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Stability studies of first-order spin-hydrodynamic frameworks

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We study the stability of first-order dissipative spin-hydrodynamic frameworks. We considered two different first-order dissipative spin-hydrodynamic frameworks. The first one considers the spin-chemical potential ($\omega^{\alpha\beta}$) to be first order [$\mathcal{O}(\partial)$] in the hydrodynamic gradient expansion. The hydrodynamic gradient ordering of the spin-chemical potential is a debatable issue within the frameworks of spin hydrodynamics. Therefore, as a second choice, we also consider the spin-hydrodynamic equations with $\omega^{\alpha\beta} \sim \mathcal{O}(1)$. We find that, for both frameworks, at the level of linear perturbations some spin modes can be unstable. To remove these generic instabilities, we consider the *Frenkel condition*. We argue that the Frenkel condition helps get rid of the unstable solutions in both cases but with a physical drawback for the case where $\omega^{\mu\nu} \sim \mathcal{O}(\partial)$.

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I. INTRODUCTION

Experimental observations of spin-polarized weakly decaying hyperons have given us a unique opportunity to explore the vortical structure of the strongly coupled plasma produced in relativistic heavy-ion experiments [1–10]. Motivated by the successes of the relativistic dissipative hydrodynamic framework in heavy-ion phenomenology [11], it is naturally desirable to generalize the hydrodynamic framework to include spin as a dynamical degree of freedom. Several spin-hydrodynamic frameworks have been developed using relativistic kinetic theory [12–25], entropy current analysis [26–31], quantum statistical density operators [32–36], effective Lagrangian approach [37–40], holography [41,42], and equilibrium partition functions [43], etc.

Spin-hydrodynamic frameworks are based on the conservation of the total angular momentum and the conservation of the total energy-momentum tensor. The quantity that separates the spin-hydrodynamic framework from the standard hydrodynamic framework is the "spin-chemical potential." In the Navier-Stokes limit of the spin-hydrodynamic framework, the spin-chemical potential is a hydrodynamic variable similar to the temperature, chemical potential, and fluid four-velocity. In local thermodynamic equilibrium, the hydrodynamic gradient ordering of the spin-chemical potential is not a settled topic, and different derivative counting schemes have been considered in the literature [26,27,29–31,44]. Hydrodynamic gradient ordering of the spin-chemical potential plays a crucial role in the thermodynamic as well as a hydrodynamic description with spin; e.g., if one considers that the spin-chemical potential is $\mathcal{O}(1)$ or $\mathcal{O}(\partial)$, then this affects the first-order spin-hydrodynamic description significantly. Since the spin-hydrodynamic framework depends on the derivative ordering of the spin-chemical potential, it is natural to investigate its effect on the propagation properties of linear perturbations.

Linear mode analysis of various hydrodynamic theories has been extensively discussed in the context of the stability and causality of fluid-dynamical theories. Generically, one performs the stability and causality analysis around a hydrostatic state, where the spatial components of the flow velocity vanish. However, it should be pointed out that even if a theory is causal and stable in the hydrostatic limit, it does not necessarily imply stability in a boosted frame. Linear mode analysis of a generic first-order theory [45–47], as well as second-order Israel-Stewart theory, have been scrutinized in various literature for a generic Lorentz-boosted frame [48–54]. Moreover, using such studies it was found that causality and stability of various hydrodynamic models are intimately related. Note that in a hydrodynamic theory instability may appear due to various factors, but it has been explicitly shown that for a parameter space where the theory gives rise to acausal mode theory also shows some instabilities [51,55]. Such theoretical intricacies associated with the mode analysis have also been studied for the relativistic magnetohydrodynamics [56], spin hydrodynamics [57–60], chiral hydrodynamics [61], etc.

In the present work, we investigate the properties of various linear perturbations of the first-order dissipative spin-hydrodynamic framework. We show that the linear mode analysis crucially depends on the spin-hydrodynamic framework and the derivative ordering of the spin-chemical potential. In our analysis, we consider two different

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spin-hydrodynamic descriptions; one considers the spin-chemical potential to be $\mathcal{O}(1)$ in the hydrodynamic gradient expansion [29], and the other theory considers the spin-chemical potential to be $\mathcal{O}(\partial)$ [26,27,30,31,44]. We also argue that both theories can give rise to linear modes which are unstable. Note that the spin-chemical potential, denoted as $\omega^{\mu\nu}$, is a two-rank antisymmetric tensor that has six independent components. Our calculations suggest that the instabilities appear in the spin-hydrodynamic description due to the boost degrees of freedom ω^{0i}. In principle, such unstable modes can be removed from the theory by suitably eliminating ω^{0i} degrees of freedom. This can be achieved by incorporating the "Frenkel condition" [62,63].

The paper is organized as follows. We begin by studying the stability of the first-order spin-hydrodynamic [26], which considers the spin-chemical potential $\omega_{\mu\nu}$ to be of the first order in gradient expansion in Sec. II. In this section, we apply linear perturbation on top of a specified global equilibrium background for the spin-hydrodynamic equations, and then we solved them in Fourier space where we find that some spin modes can be unstable. We then show that imposing the Frenkel condition gets rid of the unstable solutions while leading to some physical drawbacks to the system. In Sec. III, we repeat the same procedure but for the spin-hydrodynamic formulation [29], that follows the spin-chemical potential at the leading order in the hydrodynamic gradient expansion. We also find that some spin modes are unstable. For such a case, imposing the Frenkel condition gets rid of the unstable solution without any physical disadvantages. In Sec. IV, we summarize and conclude.

II. SPIN-CHEMICAL POTENTIAL FIRST ORDER IN THE HYDRODYNAMIC GRADIENT EXPANSION

A. Formulation

Often, it has been argued that in global equilibrium the spin-chemical potential should be proportional to the thermal vorticity [21,26]. Hence, one can consider a situation where $\omega^{\alpha\beta} \sim \mathcal{O}(\partial)$. Such a spin-hydrodynamic description has been discussed in Refs. [26,27,44]. It will be similarly interesting to study the linear modes for the spin-hydrodynamic description where $\omega^{\alpha\beta} \sim \mathcal{O}(\partial)$. It should be emphasized that, although the spin-chemical potential is argued to be $\mathcal{O}(\partial)$, the spin density is $\mathcal{O}(1)$ [31]. Considering that the spin density is proportional to the spin-chemical potential brings nontriviality to this framework. We start with the energy-momentum tensor and spin tensor having the following form [26,27,44]:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \tau^{\mu\nu} + q^\mu u^\nu - q^\nu u^\mu + \phi^{\mu\nu}, \tag{1}$$

$$S^{\mu\alpha\beta} = u^\mu S^{\alpha\beta} + S^{\mu\alpha\beta}_{(1)}. \tag{2}$$

We emphasize that in the above equation the energy-momentum tensor is not completely symmetric. Rather, it contains an antisymmetric part. Moreover, the spin tensor is antisymmetric only in the last two indices. Such a phenomenological energy-momentum tensor and spin tensor can be obtained from the canonical energy-momentum tensor and spin tensor using a proper pseudogauge transformation [44]. h^μ is the heat flow, and $\tau^{\mu\nu} = \pi^{\mu\nu} + \Pi\Delta^{\mu\nu}$ is the dissipative corrections to the symmetric part of the energy-momentum tensor. $\pi^{\mu\nu}$ is the traceless part of $\tau^{\mu\nu}$, and it is related to the shear viscosity (η). On the other hand, Π is related to bulk viscosity (ζ). The dissipative corrections to the antisymmetric part of the energy-momentum tensor are q^μ and $\phi^{\mu\nu}$. h^μ, $\tau^{\mu\nu}$, q^μ, and $\phi^{\mu\nu}$ satisfy the following conditions: $h^\mu u_\mu = 0$, $\tau^{\mu\nu} = \tau^{\nu\mu}$, $\tau^{\mu\nu}u_\nu = 0$, $q^\mu u_\mu = 0$, $\phi^{\mu\nu} = -\phi^{\nu\mu}$, and $\phi^{\mu\nu}u_\nu = 0$. The dissipative correction to the spin tensor, i.e., $S^{\lambda\mu\nu}_{(1)}$, is not fixed at the level of first-order dissipative spin hydrodynamics, as it does not contribute to the nonequilibrium entropy current[1] [26,27,44]. Various dissipative currents can be uniquely determined by using the condition that for an isolated dissipative system entropy will be produced. In terms of the hydrodynamic variables, i.e., T, u^μ, and $\omega^{\mu\nu}$, it can be shown that [26,27,44]

$$h^\mu = -\kappa(Du^\mu - \beta\nabla^\mu T), \tag{3}$$

$$q^\mu = \lambda(Du^\mu + \beta\nabla^\mu T - 4\omega^{\mu\nu}u_\nu), \tag{4}$$

$$\tau^{\mu\nu} = \eta\left(\Delta^{\mu\alpha}\partial_\alpha u^\nu + \Delta^{\nu\alpha}\partial_\alpha u^\mu - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\partial_\beta u_\alpha\right) + \zeta(\partial_\alpha u^\alpha)\Delta^{\mu\nu}, \tag{5}$$

$$\phi^{\mu\nu} = \tilde{\gamma}(\nabla^\mu u^\nu - \nabla^\nu u^\mu + 4\Delta^{\mu\alpha}\Delta^{\nu\beta}\omega_{\alpha\beta}). \tag{6}$$

