Causality and stability of relativistic spin hydrodynamics

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We study the causality and stability of relativistic hydrodynamics with the inclusion of the spin degree of freedom as a hydrodynamic field. We consider two specific models of spin hydrodynamics for this purpose. A linear mode analysis for static background shows that a first-order dissipative spin hydrodynamics remains acausal and admits instabilities. In addition, it is found that the inclusion of the spin field in hydrodynamics leads to new kinds of linear modes in the system. These new modes also exhibit instability and acausal behavior. The second model of the spin hydrodynamics that we have considered here is equivalent to a particular second-order conventional hydrodynamics with no dissipative effects. For a static background, it is found that the linear modes of this model support the sound waves only. However, when the background has constant vorticity, then the model admits instability and acausality in certain situations. It is found that the spin dynamics have an effect on the hydrodynamic response of the fluid. These findings point toward the need for a causal and stable theory with spin as a hydrodynamic field to describe the spin-polarized fluid.

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I. INTRODUCTION

Several new theoretical developments have taken place in relativistic dissipative hydrodynamics (see [1] for review), which is immensely successful in describing the data from nuclear collisions at relativistic energies [2–4]. Recently, invigorating efforts have been witnessed in the development of spin hydrodynamics [5–41] after the experimental measurement of the polarization of Λ hyperons [42,43]. In particular, it is required to know how the spin of the constituent particles is related to the fluid variables, like vorticity, symmetric gradients, or magnetic fields. The polarization of hadrons observed in noncentral collisions of heavy ions at the Relativistic Heavy Ion Collider (RHIC) at the high center of mass energies ($\sqrt{s_{NN}}$) [42,43] has been attributed to the transfer of orbital angular momentum of the fireball to the spin polarization through spin orbit coupling. However, the dependence of the local Λ spin polarization on the azimuthal angle in the transverse plane of collision observed by the STAR Collaboration [43,44] cannot be explained by hydrodynamical models based on local thermal vorticity [45–47]. The spin polarization as an independent relativistic hydrodynamic field was proposed as a possible solution to this problem, which has led to several new developments in the area of relativistic spin hydrodynamics. It was realized that also the shear stress of the fluid can give rise to spin polarization in addition to vorticity and temperature gradients [48,49]. Subsequently, it was shown that one can solve the sign problem of local Λ spin polarization by considering a possible effect of shear-induced polarization [48,50] at the constant temperature freeze-out hypersurface without incorporating any additional variables in hydrodynamics for the spin for modeling the evolution of the quark gluon plasma (QGP) phase. However, even with considering the shear-induced polarization, the (steepness of) variation of the component of the polarization along the direction of global angular momentum with azimuthal angel is not well reproduced [50]. Also the effect of polarization on the fluid dynamic evolution of QGP is not fully understood. There the incorporation of the spin density as a new field variable in the hydrodynamic setup remains relevant for understanding the spin polarization in the RHIC.

It must be emphasized that the inclusion of spin observables in hydrodynamics opens up an interesting possibility of developing a "classical" tool for studying the quantum effect in a many-body system like quark gluon plasma. Other new interesting developments are

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the chiral hydrodynamics [51,52] and the chiral vortical effects [53,54]. In condensed matter systems also hydrodynamics with spin observables have found many interesting applications (see [55] for a review).

The incorporation of the spin as a hydrodynamic field and its effect on the evolution of relativistic fluid is one of the most active areas of contemporary research [5,24,56]. The inclusion of spin in the general relativity is also a long-standing problem [57]. The evolution of the spin and other hydrodynamic fields (e.g., energy density, pressure, velocity, etc.) is governed by the conservation of the total angular momentum along with the other equations governing the conservations of energy-momentum and conserved charges (net electric charge, net baryonic charge, etc.). However, the definitions of the energy-momentum and spin tensors are not unique because of the presence of pseudo gauge transformation degrees of freedom [57]. Therefore, in spite of tremendous efforts, the formulation of relativistic dissipative spin hydrodynamics remains incomplete. One can obtain many different pairs of these tensors [57] through pseudogauge transformation. This ambiguity can be illustrated through the following situation: At the microscopic level, the energy-momentum tensor, defined for a system of particles with spin, can have symmetric and antisymmetric parts, where the antisymmetric part can be attributed to spin. Now with the help of pseudogauge transformation, one can define a new energy-momentum tensor [58], the Belinfante tensor, which is symmetric. Recently it has been shown in [7] that the entropy currents under this transformation are not equivalent in nonequilibrium situations. This is intriguing since this difference in expressions of entropy current imply that the physics of the two situations are not the same. Another interesting point of view was advanced in Ref. [8], where the authors demonstrate that the second-order conventional hydrodynamics is equivalent to spin hydrodynamics in the dissipationless limit. The demonstration, however, uses the pseudogauge transformations along with the suitable generalization of the currents associated with the entropy and number densities. However, due to this equivalence, one may think that perhaps one does not need to have spin hydrodynamics, since conventional hydrodynamics suffices, which needs to be investigated. Apart from that, we note that the energy-momentum tensor for the second-order conventional hydrodynamics contains contributions from the fluid vorticity [59–61]. But the inclusion of vorticity brings spin dynamics in the hydrodynamic theory since the presence of the finite vorticity in the system can be regarded as a source of spin polarization. In addition to that, the shear stress is also a source of spin polarization in a fluid. This points toward the requirement of a treatment, more than the conventional formulation, to account for the spin dynamics, with a spin density as an independent hydrodynamic field.

