Quark hierarchical structures in modular symmetric flavor models at level 6

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We study modular symmetric quark flavor models without fine-tuning. Mass matrices are written in terms of modular forms, and modular forms in the vicinity of the modular fixed points become hierarchical depending on their residual charges. Thus modular symmetric flavor models in the vicinity of the modular fixed points have a possibility to describe mass hierarchies without fine-tuning. Since describing quark hierarchies without fine-tuning requires Z_n residual symmetry with $n \geq 6$, we focus on Γ_6 modular symmetry in the vicinity of the cusp $\tau = i\infty$ where Z_6 residual symmetry remains. We use only modular forms belonging to singlet representations of Γ_6 to make our analysis simple. Consequently, viable quark flavor models are obtained without fine-tuning.

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I. INTRODUCTION

The origin of quark and lepton flavor structures such as hierarchical masses and mixing angles is one of challenging issues in particle physics. Indeed, many works were done in order to solve the problem. Among such works, modular symmetric flavor models are interesting. In these flavor models, the quark and lepton mass matrices are written in terms of modular forms, which are holomorphic functions of the modulus τ [1].[1] It is well known that the finite modular groups Γ_N for $N = 2, 3, 4, 5$ are isomorphic to the non-Abelian finite groups S_3, A_4, S_4, and A_5, respectively [18]. This is interesting since the non-Abelian finite groups are long familiar in flavor models for quarks and leptons [19–29]. Inspired by this point, the modular symmetric lepton flavor models have been proposed in $\Gamma_2 \simeq S_3$ [30], $\Gamma_3 \simeq A_4$ [1], $\Gamma_4 \simeq S_4$ [31], and $\Gamma_5 \simeq A_5$ [32,33]. In addition, modular symmetries at levels 6 [34] and 7 [35] were studied. Furthermore, modular forms of other groups were also studied [8,36–39].

Using these various modular forms, the mass ratios and flavor mixing of quarks and leptons have been discussed successfully in these years. Phenomenological studies have been developed in many works and interesting results have been obtained [40–79]. However, one needs to fine-tune coefficients of modular forms in Yukawa couplings in order to describe the hierarchical structure of fermion masses, in particular quark mass hierarchies.

Describing the lepton flavors without fine-tuning on modular invariant models was proposed in Refs. [80,81]. Authors focused on the vicinity of three modular fixed points, $\tau = i$, $\omega (= e^{2\pi i/3})$ and $i\infty$ where residual symmetries remain [43]. The values of modular forms become hierarchical as close to these modular fixed points due to approximate residual symmetries. Then, the hierarchy among values of the modular forms is determined by charges of residual symmetries at the modular fixed points. For instance the modular forms of $\Gamma_4 \simeq S_4$ with Z_4 (T-transformation) charges 0, 1, 2 and 3 can take the sizes 1, ε, ε^2 and ε^3, respectively in the vicinity of $\tau = i\infty$, where ε expresses the deviation from the modular fixed points. Indeed viable lepton models on the double covering groups of Γ_N, $\Gamma'_3 \simeq A'_4$, $\Gamma'_4 \simeq S'_4$, and $\Gamma_5 \simeq A'_5$, were studied in Ref. [81]. This is one successful way to generate hierarchical structures without fine-tuning. Nevertheless the realization of the quark flavor structure is not straightforward. Experiments show mass hierarchies of up sector quarks as $m_u/m_t \sim 10^{-5}$ and $m_c/m_t \sim 10^{-2} - 10^{-3}$ and ones of down sector quarks as $m_d/m_b \sim 10^{-3}$ and $m_s/m_b \sim 10^{-2}$ [82]. Suppose that $\varepsilon = \mathcal{O}(0.1)$. Then, we could explain these mass ratios except $m_u/m_t \sim 10^{-5}$. However, ε^5 does not appear in the framework of the finite modular group of level N less than 6 since the present residual symmetries Z_2, Z_3, and Z_N at $\tau = i$, ω and $i\infty$ do not possess the charge larger than 4. Thus describing the quark flavor structure without fine-tuning requires the way generating hierarchical mass ratios including $\varepsilon^5 = \mathcal{O}(10^{-5})$.

One way is to relax the quark mass eigenvalues by tuning the values of coupling constants in Yukawa couplings. In Ref. [83], the quark flavor model with A_4 modular

[1]The modular flavor symmetry was also studied from the top-down approach such as string theory [2–17].

symmetry was studied and succeeded to generate both up and down sector quark mass hierarchies by adjusting one coupling constant ratio denoted by g_u/g_d to $\mathcal{O}(10)$. As a result, quark mass hierarchies originate from following two origins,

(i) The vacuum expectation value (VEV) of the modulus τ (the deviation from the modular fixed points).
(ii) The coupling constants in Yukawa couplings.

Another way is to introduce the finite modular symmetry including Z_n residual symmetry with $n \geq 6$. In such models, the modular forms in the vicinity of the symmetric points can take the sizes $1, \varepsilon, \cdots, \varepsilon^{n-1}$ depending on their residual charges. Hence, it may be possible to generate mass hierarchies in both up and down sector quarks without fine-tuning using the hierarchical values of the modular forms up to ε^5. Note that in this way quark mass hierarchies simply originate from (i) above. In this paper, we study the modular symmetric quark flavor model with the finite modular group of level 6, $\Gamma_6 \simeq S_3 \times A_4$. The finite modular symmetry Γ_6 breaks into Z_6 (T-transformation) residual symmetry at $\tau = i\infty$ and can generate the hierarchical values of the modular forms up to ε^5 in the vicinity of $\tau = i\infty$.

This paper is organized as follows. In Sec. II, we study generic aspects for the modular symmetric quark flavor models being able to realize both up and down sector quark mass hierarchies without fine-tuning along the lines proposed in Ref. [81]. In Sec. III, we study quark flavor models with the finite modular group Γ_6. Section IV is our conclusion. We summarize group theoretical aspects of Γ_6 in Appendix A and the modular forms of level 6 in Appendix B.

II. HIERARCHICAL QUARK MASS MATRICES WITHOUT FINE-TUNING

In this section, we present modular symmetric quark flavor models without fine-tuning. We start from the following assignment of modular weights to supermultiplets:

(i) Quark doublets $Q = (Q^1, Q^2, Q^3)$ are assigned into three-dimensional (reducible or irreducible) representation of a finite modular group with weight $-k_Q$.
(ii) Up sector quark singlets $u_R = (u_R^1, u_R^2, u_R^3)$ are assigned into three-dimensional (reducible or irreducible) representation of a finite modular group with weight $-k_u$.
(iii) Down sector quark singlets $d_R = (d_R^1, d_R^2, d_R^3)$ are assigned into three-dimensional (reducible or irreducible) representation of a finite modular group with weight $-k_d$.
(iv) Each of up and down sector Higgs fields $H_{u,d}$ is assigned into one-dimensional representations of a finite modular group with weight $-k_{H_{u,d}}$.

Note that three-dimensional representations are constructed by combining singlets, doublets and triplets of any finite modular groups. The most general form of the superpotential relevant to up sector quark masses is written as

$$W_u = \sum_{\boldsymbol{r}_i} \left[Y_{\boldsymbol{r}_i}^{(k_{Y_u})} (Q^1 \; Q^2 \; Q^3) \begin{pmatrix} \alpha_{\boldsymbol{r}_i}^{11} & \alpha_{\boldsymbol{r}_i}^{12} & \alpha_{\boldsymbol{r}_i}^{13} \\ \alpha_{\boldsymbol{r}_i}^{21} & \alpha_{\boldsymbol{r}_i}^{22} & \alpha_{\boldsymbol{r}_i}^{23} \\ \alpha_{\boldsymbol{r}_i}^{31} & \alpha_{\boldsymbol{r}_i}^{32} & \alpha_{\boldsymbol{r}_i}^{33} \end{pmatrix} \begin{pmatrix} u_R^1 \\ u_R^2 \\ u_R^3 \end{pmatrix} H_u \right]_{\mathbf{1}}, \tag{1}$$

where $Y_{\boldsymbol{r}_i}^{(k_{Y_u})}$ denotes the modular forms of irreducible representation $\boldsymbol{r}_i$ for weight $k_{Y_u} = k_Q + k_u + k_{H_u}$. Some of coupling constants α^{ij} may be related each other when quark doublets Q and/or up sector quark singlets u_R belong to multiplets. Similarly the superpotential relevant to down sector quark masses is written as

$$W_d = \sum_{\boldsymbol{r}_i} \left[Y_{\boldsymbol{r}_i}^{(k_{Y_d})} (Q^1 \; Q^2 \; Q^3) \begin{pmatrix} \beta_{\boldsymbol{r}_i}^{11} & \beta_{\boldsymbol{r}_i}^{12} & \beta_{\boldsymbol{r}_i}^{13} \\ \beta_{\boldsymbol{r}_i}^{21} & \beta_{\boldsymbol{r}_i}^{22} & \beta_{\boldsymbol{r}_i}^{23} \\ \beta_{\boldsymbol{r}_i}^{31} & \beta_{\boldsymbol{r}_i}^{32} & \beta_{\boldsymbol{r}_i}^{33} \end{pmatrix} \begin{pmatrix} d_R^1 \\ d_R^2 \\ d_R^3 \end{pmatrix} H_d \right]_{\mathbf{1}}, \tag{2}$$

with $k_{Y_d} = k_Q + k_d + k_{H_d}$. They lead to the up and down sector quark mass matrices, M_u and M_d,

$$(Q^1 \quad Q^2 \quad Q^3) M_u \begin{pmatrix} u_R^1 \\ u_R^2 \\ u_R^3 \end{pmatrix} = \sum_{\boldsymbol{r}_i} \left[Y_{\boldsymbol{r}_i}^{(k_{Y_u})} (Q^1 \quad Q^2 \quad Q^3) \begin{pmatrix} \alpha_{\boldsymbol{r}_i}^{11} & \alpha_{\boldsymbol{r}_i}^{12} & \alpha_{\boldsymbol{r}_i}^{13} \\ \alpha_{\boldsymbol{r}_i}^{21} & \alpha_{\boldsymbol{r}_i}^{22} & \alpha_{\boldsymbol{r}_i}^{23} \\ \alpha_{\boldsymbol{r}_i}^{31} & \alpha_{\boldsymbol{r}_i}^{32} & \alpha_{\boldsymbol{r}_i}^{33} \end{pmatrix} \begin{pmatrix} u_R^1 \\ u_R^2 \\ u_R^3 \end{pmatrix} \langle H_u \rangle \right]_{\mathbf{1}}, \tag{3}$$

$$(Q^1 \quad Q^2 \quad Q^3) M_d \begin{pmatrix} d_R^1 \\ d_R^2 \\ d_R^3 \end{pmatrix} = \sum_{\boldsymbol{r}_i} \left[Y_{\boldsymbol{r}_i}^{(k_{Y_d})} (Q^1 \quad Q^2 \quad Q^3) \begin{pmatrix} \beta_{\boldsymbol{r}_i}^{11} & \beta_{\boldsymbol{r}_i}^{12} & \beta_{\boldsymbol{r}_i}^{13} \\ \beta_{\boldsymbol{r}_i}^{21} & \beta_{\boldsymbol{r}_i}^{22} & \beta_{\boldsymbol{r}_i}^{23} \\ \beta_{\boldsymbol{r}_i}^{31} & \beta_{\boldsymbol{r}_i}^{32} & \beta_{\boldsymbol{r}_i}^{33} \end{pmatrix} \begin{pmatrix} d_R^1 \\ d_R^2 \\ d_R^3 \end{pmatrix} \langle H_d \rangle \right]_{\mathbf{1}}. \tag{4}$$

We expect that the coefficients α^{ij} and β^{ij} are of $\mathcal{O}(1)$, because we do not explain quark mass hierarchies by using hierarchies of these coefficients. In particular, we restrict all coupling constants α^{ij} and β^{ij} to ± 1 and study the realization of the orders of mass ratios and the Cabibbo, Kobayashi, Maskawa (CKM) matrix elements. Then free parameter is only the value of the modulus τ (and the choices of $+1$ or -1 in coupling constants α^{ij} and β^{ij}).

The reference values of up and down quark mass ratios are shown in Table I. Values at a high scale energy include renormalization group effects, which depend on the scenario. We use the values of Refs. [84,85] at the grand unified theory (GUT) scale in the minimal supersymmetric standard model with $\tan\beta = 5$.

In order to realize hierarchical quark masses as shown in Table I without fine-tuning, it is necessary to generate hierarchies by values of the modular forms. Actually such hierarchical values of the modular forms can be realized in a vicinity of three modular fixed points, $\tau = i$, ω and $i\infty$. This can be understood as follows. As an example let us consider Z_n symmetric point and quark doublets Q, up sector quark singlets u_R, and up-type Higgs field H_u with the following Z_n residual charges,

$$Q\colon\ (1, n-1, 0), \qquad u_R\colon\ (1, 0, 0), \qquad H_u\colon\ 0. \tag{5}$$

Then the entries of the up sector quark mass matrix, M_u^{ij}, must have the following Z_n residual charges to make Lagrangian modular invariant,

$$M_u^{ij}\colon \begin{pmatrix} n-2 & n-1 & n-1 \\ 0 & 1 & 1 \\ n-1 & 0 & 0 \end{pmatrix}. \tag{6}$$

In the vicinity of Z_n symmetric point, the modular forms with Z_n residual charge q, $f(\tau)$, can be expanded by the deviation from the symmetric point to the power of q [81]:

(1) $\tau \sim i$: $f(\tau) \sim \varepsilon^q$, $\varepsilon \equiv \frac{\tau - i}{\tau + i}$.

(2) $\tau \sim \omega$: $f(\tau) \sim \varepsilon^q$, $\varepsilon \equiv \frac{\tau - \omega}{\tau - \omega^2}$.

(3) $\tau \sim i\infty$: $f(\tau) \sim \varepsilon^q$, $\varepsilon \equiv e^{-2\pi \mathrm{Im}\tau/N}$ (N is a level of the finite modular group).

Thus the above up sector quark mass matrix can be evaluated as

$$M_u^{ij} \sim \begin{pmatrix} \varepsilon^{n-2} & \varepsilon^{n-1} & \varepsilon^{n-1} \\ 1 & \varepsilon & \varepsilon \\ \varepsilon^{n-1} & 1 & 1 \end{pmatrix}, \tag{7}$$

in the vicinity of Z_n symmetric point. Similarly, for the down sector quark mass matrix as well as lepton mass matrices, the modular forms take hierarchical values depending on their residual charges and lead to hierarchical mass matrices as close to the modular fixed points. In Ref. [81], lepton flavor models without fine-tuning around the vicinity of the modular fixed points was studied.

On the other hand, it is difficult to realize quark mass hierarchies by the values of the modular forms in the vicinity of $\tau = i$ and ω. To realize both up and down sector quark mass hierarchies in Table I simultaneously, we may need ε to the fifth power, when $\varepsilon = \mathcal{O}(0.1)$. Hence we need five different residual charges. This requirement excludes the vicinity of $\tau = i$ and ω since they correspond to Z_2 and Z_3 symmetries, respectively. In other words, such hierarchical masses can be realized in the vicinity of the cusp $\tau = i\infty$ with Z_N charge for $N \geq 6$.