Here, $D \equiv u^\mu\partial_\mu$, $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$, and $\tilde{\gamma} = \beta\gamma/2$. Note that h^μ, q^μ, $\tau^{\mu\nu}$, and $\phi^{\mu\nu}$ are $\mathcal{O}(\partial)$ in the hydrodynamic gradient expansion. Dissipative currents h^μ and q^μ as given in Eqs. (3) and (4) can be further simplified by using the leading-order hydrodynamic equations. Conservation of $T^{\mu\nu}_{(0)} \equiv \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu}$ gives us

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[1] Recall that the ansatz for the nonequilibrium entropy current can be expressed as $\mathcal{S}^\mu = \beta_\nu T^{\mu\nu} + p\beta^\mu - \beta\omega_{\alpha\beta}S^{\mu\alpha\beta}$. For the first-order dissipative spin-hydrodynamic framework, $\mathcal{S}^\mu$ can contain terms up to the order of $\mathcal{O}(\partial)$. Such $\mathcal{O}(\partial)$ terms can come from both $T^{\mu\nu}$ and $S^{\mu\alpha\beta}$. But if we consider $\omega^{\alpha\beta}$ to be of the order of $\mathcal{O}(\partial)$, then $\omega_{\alpha\beta}S^{\mu\alpha\beta}_{(1)}$ terms will be of the order of $\mathcal{O}(\partial^2)$. Therefore, we can conclude that if $\omega^{\alpha\beta}$ is of the order of $\mathcal{O}(\partial)$, then the dissipative part of the spin tensor would not contribute to the entropy current. On the other hand, if $\omega^{\alpha\beta}$ is of the order of $\mathcal{O}(1)$, then $S^{\mu\alpha\beta}_{(1)}$ will contribute to the first-order spin-hydrodynamic framework. This is the most striking difference between various frameworks considered here.

$$u^\mu \partial_\mu \varepsilon + (\varepsilon + p)(\partial_\mu u^\mu) = 0, \tag{7}$$

$$(\varepsilon + p) D u^\alpha - \nabla^\alpha p = 0. \tag{8}$$

Using Eq. (8) back into Eq. (3), it can be easily shown that

$$h^\mu = 0 + \mathcal{O}(\partial^2). \tag{9}$$

In order to obtain the above equation, we have used the thermodynamic relation $Ts + S^{\alpha\beta}\omega_{\alpha\beta} = \varepsilon + p$, $dp = s dT + S^{\mu\nu} d\omega_{\mu\nu}$. Here, we consider $\omega^{\mu\nu} \sim \mathcal{O}(\partial)$, and $S^{\mu\nu} \sim \mathcal{O}(1)$ in the hydrodynamic gradient expansion [26,27,44]. Furthermore,

$$\begin{aligned} q^\mu &= \lambda(Du^\mu + \beta\nabla^\mu T - 4\omega^{\mu\nu}u_\nu) \\ &= \lambda\left(2\frac{\nabla^\mu p}{\varepsilon + p} - 4\omega^{\mu\nu}u_\nu\right) + \mathcal{O}(\partial^2). \end{aligned} \tag{10}$$

To obtain the linear-order hydrodynamic perturbation with respect to a global equilibrium, we consider $u^\mu_{(0)} \equiv (1, 0, 0, 0)$, $\omega^{\mu\nu}_{(0)} = 0$, and $S^{\mu\nu}_{(0)} = 0$ [26].[2] Some comments on the choice of global equilibrium are in order here. Note that for a (most general) global equilibrium the following conditions must be fulfilled: $\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$, $\beta_\nu = b^0_\nu + \varpi_{\nu\lambda} x^\lambda$, and $\varpi_{\mu\nu} = (\partial_\nu \beta_\mu - \partial_\mu \beta_\nu)/2 = \text{const}$ [15]. Here, $\beta_\nu \equiv u_\nu / T$, b^0_ν is a constant four-vector, and $\varpi_{\nu\lambda}$ is the thermal vorticity. In terms of hydrodynamic gradient expansion, thermal vorticity is $\mathcal{O}(\partial)$. Therefore, a generic global equilibrium allows for $\mathcal{O}(\partial)$ terms. This is one of the most nontrivial features of spin-hydrodynamic frameworks as compared to the hydrodynamics for "spinless" fluids, where all the $\mathcal{O}(\partial)$ terms vanish in global equilibrium [45]. If we identify the spin-chemical potential with thermal vorticity in global equilibrium, then it is natural to consider the spin-chemical potential to be $\mathcal{O}(\partial)$ in hydrodynamic gradient expansion. However, such identification crucially depends on the symmetry of the energy-momentum tensor [26,29]. One could, in principle, consider the spin-chemical potential to be either $\mathcal{O}(1)$ [29] or $\mathcal{O}(\partial)$ [26] in the hydrodynamic gradient expansion. Similar to other thermodynamic quantities, e.g., temperature (T) or chemical potential (μ), the value of the spin-chemical potential can also be considered to be zero in global equilibrium (analogous to $\mu = 0$ corresponding to a baryon-free QCD plasma). Zero spin-chemical potential also implies zero spin density. Such a situation represents unpolarized (zero spin polarization) global equilibrium, as considered in the present article. All perturbations are considered around the unpolarized global equilibrium.

For such a global equilibrium configuration, various dissipative currents vanish, i.e., $q^\mu_{(0)} = 0$, $\phi^{\mu\nu}_{(0)} = 0$, and $\tau^{\mu\nu}_{(0)} = 0$. Note that $\tau^{\mu\nu}$, q^μ, and $\phi^{\mu\nu}$ are already $\mathcal{O}(\partial)$. Therefore, we consider $\delta\tau^{\mu\nu}$, δq^μ, and $\delta\phi^{\mu\nu}$ up to $\mathcal{O}(\partial^2)$, and we neglect all higher-order terms:

$$\begin{aligned} \delta\tau^{\mu\nu} = \eta\Big(&\Delta^{\mu\alpha}_{(0)}\partial_\alpha \delta u^\nu + \Delta^{\nu\alpha}_{(0)}\partial_\alpha \delta u^\mu - \frac{2}{3}\Delta^{\mu\nu}_{(0)}\Delta^{\alpha\beta}_{(0)}\partial_\beta \delta u_\alpha\Big) \\ &+ \zeta(\partial_\alpha \delta u^\alpha)\Delta^{\mu\nu}_{(0)} + \mathcal{O}(\partial^3), \end{aligned} \tag{11}$$

$$\delta q^\mu = \lambda\left(2\frac{\Delta^{\mu\alpha}_{(0)}\partial_\alpha \delta p}{\varepsilon_{(0)} + p_{(0)}} - 4\delta\omega^{\mu\nu}u^{(0)}_\nu\right) + \mathcal{O}(\partial^3), \tag{12}$$

$$\begin{aligned} \delta\phi^{\mu\nu} = \tilde{\gamma}(&\Delta^{\mu\alpha}_{(0)}\partial_\alpha \delta u^\nu - \Delta^{\nu\alpha}_{(0)}\partial_\alpha \delta u^\mu \\ &+ 4\Delta^\mu{}_{\rho(0)}\Delta^\nu{}_{\lambda(0)}\delta\omega^{\rho\lambda}) + \mathcal{O}(\partial^3). \end{aligned} \tag{13}$$

In global equilibrium, it is easy to show that $T^{0i}_{(0)} = 0$. The perturbation δT^{0i} can be expressed as

$$\delta T^{0i} = (\varepsilon_{(0)} + p_{(0)})\delta u^i + \delta\tau^{0i} - \delta q^i + \delta\phi^{0i}. \tag{14}$$

For the flow perturbation of the form $\delta u^\mu = (0, \delta u^i)$, using Eqs. (11)–(13) it can be shown that $\delta\tau^{0i} = 0$, $\delta q^0 = 0$, and $\delta\phi^{0i} = 0$. But δq^i is nonvanishing, and it can be expressed as

$$\begin{aligned} \delta q^i &= \lambda\left(2\frac{\nabla^i_{(0)}\delta p}{\varepsilon_{(0)} + p_{(0)}} - 4\delta\omega^{i\nu}u^{(0)}_\nu\right) + \mathcal{O}(\partial^3) \\ &= \lambda' c_s^2 \partial^i \delta\varepsilon - \frac{4\lambda}{\chi_b}\delta S^{i0} + \mathcal{O}(\partial^3) \\ &= \lambda' c_s^2 \partial^i \delta\varepsilon - D_b \delta S^{i0} + \mathcal{O}(\partial^3). \end{aligned} \tag{15}$$

