frac{s(r)}{r} = f(r), \tag{11}$$

where $f(r)$ is the normal component of the external force per unit of the volume. This equation describes the balance between internal forces pushing out from center and external force pulling inwards to the center. When the corresponding forces are equal, the system is at equilibrium. The von Laue stability condition for such equilibrium equation gets the following form:

$$\int d^3r p(r) = -\frac{1}{3}\int d^3r r f(r). \tag{12}$$

One can also rewrite the equilibrium equation (11) as

$$p_{\rm r}(r) + \sigma(r) = 2\int_r^\infty dx \frac{s(x)}{x}, \tag{13}$$

with $\sigma(r) = \int_r^\infty dx f(x)$. The left-hand side describes the normal component of a total force of the system acting on the infinitesimal unit area dS^i as it follows from below

$$\begin{aligned} F^i(r) &= (T^{ij}(r) + \delta^{ij}\sigma(r))dS_j \\ &= \left(\frac{2}{3}s(r) + p(r) + \int_r^\infty dx f(x)\right)dS^i \\ &= (p_{\rm r}(r) + \sigma(r))dS^i. \end{aligned} \tag{14}$$

Then for systems affected by an external force, i.e. for systems described by the equilibrium equation (11), the local stability criterium of Eq. (10) can be modified as

$$\frac{2}{3}s(r) + p(r) + \int_r^\infty dx f(x) \geq 0. \tag{15}$$

For a time-dependent system the conservation of EMT gives the equation $\partial_i T^{ij} = -\partial_0 T^{0j}$, where the left-hand side again describes internal force of the system and the right-hand side can be interpreted as an external force.

D. Mean square energy radius and mechanical radius

For a positive energy density $T_{00}(r)$ the mean square radius of the energy density can be introduced as

$$\langle r^2\rangle_E = \frac{\int d^3r r^2 T_{00}(r)}{\int d^3r T_{00}(r)}, \tag{16}$$

which characterizes the size of the system in which the energy is distributed.

According to the local stability condition, for a stable system the radial force must be positive, so the mechanical mean square radius where the normal force is distributed can be defined as in Ref. [12]; for a system with the equilibrium equation (6) it takes the form

$$\langle r^2\rangle_{\rm mech} = \frac{\int d^3r r^2 p_{\rm r}(r)}{\int d^3r p_{\rm r}(r)}, \tag{17}$$

and for the system with the equilibrium equation (11) it takes has the form

$$\langle r^2\rangle_{\rm mech} = \frac{\int d^3r r^2 (p_{\rm r}(r) + \sigma(r))}{\int d^3r (p_{\rm r}(r) + \sigma(r))}, \tag{18}$$

where $p_{\rm r}(r)$ and $\sigma(r)$ are defined above.

The local stability condition has not been mathematically proven and thereby is still questioned; see e.g. criticism in Ref. [13]. Moreover, as it was mentioned in recent studies of Refs. [14–16], the local stability condition is not satisfied in the presence of the long-range forces. Additionlly, as pointed out in Refs. [14,15,17,18], there is another problem for systems with the long-range contribution, namely, the divergence of essential quantities that describe mechanical properties of a system, such as the D term and the mean square radii corresponding to the EMT densities.

At this point, it is important to mention that the 't Hooft-Polyakov monopole is accepted to be a stable system; see arguments of Refs. [19,20]. As we will see later, the local stability condition is violated for the monopole and the dyon. At first glance, therefore, one could think that the criterium is not correct. However, the monopole and the dyon involve electromagnetic interaction, which supports the idea that the local stability condition does not apply correctly in the presence of long-range forces.

III. 't HOOFT-POLYAKOV MONOPOLE

A. Equations of motion

The grand unified theories combine the electromagnetic, weak, and strong forces into a single force. One of the first such theories was suggested by Georgi and Glashow in 1974 in Ref. [21]. Later Polyakov and 't Hooft independently found that magnetic monopoles automatically appear in all grand unified theories [1,2]. The most simple model where the magnetic monopole exists is the gauge SU(2) Georgi-Glashow model with Higgs triplet field φ^a, $a = 1, 2, 3$, which belongs to adjoint representation. The corresponding gauge invariant action of the model is

$$S = \int d^4x \left[-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{1}{2}(\mathcal{D}_\mu\varphi)^a(\mathcal{D}^\mu\varphi)^a - \frac{\lambda}{4}(\varphi^a\varphi^a - v^2)^2\right], \tag{19}$$

where λ is a dimensionless coupling constant, v^2 is the squared vacuum expectation value of the Higgs field, and μ and ν are Lorenz indices. The covariant derivative and the non-Abelian field strength tensor are defined as follows:

$$\begin{aligned} (\mathcal{D}_\mu\varphi)^a &= \partial_\mu\varphi^a + g\epsilon^{abc}A^b_\mu\varphi^c, \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu, \end{aligned} \tag{20}$$

where g is the gauge coupling constant and A_μ^a is the gauge vector field. Choosing the vacuum expectation value as $\varphi_0^a = (0, 0, v)$ and considering small fluctuations of the ground state in unitary gauge, one can show that the Georgi-Glashow model has one massive scalar field $\eta_3(x)$ with mass $m_H = \sqrt{2\lambda}v$, one massless vector field A_μ^3 corresponding to the $U(1)$ subgroup of the $SU(2)$ gauge group, and two massive vector fields A_μ^1 and A_μ^2 both with masses $m_V = gv$.

In this model we are interested in a soliton solution; in other words, in a static solution of classical field equations with finite energy. As we want to study static soliton configuration, we consider the fields $A_i^a(\vec{x})$ and $\varphi^a(\vec{x})$ independent of time. We also fix zero component of a vector field through the gauge $A_0^a = 0$ to have zero electric field. Requiring the energy to be finite the following configuration of fields can be found:

$$\varphi^a = n^a v h(r),$$
$$A_i^a = \frac{1}{gr}\epsilon_{aij} n_j (1 - F(r)), \tag{21}$$

with the unit vector $n^a = \frac{r^a}{r}$, the unknown profile functions $h(r)$ and $F(r)$, and the following boundary conditions:

$$F(r) = 0 \quad \text{at } r \to \infty, \qquad F(r) = 1 \quad \text{at } r \to 0,$$
$$h(r) = 1 \quad \text{at } r \to \infty, \qquad h(r) = 0 \quad \text{at } r \to 0. \tag{22}$$

The explicit form of the profile functions can be obtained from equations of motion which can be computed by varying the action with respect to scalar and vector fields, which for the static case reduces to the form

$$\mathcal{D}_i F_{ij}^a = g\epsilon^{abc}\varphi^b (\mathcal{D}_j\varphi)^c,$$
$$\mathcal{D}_i(\mathcal{D}_i\varphi)^a = \lambda(\varphi^b\varphi^b - v^2)\varphi^a. \tag{23}$$

In terms of the profile functions the equations transform to

$$F''(r) - \frac{F(r)(F^2(r) - 1)}{r^2} - g^2v^2F(r)h^2(r) = 0,$$
$$h''(r) + 2\frac{h'(r)}{r} - 2\frac{F^2(r)h(r)}{r^2} + \lambda v^2 h(r)(1 - h^2(r)) = 0. \tag{24}$$

Rescaling the argument r with dimensionless argument ρ as $\rho = \frac{r}{R_0}$, where $R_0 = \frac{1}{m_V} = \frac{1}{gv}$ is a typical size of the solution, and introducing the new parameter $\beta^2 = 2\frac{\lambda}{g^2} = \frac{m_H^2}{m_V^2}$ we obtain the following system of equations of motion:

$$F''(\rho) - \frac{F(\rho)(F^2(\rho) - 1)}{\rho^2} - F(\rho)h^2(\rho) = 0,$$
$$h''(\rho) + 2\frac{h'(\rho)}{\rho} - 2\frac{F^2(\rho)h(\rho)}{\rho^2} + \frac{\beta^2}{2}h(\rho)(1 - h^2(\rho)) = 0, \tag{25}$$

with the same boundary conditions as before. This system can be solved analytically only for the limit $\beta = 0$, otherwise one solves it numerically. However, one can find approximate solutions at small and large distances:

$$F(\rho) \underset{\rho\to\infty}{\simeq} C_F e^{-\rho}\left(1 - \frac{1}{2\rho} + \frac{3}{8\rho^2} + O\left(\frac{1}{\rho^3}\right)\right),$$
$$F(\rho) \underset{\rho\to 0}{\simeq} 1 + a\rho^2 + \sum_{n=2}^{\infty} c_n\rho^{2n},$$
$$h(\rho) \underset{\rho\to\infty}{\simeq} 1 - C_h\frac{e^{-\beta\rho}}{\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right) - \frac{2C_F^2}{\beta^2 - 4}\frac{e^{-2\rho}}{\rho^2}\left(1 - O\left(\frac{1}{\rho}\right)\right),$$
$$h(\rho) \underset{\rho\to 0}{\simeq} b\rho + \sum_{n=2}^{\infty} d_n\rho^{2n-1}, \tag{26}$$

where C_F, C_h, a, and b are free constants. Note that for $\beta = 2$ the constant $C_F = 0$. The coefficients c_n and d_n can be expressed in terms of a and b as

$$c_2 = \frac{1}{10}(3a^2 + b^2),$$
$$c_3 = \frac{1}{70}\left(7a^3 + 6ab^2 - \frac{b^2\beta^2}{4}\right),$$
$$d_2 = \frac{b}{10}\left(4a - \frac{\beta^2}{2}\right),$$
$$d_3 = \frac{1}{280}\left(48a^2b + 4b^3 - 4ab\beta^2 + 5b^3\beta^2 + b\frac{\beta^2}{2}\right). \tag{27}$$

In Ref. [22] another system of differential equations was obtained and analytically solved for complex non-Abelian monopole and dyon fields.

B. EMT densities

The EMT for the 't Hooft-Polyakov monopole can be obtained variating a generally covariant form of Georgi-Glashow action (19) with respect to the metric

$$T_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + F_{\mu\alpha}^a F^{\alpha a}{}_\nu + (\mathcal{D}_\mu\varphi)^a(\mathcal{D}_\nu\varphi)^a. \tag{28}$$

For the static case one gets

$$T_{00} = \frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2} (\mathcal{D}_i \varphi)^a (\mathcal{D}_i \varphi)^a + \frac{\lambda}{4} (\varphi^a \varphi^a - v^2)^2,$$
$$T_{ij} = -\frac{1}{4} \delta_{ij} F^a_{km} F^a_{km} - F^a_{ik} F^a_{kj} - \frac{1}{2} \delta_{ij} (\mathcal{D}_k \varphi)^a (\mathcal{D}_k \varphi)^a + (\mathcal{D}_i \varphi)^a (\mathcal{D}_j \varphi)^a - \frac{\lambda}{4} \delta_{ij} (\varphi^a \varphi^a - v^2)^2. \tag{29}$$

Using the 't Hooft-Polyakov ansatz given in Eq. (21) and the decomposition in Eq. (3) the EMT densities can be expressed in terms of the profile functions $F(\rho)$ and $h(\rho)$:

$$T_{00}(\rho) = \frac{1}{R_0^4 g^2} \left(\frac{F'^2}{\rho^2} + \frac{(1-F^2)^2}{2\rho^4} + \frac{1}{2} h'^2 + \frac{h^2 F^2}{\rho^2} + \frac{\beta^2}{8} (1-h^2)^2 \right),$$
$$s(\rho) = \frac{1}{R_0^4 g^2} \left(\frac{F'^2}{\rho^2} - \frac{(1-F^2)^2}{\rho^4} + h'^2 - \frac{1}{\rho^2} F^2 h^2 \right),$$
$$p(\rho) = \frac{1}{R_0^4 g^2} \left(\frac{1}{3} \frac{F'^2}{\rho^2} + \frac{(1-F^2)^2}{6\rho^4} - \frac{1}{6} h'^2 - \frac{1}{3} \frac{F^2}{\rho^2} h^2 - \frac{\beta^2}{8} (1-h^2)^2 \right). \tag{30}$$

The $T_{00}(\rho)$ component provides the information about spatial distribution of the monopole mass; $p(\rho)$ and $s(\rho)$ describe the pressure and shear force distributions inside the monopole. The first two terms in these densities originate from the term $\sim F^a_{\mu\sigma} F^{a\sigma}{}_{\nu}$ in action, the next two terms come from $\sim \mathcal{D}_\mu \varphi^a \mathcal{D}_\nu \varphi^a$, and the term proportional to β^2 comes from the Higgs potential.