Let us discuss the candidates of the modular symmetry. As mentioned above the level of the modular symmetry must be lager than 5. Here we focus on the levels 6 and 7, that is, $\Gamma_6 \simeq S_3 \times A_4$ and $\Gamma_7 \simeq PSL(2, Z_7)$ as the candidates of the modular symmetry. As the irreducible representations less than four dimension, $\Gamma_7 \simeq PSL(2, Z_7)$ has only one singlet $\mathbf{1}$ and two triplets $\mathbf{3}$ and $\bar{\mathbf{3}}$ [35]; this variety of irreducible representations may not be enough to find the models being able to realize both up and down sector quark masses. In contrast, $\Gamma_6 \simeq S_3 \times A_4$ has six singlets, $\mathbf{1}_0^0$, $\mathbf{1}_1^0$, $\mathbf{1}_2^0$, $\mathbf{1}_0^1$, $\mathbf{1}_1^1$, and $\mathbf{1}_2^1$, three doublets, $\mathbf{2}_0$, $\mathbf{2}_1$, and $\mathbf{2}_2$, and two triplets, $\mathbf{3}^0$ and $\mathbf{3}^1$ [34]. They would be sufficient to find realistic models. In the following, we consider the models with Γ_6 modular symmetry and realize quark flavors without fine-tuning in the vicinity of $\tau = i\infty$.

III. THE MODELS WITH Γ_6 MODULAR SYMMETRY

Here we study the models with Γ_6 modular symmetry and realize the quark flavor structure without fine-tuning. As we have mentioned in the previous section, we restrict all couplings α^{ij} and β^{ij} to ± 1 in quark mass matrices to avoid fine-tuning by them. We study the realization of the orders of mass ratios and mixing angles. We use only the modulus τ (and the choices of $+1$ or -1 in α^{ij} and β^{ij}) as a free parameter.

In Γ_6 modular symmetry, ε to the power up to 5 can appear in mass matrices. Indeed six Γ_6 singlets with six different T-charges correspond to different powers of ε in the vicinity of $\tau = i\infty$ as shown in Table II.

To realize the quark flavor structure, let us consider the following four types of mass matrices,

TABLE I. Quark mass ratios for observed values [82] and GUT scale values with $\tan\beta = 5$ [84,85].

	$\frac{m_u}{m_t} \times 10^6$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_s}{m_b} \times 10^2$
Observed values	12.6	7.38	11.2	2.22
GUT scale values	5.39	2.80	9.21	1.82

$$\text{Type I}:\ M_u \propto \begin{pmatrix} \varepsilon^5 & \varepsilon^{3-a+b} & \varepsilon^b \\ \pm\varepsilon^{5+a-b} & \pm\varepsilon^3 & \pm\varepsilon^a \\ \pm\varepsilon^{5-b} & \pm\varepsilon^{3-a} & \pm 1 \end{pmatrix}, \qquad M_d \propto \begin{pmatrix} \varepsilon^3 & \varepsilon^{2-a+b} & \varepsilon^b \\ \pm\varepsilon^{3+a-b} & \pm\varepsilon^2 & \pm\varepsilon^a \\ \pm\varepsilon^{3-b} & \pm\varepsilon^{2-a} & \pm 1 \end{pmatrix}, \tag{8}$$

$$\text{Type II}:\ M_u \propto \begin{pmatrix} \varepsilon^5 & \varepsilon^{3-a+b} & \varepsilon^b \\ \pm\varepsilon^{5+a-b} & \pm\varepsilon^3 & \pm\varepsilon^a \\ \pm\varepsilon^{5-b} & \pm\varepsilon^{3-a} & \pm 1 \end{pmatrix}, \qquad M_d \propto \begin{pmatrix} \varepsilon^4 & \varepsilon^{2-a+b} & \varepsilon^b \\ \pm\varepsilon^{4+a-b} & \pm\varepsilon^2 & \pm\varepsilon^a \\ \pm\varepsilon^{4-b} & \pm\varepsilon^{2-a} & \pm 1 \end{pmatrix}, \tag{9}$$

$$\text{Type III}:\ M_u \propto \begin{pmatrix} \varepsilon^5 & \varepsilon^{2-a+b} & \varepsilon^b \\ \pm\varepsilon^{5+a-b} & \pm\varepsilon^2 & \pm\varepsilon^a \\ \pm\varepsilon^{5-b} & \pm\varepsilon^{2-a} & \pm 1 \end{pmatrix}, \qquad M_d \propto \begin{pmatrix} \varepsilon^3 & \varepsilon^{2-a+b} & \varepsilon^b \\ \pm\varepsilon^{3+a-b} & \pm\varepsilon^2 & \pm\varepsilon^a \\ \pm\varepsilon^{3-b} & \pm\varepsilon^{2-a} & \pm 1 \end{pmatrix}, \tag{10}$$

$$\text{Type IV}:\ M_u \propto \begin{pmatrix} \varepsilon^5 & \varepsilon^{2-a+b} & \varepsilon^b \\ \pm\varepsilon^{5+a-b} & \pm\varepsilon^2 & \pm\varepsilon^a \\ \pm\varepsilon^{5-b} & \pm\varepsilon^{2-a} & \pm 1 \end{pmatrix}, \qquad M_d \propto \begin{pmatrix} \varepsilon^4 & \varepsilon^{2-a+b} & \varepsilon^b \\ \pm\varepsilon^{4+a-b} & \pm\varepsilon^2 & \pm\varepsilon^a \\ \pm\varepsilon^{4-b} & \pm\varepsilon^{2-a} & \pm 1 \end{pmatrix}, \tag{11}$$

where $\pm$ corresponds any possible combinations of signs and $a, b \in \{0,1,2,3,4,5\}$. Note that it is always possible to fix the signs of (1,1), (1,2), and (1,3) components to $+1$ by the basis transformation for right-handed quarks. We set powers of ε on diagonal components in up and down sector quark mass matrices to (5,3,0) and (3,2,0) for type I, (5,3,0) and (4,2,0) for type II, (5,2,0) and (3,2,0) for type III and (5,2,0), and (4,2,0) for type IV in order to realize their hierarchical masses. Here, we use only six Γ_6 singlets, $\mathbf{1}_\mathbf{0}^\mathbf{0}$, $\mathbf{1}_\mathbf{1}^\mathbf{0}$, $\mathbf{1}_\mathbf{2}^\mathbf{0}$, $\mathbf{1}_\mathbf{0}^\mathbf{1}$, $\mathbf{1}_\mathbf{1}^\mathbf{1}$, and $\mathbf{1}_\mathbf{2}^\mathbf{1}$, as irreducible representations to make our analysis simple. Note that again powers of ε in mass matrices are determined by Z_6 charges of entries of mass matrices. Thus mass matrices of each type can be led by the following assignments,

$$\text{Type I}:\ Q = (\mathbf{1}_{b \bmod 3}^{b \bmod 2}, \mathbf{1}_{a \bmod 3}^{a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \qquad u_R = (\mathbf{1}_{5-b \bmod 3}^{5-b \bmod 2}, \mathbf{1}_{3-a \bmod 3}^{3-a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \qquad d_R = (\mathbf{1}_{3-b \bmod 3}^{3-b \bmod 2}, \mathbf{1}_{2-a \bmod 3}^{2-a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \tag{12}$$

$$\text{Type II}:\ Q = (\mathbf{1}_{b \bmod 3}^{b \bmod 2}, \mathbf{1}_{a \bmod 3}^{a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \qquad u_R = (\mathbf{1}_{5-b \bmod 3}^{5-b \bmod 2}, \mathbf{1}_{3-a \bmod 3}^{3-a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \qquad d_R = (\mathbf{1}_{4-b \bmod 3}^{4-b \bmod 2}, \mathbf{1}_{2-a \bmod 3}^{2-a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \tag{13}$$

$$\text{Type III}:\ Q = (\mathbf{1}_{b \bmod 3}^{b \bmod 2}, \mathbf{1}_{a \bmod 3}^{a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \qquad u_R = (\mathbf{1}_{5-b \bmod 3}^{5-b \bmod 2}, \mathbf{1}_{2-a \bmod 3}^{2-a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \qquad d_R = (\mathbf{1}_{3-b \bmod 3}^{3-b \bmod 2}, \mathbf{1}_{2-a \bmod 3}^{2-a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \tag{14}$$

$$\text{Type IV}:\ Q = (\mathbf{1}_{b \bmod 3}^{b \bmod 2}, \mathbf{1}_{a \bmod 3}^{a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \qquad u_R = (\mathbf{1}_{5-b \bmod 3}^{5-b \bmod 2}, \mathbf{1}_{2-a \bmod 3}^{2-a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}), \qquad d_R = (\mathbf{1}_{4-b \bmod 3}^{4-b \bmod 2}, \mathbf{1}_{2-a \bmod 3}^{2-a \bmod 2}, \mathbf{1}_\mathbf{0}^\mathbf{0}). \tag{15}$$

On the other hand, it is not always true that mass matrices in four types are definitely realized by the above assignments. It depends on weights of the Yukawa couplings. All of the singlet modular forms of Γ_6 with certain Z_6 charges do not exist for weights less than 14 as shown in Appendix B. For instance, the modular forms of weight 12 belong to $\mathbf{1}_\mathbf{1}^\mathbf{1}$ do not exist. Yukawa couplings of the weights less than 14 can lead to mass matrices with some zeros due to this shortage of the modular forms on low weights. We study the case of Yukawa couplings of the weight 14 in Sec. III A and one of the weights less than 14 in Sec. III B.

TABLE II. T-charges of six Γ_6 singlets and their orders in the vicinity of $\tau = i\infty$.

Singlet	$\mathbf{1}_\mathbf{0}^\mathbf{0}$	$\mathbf{1}_\mathbf{2}^\mathbf{1}$	$\mathbf{1}_\mathbf{1}^\mathbf{0}$	$\mathbf{1}_\mathbf{0}^\mathbf{1}$	$\mathbf{1}_\mathbf{2}^\mathbf{0}$	$\mathbf{1}_\mathbf{1}^\mathbf{1}$
T-charge	0	1	2	3	4	5
order	1	ε	ε^2	ε^3	ε^4	ε^5

A. Weight 14

First of all, we study the models with Yukawa couplings of weight 14 to avoid zero textures in mass matrices of four types. We choose $\tau = 3.2i$ as a benchmark point of the modulus. At weight 14, seven singlet modular forms, $Y^{(14)}_{\mathbf{1}_\mathbf{0}^\mathbf{0}}$, $Y^{(14)}_{\mathbf{1}_\mathbf{2}^\mathbf{1} i}$, $Y^{(14)}_{\mathbf{1}_\mathbf{1}^\mathbf{0}}$, $Y^{(14)}_{\mathbf{1}_\mathbf{0}^\mathbf{1}}$, $Y^{(14)}_{\mathbf{1}_\mathbf{2}^\mathbf{0}}$, $Y^{(14)}_{\mathbf{1}_\mathbf{1}^\mathbf{1}}$, and $Y^{(14)}_{\mathbf{1}_\mathbf{2}^\mathbf{1} ii}$, exist and they are approximated by ε as

$$
\begin{aligned}
&Y^{(14)}_{\mathbf{1}^0_0}/Y^{(14)}_{\mathbf{1}^0_0} = 1 \to 1, \qquad Y^{(14)}_{\mathbf{1}^1_2 i}/Y^{(14)}_{\mathbf{1}^0_0} = 0.172 \to \varepsilon,\\
&Y^{(14)}_{\mathbf{1}^0_1}/Y^{(14)}_{\mathbf{1}^0_0} = 0.0208 \to \varepsilon^2,\\
&Y^{(14)}_{\mathbf{1}^1_0}/Y^{(14)}_{\mathbf{1}^0_0} = 0.00358 \to \varepsilon^3,\\
&Y^{(14)}_{\mathbf{1}^0_2}/Y^{(14)}_{\mathbf{1}^0_0} = 0.000435 \to \varepsilon^4,\\
&Y^{(14)}_{\mathbf{1}^1_1}/Y^{(14)}_{\mathbf{1}^0_0} = 0.0000746 \to \varepsilon^5,\\
&Y^{(14)}_{\mathbf{1}^1_2 ii}/Y^{(14)}_{\mathbf{1}^0_0} = 0.00000156 \to \varepsilon^7,
\end{aligned} \tag{16}
$$

at $\tau = 3.2i$. Note that $Y^{(14)}_{\mathbf{1}^1_2 ii} \sim \varepsilon^7$ originates from $Y^{(6)}_{\mathbf{1}^1_0} Y^{(8)}_{\mathbf{1}^0_2} \sim \varepsilon^3 \cdot \varepsilon^4$ while $Y^{(14)}_{\mathbf{1}^1_2 i} \sim \varepsilon$ originates from $Y^{(6)}_{\mathbf{1}^1_2} Y^{(8)}_{\mathbf{1}^0_0} \sim \varepsilon \cdot 1$. ε^n for $n > 5$ can appear when the different modular forms of the same irreducible representations exist. In what follows, we ignore $Y^{(14)}_{\mathbf{1}^1_2 ii}$ because it belongs to the same representation as $Y^{(14)}_{\mathbf{1}^1_2 i}$ and $Y^{(14)}_{\mathbf{1}^1_2 i} \gg Y^{(14)}_{\mathbf{1}^1_2 ii}$.