Here, we have defined $\lambda' = \frac{2\lambda}{\varepsilon_{(0)} + p_{(0)}}$, $c_s^2 = \partial p / \partial\varepsilon$, $D_b = 4\lambda/\chi_b$, and $\chi_b = \partial S^{i0}/\partial\omega^{i0}$. We consider c_s^2, χ_b, and D_b to be constants, as any space-time derivative of these quantities will give rise to higher-order terms. Therefore, the perturbation $\delta T^{0i} \equiv \delta\pi^i$ can be expressed as

$$\delta\pi^i = (\varepsilon_{(0)} + p_{(0)})\delta u^i - \lambda' c_s^2 \partial^i \delta\varepsilon + D_b \delta S^{i0} + \mathcal{O}(\partial^3). \tag{16}$$

Note that $\delta\pi^i$ contains terms of the order of $\mathcal{O}(\partial)$ and higher. Moreover, λ' and D_b originate from the antisymmetric part of the energy-momentum tensor. Conservation of the total angular momentum can be used to write the evolution equation for the spin tensor:

$$u^\mu \partial_\mu S^{\alpha\beta} + S^{\alpha\beta}\partial_\mu u^\mu = -2(q^\alpha u^\beta - q^\beta u^\alpha + \phi^{\alpha\beta}). \tag{17}$$

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[2] Throughout the calculation, we have considered $S^{\mu\nu} \sim \mathcal{O}(1)$. However, the spin-chemical potential can be either considered as $\mathcal{O}(1)$ or $\mathcal{O}(\partial)$ in the hydrodynamic gradient expansion depending on the symmetry of the energy-momentum tensor. Such a choice of global equilibrium configuration is independent of the hydrodynamic gradient ordering of $S^{\mu\nu}$ and $\omega^{\mu\nu}$. As a result, we can choose $\omega^{\mu\nu}_{(0)} = 0$ for $\omega^{\mu\nu} \sim \mathcal{O}(1)$ as well as for $\omega^{\mu\nu} \sim \mathcal{O}(\partial)$.

Noting that we are considering the global equilibrium with $S^{\mu\nu}_{(0)} = 0$ and $\omega^{\mu\nu}_{(0)} = 0$, at the level of linear-order perturbation we can write

$$\partial_0 \delta S^{\alpha\beta} = -2(\delta q^\alpha u^\beta_{(0)} - \delta q^\beta u^\alpha_{(0)} + \delta\phi^{\alpha\beta}) + \mathcal{O}(\partial^3). \tag{18}$$

Using the above equation, we can obtain the evolution equation for δS^{0i} and δS^{ij}. The evolution equation of δS^{0i} is

$$\begin{aligned}\partial_0 \delta S^{0i} &= -2(\delta q^0 u^i_{(0)} - \delta q^i u^0_{(0)} + \delta\phi^{0i}) + \mathcal{O}(\partial^3)\\ &= 2\delta q^i + \mathcal{O}(\partial^3)\\ &= 2\lambda' c_s^2 \partial^i \delta\varepsilon - 2D_b \delta S^{i0} + \mathcal{O}(\partial^3).\end{aligned} \tag{19}$$

Using Eq. (18), the evolution equation of δS^{ij} can be written as

$$\begin{aligned}\partial_0 \delta S^{ij} &= -2\delta\phi^{ij} + \mathcal{O}(\partial^3)\\ &= -2D_s \delta S^{ij} - 2\gamma'(\partial^i \delta\pi^j - \partial^j \delta\pi^i) + \mathcal{O}(\partial^3).\end{aligned} \tag{20}$$

Here, $D_s = 4\tilde{\gamma}/\chi_s$ and $\chi_s = \partial S^{ij}/\partial\omega^{ij}$. The longitudinal projection of the conservation of the total energy-momentum tensor $u_\nu \partial_\mu T^{\mu\nu} = 0$ implies

$$u^\mu \partial_\mu \varepsilon + (\varepsilon + p)\partial_\mu u^\mu = -u_\nu \partial_\mu T^{\mu\nu}_{(1)}. \tag{21}$$

Note that the lhs of the above equation is $\mathcal{O}(\partial)$, but the rhs is $\mathcal{O}(\partial^2)$. Therefore, for the perturbation equation, the lhs will be up to the order of $\mathcal{O}(\partial^2)$ and the rhs will be up to $\mathcal{O}(\partial^3)$. Such a perturbation equation can be expressed as

$$\begin{aligned}&\partial_0 \delta\varepsilon + (\varepsilon_{(0)} + p_{(0)})\partial_i \delta u^i\\ &\quad = -\partial_0 \delta\tau^{00} - \partial_i[\delta\tau^{i0} + \delta q^i + \delta\phi^{i0}].\end{aligned} \tag{22}$$

In order to obtain Eq. (22), we drop some terms, e.g., $\delta u^\mu \partial_\mu \delta\varepsilon$ and $(\delta\varepsilon + \delta p)\partial_\mu \delta u^\mu$, which are $\mathcal{O}(\partial^3)$. Since we are restricting our analysis for only linear modes, we have dropped terms which are nonlinear in perturbations. Using the conditions that $\delta\tau^{00} = 0$, $\delta\tau^{0i} = 0$, and $\delta\phi^{i0} = 0$ and the expression of δq^i back into Eq. (22), it can be shown that

$$\partial_0 \delta\varepsilon + \partial_i \delta\pi^i + 2(\lambda' c_s^2 \partial_i \partial^i \delta\varepsilon - D_b \partial_i \delta S^{i0}) = 0. \tag{23}$$

In the above equation, we have not considered $\mathcal{O}(\partial^4)$ terms. Taking the normal projection of total energy-momentum tensor $\Delta^\alpha_\nu \partial_\mu T^{\mu\nu}$, we find

$$\begin{aligned}&(\varepsilon + p)Du^\alpha - \Delta^{\alpha\beta}\partial_\beta p + \Delta^\alpha{}_\nu \partial_\mu \tau^{\mu\nu} + \Delta^\alpha{}_\nu \partial_\mu (q^\mu u^\nu)\\ &\quad - \Delta^\alpha{}_\nu \partial_\mu (q^\nu u^\mu) + \Delta^\alpha{}_\nu \partial_\mu \phi^{\mu\nu} = 0.\end{aligned} \tag{24}$$

We should emphasize that Eq. (24) contains terms up to $\mathcal{O}(\partial^2)$. Therefore, the linear-order perturbation of Eq. (24) must contain terms up to the order of $\mathcal{O}(\partial^3)$, and we can neglect higher-order terms. The perturbation equation associated with Eq. (24) can be expressed as

$$\begin{aligned}&(\varepsilon_{(0)} + p_{(0)})\partial_0 \delta u^\alpha - \Delta^{\alpha\beta}_{(0)} \partial_\beta \delta p\\ &\quad + \eta \Delta^\alpha{}_{\nu(0)} \Delta^{\mu\beta}_{(0)} \partial_\mu \partial_\beta \delta u^\nu + \eta \Delta^{\alpha\beta}_{(0)} \partial_\mu \partial_\beta \delta u^\mu\\ &\quad - \frac{2}{3}\eta \Delta^{\alpha\mu}_{(0)} \partial_\mu \partial_\delta \delta u^\delta + \zeta \Delta^{\alpha\mu}_{(0)} \partial_\mu \partial_\delta \delta u^\delta\\ &\quad - \Delta^\alpha{}_{\nu(0)} \partial_0 \delta q^\nu + \Delta^\alpha{}_{\nu(0)} \partial_\mu \delta\phi^{\mu\nu} = 0.\end{aligned} \tag{25}$$

For $\alpha = 0$, the lhs of the above equation identically vanishes. For $\alpha = i$, we find

$$\begin{aligned}&\partial_0 \delta\pi^i - c_s^2 \partial^i \delta\varepsilon + (\gamma_\perp + \gamma')(\delta^i{}_j \partial^k \partial_k - \partial^i \partial_j)\delta\pi^j\\ &\quad + \gamma_\parallel \partial^i \partial_k \delta\pi^k + D_s \partial_k \delta S^{ki} = 0.\end{aligned} \tag{26}$$

Throughout the derivation, we use the following notations:

$$\begin{aligned}&c_s^2 \equiv \frac{\partial p}{\partial\varepsilon}, \qquad \chi_s \equiv \frac{\partial S^{ij}}{\partial\omega^{ij}}, \qquad D_s \equiv \frac{4\tilde{\gamma}}{\chi_s}, \qquad \gamma' \equiv \frac{\tilde{\gamma}}{\varepsilon_{(0)} + p_{(0)}},\\ &\chi_b \equiv \frac{\partial S^{i0}}{\partial\omega^{i0}}, \qquad D_b \equiv \frac{4\lambda}{\chi_b}, \qquad \lambda' \equiv \frac{2\lambda}{\varepsilon_{(0)} + p_{(0)}},\\ &\gamma_\parallel \equiv \frac{1}{\varepsilon_{(0)} + p_{(0)}}\left(\zeta + \frac{4}{3}\eta\right), \qquad \gamma_\perp \equiv \frac{\eta}{\varepsilon_{(0)} + p_{(0)}}.\end{aligned} \tag{27}$$

Equations (19), (20), (23), and (26) along with Eq. (16) are the main perturbation equations. We emphasize that for this framework the standard fluid perturbations, i.e., $\delta\varepsilon$ and δu^i, are coupled with the spin perturbation δS^{0i} and δS^{ij}.