It is well known that the straightforward generalization of the Navier-Stokes (NS) equation to the relativistic domain is problematic because it admits acausal and unstable solutions [62]. It is also known that these issues can be remedied by incorporating second-order corrections to the NS equation [61] if certain conditions are satisfied. It is to be noted that this approach is not unique and there exists a variety of other approaches to address the issues related to the relativistic generalization of the NS equation [63]. In the present work, we systematically analyze the issues related to causality and instability in the spin hydrodynamics presented in Refs. [6,8]. The equations of spin hydrodynamics presented in Refs. [6,8] have very different structures and support different modes. In Ref. [32], it is shown that the causality for a particular kind of spin hydrodynamics can be restored only with a second-order term like the Israel-Stewart theory [61].

The paper is organized as follows: In the next section, we first briefly introduce the dissipative spin hydrodynamics equations, and for a simple initial state, we provide a linear mode analysis. In Sec. III, we briefly introduce the convention of second-order hydrodynamics and its equivalence with spin hydrodynamics in the dissipationless limit. This section also includes the linear mode analysis for the two initial states. The first case corresponds to the stationary fluid, while the second initial state has nonzero but constant vorticity in x and y directions. Section IV is devoted to the summary and discussions.

II. DISSIPATIVE SPIN HYDRODYNAMICS

A. Structure

There are several ways to obtain the equations of spin hydrodynamics. The methods based on effective field theory [30,31], the entropy current analysis approach [5], and the method of moments [20] were used to derive the equation of relativistic spin hydrodynamics. In the present work, we closely follow the approach adopted in Ref. [6]. The conventional way is to define the energy-momentum tensor $\Theta^{\mu\nu}$ and the conserved "currents" of the fluid under consideration. To incorporate spin within the hydrodynamic framework, one must consider the total angular momentum $J^{\mu\alpha\beta}$ as one of the conserved currents. The Noether current $J^{\mu\alpha\beta}$ associated with Lorentz transformation can be decomposed into spin and orbital angular momentum as follows:

$$J^{\mu\alpha\beta} = (x^{\alpha}\Theta^{\mu\beta} - x^{\beta}\Theta^{\mu\alpha}) + \Sigma^{\mu\alpha\beta}, \tag{1}$$

where $\Theta^{\mu\beta}$ is the canonical energy-momentum tensor (EMT), x^{α} is the space-time four-vector, and $\Sigma^{\mu\alpha\beta}$ is the spin tensor. The first term within the bracket on the right-hand side of Eq. (1) represents the contribution from the orbital angular momentum, which is conserved for symmetric $\Theta^{\mu\beta}$. All the dissipative fluxes that one may encounter in the formulation of dissipative hydrodynamics will be denoted with a prefix, Δ. Henceforth, the contribution from the gradients of hydrodynamic fields to $\Theta^{\mu\nu}$

will be denoted by $\Delta\Theta^{\mu\nu}$ and decomposed into symmetric ($\Delta\Theta_s^{\mu\nu}$) and antisymmetric ($\Delta\Theta_a^{\mu\nu}$) parts as follows:

$$\Delta\Theta^{\mu\nu} = \Delta\Theta_s^{\mu\nu} + \Delta\Theta_a^{\mu\nu}. \tag{2}$$

Both the symmetric and the antisymmetric parts of the canonical EMT contain information about the dissipation and transport coefficients. The mathematical form of $\Delta\Theta^{\mu\nu}$ can be determined with the help of the second law of thermodynamics. The second term on the right-hand side of Eq. (1) is the spin term, which arises due to the invariance of the underlying field under Lorentz transformation [6] and can be identified with the internal degrees of freedom. It is required that the spin term satisfy the condition $\Sigma^{\mu\alpha\beta} = -\Sigma^{\mu\beta\alpha}$.