1. Type I: (5,3,0) and (3,2,0)

The mass matrices of type I are given by

$$
M_u = \langle H_u \rangle \begin{pmatrix}
\alpha^{11} Y^{(14)}_{\mathbf{1}^1_1} & \alpha^{12} Y^{(14)}_{\mathbf{1}^{3+a-b \bmod 2}_{3+a-b \bmod 3}} & \alpha^{13} Y^{(14)}_{\mathbf{1}^{6-b \bmod 2}_{6-b \bmod 3}} \\
\alpha^{21} Y^{(14)}_{\mathbf{1}^{1-a+b \bmod 2}_{1-a+b \bmod 3}} & \alpha^{22} Y^{(14)}_{\mathbf{1}^1_0} & \alpha^{23} Y^{(14)}_{\mathbf{1}^{6-a \bmod 2}_{6-a \bmod 3}} \\
\alpha^{31} Y^{(14)}_{\mathbf{1}^{1+b \bmod 2}_{1+b \bmod 3}} & \alpha^{32} Y^{(14)}_{\mathbf{1}^{3+a \bmod 2}_{3+a \bmod 3}} & \alpha^{33} Y^{(14)}_{\mathbf{1}^0_0}
\end{pmatrix}, \tag{17}
$$

$$
M_d = \langle H_d \rangle \begin{pmatrix}
\beta^{11} Y^{(14)}_{\mathbf{1}^1_0} & \beta^{12} Y^{(14)}_{\mathbf{1}^{4+a-b \bmod 2}_{4+a-b \bmod 3}} & \beta^{13} Y^{(14)}_{\mathbf{1}^{6-b \bmod 2}_{6-b \bmod 3}} \\
\beta^{21} Y^{(14)}_{\mathbf{1}^{3-a+b \bmod 2}_{3-a+b \bmod 3}} & \beta^{22} Y^{(14)}_{\mathbf{1}^0_1} & \beta^{23} Y^{(14)}_{\mathbf{1}^{6-a \bmod 2}_{6-a \bmod 3}} \\
\beta^{31} Y^{(14)}_{\mathbf{1}^{3+b \bmod 2}_{3+b \bmod 3}} & \beta^{32} Y^{(14)}_{\mathbf{1}^{4+a \bmod 2}_{4+a \bmod 3}} & \beta^{33} Y^{(14)}_{\mathbf{1}^0_0}
\end{pmatrix}, \tag{18}
$$

where α^{ij} and β^{ij} are coupling constants which we restrict to ± 1. The hierarchical mass matrices in Eq. (8) can be obtained by choosing $+1$ or -1 appropriately in α^{ij} and β^{ij}. As a result, we find best-fit mass matrices at $\tau = 3.2i$,

$$
\begin{aligned}
M_u/(Y^{(14)}_{\mathbf{1}^0_0}\langle H_u \rangle) &= \begin{pmatrix}
Y^{(14)}_{\mathbf{1}^1_1} & Y^{(14)}_{\mathbf{1}^0_2} & Y^{(14)}_{\mathbf{1}^1_0} \\
Y^{(14)}_{\mathbf{1}^0_2} & Y^{(14)}_{\mathbf{1}^1_0} & Y^{(14)}_{\mathbf{1}^0_1} \\
-Y^{(14)}_{\mathbf{1}^0_1} & -Y^{(14)}_{\mathbf{1}^1_2 i} & Y^{(14)}_{\mathbf{1}^0_0}
\end{pmatrix} / (Y^{(14)}_{\mathbf{1}^0_0}\langle H_u \rangle) \\
&= \begin{pmatrix}
0.0000746 & 0.000435 & 0.00358 \\
0.000435 & 0.00358 & 0.0208 \\
-0.0208 & -0.172 & 1
\end{pmatrix} \\
&\sim \begin{pmatrix}
\varepsilon^5 & \varepsilon^4 & \varepsilon^3 \\
\varepsilon^4 & \varepsilon^3 & \varepsilon^2 \\
-\varepsilon^2 & -\varepsilon & 1
\end{pmatrix},
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
M_d/(Y^{(14)}_{\mathbf{1}^0_0}\langle H_d \rangle) &= \begin{pmatrix}
Y^{(14)}_{\mathbf{1}^1_0} & Y^{(14)}_{\mathbf{1}^1_0} & Y^{(14)}_{\mathbf{1}^1_0} \\
Y^{(14)}_{\mathbf{1}^0_1} & -Y^{(14)}_{\mathbf{1}^0_1} & -Y^{(14)}_{\mathbf{1}^0_1} \\
-Y^{(14)}_{\mathbf{1}^0_0} & Y^{(14)}_{\mathbf{1}^0_0} & -Y^{(14)}_{\mathbf{1}^0_0}
\end{pmatrix} / (Y^{(14)}_{\mathbf{1}^0_0}\langle H_d \rangle) \\
&= \begin{pmatrix}
0.00358 & 0.00358 & 0.00358 \\
0.0208 & -0.0208 & -0.0208 \\
-1 & 1 & -1
\end{pmatrix} \\
&\sim \begin{pmatrix}
\varepsilon^3 & \varepsilon^3 & \varepsilon^3 \\
\varepsilon^2 & -\varepsilon^2 & -\varepsilon^2 \\
-1 & 1 & -1
\end{pmatrix}.
\end{aligned} \tag{20}
$$

These mass matrices correspond to $a = 2$, $b = 3$ and can be realized by

$$
Q = (\mathbf{1}^1_0, \mathbf{1}^0_2, \mathbf{1}^0_0), \quad u_R = (\mathbf{1}^0_2, \mathbf{1}^1_1, \mathbf{1}^0_0), \quad d_R = (\mathbf{1}^0_0, \mathbf{1}^0_0, \mathbf{1}^0_0), \tag{21}
$$

and their mass matrices are written by,

$$
\begin{aligned}
M_u &= \langle H_u \rangle \begin{pmatrix}
\alpha^{11} Y^{(14)}_{\mathbf{1}^1_1} & \alpha^{12} Y^{(14)}_{\mathbf{1}^0_2} & \alpha^{13} Y^{(14)}_{\mathbf{1}^1_0} \\
\alpha^{21} Y^{(14)}_{\mathbf{1}^0_2} & \alpha^{22} Y^{(14)}_{\mathbf{1}^1_0} & \alpha^{23} Y^{(14)}_{\mathbf{1}^0_1} \\
\alpha^{31} Y^{(14)}_{\mathbf{1}^0_1} & \alpha^{32} Y^{(14)}_{\mathbf{1}^1_2 i} & \alpha^{33} Y^{(14)}_{\mathbf{1}^0_0}
\end{pmatrix}, \\
M_d &= \langle H_d \rangle \begin{pmatrix}
\beta^{11} Y^{(14)}_{\mathbf{1}^1_0} & \beta^{12} Y^{(14)}_{\mathbf{1}^1_0} & \beta^{13} Y^{(14)}_{\mathbf{1}^1_0} \\
\beta^{21} Y^{(14)}_{\mathbf{1}^0_1} & \beta^{22} Y^{(14)}_{\mathbf{1}^0_1} & \beta^{23} Y^{(14)}_{\mathbf{1}^0_1} \\
\beta^{31} Y^{(14)}_{\mathbf{1}^0_0} & \beta^{32} Y^{(14)}_{\mathbf{1}^0_0} & \beta^{33} Y^{(14)}_{\mathbf{1}^0_0}
\end{pmatrix},
\end{aligned} \tag{22}
$$

with the following choices of $+1$ or -1 in coupling constants,

$$\begin{pmatrix} \alpha^{11} & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ \alpha^{31} & \alpha^{32} & \alpha^{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} \beta^{11} & \beta^{12} & \beta^{13} \\ \beta^{21} & \beta^{22} & \beta^{23} \\ \beta^{31} & \beta^{32} & \beta^{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix}. \tag{23}$$

They lead to the following quark mass ratios,

$$(m_u, m_c, m_t)/m_t = (2.11 \times 10^{-5}, 7.07 \times 10^{-3}, 1), \tag{24}$$

$$(m_d, m_s, m_b)/m_b = (2.91 \times 10^{-3}, 1.97 \times 10^{-2}, 1), \tag{25}$$

and the absolute values of the CKM matrix elements,

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.973 & 0.231 & 0.000681 \\ 0.231 & 0.973 & 0.0270 \\ 0.00690 & 0.0261 & 1.00 \end{pmatrix}. \tag{26}$$

Results are shown in Table III. Recall that our purpose is to realize the order of quark mass rations and mixing angles without fine-tuning. For this purpose, we have fixed the coefficients, $\alpha^{ij}, \beta^{ij} = \pm 1$ to make our point clear. We could obtain more realistic values when we vary $\alpha^{ij}, \beta^{ij} = \mathcal{O}(1)$. Also other models in this type could be realistic when we vary $\alpha^{ij}, \beta^{ij} = \mathcal{O}(1)$. In addition, we have a remark on normalization of modular forms. The normalization of modular forms has ambiguity, but we expect naturally that such normalization would not lead to a large hierarchy. Our models may originate from compactification of higher dimensional field theory or superstring theory. In that case, values in our models appear in high energy scale such as the GUT scale. Renormalization group effects change values by some factors, although those effects depend on the scenario. For example, renormalization group effects in the minimal supersymmetric scenario were studied in Refs. [84,85]. Table III also shows those values at the GUT scale for $\tan\beta = 5$ as reference values.

As mentioned above, when we vary α^{ij} and β^{ij}, we can obtain more realistic values. For example, we set

$$\begin{pmatrix} \alpha^{11} & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ \alpha^{31} & \alpha^{32} & \alpha^{33} \end{pmatrix} = \begin{pmatrix} 2.547 & 1.987 & 1.052 \\ 1.124 & 1.000 & 2.998 \\ -2.511 & -1.001 & 2.754 \end{pmatrix},$$
$$\begin{pmatrix} \beta^{11} & \beta^{12} & \beta^{13} \\ \beta^{21} & \beta^{22} & \beta^{23} \\ \beta^{31} & \beta^{32} & \beta^{33} \end{pmatrix} = \begin{pmatrix} 1.149 & 1.000 & 1.405 \\ 2.997 & -2.999 & -1.504 \\ -2.961 & 1.664 & -1.494 \end{pmatrix}. \tag{27}$$

Then, we obtain the following quark mass ratios,

$$(m_u, m_c, m_t)/m_t = (5.39 \times 10^{-5}, 2.80 \times 10^{-3}, 1), \tag{28}$$

$$(m_d, m_s, m_b)/m_b = (9.21 \times 10^{-4}, 1.82 \times 10^{-2}, 1), \tag{29}$$

and the absolute values of the CKM matrix elements,

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.974 & 0.225 & 0.00353 \\ 0.225 & 0.974 & 0.0400 \\ 0.00556 & 0.0398 & 0.999 \end{pmatrix}. \tag{30}$$

Results are shown in Table IV.

TABLE III. The mass ratios of the quarks and the absolute values of the CKM matrix elements at the benchmark point $\tau = 3.2i$ in the best-fit model by Eqs. (21) and (23) of type I with Yukawa couplings of weight 14. Observed values Ref. [82] and GUT scale values with $\tan\beta = 5$ [84,85] are shown.

	$\frac{m_u}{m_t} \times 10^6$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_s}{m_b} \times 10^2$	$\lvert V_{\mathrm{CKM}}^{us}\rvert$	$\lvert V_{\mathrm{CKM}}^{cb}\rvert$	$\lvert V_{\mathrm{CKM}}^{ub}\rvert$
Obtained values	21.1	7.07	29.1	1.97	0.231	0.0270	0.000681
Observed values	12.6	7.38	11.2	2.22	0.227	0.0405	0.00361
GUT scale values	5.39	2.80	9.21	1.82	0.225	0.0400	0.00353

TABLE IV. The mass ratios of the quarks and the absolute values of the CKM matrix elements at the benchmark point $\tau = 3.2i$ in the best-fit model by Eqs. (21) and (27) of type I with Yukawa couplings of weight 14. Observed values Ref. [82] and GUT scale values with $\tan\beta = 5$ [84,85] are shown.

	$\frac{m_u}{m_t} \times 10^6$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_s}{m_b} \times 10^2$	$\lvert V_{\mathrm{CKM}}^{us}\rvert$	$\lvert V_{\mathrm{CKM}}^{cb}\rvert$	$\lvert V_{\mathrm{CKM}}^{ub}\rvert$
Obtained values	5.39	2.80	9.21	1.82	0.225	0.0400	0.00353
Observed values	12.6	7.38	11.2	2.22	0.227	0.0405	0.00361
GUT scale values	5.39	2.80	9.21	1.82	0.225	0.0400	0.00353

2. Type II: (5,3,0) and (4,2,0)

The mass matrices of type II are given by Eq. (17) and

$$M_d = \langle H_d\rangle \begin{pmatrix} \beta^{11} Y^{(14)}_{\mathbf{1}^0_2} & \beta^{12} Y^{(14)}_{\mathbf{1}^{4+a-b \bmod 2}_{4+a-b \bmod 3}} & \beta^{13} Y^{(14)}_{\mathbf{1}^{6-b \bmod 2}_{6-b \bmod 3}} \\ \beta^{21} Y^{(14)}_{\mathbf{1}^{2-a+b \bmod 2}_{2-a+b \bmod 3}} & \beta^{22} Y^{(14)}_{\mathbf{1}^0_1} & \beta^{23} Y^{(14)}_{\mathbf{1}^{6-a \bmod 2}_{6-a \bmod 3}} \\ \beta^{31} Y^{(14)}_{\mathbf{1}^{2+b \bmod 2}_{2+b \bmod 3}} & \beta^{32} Y^{(14)}_{\mathbf{1}^{4+a \bmod 2}_{4+a \bmod 3}} & \beta^{33} Y^{(14)}_{\mathbf{1}^0_0} \end{pmatrix}. \tag{31}$$

The hierarchical mass matrices in Eq. (9) can be obtained by choosing $+1$ or -1 appropriately in α^{ij} and β^{ij}. As a result, we find best-fit mass matrices at $\tau = 3.2i$,

$$\begin{aligned} M_u/(Y^{(14)}_{\mathbf{1}^0_0}\langle H_u\rangle) &= \begin{pmatrix} Y^{(14)}_{\mathbf{1}^1_1} & Y^{(14)}_{\mathbf{1}^0_2} & Y^{(14)}_{\mathbf{1}^1_0} \\ Y^{(14)}_{\mathbf{1}^0_2} & Y^{(14)}_{\mathbf{1}^1_0} & -Y^{(14)}_{\mathbf{1}^0_1} \\ -Y^{(14)}_{\mathbf{1}^0_1} & -Y^{(14)}_{\mathbf{1}^1_2 i} & -Y^{(14)}_{\mathbf{1}^0_0} \end{pmatrix} / (Y^{(14)}_{\mathbf{1}^0_0}\langle H_u\rangle) \\ &= \begin{pmatrix} 0.0000746 & 0.000435 & 0.00358 \\ 0.000435 & 0.00358 & -0.0208 \\ -0.0208 & -0.172 & -1 \end{pmatrix} \\ &\sim \begin{pmatrix} \varepsilon^5 & \varepsilon^4 & \varepsilon^3 \\ \varepsilon^4 & \varepsilon^3 & -\varepsilon^2 \\ -\varepsilon^2 & -\varepsilon & -1 \end{pmatrix}, \end{aligned} \tag{32}$$

$$\begin{aligned} M_d/(Y^{(14)}_{\mathbf{1}^0_0}\langle H_d\rangle) &= \begin{pmatrix} Y^{(14)}_{\mathbf{1}^0_2} & Y^{(14)}_{\mathbf{1}^1_0} & Y^{(14)}_{\mathbf{1}^1_0} \\ -Y^{(14)}_{\mathbf{1}^1_0} & Y^{(14)}_{\mathbf{1}^0_1} & Y^{(14)}_{\mathbf{1}^0_1} \\ Y^{(14)}_{\mathbf{1}^1_2 i} & Y^{(14)}_{\mathbf{1}^0_0} & -Y^{(14)}_{\mathbf{1}^0_0} \end{pmatrix} / (Y^{(14)}_{\mathbf{1}^0_0}\langle H_d\rangle) \\ &= \begin{pmatrix} 0.000435 & 0.00358 & 0.00358 \\ -0.00358 & 0.0208 & 0.0208 \\ 0.172 & 1 & -1 \end{pmatrix} \\ &\sim \begin{pmatrix} \varepsilon^4 & \varepsilon^3 & \varepsilon^3 \\ -\varepsilon^3 & \varepsilon^2 & \varepsilon^2 \\ \varepsilon & 1 & -1 \end{pmatrix}. \end{aligned} \tag{33}$$