B. Fourier space equations

Equations (19), (20), (23), and (26) can be solved in momentum space to obtain different dispersion relations associated with different perturbation modes. Various perturbations $\delta\varepsilon$, $\delta\pi^k$, δS^{ij}, and δS^{0i} can be expressed as plane waves in the following way:

$$\begin{aligned}\delta\varepsilon &= \tilde{\delta\varepsilon} e^{-i\omega t + i\vec{k}\cdot\vec{x}},\\ \delta\pi^k &= \widetilde{\delta\pi^k} e^{-i\omega t + i\vec{k}\cdot\vec{x}},\\ \delta S^{ij} &= \widetilde{\delta S^{ij}} e^{-i\omega t + i\vec{k}\cdot\vec{x}},\\ \delta S^{0i} &= \widetilde{\delta S^{0i}} e^{-i\omega t + i\vec{k}\cdot\vec{x}}.\end{aligned} \tag{28}$$

Because of the rotational symmetry of the system, we can consider waves which are propagating only along the z direction, i.e., $\vec{k} = (0, 0, k_z)$. For such a choice of plane wave, Eqs. (19), (20), (23), and (26) become

$$-i\omega\tilde{\delta\varepsilon} + ik_z \widetilde{\delta\pi^z} + 2\lambda' c_s^2 k_z^2 \tilde{\delta\varepsilon} + 2D_b ik_z \widetilde{\delta S^{0z}} = 0, \tag{29}$$

$$-i\omega\widetilde{\delta\pi^x} + (\gamma_\perp + \gamma')k_z^2 \widetilde{\delta\pi^x} + ik_z D_s \widetilde{\delta S^{zx}} = 0, \tag{30}$$

$$-i\omega\widetilde{\delta\pi^y} + (\gamma_\perp + \gamma')k_z^2 \widetilde{\delta\pi^y} + ik_z D_s \widetilde{\delta S^{zy}} = 0, \tag{31}$$

$$-i\omega\widetilde{\delta\pi^z} + ik_z c_s^2 \tilde{\delta\varepsilon} + \gamma_\parallel k_z^2 \widetilde{\delta\pi^z} = 0, \tag{32}$$

$$-i\omega\widetilde{\delta S^{xy}} + 2D_s\widetilde{\delta S^{xy}} = 0, \tag{33}$$

$$-i\omega\widetilde{\delta S^{zx}} + 2D_s\widetilde{\delta S^{zx}} - 2\gamma'(ik_z)\widetilde{\delta\pi^x} = 0, \tag{34}$$

$$-i\omega\widetilde{\delta S^{yz}} + 2D_s\widetilde{\delta S^{yz}} + 2\gamma'(ik_z)\widetilde{\delta\pi^y} = 0, \tag{35}$$

$$-i\omega\widetilde{\delta S^{0x}} - 2D_b\widetilde{\delta S^{0x}} = 0, \tag{36}$$

$$-i\omega\widetilde{\delta S^{0y}} - 2D_b\widetilde{\delta S^{0y}} = 0, \tag{37}$$

$$-i\omega\widetilde{\delta S^{0z}} + 2\lambda' c_s^2(ik_z)\tilde{\delta\varepsilon} - 2D_b\widetilde{\delta S^{0z}} = 0. \tag{38}$$

These equations can be represented as a matrix equation $M \times v = 0$ such that

$$v = (\tilde{\delta\varepsilon}, \widetilde{\delta\pi^z}, \widetilde{\delta S^{0z}}, \widetilde{\delta\pi^x}, \widetilde{\delta S^{zx}}, \widetilde{\delta\pi^y}, \widetilde{\delta S^{zy}}, \widetilde{\delta S^{0x}}, \widetilde{\delta S^{0y}}, \widetilde{\delta S^{xy}})^T.$$

For such a choice, $\tilde{\delta\varepsilon}$, $\widetilde{\delta\pi^z}$, and $\widetilde{\delta S^{0z}}$ can be grouped together to write

$$\begin{pmatrix} -i\omega + 2\lambda' c_s^2 k_z^2 & ik_z & 2D_b ik_z \\ ic_s^2 k_z & -i\omega + \gamma_\parallel k_z^2 & 0 \\ 2\lambda' c_s^2 ik_z & 0 & -i\omega - 2D_b. \end{pmatrix} \begin{pmatrix} \tilde{\delta\varepsilon} \\ \widetilde{\delta\pi^z} \\ \widetilde{\delta S^{0z}} \end{pmatrix} \equiv \mathcal{A} \begin{pmatrix} \tilde{\delta\varepsilon} \\ \widetilde{\delta\pi^z} \\ \widetilde{\delta S^{0z}} \end{pmatrix}. \tag{39}$$

Similarly, $\widetilde{\delta\pi^x}$, $\widetilde{\delta S^{zx}}$ and $\widetilde{\delta\pi^y}$, $\widetilde{\delta S^{zy}}$ can be grouped together in the following way:

$$\begin{pmatrix} -i\omega + (\gamma_\perp + \gamma')k_z^2 & ik_z D_s \\ -2\gamma' ik_z & -i\omega + 2D_s \end{pmatrix} \begin{pmatrix} \widetilde{\delta\pi^x} \\ \widetilde{\delta S^{zx}} \end{pmatrix} \equiv \mathcal{B} \begin{pmatrix} \widetilde{\delta\pi^x} \\ \widetilde{\delta S^{zx}} \end{pmatrix} \tag{40}$$

and

$$\begin{pmatrix} -i\omega + (\gamma_\perp + \gamma')k_z^2 & ik_z D_s \\ -2\gamma' ik_z & -i\omega + 2D_s \end{pmatrix} \begin{pmatrix} \widetilde{\delta\pi^y} \\ \widetilde{\delta S^{zy}} \end{pmatrix} \equiv \mathcal{B} \begin{pmatrix} \widetilde{\delta\pi^y} \\ \widetilde{\delta S^{zy}} \end{pmatrix}. \tag{41}$$

The spin parts $\widetilde{\delta S^{0x}}$, $\widetilde{\delta S^{0y}}$, and $\widetilde{\delta S^{xy}}$ are not coupled to other perturbations. Therefore, Eqs. (33), (36), and (37) can be written as

$$\begin{pmatrix} -i\omega - 2D_b & 0 & 0 \\ 0 & -i\omega - 2D_b & 0 \\ 0 & 0 & -i\omega + 2D_s \end{pmatrix} \begin{pmatrix} \widetilde{\delta S^{0x}} \\ \widetilde{\delta S^{0y}} \\ \widetilde{\delta S^{xy}} \end{pmatrix} \equiv \mathcal{C} \begin{pmatrix} \widetilde{\delta S^{0x}} \\ \widetilde{\delta S^{0y}} \\ \widetilde{\delta S^{xy}} \end{pmatrix}. \tag{42}$$

Various dispersion relations can be obtained by solving $\det M = 0$, where

$$M = \begin{pmatrix} \mathcal{A} & 0_{3\times2} & 0_{3\times2} & 0_{3\times3} \\ 0_{2\times3} & \mathcal{B} & 0_{2\times2} & 0_{2\times3} \\ 0_{2\times3} & 0_{2\times2} & \mathcal{B} & 0_{2\times3} \\ 0_{3\times3} & 0_{3\times2} & 0_{3\times2} & \mathcal{C} \end{pmatrix}. \tag{43}$$

Using the advantage of M being a block diagonal matrix, we can directly obtain the determinant such that

$$\det(\mathcal{A})\det(\mathcal{B})^2(-i\omega - 2D_b)^2(-i\omega + 2D_s) = 0 \tag{44}$$

and can be summarized as

$$\omega = -2iD_s, \tag{45}$$

$$\omega = +2iD_b \quad \text{(two modes)}, \tag{46}$$

$$\omega = -2iD_s - i\gamma' k_z^2 + \mathcal{O}(k_z^4) \quad \text{(two modes)} \tag{47}$$

$$\omega = -i\gamma_\perp k_z^2 + \mathcal{O}(k_z^4) \quad \text{(two modes)}, \tag{48}$$

$$\omega = +c_s k_z - \frac{i}{2}\gamma_\parallel k_z^2 + \mathcal{O}(k_z^3), \tag{49}$$

$$\omega = -c_s k_z - \frac{i}{2}\gamma_\parallel k_z^2 + \mathcal{O}(k_z^3), \tag{50}$$

$$\omega = 2iD_b - 2ic_s^2\lambda' k_z^2 + \mathcal{O}(k_z^4). \tag{51}$$

Note that a physical plane wave of the form $e^{-i\omega t + i\vec{k}\cdot\vec{x}}$ must not give a solution which is growing with time.[3] But from

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[3] If the hydrodynamic perturbations are represented as $\exp(-i\omega t + i\vec{k}\cdot\vec{x})$, then stability implies $\mathrm{Im}\,\omega(\vec{k}) \leq 0$ for all $\vec{k}$ [45]. However, in the present work, we kept terms up to k_z^2 considering that hydrodynamics is a long-wavelength effective theory. Hence, low-momentum modes correctly capture different features of the hydrodynamic theory, e.g., diffusive modes, sound modes, etc. Moreover, if we restrict ourselves to first-order theory, then various algebraic equations in the momentum space and various dispersion relations $[\omega(k_z)]$ contain terms only up to k_z^2. Higher-order terms in various dispersion relations can originate from higher-order gradient terms in hydrodynamic constitutive relations for the energy-momentum tensor and spin tensor. In the Navier-Stokes limit, we do not consider such higher-order terms. Therefore, for consistency of hydrodynamic gradient expansion, we kept the lowest-order terms up to k_z^2. For a most general hydrodynamic theory with an arbitrary number of gradient terms, one should analyze a series of the form $\omega = \sum_{n=0}^{\infty} a_{(n)} k^n$ [64]. In the present calculation, we do not intend to study such a generalized dispersion relation.