Before we discuss the numerical results for the EMT density distributions and their properties, we compute their asymptotic behavior at large and small distances using the behavior of the profile functions from Eq. (26):

$$T_{00}(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2} \left(\frac{1}{2\rho^4} + \beta^2 C_h^2 \frac{e^{-2\beta\rho}}{\rho^2} \left[1 + O\left(\frac{1}{\rho}\right) \right] + 2 C_F^2 \frac{e^{-2\rho}}{\rho^2} \left[1 + O\left(\frac{1}{\rho}\right) \right] \right),$$
$$s(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2} \left(-\frac{1}{\rho^4} + \beta^2 C_h^2 \frac{e^{-2\beta\rho}}{\rho^2} \left[1 + O\left(\frac{1}{\rho}\right) \right] + C_F^2 \frac{e^{-2\rho}}{\rho^4} \left[1 + O\left(\frac{1}{\rho}\right) \right] \right),$$
$$p(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2} \left(\frac{1}{6\rho^4} - \frac{2}{3} \beta^2 C_h^2 \frac{e^{-2\beta\rho}}{\rho^2} \left[1 + O\left(\frac{1}{\rho}\right) \right] - \frac{2C_F^2}{3} \frac{e^{-2\rho}}{\rho^4} \left[1 + O\left(\frac{1}{\rho}\right) \right] \right), \tag{31}$$

and

$$T_{00}(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2} \left(\left[6a^2 + \frac{3b^2}{2} + \frac{\beta^2}{8} \right] + \rho^2 \left[8a^3 + 6ab^2 - \frac{\beta^2 b^2}{2} \right] + O(\rho^4) \right),$$
$$s(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2} \left([-8a^3 + 2ab^2 - \beta^2 b^2] \frac{\rho^2}{5} + O(\rho^4) \right),$$
$$p(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2} \left(\left[2a^2 - \frac{b^2}{2} - \frac{\beta^2}{8} \right] + \frac{\rho^2}{3} [8a^3 - 2ab^2 + \beta^2 b^2] + O(\rho^4) \right), \tag{32}$$

where C_h, C_F and a, b are free constants from the asymptotic behavior of profile functions in Eq. (26). From the large distances behavior, it is clear that the power-law decay ($\sim\rho^{-4}$) is dominating; this behavior corresponds to the contribution of the electromagnetic interaction.[1] This is not surprising if one remembers that the mass spectrum of the Lagrangian (19) has one massless vector particle corresponding to the $U(1)$ group.

The asymptotic behavior of the energy density is definitely positive at the infinity as well as at the origin. This hints at the fact that we are dealing with an usual system. The outer region of the pressure distribution has definitely a positive sign and that of the shear force distribution a negative one. Such behavior leads to violation of the local stability criterion (10). However, this behavior is related only to the electromagnetic contribution. Note that because of this long-range contribution both mean square radii in Eqs. (16) and (17) as well as the Druck term in Eq. (4) diverge.

[1]We will call this contribution the long-range interaction.

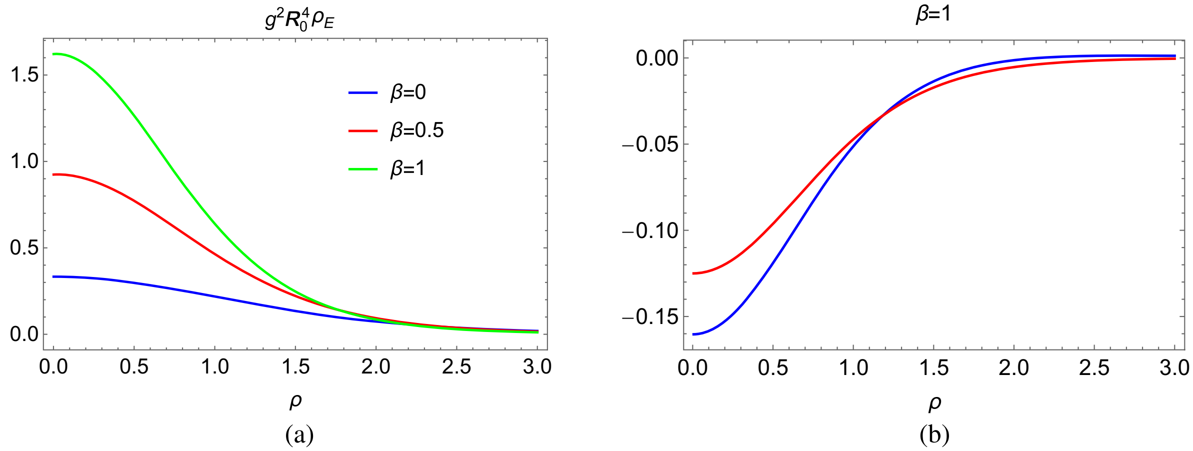

FIG. 1. (a) Energy distribution as a function of $\rho = r/R_0 = gvr$ for various values of β. (b) Blue line denotes the full pressure distribution $p(\rho)$ as a function of $\rho = r/R_0 = gvr$. Red line denotes the contribution of the Higgs potential only to the full pressure.

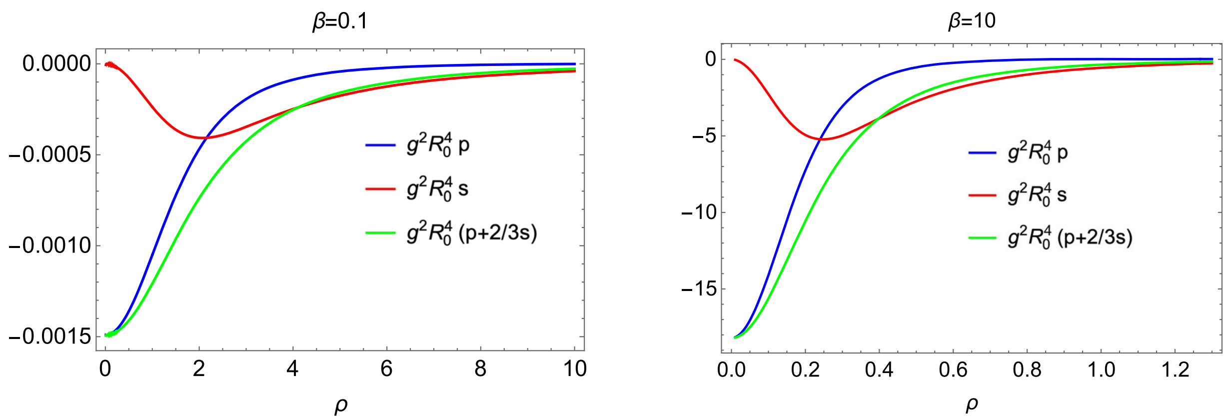

FIG. 2. Pressure distribution, shear force distribution, and stability condition from Eq. (10) as functions of $\rho = r/R_0 = gvr$ for various values of β.

In Fig. 1 the energy density for various values of β is shown. Although the energy is growing with β, according to Ref. [23] it does not diverge for $\beta \to \infty$. The distributions of the shear force, pressure, and normal force are shown in Fig. 2. Although the pressure is mostly negative, it changes the sign which allows the von Laue condition (7) to be satisfied, see e.g. Fig. 6. It is also interesting that the maximum of pressure is three times larger than the maximum of shear forces like in many other systems studied before; see e.g. Refs. [4,5,7,12,14]. From the figure one also sees that the stability condition (10) is violated everywhere and for every choice of β, although it is proved in Refs. [19,20] that the monopole is stable. Since the local stability condition is not mathematically proved, we cannot conclude from its violation that the monopole is unstable. This violation can be also explained by the fact that the stability criterion is inapplicable for systems with the long-range force that is present in the monopole. The authors of Refs. [14,15] also obtained the violation of the stability criterion for stable systems like a proton in the presence of long-range forces. All of these aspects motivate us to think that the stability condition in Eq. (10) can be applied only for the systems where only the short-range interactions are present.

C. BPS limit

The equations of motion can be solved analytically only for the limit $\beta = 0$. The solution of these equations was first obtained in Ref. [24]. In Ref. [25] Bogomolny solved these equations by reducing the system of the second order equations to the first order by considering the energy functional for the Georgi-Glashow model. By integrating the energy density over the volume one gets the energy functional

$$E_{\beta=0} = \int d^3 r T_{00}^{\beta=0}(r) = \int d^3 r \left[\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} (\mathcal{D}_i \varphi)^a (\mathcal{D}_i \varphi)^a\right]. \quad (33)$$

Rewriting it in terms of the chromomagnetic field $H_i^a = -\frac{1}{2}\epsilon_{ijk}F_{jk}^a$, Bogomolny introduced the following inequality for the monopole energy, which is also called BPS bound:

$$E_{\beta=0} \geq \int d^3 r H_i^a \mathcal{D}_i \varphi^a, \tag{34}$$

where the Bianchi identity $(\mathcal{D}_i H_i)^a = 0$ and the Gauss's theorem were used. The BPS limit gives the possible minimum of the static energy of the monopole and the equality holds for the Bogomolny equation

$$\mathcal{D}_i \varphi^a = H_i^a. \tag{35}$$

From the definition of a chromomagnetic field the following expression can be obtained:

$$H_i^a H_j^a = F_{ik}^a F_{kj}^a + \frac{1}{2}\delta_{ij} F_{km}^a F_{km}^a. \tag{36}$$

Using this expression and the Bogomolny equation (35) it can be shown that the spatial components of the EMT disappear:

$$T_{ij}(r) = 0, \tag{37}$$

independently of the choice of the ansatz in Eq. (21).

The vanishing of the pressure and shear force distributions at the BPS limit allows one to assume that the main contribution for the stress tensor T_{ij} is given by the Higgs potential. As can be seen from Fig. 1 the largest contribution to the pressure distribution is indeed due to the term originating from the Higgs potential in Eq. (19). Although the shear force distribution does not depend explicitly on β, it is related to the pressure distribution due to the differential equation (6). We think that in all models where it is possible to construct the BPS limit, the corresponding ij components of the EMT would vanish. We have also obtained such vanishing, for example, for the baby Skyrme model. The vanishing of the pressure and shear force distributions for the BSP limit indicates that the 't Hooft-Polyakov monopole in this limit has the isotropic matter distribution [12].