These mass matrices correspond to $a = 2$, $b = 3$ and can be realized by

$$Q = (\mathbf{1}^1_0, \mathbf{1}^0_2, \mathbf{1}^0_0), \quad u_R = (\mathbf{1}^0_2, \mathbf{1}^1_1, \mathbf{1}^0_0), \quad d_R = (\mathbf{1}^1_1, \mathbf{1}^0_0, \mathbf{1}^0_0), \tag{34}$$

and their mass matrices are written by,

$$\begin{aligned} M_u &= \langle H_u\rangle \begin{pmatrix} \alpha^{11} Y^{(14)}_{\mathbf{1}^1_1} & \alpha^{12} Y^{(14)}_{\mathbf{1}^0_2} & \alpha^{13} Y^{(14)}_{\mathbf{1}^1_0} \\ \alpha^{21} Y^{(14)}_{\mathbf{1}^0_2} & \alpha^{22} Y^{(14)}_{\mathbf{1}^1_0} & \alpha^{23} Y^{(14)}_{\mathbf{1}^0_1} \\ \alpha^{31} Y^{(14)}_{\mathbf{1}^0_1} & \alpha^{32} Y^{(14)}_{\mathbf{1}^1_2 i} & \alpha^{33} Y^{(14)}_{\mathbf{1}^0_0} \end{pmatrix}, \\ M_d &= \langle H_d\rangle \begin{pmatrix} \beta^{11} Y^{(14)}_{\mathbf{1}^0_2} & \beta^{12} Y^{(14)}_{\mathbf{1}^1_0} & \beta^{13} Y^{(14)}_{\mathbf{1}^1_0} \\ \beta^{21} Y^{(14)}_{\mathbf{1}^1_0} & \beta^{22} Y^{(14)}_{\mathbf{1}^0_1} & \beta^{23} Y^{(14)}_{\mathbf{1}^0_1} \\ \beta^{31} Y^{(14)}_{\mathbf{1}^1_2 i} & \beta^{32} Y^{(14)}_{\mathbf{1}^0_0} & \beta^{33} Y^{(14)}_{\mathbf{1}^0_0} \end{pmatrix}, \end{aligned} \tag{35}$$

with the following choices of $+1$ or -1 in coupling constants,

$$\begin{aligned} \begin{pmatrix} \alpha^{11} & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ \alpha^{31} & \alpha^{32} & \alpha^{33} \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \\ \begin{pmatrix} \beta^{11} & \beta^{12} & \beta^{13} \\ \beta^{21} & \beta^{22} & \beta^{23} \\ \beta^{31} & \beta^{32} & \beta^{33} \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}. \end{aligned} \tag{36}$$

They lead to the following quark mass ratios,

$$(m_u, m_c, m_t)/m_t = (2.14 \times 10^{-5}, 7.00 \times 10^{-3}, 1), \tag{37}$$

$$(m_d, m_s, m_b)/m_b = (7.16 \times 10^{-4}, 2.11 \times 10^{-2}, 1), \tag{38}$$

and the absolute values of the CKM matrix elements,

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.982 & 0.190 & 0.00309 \\ 0.190 & 0.982 & 0.0200 \\ 0.00683 & 0.0191 & 1.00 \end{pmatrix}. \tag{39}$$

Results are shown in Table V.

TABLE V. The mass ratios of the quarks and the absolute values of the CKM matrix elements at the benchmark point $\tau = 3.2i$ in the best-fit model by Eqs. (34) and (36) of type II with Yukawa couplings of weight 14. Observed values Ref. [82] and GUT scale values with $\tan\beta = 5$ [84,85] are shown.

	$\frac{m_u}{m_t} \times 10^6$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_s}{m_b} \times 10^2$	$\lvert V^{us}_{\mathrm{CKM}}\rvert$	$\lvert V^{cb}_{\mathrm{CKM}}\rvert$	$\lvert V^{ub}_{\mathrm{CKM}}\rvert$
Obtained values	21.4	7.00	7.16	2.11	0.190	0.0200	0.00309
Observed values	12.6	7.38	11.2	2.22	0.227	0.0405	0.00361
GUT scale values	5.39	2.80	9.21	1.82	0.225	0.0400	0.00353

3. Type III: (5,2,0) and (3,2,0)

The mass matrices of type III are given by Eq. (18) and

$$M_u = \langle H_u \rangle \begin{pmatrix} \alpha^{11} Y^{(14)}_{\mathbf{1}^{1}_{1}} & \alpha^{12} Y^{(14)}_{\mathbf{1}^{4+a-b \bmod 2}_{4+a-b \bmod 3}} & \alpha^{13} Y^{(14)}_{\mathbf{1}^{6-b \bmod 2}_{6-b \bmod 3}} \\ \alpha^{21} Y^{(14)}_{\mathbf{1}^{1-a+b \bmod 2}_{1-a+b \bmod 3}} & \alpha^{22} Y^{(14)}_{\mathbf{1}^{0}_{1}} & \alpha^{23} Y^{(14)}_{\mathbf{1}^{6-a \bmod 2}_{6-a \bmod 3}} \\ \alpha^{31} Y^{(14)}_{\mathbf{1}^{1+b \bmod 2}_{1+b \bmod 3}} & \alpha^{32} Y^{(14)}_{\mathbf{1}^{4+a \bmod 2}_{4+a \bmod 3}} & \alpha^{33} Y^{(14)}_{\mathbf{1}^{0}_{0}} \end{pmatrix}. \tag{40}$$

The hierarchical mass matrices in Eq. (10) can be obtained by choosing $+1$ or -1 appropriately in α^{ij} and β^{ij}. As a result, we find best-fit mass matrices at $\tau = 3.2i$,

$$\begin{aligned} M_u/(Y^{(14)}_{\mathbf{1}^{0}_{0}} \langle H_u \rangle) &= \begin{pmatrix} Y^{(14)}_{\mathbf{1}^{1}_{1}} & Y^{(14)}_{\mathbf{1}^{1}_{0}} & Y^{(14)}_{\mathbf{1}^{1}_{0}} \\ Y^{(14)}_{\mathbf{1}^{0}_{2}} & -Y^{(14)}_{\mathbf{1}^{0}_{1}} & -Y^{(14)}_{\mathbf{1}^{0}_{1}} \\ Y^{(14)}_{\mathbf{1}^{0}_{1}} & Y^{(14)}_{\mathbf{1}^{0}_{0}} & -Y^{(14)}_{\mathbf{1}^{0}_{0}} \end{pmatrix} / (Y^{(14)}_{\mathbf{1}^{0}_{0}} \langle H_u \rangle) \\ &= \begin{pmatrix} 0.0000746 & 0.00358 & 0.00358 \\ 0.000435 & -0.0208 & -0.0208 \\ 0.0208 & 1 & -1 \end{pmatrix} \\ &\sim \begin{pmatrix} \varepsilon^5 & \varepsilon^3 & \varepsilon^3 \\ \varepsilon^4 & -\varepsilon^2 & -\varepsilon^2 \\ \varepsilon^2 & 1 & -1 \end{pmatrix}, \end{aligned} \tag{41}$$

$$\begin{aligned} M_d/(Y^{(14)}_{\mathbf{1}^{0}_{0}} \langle H_d \rangle) &= \begin{pmatrix} Y^{(14)}_{\mathbf{1}^{1}_{0}} & Y^{(14)}_{\mathbf{1}^{1}_{0}} & Y^{(14)}_{\mathbf{1}^{1}_{0}} \\ Y^{(14)}_{\mathbf{1}^{0}_{1}} & Y^{(14)}_{\mathbf{1}^{0}_{1}} & -Y^{(14)}_{\mathbf{1}^{0}_{1}} \\ Y^{(14)}_{\mathbf{1}^{0}_{0}} & -Y^{(14)}_{\mathbf{1}^{0}_{0}} & Y^{(14)}_{\mathbf{1}^{0}_{0}} \end{pmatrix} / (Y^{(14)}_{\mathbf{1}^{0}_{0}} \langle H_d \rangle) \\ &= \begin{pmatrix} 0.00358 & 0.00358 & 0.00358 \\ 0.0208 & 0.0208 & -0.0208 \\ 1 & -1 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} \varepsilon^3 & \varepsilon^3 & \varepsilon^3 \\ \varepsilon^2 & \varepsilon^2 & -\varepsilon^2 \\ 1 & -1 & 1 \end{pmatrix}. \end{aligned} \tag{42}$$

These mass matrices correspond to $a = 2$, $b = 3$ and can be realized by

$$Q = (\mathbf{1}^{1}_{0}, \mathbf{1}^{0}_{2}, \mathbf{1}^{0}_{0}), \quad u_R = (\mathbf{1}^{0}_{2}, \mathbf{1}^{0}_{0}, \mathbf{1}^{0}_{0}), \quad d_R = (\mathbf{1}^{0}_{0}, \mathbf{1}^{0}_{0}, \mathbf{1}^{0}_{0}), \tag{43}$$

and their mass matrices are written by,

$$\begin{aligned} M_u &= \langle H_u \rangle \begin{pmatrix} \alpha^{11} Y^{(14)}_{\mathbf{1}^{1}_{1}} & \alpha^{12} Y^{(14)}_{\mathbf{1}^{1}_{0}} & \alpha^{13} Y^{(14)}_{\mathbf{1}^{1}_{0}} \\ \alpha^{21} Y^{(14)}_{\mathbf{1}^{0}_{2}} & \alpha^{22} Y^{(14)}_{\mathbf{1}^{0}_{1}} & \alpha^{23} Y^{(14)}_{\mathbf{1}^{0}_{1}} \\ \alpha^{31} Y^{(14)}_{\mathbf{1}^{0}_{1}} & \alpha^{32} Y^{(14)}_{\mathbf{1}^{0}_{0}} & \alpha^{33} Y^{(14)}_{\mathbf{1}^{0}_{0}} \end{pmatrix}, \\ M_d &= \langle H_d \rangle \begin{pmatrix} \beta^{11} Y^{(14)}_{\mathbf{1}^{1}_{0}} & \beta^{12} Y^{(14)}_{\mathbf{1}^{1}_{0}} & \beta^{13} Y^{(14)}_{\mathbf{1}^{1}_{0}} \\ \beta^{21} Y^{(14)}_{\mathbf{1}^{0}_{1}} & \beta^{22} Y^{(14)}_{\mathbf{1}^{0}_{1}} & \beta^{23} Y^{(14)}_{\mathbf{1}^{0}_{1}} \\ \beta^{31} Y^{(14)}_{\mathbf{1}^{0}_{0}} & \beta^{32} Y^{(14)}_{\mathbf{1}^{0}_{0}} & \beta^{33} Y^{(14)}_{\mathbf{1}^{0}_{0}} \end{pmatrix}, \end{aligned} \tag{44}$$

with the following choices of $+1$ or -1 in coupling constants,

$$\begin{aligned} \begin{pmatrix} \alpha^{11} & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ \alpha^{31} & \alpha^{32} & \alpha^{33} \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix}, \\ \begin{pmatrix} \beta^{11} & \beta^{12} & \beta^{13} \\ \beta^{21} & \beta^{22} & \beta^{23} \\ \beta^{31} & \beta^{32} & \beta^{33} \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}. \end{aligned} \tag{45}$$

They lead to the following quark mass ratios,

$$(m_u, m_c, m_t)/m_t = (1.04 \times 10^{-4}, 2.12 \times 10^{-2}, 1), \tag{46}$$

$$(m_d, m_s, m_b)/m_b = (2.91 \times 10^{-3}, 1.97 \times 10^{-2}, 1), \tag{47}$$

and the absolute values of the CKM matrix elements,

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.967 & 0.255 & 0.00000171 \\ 0.255 & 0.967 & 0.00706 \\ 0.00180 & 0.00682 & 1.00 \end{pmatrix}. \tag{48}$$

Results are shown in Table VI.

TABLE VI. The mass ratios of the quarks and the absolute values of the CKM matrix elements at the benchmark point $\tau = 3.2i$ in the best-fit model by Eqs. (43) and (45) of type III with Yukawa couplings of weight 14. Observed values Ref. [82] and GUT scale values with $\tan\beta = 5$ [84,85] are shown.

	$\frac{m_u}{m_t} \times 10^6$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_s}{m_b} \times 10^2$	$\lvert V^{us}_{\mathrm{CKM}} \rvert$	$\lvert V^{cb}_{\mathrm{CKM}} \rvert$	$\lvert V^{ub}_{\mathrm{CKM}} \rvert$
Obtained values	104	21.2	29. 1	1.97	0.255	0.00706	0.00000171
Observed values	12.6	7.38	11.2	2.22	0.227	0.0405	0.00361
GUT scale values	5.39	2.80	9.21	1.82	0.225	0.0400	0.00353

4. Type IV: (5,2,0) and (4,2,0)

The mass matrices of type IV are given by Eqs. (40) and (31). The hierarchical mass matrices in Eq. (11) can be obtained by choosing $+1$ or -1 appropriately in α^{ij} and β^{ij}. As a result, we find best-fit mass matrices at $\tau = 3.2i$,

$$M_u/(Y^{(14)}_{\mathbf{1}^0_0}\langle H_u\rangle) = \begin{pmatrix} Y^{(14)}_{\mathbf{1}^1_1} & Y^{(14)}_{\mathbf{1}^1_0} & Y^{(14)}_{\mathbf{1}^0_0} \\ Y^{(14)}_{\mathbf{1}^0_2} & Y^{(14)}_{\mathbf{1}^0_1} & Y^{(14)}_{\mathbf{1}^1_1} \\ Y^{(14)}_{\mathbf{1}^1_1} & -Y^{(14)}_{\mathbf{1}^1_0} & -Y^{(14)}_{\mathbf{1}^0_0} \end{pmatrix} /(Y^{(14)}_{\mathbf{1}^0_0}\langle H_u\rangle)$$
$$= \begin{pmatrix} 0.0000746 & 0.00358 & 1 \\ 0.000435 & 0.0208 & 0.0000746 \\ 0.0000746 & -0.00358 & -1 \end{pmatrix}$$
$$\sim \begin{pmatrix} \varepsilon^5 & \varepsilon^3 & 1 \\ \varepsilon^4 & \varepsilon^2 & \varepsilon^5 \\ \varepsilon^5 & -\varepsilon^3 & -1 \end{pmatrix}, \quad (49)$$

$$M_d/(Y^{(14)}_{\mathbf{1}^0_0}\langle H_d\rangle) = \begin{pmatrix} Y^{(14)}_{\mathbf{1}^0_2} & Y^{(14)}_{\mathbf{1}^1_0} & Y^{(14)}_{\mathbf{1}^0_0} \\ Y^{(14)}_{\mathbf{1}^1_0} & -Y^{(14)}_{\mathbf{1}^0_1} & -Y^{(14)}_{\mathbf{1}^1_1} \\ Y^{(14)}_{\mathbf{1}^0_2} & Y^{(14)}_{\mathbf{1}^1_0} & -Y^{(14)}_{\mathbf{1}^0_0} \end{pmatrix} /(Y^{(14)}_{\mathbf{1}^0_0}\langle H_d\rangle)$$
$$= \begin{pmatrix} 0.000435 & 0.00358 & 1 \\ 0.00358 & -0.0208 & -0.0000746 \\ 0.000435 & 0.00358 & -1 \end{pmatrix}$$
$$\sim \begin{pmatrix} \varepsilon^4 & \varepsilon^3 & 1 \\ \varepsilon^3 & -\varepsilon^2 & -\varepsilon^5 \\ \varepsilon^4 & \varepsilon^3 & -1 \end{pmatrix}. \quad (50)$$