Eqs. (46) and (51) it is clear that an unstable mode exists for $D_b > 0$. Therefore, the spin-hydrodynamic equations are not stable for the linear perturbation around the global equilibrium considered here. The same observation has been pointed out in Ref. [60].[4] This problem may be solved by appropriately defining D_b. There could be a physical reason to choose an appropriate sign of D_b. Throughout the calculation, D_b and D_s are considered to be positive. This implies that $\chi_b > 0$ and $\chi_s > 0$ Eq. (27). This is very natural to argue using the equation of state relating spin density tensor $S^{\mu\nu}$ and $\omega^{\mu\nu}$ [31,65]. Generically, the spin density tensor should be proportional to the spin-chemical potential, i.e., $S^{\mu\nu} \sim \omega^{\mu\nu}$. The proportionality factor should be either positive or negative. If the proportionality factor is negative, then $D_b < 0$ and $D_s < 0$. In this case, although $D_b < 0$, due to negative D_s some modes will remain unstable. On the other hand, if the proportionality factor is positive, then D_b and D_s are both positive. In that case, also the theory will give rise to unstable modes. Only if $D_b < 0$ or $\chi_b < 0$, keeping $\chi_s > 0$, then unstable modes will not appear. This implies that the proportionality factor appearing in the spin equation of state for the $0i$th components will be different from ijth components. This is rather difficult to justify physically. But this needs to be investigated thoroughly for a proper understanding of the spin perturbation equations. We consider an alternative approach to remove this instability by imposing the Frenkel condition. We know that this instability is caused by the modes associated with D_b, and it may be possible to eliminate those problematic modes using the Frenkel condition, i.e., $S^{\mu\nu}u_\nu = 0 = \omega^{\mu\nu}u_\nu$ [63].

C. Frenkel condition to solve the problem of instability

The "macroscopic" Frenkel condition $u_\mu S^{\mu\nu} = 0$ has been considered recently, for instance, in Ref. [63], to develop a spin-hydrodynamic framework with a canonical spin tensor.[5] In the canonical framework, where the spin tensor is totally antisymmetric, the Frenkel condition is naturally built in the system, i.e., $S^{\mu\nu} \equiv u_\lambda S^{\lambda\mu\nu} \Rightarrow u_\mu S^{\mu\nu} = 0$. Obviously, as $S^{\mu\nu}$ is antisymmetric, the Frenkel condition reduces the number of dynamical degrees of freedom in the hydrodynamic framework from six to three. In Ref. [44], we have argued that the energy-momentum and spin tensors considered in the present calculation can be obtained from a properly defined canonical framework using a pseudogauge transformation. $S^{\mu\nu}$ appears in thermodynamic relations including spin-chemical potential which are expected to be independent of pseudogauge transformations. Hence, we also expect the Frenkel condition to be applicable to the present calculation.

If we impose the Frenkel condition, i.e., $S^{\mu\nu}u_\nu = 0$ or $\omega^{\mu\nu}u_\nu = 0$, then various dissipative currents, i.e., h^μ and $\tau^{\mu\nu}$ as given in Eqs. (3) and (5), respectively, remain unaltered. But q^μ and $\phi^{\mu\nu}$ change to

$$q^\mu = \lambda(Du^\mu + \beta\nabla^\mu T), \tag{52}$$

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[4] Note that in Ref. [60] the authors considered a different counting scheme than what we considered here. In this reference, the authors could find next to the leading-order contribution to the spin tensor and the associated transport coefficient χ_1. In the limit $\chi_1 = 0$, various dissipative currents as obtained in Ref. [60] match with the calculation as given in Ref. [26]. In this limit, there exists an unstable mode if $D_b > 0$ or $\chi_b > 0$.

[5] The Frenkel theory was originally introduced to generalize the definition of nonrelativistic spin to a covariant expression in a quantum field theoretical system to describe the microscopic spin degree of freedom. In an operator form, the *microscopic* Frenkel condition can be expressed as $\langle p, s|\hat{P}_\mu \hat{S}^{\mu\nu}|p, s\rangle = 0$, where $|p, s\rangle$ denotes a single-particle state with momentum p and spin s [20]. In the particle rest frame (PRF), it can be argued that the expectation value of $\hat{S}^{\mu\nu}$ has only spacelike components, which for the case of a *canonical* spin tensor give rise to the nonrelativistic spin vector operator [20]. Hydrodynamic descriptions are different from single-particle dynamics. Hydrodynamics is a classical field theory resulting from the coarse-graining of quantum field dynamics. Therefore, for a hydrodynamic theory, the generalization of the concepts applicable to single-particle dynamics needs to be provided. Such a generalization of the microscopic Frenkel condition for a fluid element can be expressed as $u_\mu S^{\mu\nu} = 0$. One should emphasize that in quantum field theory $\hat{S}^{\mu\nu}$ is defined as a volume-integrated operator. On the other hand, in hydrodynamics, one deals with densities of macroscopic quantities. Note that, at the hydrodynamic level, dealing with the concept of PRF can be misleading. Rather, the notion of a fluid rest frame (FRF) is more suitable. While, in the case of massless particles, the PRF does not exist, one can certainly define the FRF. Therefore, $S^{\mu\nu}$ should be considered as the spin density tensor characterizing the fluid element, contrary to the spin tensor for an individual particle. In the FRF with $u^\mu \equiv (1, 0, 0, 0)$, only spacelike components of $S^{\mu\nu}$ survive, i.e., $S^{ij} \neq 0$. It can be considered analogous (not the same) to the PRF condition for a massive particle.

$$\phi^{\mu\nu} = \tilde{\gamma}(\nabla^\mu u^\nu - \nabla^\nu u^\mu + 4\omega^{\mu\nu}). \tag{53}$$

Neglecting all higher-order terms, δq^μ and $\delta\phi^{\mu\nu}$ up to $\mathcal{O}(\partial^2)$ can be expressed as

$$\delta q^\mu = \lambda\left(2\frac{\Delta^{\mu\alpha}_{(0)}\partial_\alpha\delta p}{\varepsilon_{(0)} + p_{(0)}}\right) + \mathcal{O}(\partial^3), \tag{54}$$

$$\delta\phi^{\mu\nu} = \tilde{\gamma}(\Delta^{\mu\alpha}_{(0)}\partial_\alpha\delta u^\nu - \Delta^{\nu\alpha}_{(0)}\partial_\alpha\delta u^\mu + 4\delta\omega^{\mu\nu}) + \mathcal{O}(\partial^3). \tag{55}$$

Using the evolution equation of $\delta S^{\alpha\beta}$ as given in Eq. (18), we find

$$\partial_0\delta S^{0i} = 2\lambda' c_s^2\partial^i\delta\varepsilon. \tag{56}$$

Using the linear-order perturbation of the Frenkel condition $S^{\mu\nu}u_\nu = 0 = \omega^{\mu\nu}u_\nu$ gives us $\delta S^{\mu\nu}u^{(0)}_\nu = 0$. Therefore, due to the use of the Frenkel condition, we find $\delta\omega^{0i} = 0 = \delta S^{0i}$. Using the condition that $\delta S^{0i} = 0$ in the above equation, we get $\partial^i\delta\varepsilon = 0$. Note that for the linear model analysis we are interested in the plane wave solution of various hydrodynamic perturbations of the form $e^{-ik\cdot x}$. Thus, $\partial^i\delta\varepsilon = 0$ implies that $\delta\varepsilon$ itself vanishes, i.e., $\delta\varepsilon = 0$ and $\delta q^i = 0$. Using Eq. (18), the evolution equation of δS^{ij} can be written as

$$\partial_0\delta S^{ij} + 2D_s\delta S^{ij} + 2\gamma'(\partial^i\delta\pi^j - \partial^j\delta\pi^i) = 0, \tag{57}$$

where $\delta\pi^i = (\varepsilon_{(0)} + p_{(0)})\delta u^i$. Now let us look into the conservation of the energy-momentum tensor. Using the conditions $\delta\varepsilon = 0$ and $\delta q^i = 0$ in the longitudinal projection of the energy-momentum tensor, i.e., Eq. (22), we find $\partial_i\delta\pi^i = 0$. On the other hand, the perturbation equation associated with the normal projection of conservation of total energy-momentum tensor gives us

$$(\varepsilon_{(0)} + p_{(0)})\partial_0\delta u^i + \eta\partial^k\partial_k\delta u^i + \partial_k\delta\phi^{ki} = 0. \tag{58}$$