D. Abelian field strength tensor and electromagnetic EMT

In the previous subsection we obtained the EMT in non-Abelian $SU(2)$ gauge theory; however, the theory has one massless vector field corresponding to the Abelian $U(1)$ subgroup. This massless vector field is responsible for the electromagnetic contribution to the EMT and for the divergence of such mechanical properties of the monopole as the D term, mean square energy, and mechanical radii. We will subtract this long-range contribution from the EMT and study the remaining short-range structure. For this we have to determine the $U(1)$ Abelian field strength tensor. Let us denote the Abelian field strength tensor as $\mathcal{F}_{\mu\nu}$. Since in the unitary gauge the massless vector field of the theory is the third component of the vector field A_3^μ (see Sec. III A), the expression for the Abelian field strength tensor must be $SU(2)$ gauge invariant and coincide with $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$ in the unitary gauge [2]. We introduce a general definition of the Abelian field strength tensor without any requirements on ϕ as

$$\mathcal{F}_{\mu\nu} = \phi^a F_{\mu\nu}^a - \frac{c_1}{g}\varepsilon^{abc}\phi^a \mathcal{D}_\mu \phi^b \mathcal{D}_\nu \phi^c. \tag{38}$$

The constant c_1 is not fixed by the above requirements for the Abelian field strength tensor $\mathcal{F}_{\mu\nu}$; however, it is needed to define the magnetic charge density in such a way that it coincides with the topological charge density. 't Hooft suggested in Ref. [2] the tensor with the unit vector $\phi^a = \hat{\varphi}^a = \frac{\varphi^a}{|\varphi|}$ and $c_1 = 1$, Faddeev offered in Ref. [26] the tensor with $\phi^a = \frac{\varphi^a}{v}$ and $c_1 = 0$, where φ^a is the ansatz from Eq. (21), and Boulware in Ref. [27] suggested the tensor with $\phi^a = \hat{\varphi}^a = \frac{\varphi^a}{|\varphi|}$ and $c_1 = 0$. Since the underlying theory combines the long- and short-range forces, it is not possible to uniquely define the $U(1)$ field strength tensor to separate the short-range interaction from the long one.

For the general choice of the Abelian field strength tensor the magnetic charge density gets the following form:

$$\begin{aligned}\rho_M &= -\frac{1}{2}\varepsilon_{ijk}\partial_i \mathcal{F}_{jk} \\ &= \frac{1}{2}\varepsilon_{ijp}(\mathcal{D}_p\phi)^a F_{ij}^a (c_1 \vec{\phi}^2 - 1) \\ &\quad + \frac{c_1}{2g}\varepsilon^{abc}\varepsilon_{ijp}(\mathcal{D}_i\phi)^a(\mathcal{D}_j\phi)^b(\mathcal{D}_p\phi)^c \\ &\quad - \frac{c_1}{2}\varepsilon_{ijp}\phi^a F_{ij}^a \frac{1}{2}\partial_p \vec{\phi}^2.\end{aligned} \tag{39}$$

One can show that for 't Hooft's definition of the Abelian field strength tensor, the magnetic and topological charge densities coincide and describe a single pointlike particle with the magnetic charge $\frac{4\pi}{g}$ at the origin; see Ref. [2]. Such monopole is called Dirac's monopole. In Ref. [28] Dirac derived the quantization of the magnetic charge as $Q_M = Q_T/g$, where Q_M is a magnetic charge and Q_T is a topological charge.

In contrast to 't Hooft's definition of the Abelian field strength tensor, Faddeev's and Boulware's definitions describe the magnetic charge density smoothly without singularities at the origin. The topological and magnetic charge densities are not equal in these cases, but the quantization of the magnetic charge is satisfied. It is also interesting to notice that the magnetic charge density in Eq. (39) for Faddeev's and Boulware's definitions of the Abelian strength tensor coincides with the energy density in the BSP limit in Eq. (34) up to some dimensional normalization factor. For the positive magnetic charge distribution, the mean square radius can be determined,

$$\langle r^2\rangle_M = \frac{\int d^3r r^2 \rho_M(r)}{\int d^3 r \rho_M(r)}. \tag{40}$$

We define the electromagnetic EMT analogously to the EMT in electrodynamics as

$$T^C_{\mu\nu} = \mathcal{F}_\mu{}^\alpha \mathcal{F}_{\alpha\nu} + \frac{1}{4}\eta_{\mu\nu}\mathcal{F}^{\alpha\beta}\mathcal{F}_{\alpha\beta}, \tag{41}$$

where $\mathcal{F}_{\mu\nu}$ is the $U(1)$ field strength tensor. As we have discussed $\mathcal{F}_{\mu\nu}$ is not uniquely defined. Since we are interested in the spatial structure of the monopole, it makes no sense to define $T^C_{\mu\nu}$ according to 't Hooft's definition of the $\mathcal{F}_{\mu\nu}$ because of singularities at the origin. Faddeev's and Boulware's smooth definitions of the Abelian strength tensor $\mathcal{F}_{\mu\nu}$ could be a better candidate to define the $T^C_{\mu\nu}$; however, we suggest another definition of the $U(1)$ field strength tensor,

$$\mathcal{F}_{\mu\nu} = \frac{\varphi^a}{v}F^a_{\mu\nu} - \frac{1}{gv^3}\varepsilon^{abc}\varphi^a\mathcal{D}_\mu\varphi^b\mathcal{D}_\nu\varphi^c, \tag{42}$$

which is also smooth as Faddeev's and Boulware's definitions are and it coincides with the 't Hooft's definition at long distances. For this definition of the Abelian strength tensor the electromagnetic EMT distributions have the following form:

$$T^C_{00}(\rho) = \frac{1}{R_0^4 g^2}\frac{Q^2(\rho)}{2\rho^4},$$
$$p^C(\rho) = \frac{1}{R_0^4 g^2}\frac{Q^2(\rho)}{6\rho^4},$$
$$s^C(\rho) = -\frac{1}{R_0^4 g^2}\frac{Q^2(\rho)}{\rho^4}, \tag{43}$$

where $Q(r) = h(r)[1 - F^2(r)(1 - h(r)^2)]$. The function $Q(r)$ is directly related to the magnetic charge density through

$$\rho_M(\rho) = \frac{1}{gR_0^3}\frac{1}{\rho^2}\frac{d}{d\rho}Q(\rho). \tag{44}$$

The electromagnetic EMT densities are shown in Fig. 3. The corresponding asymptotic behavior can be found in Appendix (A3). Note that since the full spatial part of the EMT in the BSP limit where $\beta = 0$ vanishes, see Sec. III C, and the electromagnetic EMT does not depend on β (and thereby does not vanish in this limit) the short-range part of the EMT in Eq. (45) has to be equal to the long-range part. This means that the scalar interaction in the BSP limit becomes long-range interacting.

E. Short-range part of the EMT and its consequences

We denote the short-range part of the EMT as $T^{SR}_{\mu\nu}$ and define it as

$$T^{SR}_{\mu\nu} = T_{\mu\nu} - T^C_{\mu\nu}, \tag{45}$$

where the full EMT $T_{\mu\nu}$ is obtained in Eq. (30) and the electromagnetic EMT in Eq. (43). After some simple algebraic calculation the following expressions for the short-range part of the EMT densities can be found:

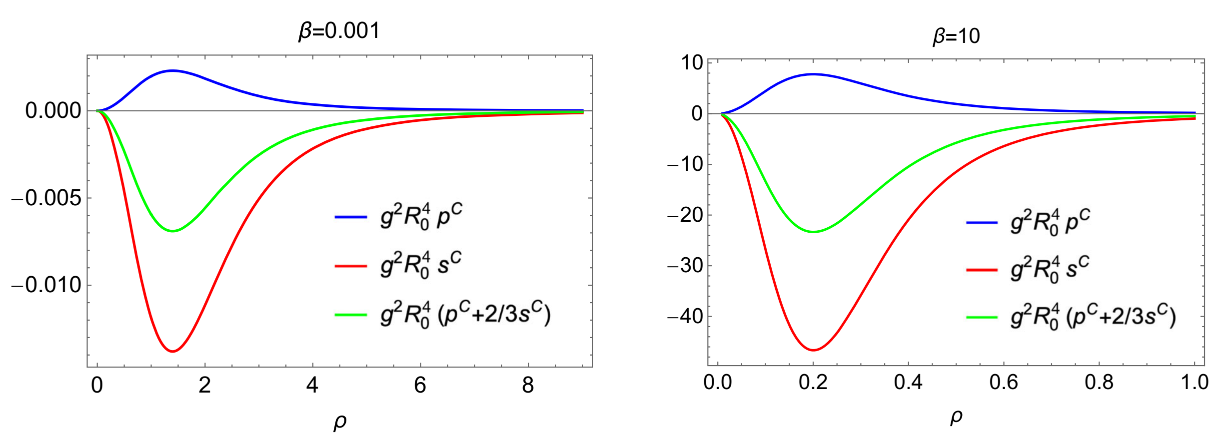

FIG. 3. The electromagnetic pressure, share force and normal force distributions as functions of $\rho = r/R_0 = gvr$ from Eq. (43) for various values of β.

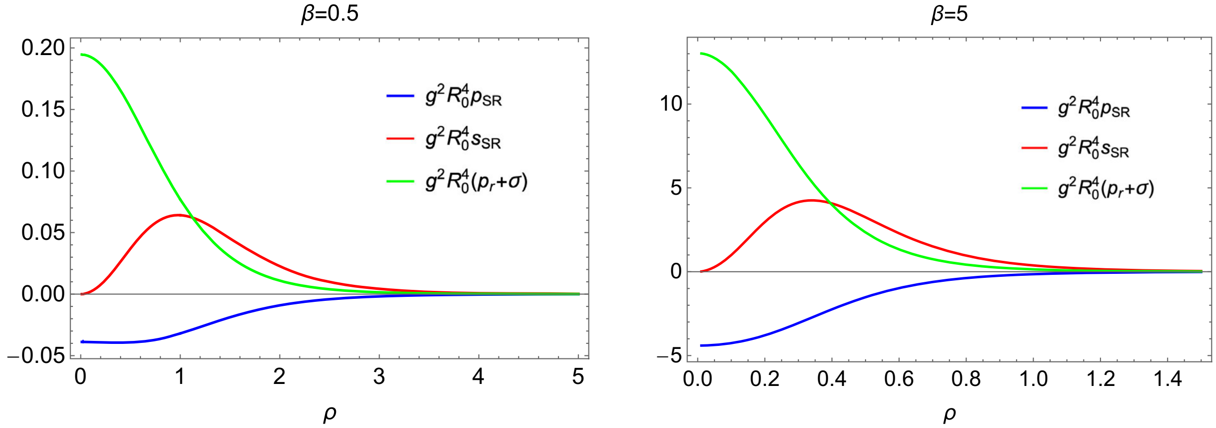

FIG. 4. Short-range part of the pressure, shear force, and total normal force distributions from Eq. (46) as a function of $\rho = r/R_0 = gvr$ for various values of β. Note that the total normal force is defined as in Eq. (15).