These mass matrices correspond to $a = 5$, $b = 0$ and can be realized by

$$Q = (\mathbf{1}^0_0, \mathbf{1}^1_2, \mathbf{1}^0_0), \quad u_R = (\mathbf{1}^1_2, \mathbf{1}^1_0, \mathbf{1}^0_0), \quad d_R = (\mathbf{1}^0_1, \mathbf{1}^1_0, \mathbf{1}^0_0), \quad (51)$$

and their mass matrices are written by,

$$M_u = \langle H_u\rangle \begin{pmatrix} \alpha^{11} Y^{(14)}_{\mathbf{1}^1_1} & \alpha^{12} Y^{(14)}_{\mathbf{1}^1_0} & \alpha^{13} Y^{(14)}_{\mathbf{1}^0_0} \\ \alpha^{21} Y^{(14)}_{\mathbf{1}^0_2} & \alpha^{22} Y^{(14)}_{\mathbf{1}^0_1} & \alpha^{23} Y^{(14)}_{\mathbf{1}^1_1} \\ \alpha^{31} Y^{(14)}_{\mathbf{1}^1_1} & \alpha^{32} Y^{(14)}_{\mathbf{1}^1_0} & \alpha^{33} Y^{(14)}_{\mathbf{1}^0_0} \end{pmatrix},$$
$$M_d = \langle H_d\rangle \begin{pmatrix} \beta^{11} Y^{(14)}_{\mathbf{1}^0_2} & \beta^{12} Y^{(14)}_{\mathbf{1}^1_0} & \beta^{13} Y^{(14)}_{\mathbf{1}^0_0} \\ \beta^{21} Y^{(14)}_{\mathbf{1}^1_0} & \beta^{22} Y^{(14)}_{\mathbf{1}^0_1} & \beta^{23} Y^{(14)}_{\mathbf{1}^1_1} \\ \beta^{31} Y^{(14)}_{\mathbf{1}^0_2} & \beta^{32} Y^{(14)}_{\mathbf{1}^1_0} & \beta^{33} Y^{(14)}_{\mathbf{1}^0_0} \end{pmatrix}, \quad (52)$$

with the following choices of $+1$ or -1 in coupling constants,

$$\begin{pmatrix} \alpha^{11} & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ \alpha^{31} & \alpha^{32} & \alpha^{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix},$$
$$\begin{pmatrix} \beta^{11} & \beta^{12} & \beta^{13} \\ \beta^{21} & \beta^{22} & \beta^{23} \\ \beta^{31} & \beta^{32} & \beta^{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix}. \quad (53)$$

They lead to the following quark mass ratios,

$$(m_u, m_c, m_t)/m_t = (7.46 \times 10^{-5}, 1.47 \times 10^{-2}, 1), \quad (54)$$

$$(m_d, m_s, m_b)/m_b = (1.01 \times 10^{-3}, 1.54 \times 10^{-2}, 1), \quad (55)$$

and the absolute values of the CKM matrix elements,

$$|V_{\text{CKM}}| = \begin{pmatrix} 0.974 & 0.226 & 0.0000000194 \\ 0.226 & 0.974 & 0.000158 \\ 0.0000358 & 0.000154 & 1.00 \end{pmatrix}. \quad (56)$$

Results are shown in Table VII.

B. Weights less than 14

Next we study the models with Yukawa couplings of weights less than 14. In this case, some of entries in mass

TABLE VII. The mass ratios of the quarks and the absolute values of the CKM matrix elements at the benchmark point $\tau = 3.2i$ in the best-fit model by Eqs. (51) and (53) of type IV with Yukawa couplings of weight 14. Observed values Ref. [82] and GUT scale values with $\tan\beta = 5$ [84,85] are shown.

	$\frac{m_u}{m_t}\times 10^6$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_s}{m_b}\times 10^2$	$\lvert V^{us}_{\text{CKM}}\rvert$	$\lvert V^{cb}_{\text{CKM}}\rvert$	$\lvert V^{ub}_{\text{CKM}}\rvert$
Obtained values	74.6	14.7	10.1	1.54	0.226	0.000158	1.94×10^{-8}
Observed values	12.6	7.38	11.2	2.22	0.227	0.0405	0.00361
GUT scale values	5.39	2.80	9.21	1.82	0.225	0.0400	0.00353

matrices vanish because there do not exist modular forms of proper weights and representations.

1. Weights 8 and 10

As an example, let us consider the case that Yukawa couplings for the up sector have the weight 8 and ones for the down sector have the weight 10. We choose $\tau = 3.7i$ as a benchmark point of the modulus. At weight 8, four singlet modular forms, $Y^{(8)}_{\mathbf{1}^0_0}$, $Y^{(8)}_{\mathbf{1}^1_2}$, $Y^{(8)}_{\mathbf{1}^0_1}$ and $Y^{(8)}_{\mathbf{1}^0_2}$, exist and they are approximated by ε as

$$\begin{aligned}
&Y^{(8)}_{\mathbf{1}^0_0}/Y^{(8)}_{\mathbf{1}^0_0} = 1 \to 1, \quad Y^{(8)}_{\mathbf{1}^1_2}/Y^{(8)}_{\mathbf{1}^0_0} = -0.0719 \to \varepsilon, \\
&Y^{(8)}_{\mathbf{1}^0_1}/Y^{(8)}_{\mathbf{1}^0_0} = 0.00732 \to \varepsilon^2, \quad Y^{(8)}_{\mathbf{1}^0_2}/Y^{(8)}_{\mathbf{1}^0_0} = 0.0000535 \to \varepsilon^4,
\end{aligned} \tag{57}$$

at $\tau = 3.7i$. At weight 10, five singlet modular forms, $Y^{(10)}_{\mathbf{1}^0_0}$, $Y^{(10)}_{\mathbf{1}^1_2}$, $Y^{(10)}_{\mathbf{1}^0_1}$, $Y^{(10)}_{\mathbf{1}^1_0}$ and $Y^{(10)}_{\mathbf{1}^1_1}$, exist and they are approximated by ε as

$$\begin{aligned}
&Y^{(10)}_{\mathbf{1}^0_0}/Y^{(10)}_{\mathbf{1}^0_0} = 1 \to 1, \qquad Y^{(10)}_{\mathbf{1}^1_2}/Y^{(10)}_{\mathbf{1}^0_0} = 0.102 \to \varepsilon, \\
&Y^{(10)}_{\mathbf{1}^0_1}/Y^{(10)}_{\mathbf{1}^0_0} = 0.00732 \to \varepsilon^2, \\
&Y^{(10)}_{\mathbf{1}^1_0}/Y^{(10)}_{\mathbf{1}^0_0} = 0.000744 \to \varepsilon^3, \\
&Y^{(10)}_{\mathbf{1}^1_1}/Y^{(10)}_{\mathbf{1}^0_0} = 0.00000544 \to \varepsilon^5,
\end{aligned} \tag{58}$$

at $\tau = 3.7i$. As a result, we find the following best-fit mass matrices of type III,

$$\begin{aligned}
M_u/(Y^{(8)}_{\mathbf{1}^0_0}\langle H_u\rangle) &= \begin{pmatrix} 0 & 0 & Y^{(8)}_{\mathbf{1}^0_1} \\ Y^{(8)}_{\mathbf{1}^0_2} & Y^{(8)}_{\mathbf{1}^0_1} & Y^{(8)}_{\mathbf{1}^1_2} \\ 0 & -Y^{(8)}_{\mathbf{1}^1_2} & -Y^{(8)}_{\mathbf{1}^0_0} \end{pmatrix} /(Y^{(8)}_{\mathbf{1}^0_0}\langle H_u\rangle) \\
&= \begin{pmatrix} 0 & 0 & 0.00732 \\ 0.0000535 & 0.00732 & -0.0719 \\ 0 & 0.0719 & -1 \end{pmatrix} \\
&\sim \begin{pmatrix} 0 & 0 & \varepsilon^2 \\ \varepsilon^4 & \varepsilon^2 & -\varepsilon \\ 0 & \varepsilon & -1 \end{pmatrix},
\end{aligned} \tag{59}$$

$$\begin{aligned}
M_d/(Y^{(10)}_{\mathbf{1}^0_0}\langle H_d\rangle) &= \begin{pmatrix} Y^{(10)}_{\mathbf{1}^1_0} & Y^{(10)}_{\mathbf{1}^1_0} & Y^{(10)}_{\mathbf{1}^0_1} \\ -Y^{(10)}_{\mathbf{1}^0_1} & -Y^{(10)}_{\mathbf{1}^0_1} & Y^{(10)}_{\mathbf{1}^1_2} \\ Y^{(10)}_{\mathbf{1}^1_2} & -Y^{(10)}_{\mathbf{1}^1_2} & Y^{(10)}_{\mathbf{1}^0_0} \end{pmatrix} /(Y^{(10)}_{\mathbf{1}^0_0}\langle H_d\rangle) \\
&= \begin{pmatrix} 0.000744 & 0.000744 & 0.00732 \\ -0.00732 & -0.00732 & 0.102 \\ 0.102 & -0.102 & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} \varepsilon^3 & \varepsilon^3 & \varepsilon^2 \\ -\varepsilon^2 & -\varepsilon^2 & \varepsilon \\ \varepsilon & -\varepsilon & 1 \end{pmatrix}.
\end{aligned} \tag{60}$$

These mass matrices correspond to $a = 1$, $b = 2$ and can be realized by

$$Q = (\mathbf{1}^0_2, \mathbf{1}^1_1, \mathbf{1}^0_0), \quad u_R = (\mathbf{1}^1_0, \mathbf{1}^1_1, \mathbf{1}^0_0), \quad d_R = (\mathbf{1}^1_1, \mathbf{1}^1_1, \mathbf{1}^0_0), \tag{61}$$

and their mass matrices,

$$\begin{aligned}
M_u &= \langle H_u\rangle \begin{pmatrix} 0 & 0 & \alpha^{13} Y^{(8)}_{\mathbf{1}^0_1} \\ \alpha^{21} Y^{(8)}_{\mathbf{1}^0_2} & \alpha^{22} Y^{(8)}_{\mathbf{1}^0_1} & \alpha^{23} Y^{(8)}_{\mathbf{1}^1_2} \\ 0 & \alpha^{32} Y^{(8)}_{\mathbf{1}^1_2} & \alpha^{33} Y^{(8)}_{\mathbf{1}^0_0} \end{pmatrix}, \\
M_d &= \langle H_d\rangle \begin{pmatrix} \beta^{11} Y^{(10)}_{\mathbf{1}^1_0} & \beta^{12} Y^{(10)}_{\mathbf{1}^1_0} & \beta^{13} Y^{(10)}_{\mathbf{1}^0_1} \\ \beta^{21} Y^{(10)}_{\mathbf{1}^0_1} & \beta^{22} Y^{(10)}_{\mathbf{1}^0_1} & \beta^{23} Y^{(10)}_{\mathbf{1}^1_2} \\ \beta^{31} Y^{(10)}_{\mathbf{1}^1_2} & \beta^{32} Y^{(10)}_{\mathbf{1}^1_2} & \beta^{33} Y^{(10)}_{\mathbf{1}^0_0} \end{pmatrix},
\end{aligned} \tag{62}$$

with the following choices of $+1$ or -1 in coupling constants,

$$\begin{aligned}
&\begin{pmatrix} - & - & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ - & \alpha^{32} & \alpha^{33} \end{pmatrix} = \begin{pmatrix} - & - & 1 \\ 1 & 1 & 1 \\ - & -1 & -1 \end{pmatrix}, \\
&\begin{pmatrix} \beta^{11} & \beta^{12} & \beta^{13} \\ \beta^{21} & \beta^{22} & \beta^{23} \\ \beta^{31} & \beta^{32} & \beta^{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}.
\end{aligned} \tag{63}$$

They lead to the following up quark and down quark mass ratios,

$$(m_u, m_c, m_t)/m_t = (1.27 \times 10^{-5}, 2.18 \times 10^{-3}, 1), \tag{64}$$

$$(m_d, m_s, m_b)/m_b = (1.44 \times 10^{-3}, 1.74 \times 10^{-2}, 1), \tag{65}$$

TABLE VIII. The mass ratios of the quarks and the absolute values of the CKM matrix elements at the benchmark point $\tau = 3.7i$ in the best-fit model by Eqs. (61) and (63) of type III with up sector Yukawa couplings of weight 8 and down sector Yukawa couplings of weight 10. Observed values Ref. [82] and GUT scale values with $\tan\beta = 5$ [84,85] are shown.

	$\frac{m_u}{m_t} \times 10^6$	$\frac{m_c}{m_t} \times 10^3$	$\frac{m_d}{m_b} \times 10^4$	$\frac{m_s}{m_b} \times 10^2$	$\lvert V_{\mathrm{CKM}}^{us}\rvert$	$\lvert V_{\mathrm{CKM}}^{cb}\rvert$	$\lvert V_{\mathrm{CKM}}^{ub}\rvert$
Obtained values	12.7	2.18	14.4	1.74	0.227	0.0300	0.00741
Observed values	12.6	7.38	11.2	2.22	0.227	0.0405	0.00361
GUT scale values	5.39	2.80	9.21	1.82	0.225	0.0400	0.00353

and the absolute values of the CKM matrix elements,

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.974 & 0.227 & 0.00741 \\ 0.227 & 0.973 & 0.0300 \\ 0.0140 & 0.0276 & 1.00 \end{pmatrix}. \tag{66}$$

Results are shown in Table VIII.