Using the expression of $\delta\phi^{\mu\nu}$ as given in Eq. (55), it can be shown that

$$\partial_k\delta\phi^{ki} = \tilde{\gamma}\partial_k\partial^k\delta u^i + D_s\partial_k\delta S^{ki}. \tag{59}$$

Using Eq. (59) back into Eq. (58), we obtain

$$\partial_0\delta\pi^i + (\gamma_\perp + \gamma')\partial_k\partial^k\delta\pi^i + D_s\partial_k\delta S^{ki} = 0. \tag{60}$$

In summary, considering the linear-order perturbations of various hydrodynamic variables, the conservation of the energy-momentum tensor and the total angular momentum tensor gives us

$$\partial_0\delta\pi^i + (\gamma_\perp + \gamma')\partial^k\partial_k\delta\pi^i + D_s\partial_k\delta S^{ki} = 0, \tag{61}$$

$$\partial_0\delta S^{ij} + 2D_s\delta S^{ij} + 2\gamma'(\partial^i\delta\pi^j - \partial^j\delta\pi^i) = 0, \tag{62}$$

$$\partial_i\delta\pi^i = 0. \tag{63}$$

Once again without the loss of generality, we consider plane wave representation of various perturbations in the momentum space:

$$\delta\pi^k = \widetilde{\delta\pi^k}e^{-i\omega t + ik_z z}, \tag{64}$$

$$\delta S^{ij} = \widetilde{\delta S^{ij}}e^{-i\omega t + ik_z z}. \tag{65}$$

In momentum space, Eq. (63) implies $\delta\pi^z = 0$. Therefore, the perturbations $\widetilde{\delta\varepsilon}$, $\widetilde{\delta\pi^z}$, $\widetilde{\delta S^{0x}}$, $\widetilde{\delta S^{0y}}$, and $\widetilde{\delta S^{0z}}$ decouple from the theory, and the remaining nontrivial equations are

$$-i\omega\widetilde{\delta\pi^x} + (\gamma_\perp + \gamma')k_z^2\widetilde{\delta\pi^x} + ik_zD_s\widetilde{\delta S^{zx}} = 0, \tag{66}$$

$$-i\omega\widetilde{\delta\pi^y} + (\gamma_\perp + \gamma')k_z^2\widetilde{\delta\pi^y} + ik_zD_s\widetilde{\delta S^{zy}} = 0, \tag{67}$$

$$-i\omega\widetilde{\delta S^{xy}} + 2D_s\widetilde{\delta S^{xy}} = 0, \tag{68}$$

$$-i\omega\widetilde{\delta S^{zx}} + 2D_s\widetilde{\delta S^{zx}} - 2\gamma'(ik_z)\widetilde{\delta\pi^x} = 0, \tag{69}$$

$$-i\omega\widetilde{\delta S^{yz}} + 2D_s\widetilde{\delta S^{yz}} + 2\gamma'(ik_z)\widetilde{\delta\pi^y} = 0. \tag{70}$$

$\widetilde{\delta\pi^x}$, $\widetilde{\delta S^{zx}}$ and $\widetilde{\delta\pi^y}$, $\widetilde{\delta S^{zy}}$ can be grouped together as given in Eqs. (40) and (41). But $\widetilde{\delta S^{xy}}$ does not couple to any other perturbation. Various dispersion relations can be obtained by solving the equation $\det(\mathcal{B})^2(-i\omega + 2D_s) = 0$ and can be summarized as

$$\omega = -2iD_s, \tag{71}$$

$$\omega = -2iD_s - i\gamma' k_z^2 + \mathcal{O}(k_z^2) \quad \text{(two modes)}, \tag{72}$$

$$\omega = -i\gamma_\perp k_z^2 + \mathcal{O}(k_z^4) \quad \text{(two modes)}. \tag{73}$$

Note that, unlike the previous case, the imaginary part of the various dispersion relation is always negative, and various perturbations will not grow with time. But this comes with a drawback. Because of the use of the Frenkel condition in this case, the standard hydrodynamic perturbations, i.e., $\delta\varepsilon$ and $\delta\pi^z$, do not appear in the theory, which is not physically appealing.

III. SPIN-CHEMICAL POTENTIAL LEADING ORDER IN THE HYDRODYNAMIC GRADIENT EXPANSION

A. Formulation

We start our discussion with the hydrodynamic framework with angular momentum where the spin-chemical potential ($\omega^{\mu\nu}$) is $\mathcal{O}(1)$ in the hydrodynamic gradient expansion [29]. In this framework, the energy-momentum tensor is totally symmetric. Note that, in the absence of any antisymmetric component of the energy-momentum tensor, the spin angular momentum is separately conserved even in the presence of interactions, which is otherwise not possible. This drastically affects the dissipative spin-hydrodynamic description, particularly various dissipative currents in the energy-momentum tensor and spin tensor [29]. Here, we proceed one step further to discuss spin-hydrodynamic modes of this theory by performing the linear mode analysis for the spin-hydrodynamic framework. We consider the metric convention with $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, and the projector orthogonal to the fluid four-velocity u^μ is $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ with $u^\mu u_\mu = 1$. For the Landau frame choice, the energy-momentum tensor ($T^{\mu\nu}$) and the spin tensor ($S^{\mu\alpha\beta}$) can be expressed as [29]

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \tau^{\mu\nu}, \tag{74}$$

$$\tau^{\mu\nu} = \pi^{\mu\nu} + \Pi\Delta^{\mu\nu}, \tag{75}$$

$$S^{\mu\alpha\beta} = u^\mu S^{\alpha\beta} + S^{\mu\alpha\beta}_{(1)}. \tag{76}$$

$\tau^{\mu\nu} = \tau^{\nu\mu}$ is the dissipative part of the energy-momentum tensor which contains the shear and bulk viscous terms. $\tau^{\mu\nu}$ satisfies the following condition: $\tau^{\mu\nu}u_\mu = 0$. Using the entropy current analysis for the Navier-Stokes theory, shear and bulk viscous terms can be expressed as $\tau^{\mu\nu} = \pi^{\mu\nu} + \Pi\Delta^{\mu\nu}$,

$$\pi^{\mu\nu} = \eta\left[\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\beta u^\beta\right], \tag{77}$$

$$\Pi = \zeta(\partial_\alpha u^\alpha). \tag{78}$$

Here, $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$, and η and ζ are the coefficients of the shear and bulk viscosity, respectively. Both η and ζ are positive definite. The spin tensor $S^{\mu\alpha\beta}$ is antisymmetric only in the last two indices. $S^{\mu\nu}$ can be considered as the spin density, i.e., $S^{\alpha\beta} \equiv u_\mu S^{\mu\alpha\beta}$, and $S^{\mu\alpha\beta}_{(1)}$ is the dissipative part of the spin tensor. The thermodynamic relations are given as $Ts + S^{\alpha\beta}\omega_{\alpha\beta} = \varepsilon + p$, $d\varepsilon = Tds + \omega_{\alpha\beta}dS^{\alpha\beta}$. The dissipative part of the spin tensor $S^{\mu\alpha\beta}_{(1)} \sim \mathcal{O}(\partial)$ can be expressed as[6] [29,66]

$$S^{\mu\alpha\beta}_{(1)} = -\frac{\mathbf{q}^\mu}{\varepsilon + p}S^{\alpha\beta} + u^\alpha\Delta^{\mu\beta}\Phi - u^\beta\Delta^{\mu\alpha}\Phi + u^\alpha\tau^{\mu\beta}_{(s)} - u^\beta\tau^{\mu\alpha}_{(s)} + u^\alpha\tau^{\mu\beta}_{(a)} - u^\beta\tau^{\mu\alpha}_{(a)} + \Theta^{\mu\alpha\beta}. \tag{79}$$