$$T_{00}^{SR}(\rho) = \frac{1}{R_0^4 g^2}\left(\frac{F'^2}{\rho^2} + \frac{(1-h^2)(1-F^2)^2}{2\rho^4} + \frac{1}{2}h'^2 + \frac{h^2F^2}{\rho^2}\left(1 - \frac{h^2}{\rho^2}\left[1 - F^2\left(1 - \frac{h^2}{2}\right)\right]\right) + \frac{\beta^2}{8}(1-h^2)^2\right),$$
$$p^{SR}(\rho) = \frac{1}{R_0^4 g^2}\left(\frac{1}{3}\frac{F'^2}{\rho^2} + \frac{(1-h^2)(1-F^2)^2}{6\rho^4} - \frac{1}{6}h'^2 - \frac{1}{3}\frac{F^2h^2}{\rho^2}\left(1 + \frac{h^2}{\rho^2}\left[1 - F^2\left(1 - \frac{h^2}{2}\right)\right]\right) - \frac{\beta^2}{8}(1-h^2)^2\right),$$
$$s^{SR}(\rho) = \frac{1}{R_0^4 g^2}\left(\frac{F'^2}{\rho^2} - \frac{(1-h^2)(1-F^2)^2}{\rho^4} + h'^2 - \frac{F^2h^2}{\rho^2}\left(1 - 2\frac{h^2}{\rho^2}\left[1 - F^2\left(1 - \frac{h^2}{2}\right)\right]\right)\right). \quad (46)$$

The corresponding asymptotic behavior can be found in Appendix (A4). The short-range part of the pressure and shear force distributions are presented in Fig. 4. The short-range part of the pressure distribution is negative and the short-range part of the shear force distribution is positive for every choice of β in the considered range of r. The total normal force distribution is positive for any value of β, which satisfies the stability criterium (15). It is also interesting to notice that the pressure distribution of the monopole does not change drastically after the subtraction of the long-range part; however, the shear force distribution does.

As we have already discussed in the first section, the static EMT according to the Noether theorem must be conserved. The conservation of EMT couples pressure and shear force densities by the differential equation (6). However, after the decomposition of the EMT into the short- and long-range parts, the separate EMTs are not conserved and the corresponding short- and long-range parts of the pressure and shear force densities couple due to the new equilibrium equations:

$$\frac{d}{dr}\left(\frac{2}{3}s^C(r) + p^C(r)\right) + 2\frac{s^C(r)}{r} = -\frac{Q_M(r)\rho_M(r)}{4\pi r^2},$$
$$\frac{d}{dr}\left(\frac{2}{3}s^{SR}(r) + p^{SR}(r)\right) + 2\frac{s^{SR}(r)}{r} = \frac{Q_M(r)\rho_M(r)}{4\pi r^2}, \quad (47)$$

where $Q_M(r) = \int_{|\vec{x}|<r} d^3x \rho_M(\vec{x}) = \frac{4\pi}{g} Q(r)$ is the magnetic charge contained in a sphere of radius r. So the right-hand side describes the "Coulomb force" of the magnetically charged sphere acting on the magnetic charge density.[2] Thereby the first equation is the equation of magnetostatic equilibrium between the "Coulomb stress" pushing the monopole outwards and the magnetic Coulomb force pulling the monopole inward to the center. In contrast, the second equation describes the balance between the "short-range stress" pulling the monopole inward to the center and the repulsive magnetic Coulomb force pushing the monopole outward. We notice that since the right-hand sides of the equilibrium equations are associated with the magnetic charge density, it is not possible to define the long- and short-range parts of the EMT using the ambiguity of the $\mathcal{F}_{\mu\nu}$ in such a way that the decomposed EMTs would be conserved separately, i.e. the right-hand sides would vanish, except the case with zero magnetic charge density.

As we discussed, the D term of the monopole diverges because of the long-range contribution. After subtracting the long-range part from the EMT, the corresponding D term still diverges for small values of β. We illustrate this divergence in the following. The D term is defined through the shear force distribution as in Eq. (4). Since for small

[2]Modulus of the magnetic field of the magnetically charged sphere is $\mathcal{H} = \frac{Q_M(r)}{4\pi r^2}$.

TABLE I. Numerical values of D terms, various radii, and different contributions to the mass for varied choice of β, where $\varepsilon \ll 1$. The estimation of the accuracy of our numerical method is about 95% for $0 \leq \beta \leq 1$ and about 85% for $\beta > 1$.

β	$M\frac{R_0 g^2}{4\pi}$	$M_C\frac{R_0 g^2}{4\pi}$	$M_{SR}\frac{R_0 g^2}{4\pi}$	$-\frac{15g^4}{64\pi^2}D$	$\frac{\langle r^2\rangle_M}{R_0^2}$	$\frac{\langle r^2\rangle_E}{R_0^2}$	$\frac{\langle r^2\rangle_{\mathrm{mech}}}{R_0^2}$
ε	0.993	0.153	0.841	∞	∞	∞	∞
0.1	1.036	0.187	0.849	7.909	20.517	12.157	20.255
0.5	1.133	0.267	0.865	1.022	4.287	3.710	3.379
1	1.204	0.336	0.869	0.43	2.168	2.436	1.545
5	1.438	0.794	0.644	0.293	0.373	1.602	0.445
8	1.457	1.084	0.373	0.213	0.205	2.328	0.307
10	1.461	1.289	0.172	0.108	0.151	4.479	0.254

values of $\beta \ll 1$ the main contribution to the short-range part of the shear force density is provided by the asymptotic behavior at large distances, see Appendix (A4), the D term diverges for the large enough R as

$$D \simeq -\frac{16\pi M_{SR} R_0}{15 g^2}\int_R^\infty d\rho \rho^4\left(C_h^2\beta^2\frac{e^{-2\beta\rho}}{\rho^2} - 2C_h\frac{e^{-\beta\rho}}{\rho^5}\right) \sim -\int_R^{1/\beta} d\rho\left(\beta^2\rho^2 - \frac{1}{\rho}\right) \sim -\frac{1}{\beta} - \ln\beta. \tag{48}$$

The mean square magnetic charge radius in Eq. (40), the mean square energy radius of the short-range part in Eq. (16), and the mean square mechanical radius in Eq. (18) also diverge for small values of β; see Appendix B.

In the end of this section we present numerical results for masses, D terms, and mean square radii of the 't Hooft-Polyakov monopole; see Table I. We use the following notations: the full mass of the monopole M is computed with the help of the energy density from Eq. (30), the electromagnetic contribution to the full mass M_C from Eq. (43), and the short-range contribution to the full mass M_{SH} is computed with the help of the expression in Eq. (46). The values of D terms have been calculated with the help of the short-range part of the shear force distribution in Eq. (46) according to (4), the mean square radius of the magnetic charge density is computed due to the definition in Eq. (40), the mean square radius of the short-range part of the energy distribution is computed according to the definition in Eq. (16), and the short-range part of the mechanical square radius is computed according to Eq. (18). It is remarkable that for the small values of β the main contribution to the monopole mass gives the short-range part and for the large values of $\beta \gtrsim 5$ the long range part. It would be interesting to find the analytic form of the pressure and shear force densities depending on the parameter β like was done in Refs. [23,29,30] for the static energy.

IV. JULIA-ZEE DYON

A. Equation of motion

In the previous section we considered a monopole solution carrying a magnetic charge. In this part we will consider a dyon, that is a hypothetical particle carrying both electric and magnetic charges; such particle was first suggested by Schwinger in 1969 in Ref. [31]. The dyon solution in the Georgi-Glashow model was first obtained in 1975 by Zee and Julia in Ref. [3]. We will use the same Georgi-Glashow model as we did in the first part with the same action (19). However, we will not fix the zero component of the vector field as we did it for the monopole case, $A_0^a(\vec{x}) \neq 0$. We are again interested in the static soliton solution with finite energy. Thereby, we choose the same spherically symmetric ansatz as we chose for the 't Hooft-Polyakov monopole in Eq. (21) with the same boundary condition as in Eq. (22). In contrast to the monopole case, for the dyon the A_0^a is nonzero and one chooses the ansatz

$$A_0^a = \frac{J(r)}{gr}\frac{r^a}{r}, \tag{49}$$

which follows from requiring the energy to be finite and definition of the Abelian field strength tensor as in Eq. (42). The boundary conditions for the function $J(r)$ are

$$J(r) \underset{r\to 0}{\simeq} 0, \qquad J(r) \underset{r\to\infty}{\simeq} -\frac{Q_D}{Q_M} + mgr, \tag{50}$$

where Q_D is an electric charge of a dyon, $Q_M = \frac{4\pi}{g}$ is a monopole charge, and m is a free constant that has dimension of mass.

From the variation of the action follows the equations of motion

$$\mathcal{D}_0 F_{00}^a - \mathcal{D}_i F_{i0}^a + g\epsilon^{abc}\varphi^b \mathcal{D}_0\varphi^c = 0,$$
$$\mathcal{D}_0 F_{0k}^a - \mathcal{D}_i F_{ik}^a - g\epsilon^{abc}\varphi^c \mathcal{D}_k\varphi^b = 0,$$
$$\mathcal{D}_0\mathcal{D}_0\varphi^a - \mathcal{D}_i\mathcal{D}_i\varphi^a + \lambda(\varphi^b\varphi^b - v^2)\varphi^a = 0. \tag{51}$$

The equations for the profile functions in dimensionless variable $\rho = gvr$ are

$$F''\rho^2 = (F^2-1)F - (J^2 - \rho^2 h^2)F,$$
$$h''\rho^2 + 2h'\rho = 2F^2 h + \frac{\beta^2}{2}h(h^2-1)\rho^2,$$
$$J''\rho^2 = 2JF^2. \tag{52}$$

Note that the boundary condition for $J(\rho)$ also changes: $J(\rho) \underset{\rho\to\infty}{\simeq} -\frac{Q_D}{Q_M} + C\rho$, where $C = m/v$ is a dimensionless free constant.

The approximate solution for small and large distances can be found as

$$
\begin{aligned}
F(\rho) &\underset{\rho\to 0}{\simeq} 1 + a\rho^2 + \sum_{n=2}^{\infty} a_{2n}\rho^{2n},\\
F(\rho) &\underset{\rho\to\infty}{\simeq} C_F e^{-\sqrt{1-C^2}\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right),\\
h(\rho) &\underset{\rho\to 0}{\simeq} b\rho + \sum_{n=1}^{\infty} b_{2n+1}\rho^{2n+1},\\
h(\rho) &\underset{\rho\to\infty}{\simeq} 1 - C_h \frac{e^{-\beta\rho}}{\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right)\\
&\quad - \frac{2C_F^2}{\beta^2 + 4C^2 - 4}\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right),\\
\tilde{J}(\rho) &\underset{\rho\to 0}{\simeq} c\rho + \sum_{n=1} c_{2n+1}\rho^{2n+1},\\
\tilde{J}(\rho) &\underset{\rho\to\infty}{\simeq} C - \frac{Q_D}{Q_M}\frac{1}{\rho} + \frac{CC_F^2}{2(1-C^2)}\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right).
\end{aligned}
\tag{53}
$$

The a, b, c are free constants, and all other constants for the small r behavior can be expressed in terms of these constants, for example,

$$
\begin{aligned}
a_4 &= \frac{1}{10}(3a^2 + b^2 - c^2),\\
b_3 &= \frac{1}{10}\left(4ab - \frac{b\beta^2}{2}\right),\\
c_3 &= \frac{2ac}{5}.
\end{aligned}
\tag{54}
$$

Since we searched the asymptotic behavior at large distances for real and falling functions, we define a new function $J(\rho) = \tilde{J}(\rho)\rho$ and find that the constant C has a restricted region: $0 \le C < 1$. From numerical analysis of the solutions we obtain that the charge ratio is restricted in the following range: $0 \le \frac{Q_D}{Q_M} \le 1$. More detailed analysis can be found in Ref. [32], where also the dependence of the parameter C on the charge ratio $\frac{Q_D}{Q_M}$ is found.