2. Weights 8 and 12

Next, let us consider the case that Yukawa couplings for the up sector have the weight 8 and ones for the down sector have the weight 12. The benchmark point of the modulus is $\tau = 3.7i$. Four singlet modular forms of weight 8, $Y^{(8)}_{\mathbf{1}^0_0}$, $Y^{(8)}_{\mathbf{1}^1_2}$, $Y^{(8)}_{\mathbf{1}^0_1}$, and $Y^{(8)}_{\mathbf{1}^0_2}$ at $\tau = 3.7i$ are given by Eq. (57). At weight 12, six singlet modular forms, $Y^{(12)}_{\mathbf{1}^0_{0i}}$, $Y^{(12)}_{\mathbf{1}^1_2}$, $Y^{(12)}_{\mathbf{1}^0_1}$, $Y^{(12)}_{\mathbf{1}^1_0}$, $Y^{(12)}_{\mathbf{1}^0_2}$, and $Y^{(12)}_{\mathbf{1}^0_{0ii}}$, exist and they are approximated by ε as

$$\begin{aligned}
&Y^{(12)}_{\mathbf{1}^0_{0i}}/Y^{(12)}_{\mathbf{1}^0_{0i}} = 1 \to 1, \qquad Y^{(12)}_{\mathbf{1}^1_2}/Y^{(12)}_{\mathbf{1}^0_{0i}} = 0.102 \to \varepsilon,\\
&Y^{(12)}_{\mathbf{1}^0_1}/Y^{(12)}_{\mathbf{1}^0_{0i}} = 0.0103 \to \varepsilon^2,\\
&Y^{(12)}_{\mathbf{1}^1_0}/Y^{(12)}_{\mathbf{1}^0_{0i}} = 0.000744 \to \varepsilon^3,\\
&Y^{(12)}_{\mathbf{1}^0_2}/Y^{(12)}_{\mathbf{1}^0_{0i}} = 0.0000757 \to \varepsilon^4,\\
&Y^{(12)}_{\mathbf{1}^0_{0ii}}/Y^{(12)}_{\mathbf{1}^0_{0i}} = 0.000000554 \to \varepsilon^6,
\end{aligned} \tag{67}$$

at $\tau = 3.7i$. Note that $Y^{(12)}_{\mathbf{1}^0_{0ii}} \sim \varepsilon^6$ originates from $Y^{(6)}_{\mathbf{1}^1_0} Y^{(6)}_{\mathbf{1}^1_0} \sim \varepsilon^3 \cdot \varepsilon^3$ while $Y^{(12)}_{\mathbf{1}^0_{0i}} \sim 1$ originates from $Y^{(6)}_{\mathbf{1}^0_0} Y^{(6)}_{\mathbf{1}^0_0} \sim 1 \cdot 1$. In what follows, we ignore $Y^{(12)}_{\mathbf{1}^0_{0ii}}$ because it belongs to the same representation as $Y^{(12)}_{\mathbf{1}^0_{0i}}$ and $Y^{(12)}_{\mathbf{1}^0_{0i}} \gg Y^{(12)}_{\mathbf{1}^0_{0ii}}$. As a result, we find the following best-fit mass matrices of type III,

$$\begin{aligned}
M_u/(Y^{(8)}_{\mathbf{1}^0_0}\langle H_u\rangle) &= \begin{pmatrix} 0 & 0 & Y^{(8)}_{\mathbf{1}^0_1} \\ Y^{(8)}_{\mathbf{1}^0_2} & Y^{(8)}_{\mathbf{1}^0_1} & Y^{(8)}_{\mathbf{1}^1_2} \\ 0 & -Y^{(8)}_{\mathbf{1}^1_2} & -Y^{(8)}_{\mathbf{1}^0_0} \end{pmatrix} / (Y^{(8)}_{\mathbf{1}^0_0}\langle H_u\rangle) \\
&= \begin{pmatrix} 0 & 0 & 0.00732 \\ 0.0000535 & 0.00732 & -0.0719 \\ 0 & 0.0719 & -1 \end{pmatrix} \\
&\sim \begin{pmatrix} 0 & 0 & \varepsilon^2 \\ \varepsilon^4 & \varepsilon^2 & -\varepsilon \\ 0 & \varepsilon & -1 \end{pmatrix},
\end{aligned} \tag{68}$$

$$\begin{aligned}
M_d/(Y^{(12)}_{\mathbf{1}^0_{0i}}\langle H_d\rangle) &= \begin{pmatrix} Y^{(12)}_{\mathbf{1}^1_0} & Y^{(12)}_{\mathbf{1}^1_0} & Y^{(12)}_{\mathbf{1}^0_1} \\ -Y^{(12)}_{\mathbf{1}^0_1} & -Y^{(12)}_{\mathbf{1}^0_1} & Y^{(12)}_{\mathbf{1}^1_2} \\ Y^{(12)}_{\mathbf{1}^1_2} & -Y^{(12)}_{\mathbf{1}^1_2} & Y^{(12)}_{\mathbf{1}^0_{0i}} \end{pmatrix} / (Y^{(12)}_{\mathbf{1}^0_{0i}}\langle H_d\rangle) \\
&= \begin{pmatrix} 0.000744 & 0.000744 & 0.0103 \\ -0.0103 & -0.0103 & 0.102 \\ 0.102 & -0.102 & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} \varepsilon^3 & \varepsilon^3 & \varepsilon^2 \\ -\varepsilon^2 & -\varepsilon^2 & \varepsilon \\ \varepsilon & -\varepsilon & 1 \end{pmatrix}.
\end{aligned} \tag{69}$$

These mass matrices correspond to $a = 1$, $b = 2$ and can be realized by

$$Q = (\mathbf{1}^0_2, \mathbf{1}^1_1, \mathbf{1}^0_0), \quad u_R = (\mathbf{1}^1_0, \mathbf{1}^1_1, \mathbf{1}^0_0), \quad d_R = (\mathbf{1}^1_1, \mathbf{1}^1_1, \mathbf{1}^0_0), \tag{70}$$

and their mass matrices,

$$M_u = \langle H_u\rangle \begin{pmatrix} 0 & 0 & \alpha^{13} Y^{(8)}_{\mathbf{1}^0_1} \\ \alpha^{21} Y^{(8)}_{\mathbf{1}^0_2} & \alpha^{22} Y^{(8)}_{\mathbf{1}^0_1} & \alpha^{23} Y^{(8)}_{\mathbf{1}^1_2} \\ 0 & \alpha^{32} Y^{(8)}_{\mathbf{1}^1_2} & \alpha^{33} Y^{(8)}_{\mathbf{1}^0_0} \end{pmatrix},$$

$$M_d = \langle H_d\rangle \begin{pmatrix} \beta^{11} Y^{(12)}_{\mathbf{1}^1_0} & \beta^{12} Y^{(12)}_{\mathbf{1}^1_0} & \beta^{13} Y^{(12)}_{\mathbf{1}^0_1} \\ \beta^{21} Y^{(12)}_{\mathbf{1}^0_1} & \beta^{22} Y^{(12)}_{\mathbf{1}^0_1} & \beta^{23} Y^{(12)}_{\mathbf{1}^1_2} \\ \beta^{31} Y^{(12)}_{\mathbf{1}^1_2} & \beta^{32} Y^{(12)}_{\mathbf{1}^1_2} & \beta^{33} Y^{(12)}_{\mathbf{1}^0_{0i}} \end{pmatrix}, \tag{71}$$

TABLE IX. The mass ratios of the quarks and the absolute values of the CKM matrix elements at the benchmark point $\tau = 3.7i$ in the best-fit model by Eqs. (70) and (72) of type III with up sector Yukawa couplings of weight 8 and down sector Yukawa couplings of weight 12. Observed values Ref. [82] and GUT scale values with $\tan\beta = 5$ [84,85] are shown.

	$\frac{m_u}{m_t}\times 10^6$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_s}{m_b}\times 10^2$	$\lvert V_{\mathrm{CKM}}^{us}\rvert$	$\lvert V_{\mathrm{CKM}}^{cb}\rvert$	$\lvert V_{\mathrm{CKM}}^{ub}\rvert$
Obtained values	12.7	2.18	17.6	2.02	0.227	0.0308	0.0103
Observed values	12.6	7.38	11.2	2.22	0.227	0.0405	0.00361
GUT scale values	5.39	2.80	9.21	1.82	0.225	0.0400	0.00353

with the following choices of $+1$ or -1 in coupling constants,

$$\begin{pmatrix} \alpha^{11} & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ \alpha^{31} & \alpha^{32} & \alpha^{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix},$$
$$\begin{pmatrix} \beta^{11} & \beta^{12} & \beta^{13} \\ \beta^{21} & \beta^{22} & \beta^{23} \\ \beta^{31} & \beta^{32} & \beta^{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}. \tag{72}$$

They lead to the following up quark and down quark mass ratios,

$$(m_u, m_c, m_t)/m_t = (1.27\times 10^{-5}, 2.18\times 10^{-3}, 1), \tag{73}$$

$$(m_d, m_s, m_b)/m_b = (1.76\times 10^{-3}, 2.02\times 10^{-2}, 1), \tag{74}$$

and the absolute values of the CKM matrix elements,

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.974 & 0.227 & 0.0103 \\ 0.226 & 0.974 & 0.0308 \\ 0.0170 & 0.0276 & 0.999 \end{pmatrix}. \tag{75}$$

Results are shown in Table IX.

Thus it is also possible to realize a realistic quark flavor structure in the models with Yukawa couplings of weights less than 14 despite some zeros in mass matrices. Here we studied two cases that Yukawa couplings for the up sector have weight 8 and ones for the down sector have weights 10 or 12 but other cases may be available for realization of the quark flavor structure.

C. Comment on the origin of Γ_6 modular symmetry

Here we comment on a plausible origin of Γ_6 modular symmetry of the theories. For example, some modular forms are derived from the torus compactification $T_1^2 \times T_2^2 \times T_3^2$ of the low-energy effective theory of the superstring theory with magnetic flux background [7–12]. The group Γ_6 may originate from one of T_i^2, while the others T_j^2 lead to a trivial symmetry. Alternatively, since $\Gamma_6 \simeq S_3 \times A_4 \simeq \Gamma_2 \times \Gamma_3$, it may be expected that $\Gamma_2 \simeq S_3$ originates from one torus T_1^2 and $\Gamma_3 \simeq A_4$ originates from another torus T_2^2 with the moduli stabilization $\tau_1 = \tau_2 \equiv \tau$. Then T_3^2 contributes to the group symmetry trivially.

IV. CONCLUSION

We have discussed the possibility to describe mass hierarchies of both up and down sector quarks as well as mixing angles without fine-tuning. Describing the quark flavor structure requires Z_n residual symmetry with $n \geq 6$. We have studied the modular symmetric quark flavor models of $\Gamma_6 \simeq S_3 \times A_4$ in the vicinity of the cusp $\tau = i\infty$ where Z_6 residual symmetry remains. Then the values of the modular forms become hierarchical as close to the cusp depending on their Z_6 residual charges.

In order to obtain viable models, we consider four types of quark mass matrices; the diagonal components in up and down sector quark mass matrices are written by ε with the powers of (5,3,0) and (3,2,0) respectively for type I, (5,3,0) and (4,2,0) for type II, (5,2,0) and (3,2,0) for type III, and (5,2,0) and (4,2,0) for type IV. The powers of nondiagonal components in up and down sector quark mass matrices have been treated as model depending values. When we assign the irreducible representations into quarks and Higgs fields, powers of ε in mass matrix components are determined by residual charges of mass matrix components. For simplicity, we have used only six singlets $\mathbf{1_0^0}$, $\mathbf{1_1^0}$, $\mathbf{1_2^0}$, $\mathbf{1_0^1}$, $\mathbf{1_1^1}$, and $\mathbf{1_2^1}$ as the irreducible representations of Γ_6. In addition, we have restricted the values of the coupling constants to ± 1 to avoid fine-tuning by them.

First, we have investigated the case that up and down sector Yukawa couplings have weight 14. In such cases, mass matrices have no zeros, that is, all of their components are written in terms of the modular forms for Γ_6 of weight 14. Consequently, we have obtained viable models at $\tau = 3.2i$ for each type without fine-tuning.

Second, we have shown the viable model in the case that Yukawa couplings of the up sector have weight 8 and ones of the down sector have weight 10 as well as weight 12. In this case some components of mass matrices can become zero because there do not exist modular forms of proper weights and representations. As a result, we have obtained the viable model at $\tau = 3.7i$ despite three zeros in the up quark mass matrix, Eq. (59).

Thus, the modular symmetric quark flavor models based on Γ_6 in the vicinity of the cusp $\tau = i\infty$ lead to successful quark mass matrices without fine-tuning. As we have commented in the end of previous section, $\Gamma_6 \simeq S_3 \times A_4 \simeq \Gamma_2 \times \Gamma_3$ may originate from the torus compactification $T_1^2 \times T_2^2 \times T_3^2$ of the low-energy effective theory of superstring theory. Motivated this point, the modular flavor models based on the direct product of finite modular groups, $\Gamma_{N_1} \times \Gamma_{N_2} \times \Gamma_{N_3}$, may be interesting. Also we can extend our analysis to the lepton sector. We will study them in the near future.

In our models, the important parameter is the modulus τ. It must be stabilized such that the proper mass hierarchies are realized. We would study such modulus stabilization elsewhere.[2]

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APPENDIX A: TENSOR PRODUCT OF Γ_6 GROUP

Here, we give a review on group theoretical aspects of Γ_6. The generators of Γ_6 are denoted by S and T, and they satisfy the following algebraic relations:

$$S^2 = (ST)^3 = T^6 = ST^2ST^3ST^4ST^3 = 1. \tag{A1}$$

In Γ_6 group, there are 12 irreducible representations, six singlets $\mathbf{1_0^0}$, $\mathbf{1_1^0}$, $\mathbf{1_2^0}$, $\mathbf{1_0^1}$, $\mathbf{1_1^1}$, and $\mathbf{1_2^1}$, three doublets $\mathbf{2_0}$, $\mathbf{2_1}$, and $\mathbf{2_2}$, two triplets $\mathbf{3^0}$ and $\mathbf{3^1}$ and one six-dimensional representation $\mathbf{6}$. Each irreducible representation is given by

$$\mathbf{1}_k^r\colon\ S = (-1)^r, \qquad T = (-1)^r \omega^k, \tag{A2}$$

$$\mathbf{2}_k\colon\ S = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \qquad T = \omega^k \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A3}$$

$$\mathbf{3}^r\colon\ (-1)^r \boldsymbol{a}_3, \qquad (-1)^r \boldsymbol{b}_3, \tag{A4}$$

$$\mathbf{6}\colon\ \frac{1}{2}\begin{pmatrix} -\boldsymbol{a}_3 & \sqrt{3}\boldsymbol{a}_3 \\ \sqrt{3}\boldsymbol{a}_3 & \boldsymbol{a}_3 \end{pmatrix}, \qquad T = \begin{pmatrix} \boldsymbol{b}_3 & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{b}_3 \end{pmatrix}, \tag{A5}$$

where $r = 0, 1$, $k = 0, 1, 2$ and

$$\boldsymbol{a}_3 = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \boldsymbol{b}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \tag{A6}$$

[2]See for modulus stabilization in modular flavor symmetric models Refs. [86–90].