Similar to $S^{\mu\alpha\beta}$, the dissipative part $S^{\mu\alpha\beta}_{(1)}$ is also antisymmetric in the last two indices. Various dissipative parts in the spin tensor, i.e., $\mathbf{q}^\mu$, Φ, $\tau^{\mu\nu}_{(s)}$, $\tau^{\mu\nu}_{(a)}$, and $\Theta^{\mu\alpha\beta}$, are first order in the hydrodynamic gradient expansion. These dissipative currents satisfy the following properties: $u_\mu\mathbf{q}^\mu = u_\mu\tau^{\mu\beta}_{(s)} = u_\mu\tau^{\mu\beta}_{(a)} = u_\mu\Theta^{\mu\alpha\beta} = 0$; $\tau^{\mu\beta}_{(s)} = \tau^{\beta\mu}_{(s)}$, $\tau^{\mu\beta}_{(a)} = -\tau^{\beta\mu}_{(a)}$, $\Theta^{\mu\alpha\beta} = -\Theta^{\mu\beta\alpha}$; $\mathrm{tr}(\tau^{\beta\mu}_{(s)}) = 0$. Note that $\tau^{\mu\nu}_{(s)}$ is a symmetric tensor. Hence, in general, it can be decomposed into a trace part and a traceless part. The trace part of $\tau^{\mu\nu}_{(s)}$ can be absorbed in Φ; hence, we consider $\tau^{\mu\nu}_{(s)}$ to be traceless. Considering that for a dissipative system entropy will be produced, the analytic expressions for $\mathbf{q}^\mu$, Φ, $\tau^{\mu\nu}_{(s)}$, $\tau^{\mu\nu}_{(a)}$, and $\Theta^{\mu\alpha\beta}$ can be given as [29]

$$\mathbf{q}^\mu = \lambda_q T\left(\frac{\nabla^\mu T}{T} - Du^\mu\right), \tag{80}$$

$$\Phi = -\chi_1 u^\alpha\nabla^\beta(\beta\omega_{\alpha\beta}), \tag{81}$$

$$\tau^{\mu\beta}_{(s)} = -\chi_2 u^\alpha\left[\Delta^{\beta\rho}\Delta^{\mu\gamma} + \Delta^{\mu\rho}\Delta^{\beta\gamma} - \frac{2}{3}\Delta^{\mu\beta}\Delta^{\rho\gamma}\right]\nabla_\gamma(\beta\omega_{\alpha\rho}), \tag{82}$$

$$\tau^{\mu\beta}_{(a)} = -\chi_3 u^\alpha(\Delta^{\beta\rho}\Delta^{\mu\gamma} - \Delta^{\mu\rho}\Delta^{\beta\gamma})\nabla_\gamma(\beta\omega_{\alpha\rho}), \tag{83}$$

$$\Theta^{\mu\alpha\beta} = -\chi_4[u^\beta u^\rho\Delta^{\alpha\delta} - u^\alpha u^\rho\Delta^{\beta\delta}]\Delta^{\mu\gamma}\nabla_\gamma(\beta\omega_{\delta\rho}) + \chi_5\Delta^{\alpha\delta}\Delta^{\beta\rho}\Delta^{\mu\gamma}\nabla_\gamma(\beta\omega_{\delta\rho}). \tag{84}$$

Here, $D \equiv u^\mu\partial_\mu$. Various transport coefficients λ_q, χ_1, χ_2, χ_3, χ_4, and χ_5 are all positive. We should emphasize the relative sign difference between two terms in the expression of $\Theta^{\mu\alpha\beta}$. Although $\Theta^{\mu\alpha\beta}$ is orthogonal to the fluid four-velocity, in the fluid rest frame it is not totally spacelike. In the fluid rest frame, terms associated with χ_5 have only spacelike indices, but the terms associated with χ_4 can have spacelike and timelike indices. This gives rise to a relative sign difference.

To study the linear mode analysis of the spin-hydrodynamic description, we consider the global equilibrium background flow $u^\mu_{(0)} \equiv (1,0,0,0)$, with $S^{\mu\nu}_{(0)} = 0$ and $\omega^{\mu\nu}_{(0)} = 0$. All other dissipative currents vanish in the global equilibrium background. Fluid flow perturbation is given as $\delta u^\mu = (0, \delta u^i)$. Taking the projection of the conservation of $T^{\mu\nu}$ along with the fluid velocity and orthogonal to the fluid velocity, we find

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[6] In the standard hydrodynamic framework without a dynamical spin degree of freedom, the nonequilibrium entropy current can be expressed as $\mathcal{S}^\mu = p\beta u^\mu + \beta u_\nu T^{\mu\nu}$. In spin hydrodynamics, the ansatz for the nonequilibrium entropy current can be generalized to $\mathcal{S}^\mu = p\beta^\mu + \beta_\nu T^{\mu\nu} - \beta\omega_{\alpha\beta}S^{\mu\alpha\beta}$. If we consider $\omega^{\mu\nu} \sim \mathcal{O}(1)$, then for the first-order theory of dissipative spin hydrodynamics $S^{\mu\alpha\beta}_{(1)}$ contributes to the entropy current.

$$u^\mu\partial_\mu\varepsilon + (\varepsilon + p)(\partial_\mu u^\mu) = -u_\nu\partial_\mu\tau^{\mu\nu}, \tag{85}$$

$$(\varepsilon + p)Du^\alpha - \Delta^{\alpha\beta}\partial_\beta p + \Delta^\alpha_\nu\partial_\mu\tau^{\mu\nu} = 0. \tag{86}$$

The corresponding linear-order perturbation equations can be expressed as

$$\partial_0\delta\varepsilon + \partial_i\delta\pi^i = 0, \tag{87}$$

$$\partial_0\delta\pi^i - c_s^2\partial^i\delta\varepsilon + \gamma_\perp[\delta^i_l\partial^k\partial_k - \partial^i\partial_l]\delta\pi^l + \gamma_\parallel\partial^i\partial_k\delta\pi^k = 0. \tag{88}$$

Here, $\delta\pi^i = (\varepsilon_{(0)} + p_{(0)})\delta u^i$, $\gamma_\perp \equiv \eta/(\varepsilon_{(0)} + p_{(0)})$, and $\gamma_\parallel \equiv (\frac{4}{3}\eta + \zeta)/(\varepsilon_{(0)} + p_{(0)})$, $c_s^2 = \partial p/\partial\varepsilon$. $\varepsilon_{(0)}$ and $p_{(0)}$ are the background energy density and pressure, respectively. To obtain the above perturbation equations, we have used the following equation:

$$\delta\tau^{\mu\nu} = \eta\left[\Delta^{\mu\alpha}_{(0)}\partial_\alpha\delta u^\nu + \Delta^{\nu\alpha}_{(0)}\partial_\alpha\delta u^\mu - \frac{2}{3}\Delta^{\mu\nu}_{(0)}\partial_\alpha\delta u^\alpha\right] + \zeta\Delta^{\mu\nu}_{(0)}\partial_\alpha\delta u^\alpha + \mathcal{O}(\partial^3). \tag{89}$$

It should be emphasized that, for the background flow with $u^\mu_{(0)} \equiv (1, 0, 0, 0)$, $\delta\tau^{00} = 0$ as well as $\delta\tau^{i0} = 0$. Since the energy-momentum tensor does not have any antisymmetric part, the spin tensor $S^{\mu\alpha\beta}$ is separately conserved. This follows from the conservation of the total angular momentum, i.e.,

$$u^\mu\partial_\mu S^{\alpha\beta} + S^{\alpha\beta}(\partial_\mu u^\mu) + \partial_\mu S^{\mu\alpha\beta}_{(1)} = 0. \tag{90}$$

For the global equilibrium condition with $S^{\alpha\beta}_{(0)} = 0$, the perturbation equation corresponding to the conservation of the spin tensor can be expressed as

$$u^\mu_{(0)}\partial_\mu\delta S^{\alpha\beta} + \partial_\mu\delta S^{\mu\alpha\beta}_{(1)} = 0. \tag{91}$$

In the above equation, $\delta S^{\mu\alpha\beta}_{(1)}$ can be obtained systematically by taking the linear-order perturbation of Eq. (79) for the given global equilibrium conditions. To obtain $\delta S^{\mu\alpha\beta}_{(1)}$, we use the following relations:

$$\delta\Phi = -\chi_1\beta_{(0)}\partial_i\delta\omega^{0i}, \tag{92}$$

$$\delta\tau^{\mu\alpha}_s = -\beta_{(0)}\chi_2 u^a_{(0)}\left[\Delta^{\alpha\rho}_{(0)}\Delta^{\mu\gamma}_{(0)} + \Delta^{\mu\rho}_{(0)}\Delta^{\alpha\gamma}_{(0)} - \frac{2}{3}\Delta^{\mu\alpha}_{(0)}\Delta^{\rho\gamma}_{(0)}\right]\nabla^{(0)}_\gamma\delta\omega_{a\rho}, \tag{93}$$

$$\delta\tau^{\mu\beta}_{(a)} = -\beta_{(0)}\chi_3 u^\alpha_{(0)}(\Delta^{\beta\rho}_{(0)}\Delta^{\mu\gamma}_{(0)} - \Delta^{\mu\rho}_{(0)}\Delta^{\beta\gamma}_{(0)})\nabla^{(0)}_\gamma\delta\omega_{\alpha\rho}, \tag{94}$$

$$\delta\Theta^{\mu\alpha\beta} = -\chi_4\beta_{(0)}[u^\beta_{(0)}u^\rho_{(0)}\Delta^{\alpha\delta}_{(0)} - u^\alpha_{(0)}u^\rho_{(0)}\Delta^{\beta\delta}_{(0)}]\Delta^{\mu\gamma}_{(0)}\nabla^{(0)}_\gamma\delta\omega_{\delta\rho} + \chi_5\beta_{(0)}\Delta^{\alpha\delta}_{(0)}\Delta^{\beta\rho}_{(0)}\Delta^{\mu\gamma}_{(0)}\nabla^{(0)}_\gamma\delta\omega_{\delta\rho}. \tag{95}$$