B. EMT densities

The EMT for the Julia-Zee dyon can be obtained analogically to the 't Hooft-Polyakov monopole by variating the generally covariant form of the Georgi-Glashow action (19) with respect to the metric and it can be decomposed into magnetic and electric parts:

$$
\begin{aligned}
T_{00}(r) &= T^M_{00}(r) + T^E_{00}(r),\\
T_{ij}(r) &= T^M_{ij}(r) + T^E_{ij}(r),
\end{aligned}
\tag{55}
$$

where the magnetic part equals the EMT of a monopole which we have already computed in the previous section; see Eqs. (29), (30), and (31). The electric part of the EMT is

$$
\begin{aligned}
T^E_{00}(r) &= \frac{1}{2}(\mathcal{D}_0\varphi^a\mathcal{D}_0\varphi^a + F^a_{0i}F^a_{0i}),\\
T^E_{ij}(r) &= \frac{1}{2}\left[\delta_{ij}(F^a_{0k}F^a_{0k} + \mathcal{D}_0\varphi^a\mathcal{D}_0\varphi^a) - 2F^a_{i0}F^a_{j0}\right].
\end{aligned}
\tag{56}
$$

Note that same as for the monopole case the spatial components of the EMT for the dyon vanish in the BSP limit, where $\beta = 0$. For the electric part of the energy density, pressure, and shear force distributions of the dyon in dimensionless variable ρ, the following expressions are obtained:

$$
\begin{aligned}
T^E_{00}(\rho) &= \frac{1}{R_0^4 g^2}\left(\frac{\tilde{J}'^2}{2} + \frac{\tilde{J}^2F^2}{\rho^2}\right),\\
p^E(\rho) &= \frac{1}{R_0^4 g^2}\left(\frac{\tilde{J}'^2}{6} + \frac{\tilde{J}^2F^2}{3\rho^2}\right),\\
s^E(\rho) &= \frac{1}{R_0^4 g^2}\left(-\tilde{J}'^2 + \frac{\tilde{J}^2F^2}{\rho^2}\right).
\end{aligned}
\tag{57}
$$

With the help of the asymptotic behavior of the profile functions in Eq. (53), the asymptotic behavior of the EMT for the dyon can be obtained:

$$
\begin{aligned}
T_{00}(\rho) &\underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2}\left[\frac{1}{2}\left(1 + \left(\frac{Q_D}{Q_M}\right)^2\right)\frac{1}{\rho^4} + 2C_F^2\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right) + \beta^2C_F^2\frac{e^{-2\beta\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right)\right],\\
p(\rho) &\underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2}\left[\frac{1}{6}\left(1 + \left(\frac{Q_D}{Q_M}\right)^2\right)\frac{1}{\rho^4} - \frac{2}{3}\beta^2C_F^2\frac{e^{-2\beta\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right) + O\left(\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^3}\right)\right],\\
s(\rho) &\underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2}\left[-\left(1 + \left(\frac{Q_D}{Q_M}\right)^2\right)\frac{1}{\rho^4} + \beta^2C_F^2\frac{e^{-2\beta\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right) + O\left(\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^3}\right)\right].
\end{aligned}
\tag{58}
$$

From this behavior one sees that the long-range contribution presented in the dyon is even stronger than in the monopole case; see Eq. (31). In Appendix A 2 one can find the asymptotic behavior of the EMT near the origin as well as the behavior of the electric part of the EMT only. Again already from the asymptotic behavior it is clear that the stability condition of

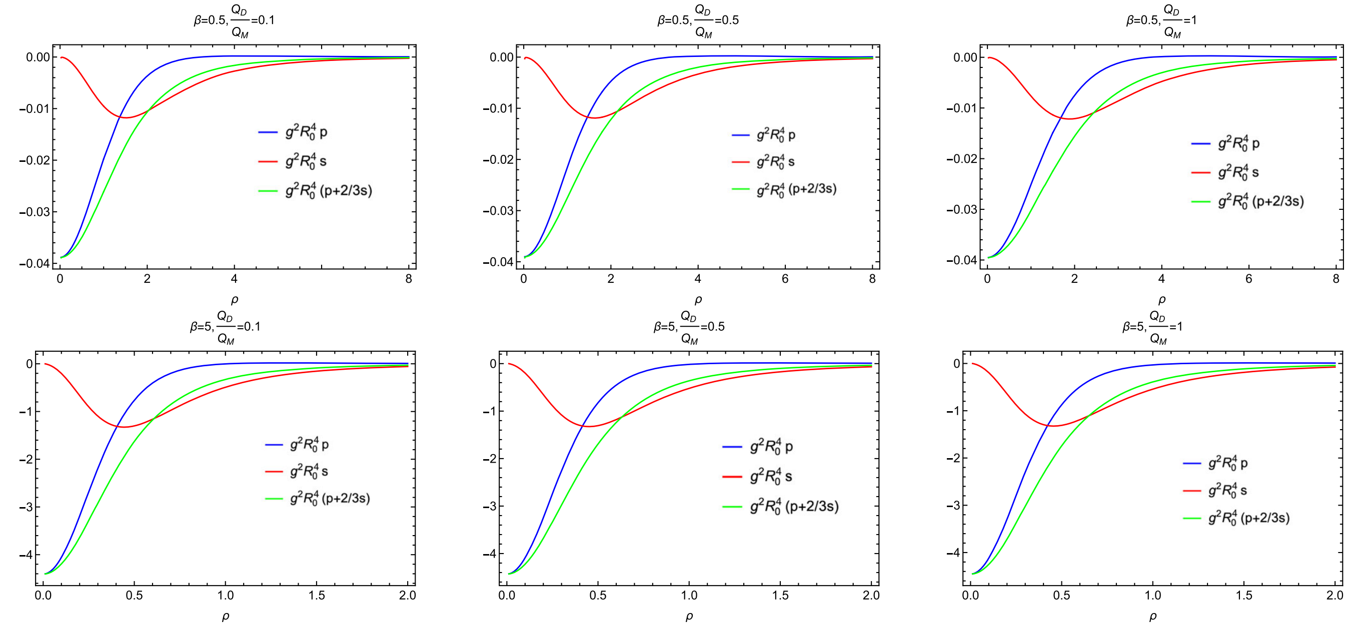

FIG. 5. The mechanical properties of dyons from Eq. (55) as functions of $\rho = r/R_0 = gvr$ for varied value of β and charge ratio $\frac{Q_D}{Q_M}$.

Eq. (10) is violated. In Fig. 5 the full pressure and shear force distributions of the dyon are presented. The shear force and pressure distributions have very similar form as it was in the monopole case; see Fig. 2. These distributions are mostly negative for every choice of parameter. However, pressure distribution changes its sign, which allows the von Laue condition (7) to be satisfied; see e.g. Fig. 6. It is also interesting to notice that the main contribution to the dyon energy comes from the monopole part as can be seen in Fig. 7. Moreover, for the growing β this contribution is increasing.

C. Electromagnetic and short-range parts of the EMT

According to the definition of the electromagnetic EMT in Eq. (41) the long-range part of the EMT for a dyon is

$$T_{00}^C = \frac{1}{4}\mathcal{F}_{ij}\mathcal{F}_{ij} + \frac{1}{2}\mathcal{F}_{0i}\mathcal{F}_{0i},$$
$$T_{ij}^C = -\frac{1}{4}\delta_{ij}\mathcal{F}_{mn}\mathcal{F}_{mn} - \mathcal{F}_{ik}\mathcal{F}_{kj} + \frac{1}{2}\delta_{ij}\mathcal{F}_{0k}\mathcal{F}_{0k} - \mathcal{F}_{i0}\mathcal{F}_{j0}. \tag{59}$$

Using the modified 't Hooft Abelian field strength tensor in Eq. (42) one gets the following expressions for the electromagnetic EMT densities of a dyon:

$$T_{00}^C(\rho) = \frac{1}{2}\frac{1}{g^2R_0^4}\frac{1}{\rho^4}(Q^2(\rho) + \tilde{Q}^2(\rho)),$$
$$p^C(\rho) = \frac{1}{6}\frac{1}{g^2R_0^4}\frac{1}{\rho^4}(Q^2(\rho) + \tilde{Q}^2(\rho)),$$
$$s^C(\rho) = -\frac{1}{g^2R_0^4}\frac{1}{\rho^4}(Q^2(\rho) + \tilde{Q}^2(\rho)), \tag{60}$$

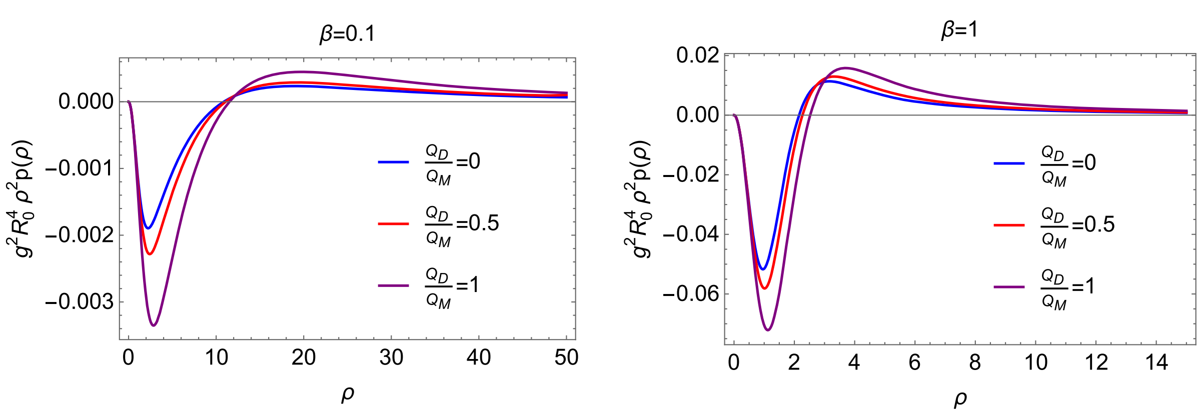

FIG. 6. The pressure distributions of the monopole ($\frac{Q_D}{Q_M} = 0$) and the dyon for various values of β and charge ratio $\frac{Q_D}{Q_M}$. The pressure distribution has one mode for any choice of the parameters.