In this basis, the Kronecker products between irreducible representations are

$$\begin{aligned} &\mathbf{1}_i^r \otimes \mathbf{1}_j^s = \mathbf{1}_m^t, \qquad \mathbf{1}_i^r \otimes \mathbf{2}_j = \mathbf{2}_m, \\ &\mathbf{1}_i^r \otimes \mathbf{3}^s = \mathbf{3}^t, \qquad \mathbf{1}_i^r \otimes \mathbf{6} = \mathbf{6}, \end{aligned} \tag{A7}$$

$$\begin{aligned} &\mathbf{2}_i \otimes \mathbf{2}_j = \mathbf{1}_m^0 \oplus \mathbf{1}_m^1 \oplus \mathbf{2}_m, \qquad \mathbf{2}_i \otimes \mathbf{3}^r = \mathbf{6}, \\ &\mathbf{2}_i \otimes \mathbf{6} = \mathbf{3}^0 \oplus \mathbf{3}^1 \oplus \mathbf{6}, \end{aligned} \tag{A8}$$

$$\begin{aligned} &\mathbf{3}^r \otimes \mathbf{3}^s = \mathbf{1}_0^t \oplus \mathbf{1}_1^t \oplus \mathbf{1}_2^t \oplus \mathbf{3}_1^t \oplus \mathbf{3}_2^t, \\ &\mathbf{3}^r \otimes \mathbf{6} = \mathbf{2}_0 \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{6} \oplus \mathbf{6}, \end{aligned} \tag{A9}$$

$$\begin{aligned} \mathbf{6} \otimes \mathbf{6} = {} & \mathbf{1}_0^0 \oplus \mathbf{1}_1^0 \oplus \mathbf{1}_2^0 \oplus \mathbf{1}_0^1 \oplus \mathbf{1}_1^1 \oplus \mathbf{1}_2^1 \oplus \mathbf{2}_0 \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \\ & \oplus \mathbf{3}^0 \oplus \mathbf{3}^0 \oplus \mathbf{3}^1 \oplus \mathbf{3}^1 \oplus \mathbf{6} \oplus \mathbf{6}, \end{aligned} \tag{A10}$$

where $i,\ j = 0, 1, 2,\ r,\ s = 0, 1,\ m = i + j (\mathrm{mod}\ 3)$, and $t = r + s (\mathrm{mod}\ 2)$. In the following, we show the Clebsch-Gordon (CG) coefficients of these products.

$$(\alpha_1)_{\mathbf{1}_i^r} \otimes \begin{pmatrix} \beta^1 \\ \beta_2 \end{pmatrix}_{\mathbf{2}_j} = \alpha_1 P_2^r \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}_{\mathbf{2}_m},$$

$$(\alpha_1)_{\mathbf{1}_i^r} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{\mathbf{3}^s} = \alpha_1 P_3^i \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{\mathbf{3}^t},$$

$$(\alpha_1)_{\mathbf{1}_i^r} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix}_{\mathbf{6}} = \alpha_1 P_6(r, i) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix}_{\mathbf{6}},$$

$$\begin{aligned} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_{\mathbf{2}_i} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}_{\mathbf{2}_j} = {} & \frac{1}{\sqrt{2}}(\alpha_1\beta_1 + \alpha_2\beta_2)_{\mathbf{1}_m^0} \\ & \oplus \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \alpha_2\beta_1)_{\mathbf{1}_m^1} \\ & \oplus \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_1\beta_1 - \alpha_2\beta_2 \\ -\alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix}_{\mathbf{2}_m}, \end{aligned}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}_{\mathbf{2}_i} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}_{\mathbf{3}^r} = P_6(r, i) \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_1\beta_3 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \\ \alpha_2\beta_3 \end{pmatrix}_{\mathbf{6}},$$

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}_{\mathbf{2}_i} \otimes \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\\ \beta_6\end{pmatrix}_{\mathbf{6}} = \frac{P_3^i}{\sqrt{2}}\begin{pmatrix}\alpha_1\beta_1+\alpha_2\beta_4\\ \alpha_1\beta_2+\alpha_2\beta_5\\ \alpha_1\beta_3+\alpha_2\beta_6\end{pmatrix}_{\mathbf{3}^0} \oplus \frac{P_3^i}{\sqrt{2}}\begin{pmatrix}\alpha_1\beta_4-\alpha_2\beta_1\\ \alpha_1\beta_5-\alpha_2\beta_2\\ \alpha_1\beta_6-\alpha_2\beta_3\end{pmatrix}_{\mathbf{3}^1} \oplus \frac{P_6(0,i)}{\sqrt{2}}\begin{pmatrix}\alpha_1\beta_1-\alpha_2\beta_4\\ \alpha_1\beta_2-\alpha_2\beta_5\\ \alpha_1\beta_3-\alpha_2\beta_6\\ -\alpha_1\beta_4-\alpha_2\beta_1\\ -\alpha_1\beta_5-\alpha_2\beta_2\\ -\alpha_1\beta_6-\alpha_2\beta_3\end{pmatrix}_{\mathbf{6}},$$

$$\begin{aligned}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}_{\mathbf{3}^r} \otimes \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}_{\mathbf{3}^s} &= \frac{1}{\sqrt{3}}(\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2)_{\mathbf{1}^t_0} \oplus \frac{1}{\sqrt{3}}(\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3)_{\mathbf{1}^t_1} \oplus \frac{1}{\sqrt{3}}(\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1)_{\mathbf{1}^t_2}\\ &\oplus \frac{1}{\sqrt{3}}\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\ -\alpha_1\beta_2-\alpha_2\beta_1+2\alpha_3\beta_3\\ -\alpha_1\beta_3+2\alpha_2\beta_2-\alpha_3\beta_1\end{pmatrix}_{\mathbf{3}^t_1} \oplus \frac{1}{\sqrt{2}}\begin{pmatrix}-\alpha_2\beta_3+\alpha_3\beta_2\\ -\alpha_1\beta_2+\alpha_2\beta_1\\ \alpha_1\beta_3-\alpha_3\beta_1\end{pmatrix}_{\mathbf{3}^t_2},\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}_{\mathbf{3}^r} \otimes \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\\ \beta_6\end{pmatrix}_{\mathbf{6}} &= \frac{P_2^r}{\sqrt{3}}\begin{pmatrix}\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\\ \alpha_1\beta_4+\alpha_2\beta_6+\alpha_3\beta_5\end{pmatrix}_{\mathbf{2}_0} \oplus \frac{P_2^r}{\sqrt{3}}\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3\\ \alpha_1\beta_5+\alpha_2\beta_4+\alpha_3\beta_6\end{pmatrix}_{\mathbf{2}_1} \oplus \frac{P_2^r}{\sqrt{3}}\begin{pmatrix}\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1\\ \alpha_1\beta_6+\alpha_2\beta_5+\alpha_3\beta_4\end{pmatrix}_{\mathbf{2}_2}\\ &\oplus \frac{P_6(r,0)}{\sqrt{2}}\begin{pmatrix}\alpha_1\beta_1-\alpha_3\beta_2\\ -\alpha_2\beta_1+\alpha_3\beta_3\\ -\alpha_1\beta_3+\alpha_2\beta_2\\ \alpha_1\beta_4-\alpha_3\beta_5\\ -\alpha_2\beta_4+\alpha_3\beta_6\\ -\alpha_1\beta_6+\alpha_2\beta_5\end{pmatrix}_{\mathbf{6}} \oplus \frac{P_6(r,0)}{\sqrt{2}}\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_1\beta_2-\alpha_2\beta_1\\ -\alpha_1\beta_3+\alpha_3\beta_1\\ \alpha_2\beta_6-\alpha_3\beta_5\\ \alpha_1\beta_5-\alpha_2\beta_4\\ -\alpha_1\beta_6+\alpha_3\beta_4\end{pmatrix}_{\mathbf{6}},\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6\end{pmatrix}_{\mathbf{6}} \otimes \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\\ \beta_6\end{pmatrix}_{\mathbf{6}} = &\ \tfrac{1}{\sqrt{6}}(\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2+\alpha_4\beta_4+\alpha_5\beta_6+\alpha_6\beta_5)_{\mathbf{1}^0_0}\\ &\oplus \tfrac{1}{\sqrt{6}}(\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3+\alpha_4\beta_5+\alpha_5\beta_4+\alpha_6\beta_6)_{\mathbf{1}^0_1}\\ &\oplus \tfrac{1}{\sqrt{6}}(\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1+\alpha_4\beta_6+\alpha_5\beta_5+\alpha_6\beta_4)_{\mathbf{1}^0_2}\\ &\oplus \tfrac{1}{\sqrt{6}}(\alpha_1\beta_4+\alpha_2\beta_6+\alpha_3\beta_5-\alpha_4\beta_1-\alpha_5\beta_3-\alpha_6\beta_2)_{\mathbf{1}^1_0}\\ &\oplus \tfrac{1}{\sqrt{6}}(\alpha_1\beta_5+\alpha_2\beta_4+\alpha_3\beta_6-\alpha_4\beta_2-\alpha_5\beta_1-\alpha_6\beta_3)_{\mathbf{1}^1_1}\\ &\oplus \tfrac{1}{\sqrt{6}}(\alpha_1\beta_6+\alpha_2\beta_5+\alpha_3\beta_4-\alpha_4\beta_3-\alpha_5\beta_2-\alpha_6\beta_1)_{\mathbf{1}^1_2}\\ &\oplus \frac{1}{\sqrt{6}}\begin{pmatrix}\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2-\alpha_4\beta_4-\alpha_5\beta_6-\alpha_6\beta_5\\ -(\alpha_1\beta_4+\alpha_2\beta_6+\alpha_3\beta_5+\alpha_4\beta_1+\alpha_5\beta_3+\alpha_6\beta_2)\end{pmatrix}_{\mathbf{2}_0}\\ &\oplus \frac{1}{\sqrt{6}}\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3-\alpha_4\beta_5-\alpha_5\beta_4-\alpha_6\beta_6\\ -(\alpha_1\beta_5+\alpha_2\beta_4+\alpha_3\beta_6+\alpha_4\beta_2+\alpha_5\beta_1+\alpha_6\beta_3)\end{pmatrix}_{\mathbf{2}_1}\\ &\oplus \frac{1}{\sqrt{6}}\begin{pmatrix}\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1-\alpha_4\beta_6-\alpha_5\beta_5-\alpha_6\beta_4\\ -(\alpha_1\beta_6+\alpha_2\beta_5+\alpha_3\beta_4+\alpha_4\beta_3+\alpha_5\beta_2+\alpha_6\beta_1)\end{pmatrix}_{\mathbf{2}_2}\end{aligned}$$

$$\oplus\frac{1}{2\sqrt{3}}\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2+2\alpha_4\beta_4-\alpha_5\beta_6-\alpha_6\beta_5\\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1+2\alpha_6\beta_6-\alpha_4\beta_5-\alpha_5\beta_4\\ 2\alpha_2\beta_2-\alpha_1\beta_3-\alpha_3\beta_1+2\alpha_5\beta_5-\alpha_4\beta_6-\alpha_6\beta_4\end{pmatrix}_{\mathbf{3^0}}\oplus\frac{1}{2}\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2+\alpha_5\beta_6-\alpha_6\beta_5\\ \alpha_1\beta_2-\alpha_2\beta_1+\alpha_4\beta_5-\alpha_5\beta_4\\ -\alpha_1\beta_3+\alpha_3\beta_1-\alpha_4\beta_6+\alpha_6\beta_4\end{pmatrix}_{\mathbf{3^0}}$$
$$\oplus\frac{1}{2}\begin{pmatrix}\alpha_2\beta_6-\alpha_3\beta_5-\alpha_5\beta_3+\alpha_6\beta_2\\ \alpha_1\beta_5-\alpha_2\beta_4-\alpha_4\beta_2+\alpha_5\beta_1\\ -\alpha_1\beta_6+\alpha_3\beta_4+\alpha_4\beta_3-\alpha_6\beta_1\end{pmatrix}_{\mathbf{3^1}}\oplus\frac{1}{2\sqrt{3}}\begin{pmatrix}2\alpha_1\beta_4-\alpha_2\beta_6-\alpha_3\beta_5-2\alpha_4\beta_1+\alpha_5\beta_3+\alpha_6\beta_2\\ -\alpha_1\beta_5-\alpha_2\beta_4+2\alpha_3\beta_6+\alpha_4\beta_2+\alpha_5\beta_1-2\alpha_6\beta_3\\ -\alpha_1\beta_6+2\alpha_2\beta_5-\alpha_3\beta_4+\alpha_4\beta_3-2\alpha_5\beta_2+\alpha_6\beta_1\end{pmatrix}_{\mathbf{3^1}}$$
$$\oplus\frac{1}{2\sqrt{3}}\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2-2\alpha_4\beta_4+\alpha_5\beta_6+\alpha_6\beta_5\\ -\alpha_1\beta_2-\alpha_2\beta_1+2\alpha_3\beta_3+\alpha_4\beta_5+\alpha_5\beta_4-2\alpha_6\beta_6\\ -\alpha_1\beta_3+2\alpha_2\beta_2-\alpha_3\beta_1+\alpha_4\beta_6-2\alpha_5\beta_5+\alpha_6\beta_4\\ -2\alpha_1\beta_4+\alpha_2\beta_6+\alpha_3\beta_5-2\alpha_4\beta_1+\alpha_5\beta_3+\alpha_6\beta_2\\ \alpha_1\beta_5+\alpha_2\beta_4-2\alpha_3\beta_6+\alpha_4\beta_2+\alpha_5\beta_1-2\alpha_6\beta_3\\ \alpha_1\beta_6-2\alpha_2\beta_5+\alpha_3\beta_4+\alpha_4\beta_3-2\alpha_5\beta_2+\alpha_6\beta_1\end{pmatrix}_{\mathbf{6}}\oplus\frac{1}{2}\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2-\alpha_5\beta_6+\alpha_6\beta_5\\ \alpha_1\beta_2-\alpha_2\beta_1-\alpha_4\beta_5+\alpha_5\beta_4\\ -\alpha_1\beta_3+\alpha_3\beta_1+\alpha_4\beta_6-\alpha_6\beta_4\\ -\alpha_2\beta_6+\alpha_3\beta_5-\alpha_5\beta_3+\alpha_6\beta_2\\ -\alpha_1\beta_5+\alpha_2\beta_4-\alpha_4\beta_2+\alpha_5\beta_1\\ \alpha_1\beta_6-\alpha_3\beta_4+\alpha_4\beta_3-\alpha_6\beta_1\end{pmatrix}_{\mathbf{6}}.$$

Here we have used the notations,

$$P_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\qquad P_3=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},$$
$$P_6(r,i)=\begin{pmatrix}\mathbf{0_3}&\mathbf{1_3}\\-\mathbf{1_3}&\mathbf{0_3}\end{pmatrix}^r\begin{pmatrix}P_3&\mathbf{0_3}\\\mathbf{0_3}&P_3\end{pmatrix}^i. \tag{A11}$$

Further details can be found in Ref. [34].