The spin tensor can further be decomposed into two parts,

$$\Sigma^{\mu\alpha\beta} = S^{\alpha\beta}u^\mu + \Delta\Sigma^{\mu\alpha\beta}, \tag{3}$$

where $S^{\alpha\beta}$ is spin polarization density in the fluid rest frame and $\Delta\Sigma^{\mu\alpha\beta}$ is the spin dissipation. Moreover, the current density j^μ for conserved charges (baryonic charge for the system formed in the relativistic nuclear collision) can be written as

$$j^\mu = nu^\mu + n^\mu, \tag{4}$$

where n is the charge density at the fluid rest frame and n^μ is the charge diffusion, which vanishes in Eckart's choice of frame.

Next, one can write EMT for the fluid as

$$\Theta^{\mu\nu} = \Theta_o^{\mu\nu} + \Delta\Theta_s^{\mu\nu} + \Delta\Theta_a^{\mu\nu}, \tag{5}$$

where $\Theta_o^{\mu\nu}$ is the ideal part of the EMT, which is given by

$$\Theta_o^{\mu\nu} = \epsilon u^\mu u^\nu + \mathrm{P}\Delta^{\mu\nu}, \tag{6}$$

where ϵ, P, and u^μ denote energy density, pressure, and fluid four velocity of the fluid, respectively. The signature metric of the flat space-time is taken here as $g^{\mu\nu} = \mathrm{diag}(-,+,+,+)$ with all the nondiagonal components being zero such that the projection operator $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ satisfies the condition $\Delta^{\mu\nu}u_\mu = 0$. The velocity field u^μ satisfies the normalization condition $u^\mu u_\mu = -1$. The quantities P, ϵ, and n are related through the equation of state as $P = P(\epsilon, n)$.

Expressions for $\Delta\Theta_s^{\mu\nu}$ and $\Delta\Theta_a^{\mu\nu}$ can be decomposed as [6,64]

$$\Delta\Theta_s^{\mu\nu} = \Pi\Delta^{\mu\nu} + h^\mu u^\nu + u^\mu h^\nu + \pi^{\mu\nu}, \tag{7}$$

$$\Delta\Theta_a^{\mu\nu} = q^\mu u^\nu - u^\mu q^\nu + \phi^{\mu\nu}, \tag{8}$$

where the scalar Π, the vectors (h^μ and q^μ), and the rank-2 tensors ($\pi^{\mu\nu}$ and $\phi^{\mu\nu}$) are the dissipation fluxes. All the dissipative fluxes in the canonical EMT individually satisfy the transversality condition with respect to the hydrodynamical velocity u^μ given by $h^\mu u_\mu = q^\mu u_\mu = \Pi\Delta^{\mu\nu}u_\mu = \pi^{\mu\nu}u_\mu = u_\mu\phi^{\mu\nu} = 0$. The dissipation vector h^μ represents the contribution to the energy flow that does not depend on the spin polarization, while the vector q^μ describes the dissipation due to spin polarization. The tensor $\pi^{\mu\nu}$ is a symmetric traceless tensor representing the shear-stress tensor without any effect of the spin polarization, whereas $\phi^{\mu\nu}$ is an antisymmetric shear tensor describing the dissipation due to vorticity and spin polarization. The mathematical forms of the scalar, vector, and tensor dissipative fluxes can be constructed in terms of $g^{\mu\nu}$, the hydrodynamical fields, and the transport coefficients with the help of the second law of thermodynamics. The transport coefficients can be determined from the underlying microscopic theories.

The equations of motion of a relativistic fluid with spin degrees are given by

$$\partial_\mu\Theta^{\mu\nu} = 0, \tag{9}$$

$$\partial_\mu J^{\mu\alpha\beta} = 0, \tag{10}$$

$$\partial_\mu j^\mu = 0. \tag{11}$$

The second law of thermodynamics requires that the entropy current s^μ satisfies the following condition:

$$\partial_\mu s^\mu \geq 0. \tag{12}$$

From Eq. (10) and using the definition of total angular momentum [Eq. (1)], one gets the equation for spin dynamics as

$$\partial_\rho\Sigma^{\rho\mu\nu} = -2\Delta\Theta_a^{\mu\nu}. \tag{13}$$

This equation indicates that the evolution of the spin is governed by the antisymmetric part of the EMT.