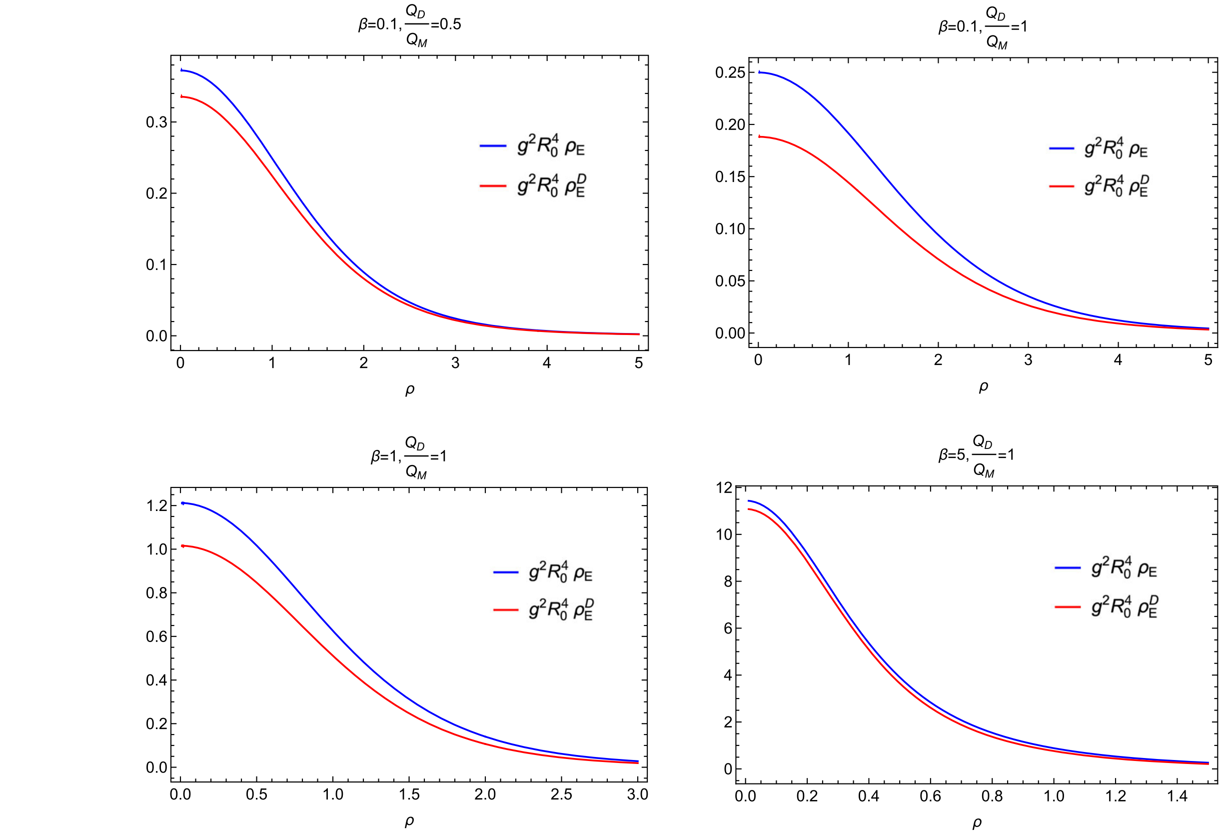

FIG. 7. Comparison of the full energy densities of a dyon and the contributions of the monopole part only for various values of β and charge ratio $\frac{Q_D}{Q_M}$.

where the function Q is defined in Eq. (43) and is related to the magnetic charge density in Eq. (44) and the function $\tilde{Q}$ is related to the electric charge density of a dyon

$$\rho_D(\rho)=\frac{1}{gR_0^3}\frac{1}{\rho^2}\frac{d\tilde{Q}(\rho)}{d\rho},\quad \text{with}\quad \tilde{Q}(\rho)=\rho^2 h(\rho)\tilde{J}'(\rho). \tag{61}$$

Comparing these expressions with the expressions for the monopole case in Eq. (43), it is clear that both have similar behavior. Hence, it is again not possible to define the various mean square radii or D terms, because they diverge.

We will exclude the long-range contribution given in Eq. (60) from the EMT of the dyon in Eq. (55) in the same way as we did for the monopole case using Eq. (45). After simple algebraic calculations we obtain the following expression for the short-range part of the EMT of the dyon:

$$\begin{aligned}
T_{00}^{\mathrm{SR}}(\rho) &= T_{00}^{\mathrm{M,SR}}(\rho)+\frac{1}{R_0^4g^2}\left(\frac{1}{2}\tilde{J}'^2(1-h^2)+\frac{\tilde{J}^2F^2}{\rho^2}\right),\\
p^{\mathrm{SR}}(\rho) &= p^{\mathrm{M,SR}}(\rho)+\frac{1}{R_0^4g^2}\left(\frac{1}{6}\tilde{J}'^2(1-h^2)+\frac{1}{3}\frac{\tilde{J}^2F^2}{\rho^2}\right),\\
s^{\mathrm{SR}}(\rho) &= s^{\mathrm{M,SR}}(\rho)-\frac{1}{R_0^4g^2}\left(\tilde{J}'^2(1-h^2)-\frac{\tilde{J}^2F^2}{\rho^2}\right).
\end{aligned} \tag{62}$$

Here $T_{00}^{\mathrm{M,SR}}$, $p^{\mathrm{M,SR}}$, and $s^{\mathrm{M,SR}}$ correspond to the short-range part of the EMT of the monopole in Eq. (46). The final short-range distributions of mechanical properties are shown in Fig. 8. These distributions have the same behavior as in the monopole case.

The equilibrium equation of the long- and short-range parts of the EMT analogously to Eq. (47) is given as

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{2}{3}s^C(r)+p^C(r)\right)+2\frac{s^C(r)}{r}\\
&\quad=-\frac{Q_M(r)\rho_M(r)}{4\pi r^2}-\frac{\tilde{Q}_D(r)\rho_D(r)}{4\pi r^2},\\
&\frac{d}{dr}\left(\frac{2}{3}s^{SR}(r)+p^{SR}(r)\right)+2\frac{s^{SR}(r)}{r}\\
&\quad=\frac{Q_M(r)\rho_M(r)}{4\pi r^2}+\frac{\tilde{Q}_D(r)\rho_D(r)}{4\pi r^2},
\end{aligned} \tag{63}$$

where $Q_M(r)$ and $\tilde{Q}_D(r)$ are magnetic and electric charges contained in the sphere of radius r:

$$\begin{aligned}
Q_M(r) &= \int_{|\vec{x}|<r} d^3x\rho_M(\vec{x})=\frac{4\pi}{g}Q(r),\\
\tilde{Q}_D(r) &= \int_{|\vec{x}|<r} d^3x\rho_D(\vec{x})=\frac{4\pi}{g}\tilde{Q}(r).
\end{aligned} \tag{64}$$

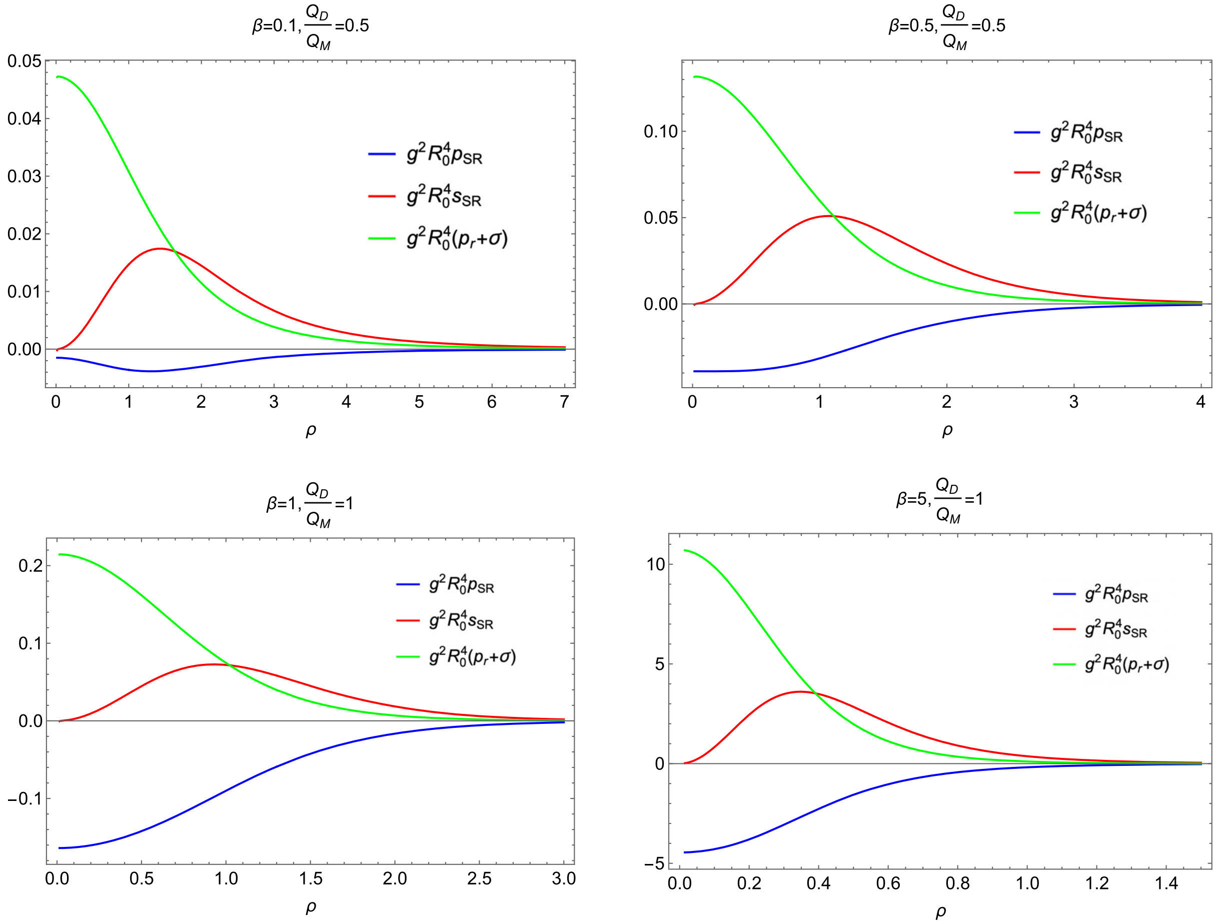

FIG. 8. Short-range mechanical properties of dyons as a functions of $\rho = r/R_0 = gvr$ from Eq. (62) for varied values of β and charge ratio $\frac{Q_D}{Q_M}$. Note that the total normal force is defined as in Eq. (15).

The first equilibrium equation describes the balance between two types of Coulomb forces, namely, magnetic and electric ones, pulling the dyon inwards to the center and the Coulomb stress pushing the dyon outwards. The second equation describes the balance between the two types of the repulsive Coulomb forces pushing the dyon outward and the short-range stress pulling the dyon inward to the center. Same as for the monopole case it is not possible to define the short-range part of the EMT using the ambiguity of the $\mathcal{F}_{\mu\nu}$ in such a way that the long- and short-range parts of the EMT are conserved separately unless the electric and magnetic charge densities both vanish.