APPENDIX B: MODULAR FORMS OF Γ_6

Here we give a review on the modular forms of Γ_6. The modular forms of level 6 of even weights can be constructed from the products of the Dedekind eta function [34],

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n),\qquad q=e^{2\pi i\tau}. \tag{B1}$$

Using η, four linearly independent modular forms of weight 2 can be written down as

$$Y^{(2)}_{\mathbf{3^0}}(\tau)=\begin{pmatrix}-Y_1^2\\\sqrt{2}Y_1Y_2\\Y_2^2\end{pmatrix},\qquad Y^{(2)}_{\mathbf{1^1_2}}(\tau)=Y_3Y_6-Y_4Y_5,$$
$$Y^{(2)}_{\mathbf{2_0}}(\tau)=\frac{1}{\sqrt{2}}\begin{pmatrix}Y_1Y_4-Y_2Y_3\\Y_1Y_6-Y_2Y_5\end{pmatrix}, \tag{B2}$$

$$Y^{(2)}_{\mathbf{6}}(\tau)=\frac{1}{\sqrt{2}}\begin{pmatrix}Y_1Y_4+Y_2Y_3\\\sqrt{2}Y_2Y_4\\-\sqrt{2}Y_1Y_3\\Y_1Y_6+Y_2Y_5\\\sqrt{2}Y_2Y_6\\-\sqrt{2}Y_1Y_5\end{pmatrix}, \tag{B3}$$

where

$$Y_1(\tau)=3\frac{\eta^3(3\tau)}{\eta(\tau)}+\frac{\eta^3(\tau/3)}{\eta(\tau)}, \tag{B4}$$

$$Y_2(\tau)=3\sqrt{2}\frac{\eta^3(3\tau)}{\eta(\tau)}, \tag{B5}$$

$$Y_3(\tau)=3\sqrt{2}\frac{\eta^3(6\tau)}{\eta(2\tau)}, \tag{B6}$$

$$Y_4(\tau)=-3\frac{\eta^3(6\tau)}{\eta(2\tau)}-\frac{\eta^3(2\tau/3)}{\eta(2\tau)}, \tag{B7}$$

$$Y_5(\tau)=\sqrt{6}\frac{\eta^3(6\tau)}{\eta(2\tau)}-\sqrt{6}\frac{\eta^3(3\tau/2)}{\eta(\tau/2)}, \tag{B8}$$

$$Y_6(\tau)=-\sqrt{3}\frac{\eta^3(6\tau)}{\eta(2\tau)}+\frac{1}{\sqrt{3}}\frac{\eta^3(\tau/6)}{\eta(\tau/2)}-\frac{1}{\sqrt{3}}\frac{\eta^3(2\tau/3)}{\eta(2\tau)}+\sqrt{3}\frac{\eta^3(3\tau/2)}{\eta(\tau/2)}. \tag{B9}$$

Then we can construct the modular forms of weight 4 by the CG coefficients shown in Appendix A as

$$Y^{(4)}_{\mathbf{1^0_0}}(\tau)=(Y^{(2)}_{\mathbf{2_0}}Y^{(2)}_{\mathbf{2_0}})_{\mathbf{1^0_0}},\qquad Y^{(4)}_{\mathbf{1^0_1}}(\tau)=(Y^{(2)}_{\mathbf{1^1_2}}Y^{(2)}_{\mathbf{1^1_2}})_{\mathbf{1^0_1}},$$
$$Y^{(4)}_{\mathbf{2_0}}(\tau)=(Y^{(2)}_{\mathbf{2_0}}Y^{(2)}_{\mathbf{2_0}})_{\mathbf{2_0}}, \tag{B10}$$

$$Y^{(4)}_{\mathbf{2_2}}(\tau)=(Y^{(2)}_{\mathbf{1^1_2}}Y^{(2)}_{\mathbf{2_0}})_{\mathbf{2_2}},\qquad Y^{(4)}_{\mathbf{3^0}}(\tau)=(Y^{(2)}_{\mathbf{2_0}}Y^{(2)}_{\mathbf{6}})_{\mathbf{3^0}},$$
$$Y^{(4)}_{\mathbf{3^1}}(\tau)=(Y^{(2)}_{\mathbf{1^1_2}}Y^{(2)}_{\mathbf{3^0}})_{\mathbf{3^1}}, \tag{B11}$$

$$Y^{(4)}_{\mathbf{6}i}(\tau)=(Y^{(2)}_{\mathbf{1^1_2}}Y^{(2)}_{\mathbf{6}})_{\mathbf{6}},\qquad Y^{(4)}_{\mathbf{6}ii}(\tau)=(Y^{(2)}_{\mathbf{2_0}}Y^{(2)}_{\mathbf{3^0}})_{\mathbf{6}}. \tag{B12}$$

TABLE X. The modular forms of level 6 of even weights up to 6.

	Modular form $Y_{\boldsymbol{r}}^{(k_Y)}$
$k_Y = 2$	$Y^{(2)}_{\mathbf{1}^1_2}$, $Y^{(2)}_{\mathbf{2}_0}$, $Y^{(2)}_{\mathbf{3}^0}$, $Y^{(2)}_{\mathbf{6}}$
$k_Y = 4$	$Y^{(4)}_{\mathbf{1}^0_0}$, $Y^{(4)}_{\mathbf{1}^0_1}$, $Y^{(4)}_{\mathbf{2}_0}$, $Y^{(4)}_{\mathbf{2}_2}$, $Y^{(4)}_{\mathbf{3}^0}$, $Y^{(4)}_{\mathbf{3}^1}$, $Y^{(4)}_{\mathbf{6}i}$, $Y^{(4)}_{\mathbf{6}ii}$
$k_Y = 6$	$Y^{(6)}_{\mathbf{1}^0_0}$, $Y^{(6)}_{\mathbf{1}^1_0}$, $Y^{(6)}_{\mathbf{1}^1_2}$, $Y^{(6)}_{\mathbf{2}_0}$, $Y^{(6)}_{\mathbf{2}_1}$, $Y^{(6)}_{\mathbf{2}_2}$, $Y^{(6)}_{\mathbf{3}^0i}$, $Y^{(6)}_{\mathbf{3}^0ii}$, $Y^{(6)}_{\mathbf{3}^1}$, $Y^{(6)}_{\mathbf{6}i}$, $Y^{(6)}_{\mathbf{6}ii}$, $Y^{(6)}_{\mathbf{6}iii}$

Note that $Y^{(4)}_{\mathbf{6}i}$ and $Y^{(4)}_{\mathbf{6}ii}$ stand for two linearly independent six-dimensional modular forms of weight 4. We use the same convention for other modular forms. Similarly, we construct the modular forms of weight 6 as

$$\begin{aligned} &Y^{(6)}_{\mathbf{1}^0_0}(\tau) = (Y^{(2)}_{\mathbf{2}_0} Y^{(4)}_{\mathbf{2}_0})_{\mathbf{1}^0_0}, \qquad Y^{(6)}_{\mathbf{1}^1_0}(\tau) = (Y^{(2)}_{\mathbf{1}^1_2} Y^{(4)}_{\mathbf{1}^0_1})_{\mathbf{1}^1_0}, \\ &Y^{(6)}_{\mathbf{1}^1_2}(\tau) = (Y^{(2)}_{\mathbf{1}^1_2} Y^{(4)}_{\mathbf{1}^0_0})_{\mathbf{1}^1_2}, \end{aligned} \tag{B13}$$

$$\begin{aligned} &Y^{(6)}_{\mathbf{2}_0}(\tau) = (Y^{(2)}_{\mathbf{2}_0} Y^{(4)}_{\mathbf{1}^0_0})_{\mathbf{2}_0}, \qquad Y^{(6)}_{\mathbf{2}_1}(\tau) = (Y^{(2)}_{\mathbf{2}_0} Y^{(4)}_{\mathbf{1}^0_1})_{\mathbf{2}_1}, \\ &Y^{(6)}_{\mathbf{2}_2}(\tau) = (Y^{(2)}_{\mathbf{1}^1_2} Y^{(4)}_{\mathbf{2}_0})_{\mathbf{2}_2}, \end{aligned} \tag{B14}$$

$$\begin{aligned} &Y^{(6)}_{\mathbf{3}^0i}(\tau) = (Y^{(2)}_{\mathbf{3}^0} Y^{(4)}_{\mathbf{1}^0_1})_{\mathbf{3}^0}, \qquad Y^{(6)}_{\mathbf{3}^0ii}(\tau) = (Y^{(2)}_{\mathbf{3}^0} Y^{(4)}_{\mathbf{1}^0_0})_{\mathbf{3}^0}, \\ &Y^{(6)}_{\mathbf{3}^1}(\tau) = (Y^{(2)}_{\mathbf{1}^1_2} Y^{(4)}_{\mathbf{3}^0})_{\mathbf{3}^1}, \end{aligned} \tag{B15}$$

$$\begin{aligned} &Y^{(6)}_{\mathbf{6}i}(\tau) = (Y^{(2)}_{\mathbf{2}_0} Y^{(4)}_{\mathbf{3}^1})_{\mathbf{6}}, \qquad Y^{(6)}_{\mathbf{6}ii}(\tau) = (Y^{(2)}_{\mathbf{6}} Y^{(4)}_{\mathbf{1}^0_0})_{\mathbf{6}}, \\ &Y^{(6)}_{\mathbf{6}iii}(\tau) = (Y^{(2)}_{\mathbf{3}^0} Y^{(4)}_{\mathbf{2}_0})_{\mathbf{6}}. \end{aligned} \tag{B16}$$

In Table X we summarize the modular forms of level 6 of even weights up to 6.

Also we construct the singlet modular forms of weights 8, 10, 12, and 14 which we have used in our analysis. First, the singlet modular forms of weight 8 are given by

$$\begin{aligned} &Y^{(8)}_{\mathbf{1}^0_0} = (Y^{(4)}_{\mathbf{1}^0_0} Y^{(4)}_{\mathbf{1}^0_0})_{\mathbf{1}^0_0}, \qquad Y^{(8)}_{\mathbf{1}^0_1} = (Y^{(4)}_{\mathbf{1}^0_0} Y^{(4)}_{\mathbf{1}^0_1})_{\mathbf{1}^0_1}, \\ &Y^{(8)}_{\mathbf{1}^0_2} = (Y^{(4)}_{\mathbf{1}^0_1} Y^{(4)}_{\mathbf{1}^0_1})_{\mathbf{1}^0_2}, \qquad Y^{(8)}_{\mathbf{1}^1_2} = (Y^{(4)}_{\mathbf{2}_0} Y^{(4)}_{\mathbf{2}_2})_{\mathbf{1}^1_2}. \end{aligned} \tag{B17}$$

The singlet modular forms of weight 10 are given by

$$\begin{aligned} &Y^{(10)}_{\mathbf{1}^0_0} = (Y^{(4)}_{\mathbf{1}^0_0} Y^{(6)}_{\mathbf{1}^0_0})_{\mathbf{1}^0_0}, \qquad Y^{(10)}_{\mathbf{1}^0_1} = (Y^{(4)}_{\mathbf{1}^0_1} Y^{(6)}_{\mathbf{1}^0_0})_{\mathbf{1}^0_1}, \\ &Y^{(10)}_{\mathbf{1}^1_0} = (Y^{(4)}_{\mathbf{1}^0_0} Y^{(6)}_{\mathbf{1}^1_0})_{\mathbf{1}^1_0}, \end{aligned} \tag{B18}$$

TABLE XI. The singlet modular forms of level 6 of weights 8, 10, 12, and 14.

	Modular form $Y_{\boldsymbol{r}}^{(k_Y)}$
$k_Y = 8$	$Y^{(8)}_{\mathbf{1}^0_0}$, $Y^{(8)}_{\mathbf{1}^0_1}$, $Y^{(8)}_{\mathbf{1}^0_2}$, $Y^{(8)}_{\mathbf{1}^1_2}$
$k_Y = 10$	$Y^{(10)}_{\mathbf{1}^0_0}$, $Y^{(10)}_{\mathbf{1}^0_1}$, $Y^{(10)}_{\mathbf{1}^1_0}$, $Y^{(10)}_{\mathbf{1}^1_1}$, $Y^{(10)}_{\mathbf{1}^1_2}$
$k_Y = 12$	$Y^{(12)}_{\mathbf{1}^0_0i}$, $Y^{(12)}_{\mathbf{1}^0_0ii}$, $Y^{(12)}_{\mathbf{1}^0_1}$, $Y^{(12)}_{\mathbf{1}^0_2}$, $Y^{(12)}_{\mathbf{1}^1_0}$, $Y^{(12)}_{\mathbf{1}^1_2}$
$k_Y = 14$	$Y^{(14)}_{\mathbf{1}^0_0}$, $Y^{(14)}_{\mathbf{1}^0_1}$, $Y^{(14)}_{\mathbf{1}^0_2}$, $Y^{(14)}_{\mathbf{1}^1_0}$, $Y^{(14)}_{\mathbf{1}^1_1}$, $Y^{(14)}_{\mathbf{1}^1_2i}$, $Y^{(14)}_{\mathbf{1}^1_2ii}$

$$Y^{(10)}_{\mathbf{1}^1_1} = (Y^{(4)}_{\mathbf{1}^0_1} Y^{(6)}_{\mathbf{1}^1_0})_{\mathbf{1}^1_1}, \qquad Y^{(10)}_{\mathbf{1}^1_2} = (Y^{(4)}_{\mathbf{1}^0_0} Y^{(6)}_{\mathbf{1}^1_2})_{\mathbf{1}^1_2}. \tag{B19}$$

The singlet modular forms of weight 12 are given by

$$\begin{aligned} &Y^{(12)}_{\mathbf{1}^0_0i} = (Y^{(6)}_{\mathbf{1}^0_0} Y^{(6)}_{\mathbf{1}^0_0})_{\mathbf{1}^0_0}, \qquad Y^{(12)}_{\mathbf{1}^0_0ii} = (Y^{(6)}_{\mathbf{1}^1_0} Y^{(6)}_{\mathbf{1}^1_0})_{\mathbf{1}^0_0}, \\ &Y^{(12)}_{\mathbf{1}^0_1} = (Y^{(6)}_{\mathbf{1}^1_2} Y^{(6)}_{\mathbf{1}^1_2})_{\mathbf{1}^0_1}, \end{aligned} \tag{B20}$$

$$\begin{aligned} &Y^{(12)}_{\mathbf{1}^0_2} = (Y^{(6)}_{\mathbf{1}^1_0} Y^{(6)}_{\mathbf{1}^1_2})_{\mathbf{1}^0_2}, \qquad Y^{(12)}_{\mathbf{1}^1_0} = (Y^{(6)}_{\mathbf{1}^0_0} Y^{(6)}_{\mathbf{1}^1_0})_{\mathbf{1}^1_0}, \\ &Y^{(12)}_{\mathbf{1}^1_2} = (Y^{(6)}_{\mathbf{1}^0_0} Y^{(6)}_{\mathbf{1}^1_2})_{\mathbf{1}^1_2}. \end{aligned} \tag{B21}$$

The singlet modular forms of weight 14 are given by

$$\begin{aligned} &Y^{(14)}_{\mathbf{1}^0_0} = (Y^{(6)}_{\mathbf{1}^0_0} Y^{(8)}_{\mathbf{1}^0_0})_{\mathbf{1}^0_0}, \qquad Y^{(14)}_{\mathbf{1}^0_1} = (Y^{(6)}_{\mathbf{1}^0_0} Y^{(8)}_{\mathbf{1}^0_1})_{\mathbf{1}^0_1}, \\ &Y^{(14)}_{\mathbf{1}^0_2} = (Y^{(6)}_{\mathbf{1}^0_0} Y^{(8)}_{\mathbf{1}^0_2})_{\mathbf{1}^0_2}, \end{aligned} \tag{B22}$$

$$\begin{aligned} &Y^{(14)}_{\mathbf{1}^1_0} = (Y^{(6)}_{\mathbf{1}^1_0} Y^{(8)}_{\mathbf{1}^0_0})_{\mathbf{1}^1_0}, \qquad Y^{(14)}_{\mathbf{1}^1_1} = (Y^{(6)}_{\mathbf{1}^1_0} Y^{(8)}_{\mathbf{1}^0_1})_{\mathbf{1}^1_1}, \\ &Y^{(14)}_{\mathbf{1}^1_2i} = (Y^{(6)}_{\mathbf{1}^1_2} Y^{(8)}_{\mathbf{1}^0_0})_{\mathbf{1}^1_2}, \end{aligned} \tag{B23}$$

$$Y^{(14)}_{\mathbf{1}^1_2ii} = (Y^{(6)}_{\mathbf{1}^1_0} Y^{(8)}_{\mathbf{1}^0_2})_{\mathbf{1}^1_2}. \tag{B24}$$

In Table XI we summarize the singlet modular forms of level 6 of weights 8, 10, 12, and 14.

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