Next by using Eqs. (9), (10) and (11) we obtain

$$D\epsilon = -(\epsilon + P)\theta + u_\nu\partial_\mu[\Delta\Theta^{\mu\nu}], \tag{14}$$

$$(\epsilon + P)Du^\mu = -\Delta^{\mu\nu}\partial_\nu P - \Delta_\nu^\mu\partial_\alpha\Delta\Theta^{\alpha\nu}, \tag{15}$$

$$DS^{\alpha\beta} = -S^{\alpha\beta}\theta - 2\Delta\Theta_a^{\alpha\beta} - \partial_\mu\Delta\Sigma^{\mu\alpha\beta}, \tag{16}$$

$$Dn = -n\theta, \tag{17}$$

where $D \equiv u^\mu\partial_\mu$ and $\theta \equiv \partial_\mu u^\mu$. The first law of thermodynamics is generalized to incorporate the spin density $S^{\mu\nu}$ as [6]

$$Tds = d\epsilon - \mu dn - \omega_{\mu\nu}dS^{\mu\nu}, \tag{18}$$

$$Ts = \epsilon + P - \mu n - \omega_{\mu\nu} S^{\mu\nu}, \tag{19}$$

where s, μ, and $\omega_{\mu\nu}$, respectively, denote entropy density, (baryonic) chemical potential, and the chemical potential corresponding to the spin tensor. This requires spin to be a conserved quantity. As described by Eq. (13), spin dynamics is governed by the antisymmetric part of the canonical EMT. Thus, the incorporation of spin degrees of freedom within a hydrodynamic framework requires that the relaxation time for spin density is longer than the mean free time related to the microscopic scattering of the fluid particles [6]. From the differential statement of the first law, one can write the space-time evolution of entropy density as

$$TDs = D\epsilon - \mu Dn - \omega_{\mu\nu} DS^{\mu\nu}. \tag{20}$$

In the presence of dissipative fluxes, one decomposes the entropy current as $s^\mu = su^\mu + \Delta s^\mu$ and the velocity projection requires that $\Delta s^\mu u_\mu = 0$ [6]. In order to apply the second law of thermodynamics, one takes divergence s^μ to get

$$\partial_\mu s^\mu = s\theta + Ds + \partial_\mu \Delta s^\mu. \tag{21}$$

Now, first, we eliminate Ds from Eq. (21) by using (20) and then use Eqs. (15)–(17) to obtain

$$\begin{aligned}\partial_\mu s^\mu = {}& \beta\theta\{Ts - (\epsilon + P) + \mu n + \omega_{\alpha\beta} S^{\alpha\beta}\} - \Delta\Theta^{\mu\nu}\partial_\mu(\beta u_\nu) \\ & - \Delta\Sigma^{\mu\alpha\beta}\partial_\mu(\beta\omega_{\alpha\beta}) + 2\beta\omega_{\alpha\beta}\Delta\Theta^{\alpha\beta}_{a} \\ & + \partial_\mu(\Delta s^\mu + \beta u_\nu \Delta\Theta^{\mu\nu} + \beta\omega_{\alpha\beta}\Delta\Sigma^{\mu\alpha\beta}),\end{aligned} \tag{22}$$

where $\beta = 1/T$ is the inverse temperature. In Eq. (22), the term within $\{\}$ vanishes due to the first law of thermodynamics given by Eq. (19). The last term on the right-hand side can be made zero by demanding

$$\Delta s^\mu = -\beta u_\nu \Delta\Theta^{\mu\nu} - \beta\omega_{\alpha\beta}\Delta\Sigma^{\mu\alpha\beta}. \tag{23}$$