In Table II the full mass of a dyon is denoted as M, the monopole contribution to the full dyon mass as M_M, the M_C and M_{SR} are the long- and short-range contributions to the full mass, correspondingly, and the D terms are computed with the help of the short-range part of the shear force distribution in Eq. (62). The following quantities can be also found in the table: the energy mean square radius $\langle r^2\rangle_E$, the magnetic and electric charge mean square radii $\langle r^2\rangle_M$ and $\langle r^2\rangle_D$, and the mechanical radius $\langle r^2\rangle_{\text{mech}}$. It is remarkable that the D term is growing with the parameter $\frac{Q_D}{Q_M}$ and decreasing with the parameter β. As it was already mentioned, the main contribution to the dyon mass is given by the monopole part of the dyon; moreover, the same as it was for the monopole: for small values of β the main contribution to the mass comes from the short-range region, while for the value $\beta \gtrsim 5$ from the long-range part. The following ratios of the radii can be noticed from the table:

$$\begin{aligned}
&\frac{\langle r^2\rangle_M}{\langle r^2\rangle_D} > 1, \quad \text{for small } \beta \text{ and every } \frac{Q_D}{Q_M},\\
&\frac{\langle r^2\rangle_M}{\langle r^2\rangle_D} < 1, \quad \text{for } \beta \gtrsim 0.1 \text{ and every } \frac{Q_D}{Q_M},\\
&\frac{\langle r^2\rangle_M}{\langle r^2\rangle_E} > 1, \quad \text{for } \beta < 1 \text{ and every } \frac{Q_D}{Q_M},\\
&\frac{\langle r^2\rangle_M}{\langle r^2\rangle_E} < 1, \quad \text{for } \beta \geq 1 \text{ and every } \frac{Q_D}{Q_M},\\
&\frac{\langle r^2\rangle_D}{\langle r^2\rangle_E} > 1, \quad \text{for every } \beta \text{ and every } \frac{Q_D}{Q_M}.
\end{aligned} \tag{65}$$

TABLE II. Numerical value of D terms, radii, and different contributions to the mass for various values of β and the charge ratio $\frac{Q_D}{Q_M}$, where $\varepsilon \ll 1$. The estimation of the accuracy of our numerical method is about 95% for $0 \le \beta \le 1$ and about 85% for $\beta > 1$.

β	$\frac{Q_D}{Q_M}$	$M_M \frac{R_0 g^2}{4\pi}$	$M \frac{R_0 g^2}{4\pi}$	$M_C \frac{R_0 g^2}{4\pi}$	$M_{SR} \frac{R_0 g^2}{4\pi}$	$-\frac{15 g^4}{64\pi^2} D$	$\frac{\langle r^2\rangle_E}{R_0^2}$	$\frac{\langle r^2\rangle_M}{R_0^2}$	$\frac{\langle r^2\rangle_D}{R_0^2}$	$\frac{\langle r^2\rangle_{\text{mech}}}{R_0^2}$
ε	0.5	0.998	1.109	0.168	0.941	∞	∞	∞	∞	∞
ε	1	1.050	1.400	0.207	1.193	∞	∞	∞	∞	∞
0.1	0.5	1.043	1.164	0.21	0.953	10.826	13.863	23	21.447	24.000
0.1	1	1.099	1.484	0.263	1.215	20.832	18.373	29.247	28.838	33.253
0.5	0.5	1.142	1.281	0.304	0.977	1.33	4.327	4.803	5.624	3.883
0.5	1	1.2	1.654	0.394	1.26	2.225	6.136	6.08	7.732	4.972
1	0.5	1.220	1.371	0.386	0.985	0.529	2.909	2.416	3.504	1.630
1	1	1.28	1.738	0.477	1.262	0.698	4.467	2.954	5.288	1.200
5	0.5	1.44	1.604	0.839	0.375	0.49	2.139	0.4	2.201	0.42
5	1	1.492	1.825	0.909	0.915	0.405	3.299	2.163	3.489	0.365

V. SUMMARY AND CONCLUSIONS

In this work the EMTs of the 't Hooft-Polyakov monopole and the Julia-Zee dyon were studied. These EMTs contain long-range contributions which present analogies to the Coulomb interaction. The local stability condition containing the pressure and shear force distributions is violated for both cases, the monopole and the dyon. Therefore, the applicability of the condition in the presence of long-range contributions is questioned. Moreover, such important quantities which give information about mechanical properties of a system as the D term, energy, and charge mean square radii of the monopole and the dyon cannot be computed due to the presence of the long-range interaction.

To shed more light on the mechanical properties of the monopole and the dyon and on the local stability condition, we exclude the long-range contribution from the EMT of the monopole and the dyon. The difficulty of such calculation is that this contribution cannot be uniquely defined. We suggest the modified 't Hooft definition of the Abelian field strength tensor for this purpose. Dealing with the separate long- and short-range parts of the EMTs, we obtained the equilibrium equations which couple the pressure, shear force distributions, and the external force acting on the system. In this line we have also modified the local mechanical stability condition for systems in the presence of external forces. According to this condition the short-range part of the 't Hooft-Polyakov monopole as well as of the Julia-Zee dyon is stable for every choice of the parameter β.

Further, in this paper numerous figures describing mechanical properties of the 't Hooft-Polyakov monopole and the Julia-Zee dyon can be found as well as tables with varied contributions to the masses, D terms, and mean square energy, magnetic, and electric charge radii.

ACKNOWLEDGMENTS

I would like to thank H. Alharazin, J. Gegelia, S. Korenblit, M. Polyakov, P. Schweitzer, and Ya. Shnir for encouraging, useful comments and numerous discussions. This work was supported in part by DFG and NSFC through funds provided to the Sino-German CRC 110 "Symmetries and the Emergence of Structure in QCD" (NSFC Grant No. 11621131001, DFG Project-ID 196253076-TRR 110).

APPENDIX A: ASYMPTOTIC BEHAVIORS

We specify here the asymptotic behaviors of the EMT which was not directly given in the main body of the paper.

1. For monopole part

The function $Q(\rho)$ has the following asymptotic behavior:

$$Q(\rho) \underset{\rho\to\infty}{\simeq} 1 - C_h \frac{e^{-\beta\rho}}{\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right) - \frac{2C_F^2}{\beta^2 - 4}\frac{e^{-2\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right),$$

$$Q(\rho) \underset{\rho\to 0}{\simeq} \rho^3(-2ab + b^3) + \frac{\rho^5}{20}(-48a^2b + 2ab(32b^2 + \beta^2) - b^3(4 + 3\beta^2)) + O(\rho^7). \tag{A1}$$

The asymptotic behavior of the magnetic charge density is

$$\rho_M(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{gR_0^3}\left[3b^3 - 6ab + \rho^2\left(-12a^2b - b^3 + 16ab^3 + \frac{1}{2}ab\beta^2 - \frac{3b^3\beta^2}{4}\right) + O(\rho^4)\right],$$
$$\rho_M(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{gR_0^3}\left[C_h\beta\frac{e^{-\beta\rho}}{\rho^3}\left(1 + O\left(\frac{1}{\rho}\right)\right) + \frac{4C_F^2}{\beta^2-4}\frac{e^{-2\rho}}{\rho^4}\left(1 + O\left(\frac{1}{\rho}\right)\right)\right]. \tag{A2}$$

The "Coulomb EMT" behaves asymptotically as

$$T_{00}^C(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4g^2}\left[\frac{1}{2\rho^4} - C_h\frac{e^{-\beta\rho}}{\rho^5}\left(1 + O\left(\frac{1}{\rho}\right)\right) + \frac{C_h^2}{2}\frac{e^{-2\beta\rho}}{\rho^6}\left(1 + O\left(\frac{1}{\rho}\right)\right) - \frac{2C_F^2}{\beta^2-4}\frac{e^{-2\rho}}{\rho^6}\left(1 + O\left(\frac{1}{\rho}\right)\right)\right],$$
$$p^C(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4g^2}\left[\frac{1}{6\rho^4} - \frac{C_h}{3}\frac{e^{-\beta\rho}}{\rho^5}\left(1 + O\left(\frac{1}{\rho}\right)\right) + \frac{C_h^2}{6}\frac{e^{-2\beta\rho}}{\rho^6}\left(1 + O\left(\frac{1}{\rho}\right)\right) - \frac{2C_F^2}{3(\beta^2-4)}\frac{e^{-2\rho}}{\rho^6}\left(1 + O\left(\frac{1}{\rho}\right)\right)\right],$$
$$s^C(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4g^2}\left[-\frac{1}{\rho^4} + 2C_h\frac{e^{-\beta\rho}}{\rho^5}\left(1 + O\left(\frac{1}{\rho}\right)\right) - C_h^2\frac{e^{-2\beta\rho}}{\rho^6}\left(1 + O\left(\frac{1}{\rho}\right)\right) + \frac{4C_F^2}{\beta^2-4}\frac{e^{-2\rho}}{\rho^6}\left(1 + O\left(\frac{1}{\rho}\right)\right)\right],$$
$$T_{00}^C(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4g^2}\left[\frac{\rho^2}{2}(-2ab + b^3)^2 + \frac{\rho^4}{20}b(-2a + b^2)(-48a^2b + 2ab(32b^2 + \beta^2) - b^3(4 + 3\beta^2)) + O(\rho^6)\right],$$
$$p^C(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4g^2}\left[\frac{\rho^2}{6}(-2ab + b^3)^2 + \frac{\rho^4}{60}b(-2a + b^2)(-48a^2b + 2ab(32b^2 + \beta^2) - b^3(4 + 3\beta^2)) + O(\rho^6)\right],$$
$$s^C(\rho) \underset{\rho\to 0}{\simeq} -\frac{1}{R_0^4g^2}\left[\rho^2(-2ab + b^3)^2 + \frac{\rho^4}{10}b(-2a + b^2)(-48a^2b + 2ab(32b^2 + \beta^2) - b^3(4 + 3\beta^2)) + O(\rho^6)\right]. \tag{A3}$$

The asymptotic behavior of the short-range part of the EMT is

$$T_{00}^{SR}(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4g^2}\left[C_h\frac{e^{-\beta\rho}}{\rho^5}\left(1 + O\left(\frac{1}{\rho}\right)\right) + C_h^2\beta^2\frac{e^{-2\beta\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right) + 2C_F^2\frac{e^{-2\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right)\right],$$
$$p^{SR}(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4g^2}\left[\frac{C_h}{3}\frac{e^{-\beta\rho}}{\rho^5}\left(1 + O\left(\frac{1}{\rho}\right)\right) - \frac{2}{3}C_h^2\beta^2\frac{e^{-2\beta\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right) - \frac{2C_F^2}{3}\frac{e^{-2\rho}}{\rho^4}\left(1 + O\left(\frac{1}{\rho}\right)\right)\right],$$
$$s^{SR}(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4g^2}\left[-2C_h\frac{e^{-\beta\rho}}{\rho^5}\left(1 + O\left(\frac{1}{\rho}\right)\right) + C_h^2\beta^2\frac{e^{-2\beta\rho}}{\rho^2}\left(1 + O\left(\frac{1}{\rho}\right)\right) + C_F^2\frac{e^{-2\rho}}{\rho^4}\left(1 + O\left(\frac{1}{\rho}\right)\right)\right],$$
$$T_{00}^{SR}(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4g^2}\left[\left(6a^2 + \frac{3}{2}b^2 + \frac{\beta^2}{8}\right) + \left(8a^3 + 6ab^2 - 2a^2b^2 + 2ab^4 - \frac{b^6}{2} - \frac{\beta^2b^2}{2}\right)\rho^2 + O(\rho^4)\right],$$
$$p^{SR}(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4g^2}\left[\left(2a^2 - \frac{b^2}{2} - \frac{\beta^2}{8}\right) + \left(8a^3 - 2ab^2 - 2a^2b^2 + 2ab^4 - \frac{b^6}{2} + \beta^2b^2\right)\frac{\rho^2}{3} + O(\rho^4)\right],$$
$$s^{SR}(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4g^2}\left[(-8a^3 + 2ab^2 + 20a^2b^2 - 20ab^4 + 5b^6 - \beta^2b)\frac{\rho^2}{5} + O(\rho^4)\right]. \tag{A4}$$