In Eq. (79), $\mathbf{q}^\mu$ appears with $S^{\alpha\beta}$. Since for global equilibrium $\mathbf{q}^\mu$ as well as $S^{\alpha\beta}$ vanishes, in $\delta S^{\mu\alpha\beta}_{(1)}$, $\delta\mathbf{q}^\mu$ does not contribute. The $0i$th component of Eq. (91) can be expressed as

$$\tilde{\chi}_b\partial_0\delta\omega^{0i} - \chi_1\beta_{(0)}\partial^i\partial_l\delta\omega^{0l} - \beta_{(0)}\chi_2\left[\partial_j\partial^j\delta\omega^{0i} + \partial_j\partial^i\delta\omega^{0j} - \frac{2}{3}\partial^i\partial_k\delta\omega^{0k}\right] - \beta_{(0)}\chi_3(\partial_j\partial^j\delta\omega^{0i} - \partial_j\partial^i\delta\omega^{0j}) - \beta_0\chi_4\partial_j\partial^j\delta\omega^{0i} = 0, \tag{96}$$

and the ijth component of Eq. (91) can be expressed as

$$\tilde{\chi}_s\partial_0\delta\omega^{ij} + \beta_{(0)}\chi_5\partial_l\partial^l\delta\omega^{ij} = 0. \tag{97}$$

In Eqs. (96) and (97), we define $\tilde{\chi}_b \equiv \partial S^{0i}/\partial\omega^{0i}$ and $\tilde{\chi}_s \equiv \partial S^{ij}/\partial\omega^{ij}$. $\beta_{(0)}$ denotes the inverse global equilibrium temperature. Equations (87), (88), (96), and (97) are the perturbation equations which we can analyze in the momentum space. From these equations, one may observe that perturbation in the standard hydrodynamic variables, i.e., $\delta\varepsilon$ and $\delta\pi^i$, are decoupled from the perturbation in the spin degree of freedom, i.e., δS^{0i}, δS^{ij} or $\delta\omega^{0i}$, $\delta\omega^{ij}$. This is the artifact of the following factors: (i) in this framework, the energy-momentum tensor is symmetric, (ii) the spin-chemical potential and spin density are both $\mathcal{O}(1)$ in the hydrodynamic gradient expansion, and (iii) we consider the global equilibrium with $S^{\mu\nu}_{(0)} = 0$ and $\omega^{\mu\nu}_{(0)} = 0$.

B. Fourier space equations

We look for the plane wave solution of the form $e^{-i\omega t + i\vec{k}\cdot\vec{x}}$. Various perturbations $\delta\varepsilon$, $\delta\pi^k$, $\delta\omega^{ij}$, and $\delta\omega^{0i}$ can be expressed as plane waves in the following way:

$$\begin{aligned}\delta\varepsilon &= \tilde{\delta\varepsilon}e^{-i\omega t + i\vec{k}\cdot\vec{x}},\\ \delta\pi^k &= \widetilde{\delta\pi}^k e^{-i\omega t + i\vec{k}\cdot\vec{x}},\\ \delta\omega^{ij} &= \widetilde{\delta\omega}^{ij}e^{-i\omega t + i\vec{k}\cdot\vec{x}},\\ \delta\omega^{0i} &= \widetilde{\delta\omega}^{0i}e^{-i\omega t + i\vec{k}\cdot\vec{x}}.\end{aligned} \tag{98}$$

Again, for the rotational symmetry of the system, one may consider a plane wave along the z direction, i.e., $\vec{k} \equiv (0, 0, k_z)$ [26]. For such a waveform, Eqs. (87), (88), (96), and (97) can be expressed as

$$-i\omega\widetilde{\delta\varepsilon} + ik_z\widetilde{\delta\pi^z} = 0, \tag{99}$$

$$-i\omega\widetilde{\delta\pi^x} + \gamma_\perp k_z^2\widetilde{\delta\pi^x} = 0, \tag{100}$$

$$-i\omega\widetilde{\delta\pi^y} + \gamma_\perp k_z^2\widetilde{\delta\pi^y} = 0, \tag{101}$$

$$-i\omega\widetilde{\delta\pi^z} + c_s^2 ik_z\widetilde{\delta\varepsilon} + \gamma_\parallel k_z^2\widetilde{\delta\pi^z} = 0, \tag{102}$$

$$-i\omega\tilde{\chi}_b\widetilde{\delta\omega^{0x}} - \beta_{(0)}k_z^2(\chi_2+\chi_3+\chi_4)\widetilde{\delta\omega^{0x}} = 0, \tag{103}$$

$$-i\omega\tilde{\chi}_b\widetilde{\delta\omega^{0y}} - \beta_{(0)}k_z^2(\chi_2+\chi_3+\chi_4)\widetilde{\delta\omega^{0y}} = 0, \tag{104}$$

$$-i\omega\tilde{\chi}_b\widetilde{\delta\omega^{0z}} - \beta_{(0)}k_z^2\left(\chi_1+\frac{4}{3}\chi_2+\chi_4\right)\widetilde{\delta\omega^{0z}} = 0, \tag{105}$$

$$-i\omega\tilde{\chi}_s\widetilde{\delta\omega^{kl}} + \chi_5\beta_{(0)}k_z^2\widetilde{\delta\omega^{kl}} = 0. \tag{106}$$

Equations (99)–(102) give us the standard hydrodynamic modes:

$$\omega = -i\gamma_\perp k_z^2 \quad (\text{two modes}), \tag{107}$$

$$\omega = \pm c_s k_z - \frac{i}{2}\gamma_\parallel k_z^2 + \mathcal{O}(k_z^3). \tag{108}$$

Similarly, the spin-wave modes associated with $\delta\omega^{0x}$ and $\delta\omega^{0y}$ are the same and can be given as

$$\omega = i\beta_{(0)}k_z^2\left(\frac{\chi_2+\chi_3+\chi_4}{\tilde{\chi}_b}\right). \tag{109}$$

The wave mode associated with $\delta\omega^{0z}$ can be expressed as

$$\omega = i\beta_{(0)}k_z^2\left(\frac{\chi_1+\frac{4}{3}\chi_2+\chi_4}{\tilde{\chi}_b}\right). \tag{110}$$

Finally, the wave modes associated with $\delta\omega^{kl}$ are

$$\omega = -i\frac{\chi_5}{\tilde{\chi}_s}\beta_{(0)}k_z^2. \tag{111}$$

Note that $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \tilde{\chi}_b$, and $\tilde{\chi}_s$ are all considered to be positive. Wave modes associated with ω^{0i} or S^{0i} have a positive imaginary part. Therefore, these modes are not stable, and the corresponding modes will not decay in time. Such growing modes are not physically appealing. Such modes can be removed if we use the Frenkel condition with $S^{\mu\nu}u_\nu = 0$. At the linear-order perturbation level, for global equilibrium with $S^{\mu\nu}_{(0)}$ the Frenkel condition gives us $\delta S^{\mu\nu}u_\nu^{(0)} = 0$. This implies $\delta S^{0i} = 0$ for the background flow with $u^\mu = (1,0,0,0)$. Interestingly, if we impose the Frenkel condition, then the problematic spin modes will no longer appear in the calculation. Moreover, this does not affect any standard hydrodynamic modes, as the standard hydrodynamic mode and the spin modes are decoupled in this framework.

IV. SUMMARY

We examine the solutions of the spin-hydrodynamic equations in Fourier space for two different first-order spin-hydrodynamic formulations. The first considers the spin-chemical potential $\omega_{\mu\nu}$ to be first order in gradient expansion [26], while the other is of leading order [29]. Our calculation suggests that unstable solutions for the spin S^{0i} component may emerge from spin-hydrodynamic equations. The Frenkel condition can be used to get rid of such generic instabilities. But when we consider the Frenkel condition along with $\omega^{\mu\nu} \sim \mathcal{O}(\partial)$, then the standard hydrodynamic modes also get affected. Such an unwanted feature is absent when we consider the case with $\omega^{\mu\nu} \sim \mathcal{O}(1)$. The stability of spin modes is not a settled issue in the literature, and some physical understanding may be required to construct a proper model of first-order spin hydrodynamics. One such possibility is to look into second-order spin-hydrodynamic formulation unless the first-order spin-hydrodynamic formulation can be shown to be stable for generic configurations [67,68]. In the context of standard hydrodynamic theory, it has been argued that instability can arise due to acausal modes. In this calculation, we have not studied the causality of linear modes. Moreover, the stability property has been considered at the linear level for an unpolarized background. It will be interesting to investigate the stability property of a generic polarized background. Finally, we mention that the stability studies shown in the present article are strictly valid in the fluid rest frame. Here, we have argued that $0i$th components (the boost degree of freedom) of spin-chemical potential are responsible for instability in linearized spin-hydrodynamic equations. With the Frenkel condition, these unstable modes can be removed in the fluid rest frame. In a boosted frame, the Frenkel condition mixes $0i$th and ijth components of spin-chemical potential. Hence, in the boosted frame, the Frenkel condition does not necessarily imply stability, and it requires further investigations. We emphasize here that spin modes can have instabilities different from standard hydrodynamic instabilities. Moreover, these instabilities seem to be very generic and intimately related to the spin equation of state (relation between spin density tensor and spin-chemical potential). To develop a phenomenologically appropriate model of spin hydrodynamics, we have to pay attention to these spin modes and the spin equation of state.

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