2. For dyon part

The function $\tilde{Q}(\rho)$ has the following asymptotic behavior:

$$\tilde{Q}(\rho) \underset{\rho\to\infty}{\simeq} gQ_D - gC_hQ_D\frac{e^{-\beta\rho}}{\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right) - \frac{CC_F^2}{\sqrt{1-C^2}}e^{-2\sqrt{1-C^2}\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right),$$
$$\tilde{Q}(\rho) \underset{\rho\to 0}{\simeq} \rho^3bc + \rho^5\left(\frac{6abc}{5} + \frac{c}{20}(8ab - b\beta^2)\right) + O(\rho^7). \tag{A5}$$

The asymptotic behavior of the electric charge density is

$$\rho_D(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{gR_0^3}\left(gQ_D C_h \beta \frac{e^{-\beta\rho}}{\rho^3}\left(1+O\left(\frac{1}{\rho}\right)\right) + 2CC_F^2 \frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^2}\left(1+O\left(\frac{1}{\rho}\right)\right)\right),$$
$$\rho_D(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{gR_0^3}\left(3bc + \rho^2\left(6abc + \frac{c}{4}(8ab - b\beta^2)\right)\right) + O(\rho^4). \tag{A6}$$

The asymptotic behavior of the full EMT of the dyon near the origin is

$$T_{00}(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2}\left[6a^2 + \frac{3b^2}{2} + \frac{\beta^2}{8} + \frac{3c^2}{2} + \rho^2\left(8a^3 + 6ab^2 - \frac{b^2\beta^2}{2} + 2ac^2\right) + O(\rho^4)\right],$$
$$p(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2}\left[2a^2 - \frac{b^2}{2} - \frac{\beta^2}{8} + \frac{c^2}{2} + \frac{\rho^2}{3}(8a^3 - 2ab^2 + b^2\beta^2 + 2ac^2) + O(\rho^4)\right],$$
$$s(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2}\left[\frac{\rho^2}{5}(-8a^3 + 2ab^2 - b^2\beta^2 - 2ac^2) + O(\rho^4)\right]. \tag{A7}$$

The asymptotic behavior of the electric part of the full EMT for the dyon is

$$T_{00}^E(\rho) \underset{\rho\to\infty}{\simeq} \frac{g^2Q_D^2}{2}\frac{1}{\rho^4} + C^2C_F^2 \frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^2}\left(1+O\left(\frac{1}{\rho}\right)\right),$$
$$p^E(\rho) \underset{\rho\to\infty}{\simeq} \frac{g^2Q_D^2}{6}\frac{1}{\rho^4} + \frac{C^2C_F^2}{3} \frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^2}\left(1+O\left(\frac{1}{\rho}\right)\right),$$
$$s^E(\rho) \underset{\rho\to\infty}{\simeq} -g^2Q_D^2\frac{1}{\rho^4} + C^2C_F^2 \frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^2}\left(1+O\left(\frac{1}{\rho}\right)\right),$$
$$T_{00}^E(\rho) \underset{\rho\to 0}{\simeq} \frac{3c^2}{2} + 4ac^2\rho^2 + O(\rho^4),$$
$$p^E(\rho) \underset{\rho\to 0}{\simeq} \frac{c^2}{2} + \frac{4}{3}ac^2\rho^2 + O(\rho^4),$$
$$s^E(\rho) \underset{\rho\to 0}{\simeq} \frac{2}{5}ac^2\rho^2 + O(\rho^4). \tag{A8}$$

The asymptotic behavior of the long-range part of the EMT is

$$T_{00}^C(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2}\left[\frac{1}{2}(1+g^2Q_D^2)\frac{1}{\rho^4} - C_h(1+g^2Q_D^2)\frac{e^{-\beta\rho}}{\rho^5}\left(1+O\left(\frac{1}{\rho}\right)\right)\right.$$
$$\left. - \frac{CC_F^2 gQ_D}{\sqrt{1-C^2}}\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^4}\left(1+O\left(\frac{1}{\rho}\right)\right) + \frac{C_h^2}{2}(1+g^2Q_D^2)\frac{e^{-2\beta\rho}}{\rho^6}\left(1+O\left(\frac{1}{\rho}\right)\right)\right],$$
$$p^C(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2}\left[\frac{1}{6}(1+g^2Q_D^2)\frac{1}{\rho^4} - \frac{C_h}{3}(1+g^2Q_D^2)\frac{e^{-\beta\rho}}{\rho^5}\left(1+O\left(\frac{1}{\rho}\right)\right)\right.$$
$$\left. - \frac{CC_F^2 gQ_D}{3\sqrt{1-C^2}}\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^4}\left(1+O\left(\frac{1}{\rho}\right)\right) + \frac{C_h^2}{2}(1+g^2Q_D^2)\frac{e^{-2\beta\rho}}{\rho^6}\left(1+O\left(\frac{1}{\rho}\right)\right)\right],$$
$$s^C(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2}\left[-(1+g^2Q_D^2)\frac{1}{\rho^4} + 2C_h(1+g^2Q_D^2)\frac{e^{-\beta\rho}}{\rho^5}\left(1+O\left(\frac{1}{\rho}\right)\right)\right.$$
$$\left. + \frac{2CC_F^2 gQ_D}{\sqrt{1-C^2}}\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^4}\left(1+O\left(\frac{1}{\rho}\right)\right) - C_h^2(1+g^2Q_D^2)\frac{e^{-2\beta\rho}}{\rho^6}\left(1+O\left(\frac{1}{\rho}\right)\right)\right],$$

$$T_{00}^{C}(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2}\left[\rho^4 \frac{1}{2} b^2((-2a+b^2)^2+c^2)+O(\rho^6)\right],$$
$$p^{C}(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2}\left[\rho^4 \frac{1}{6} b^2((-2a+b^2)^2+c^2)+O(\rho^6)\right],$$
$$s^{C}(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2}[-\rho^4 b^2((-2a+b^2)^2+c^2)+O(\rho^6)]. \tag{A9}$$

The asymptotic behavior of the short-range part of the EMT of the dyon is as follows:

$$T_{00}^{SR}(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2}\left[2C_F^2 \frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^2}\left(1+O\left(\frac{1}{\rho}\right)\right)+\beta^2 C_F^2 \frac{e^{-2\beta\rho}}{\rho^2}\left(1+O\left(\frac{1}{\rho}\right)\right)+C_h(1+g^2Q_D^2)\frac{e^{-\beta\rho}}{\rho^5}\left(1+O\left(\frac{1}{\rho}\right)\right)\right],$$
$$p^{SR}(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2}\left[-\frac{2}{3}\beta^2 C_F^2 \frac{e^{-2\beta\rho}}{\rho^2}\left(1+O\left(\frac{1}{\rho}\right)\right)+O\left(\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^3}\right)+\frac{C_h}{3}(1+g^2Q_D^2)\frac{e^{-\beta\rho}}{\rho^5}\left(1+O\left(\frac{1}{\rho}\right)\right)\right],$$
$$s^{SR}(\rho) \underset{\rho\to\infty}{\simeq} \frac{1}{R_0^4 g^2}\left[\beta^2 C_F^2 \frac{e^{-2\beta\rho}}{\rho^2}\left(1+O\left(\frac{1}{\rho}\right)\right)+O\left(\frac{e^{-2\sqrt{1-C^2}\rho}}{\rho^3}\right)-2C_h(1+g^2Q_D^2)\frac{e^{-\beta\rho}}{\rho^5}\left(1+O\left(\frac{1}{\rho}\right)\right)\right],$$
$$T_{00}^{SR}(\rho) \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2}\left(6a^2+\frac{3b^2}{2}+\frac{\beta^2}{8}+\frac{3c^2}{2}+\rho^2\left[8a^3+6ab^2-\frac{b^2\beta^2}{2}+2ac^2\right]+O(\rho^4)\right),$$
$$p(\rho)^{SR} \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2}\left(2a^2-\frac{b^2}{2}-\frac{\beta^2}{8}+\frac{c^2}{2}+\frac{\rho^2}{3}[8a^3-2ab^2+b^2\beta^2+2ac^2]+O(\rho^4)\right),$$
$$s(\rho)^{SR} \underset{\rho\to 0}{\simeq} \frac{1}{R_0^4 g^2}\left(\frac{\rho^2}{5}[-8a^3+2ab^2-b^2\beta^2-2ac^2]+O(\rho^4)\right). \tag{A10}$$

APPENDIX B: DIVERGENCE OF RADII

As follows from Eqs. (A2) and (A4), the main contribution to the magnetic charge density and to the short-range part of the energy density, respectively, is provided by the asymptotic behavior at large distances. Then the corresponding mean square radii for the small values of β and for the large enough R diverge as

$$\frac{\langle r_M^2\rangle}{R_0^2} \simeq \frac{\int_R^\infty d\rho\rho^4(C_h\beta\frac{e^{-\beta\rho}}{\rho^3})}{\int_R^\infty d\rho\rho^2(C_h\beta\frac{e^{-\beta\rho}}{\rho^3})} \sim \frac{\int_R^{1/\beta} d\rho\rho}{\int_R^{1/\beta} d\rho\frac{1}{\rho}} \sim \frac{1}{\beta^2\ln(\frac{1}{\beta})}, \tag{B1}$$

$$\frac{\langle r_E^2\rangle}{R_0^2} \simeq \frac{\int_R^\infty d\rho\rho^4(C_h\frac{e^{-\beta\rho}}{\rho^5}+C_h^2\beta^2\frac{e^{-2\beta\rho}}{\rho^2})}{\int_R^\infty d\rho\rho^2(C_h\frac{e^{-\beta\rho}}{\rho^5}+C_h^2\beta^2\frac{e^{-2\beta\rho}}{\rho^2})} \sim \frac{\int_R^{1/\beta} d\rho(\frac{1}{\rho}+\beta^2\rho^2)}{\int_R^{1/\beta} d\rho(\frac{1}{\rho^3}+\beta^2)} \sim \frac{1}{\beta^2}-\frac{\ln\beta}{\beta}. \tag{B2}$$

According to the modified definition of the mechanical radius in Eq. (18), to the external force in equilibrium equation (47) for the short-range and to the asymptotic behavior in Eqs. (A1) and (A4), the mean square mechanical radius of the monopole for small values of β diverges as

$$\frac{\langle r^2\rangle_{\text{mech}}}{R_0^2} \sim \frac{\int_R^{1/\beta} d\rho\rho^4(\frac{\beta}{\rho^4}-\frac{1}{\rho^5})}{\int_R^{1/\beta} d\rho\rho^2(\frac{\beta}{\rho^4}-\frac{1}{\rho^5})} \sim \frac{\ln\beta}{\beta^2}. \tag{B3}$$

Proceeding analogously to the monopole case, we obtain for the dyon the same divergences as in Eqs. (48), (B1), (B2), and (B3). Namely, the D term diverges as in Eq. (48), the mean square electric and magnetic charge radii as in Eq. (B1), the mean square energy radius of the short-range part as in Eq. (B2), and the mean square mechanical radius as in Eq. (B3).

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