It is straightforward to check that $\Delta s^\mu u_\mu = 0$. The mathematical forms of the scalar, vector, and tensor dissipative fluxes (Π, h^μ, q^μ, $\pi^{\mu\nu}$, and $\phi^{\mu\nu}$) appearing in Eqs. (7) and (8) are required to be constrained by the second law of thermodynamics. The appropriate form of these fluxes are found to be [6]

$$\begin{aligned}\Pi &= -\zeta\theta, \\ h^\mu &= -\kappa(Du^\mu + \beta\Delta^{\mu\rho}\partial_\rho T), \\ q^\mu &= -\lambda(-Du^\mu + \beta\Delta^{\mu\rho}\partial_\rho T - 4\omega^{\mu\nu}u_\nu), \\ \pi^{\mu\nu} &= -2\eta\Delta^{\mu\nu\alpha\rho}\partial_\alpha u_\rho, \\ \phi^{\mu\nu} &= -2\gamma\left[\frac{1}{2}(\Delta^{\mu\alpha}\partial_\alpha u^\nu - \Delta^{\nu\alpha}\partial_\alpha u^\mu) - \Delta^\mu_\rho\Delta^\nu_\lambda\omega^{\rho\lambda}\right], \\ \Delta\Sigma^{\mu\alpha\delta} &= -\chi_1\Delta^{\mu\rho}\partial_\rho(\beta\omega^{\alpha\delta}),\end{aligned} \tag{24}$$

where κ, η, and ζ, respectively, denote the coefficients of thermal conductivity, shear viscosity, and bulk viscosity, and the symmetric traceless projection normal to u^μ is defined as $\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2}(\Delta^\nu_\alpha\Delta^\mu_\beta + \Delta^\mu_\alpha\Delta^\nu_\beta - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta})$. The spin fields introduce two new transport coefficients, such as λ and γ. The coefficient λ is related with heat conduction associated with the new vector current q^μ, while coefficient γ is related with new stress tensor $\phi^{\mu\nu}$ generated due to the inclusion of spin in the hydrodynamics. The other unfamiliar transport coefficient χ_1 appears due to transport of the spin field. Moreover, q^μ gets a contribution from the spin potential $\omega^{\mu\nu}$. It is interesting to note that if one identifies $\omega^{\mu\nu}$ as a vorticity, then the spin stress $\phi^{\mu\nu}$ vanishes, but q^ν survives. Thus, the effect of spin polarization only remains in the vector current associated with q^ν.

Until now, no power counting scheme is assumed in the derivation of the fluxes with entropy that includes the effect of spin current. We will consider two schemes: (i) One as in Ref. [6], where gradients are taken as $\sim\mathcal{O}(\partial^1) = \delta_g$ and the spin chemical potential is taken as $\sim\delta$. In that case, for $\delta_g^2 \ll \delta_g \ll 1$, only $\Delta\Sigma^{\mu\alpha\delta} \equiv 0$ at first order and other fluxes remains intact in the first order in Eq. (22). The other scheme of the ordering of scale is for uniform high rotation, where the vorticity is on the order of $\delta_\omega \ll 1$ and other gradients are of different scales, but δ_ω is the highest relevant scale [8]. We discuss below how the dispersion of linear perturbations is shaped for these two types of the ordering of scales, both for the first-order spin hydrodynamics and the equivalent conventional second-order hydrodynamic theory.

B. Linear analysis

To understand the stability and causality issues, first we consider an equilibrium background with flow velocity $u^\mu_0 \equiv (-1, 0, 0, 0)$. Here the subscript 0 denotes the value of a physical quantity of the background on which perturbation is placed. In addition, the background is assumed to be static and homogeneous and values of spin polarization and spin-potential tensors are considered to be zero. The background equilibrium state is the same as the one considered in Ref. [6]. We use $\mathcal{Q}$ as the generic notation for the hydrodynamic field, with $\mathcal{Q}_0$ and $\delta\mathcal{Q}$ representing the mean values and fluctuation, respectively, where $\delta\mathcal{Q}$ is a function of space and time. In this scheme, one can write the perturbed velocity vector as $\delta u^\mu \equiv (0, \delta\mathbf{u})$. In the following, we consider the spin chemical potential of order $\sim\mathcal{O}(\partial^1)$, i.e., on the order of other gradients (of u^μ, T, μ). This allows us to keep the order of vorticity the same as the order of the derivatives of other perturbed quantities like δu^μ or δT. This power counting scheme is different than the one used in Sec. III following Ref. [8].

Here, first we note that, in the absence of any spin dynamics and conventional dissipation fluxes, only sound waves are supported in the linear perturbations scheme.

On retaining terms in linear order in perturbed quantities, we get the following set of equations:

$$0 = \frac{\partial \delta\epsilon}{\partial t} + h_0 \boldsymbol{\nabla}\cdot\boldsymbol{\delta u} - (\kappa-\lambda)\frac{\partial}{\partial t}(\boldsymbol{\nabla})\cdot\delta\boldsymbol{u} - \left(\frac{\kappa+\lambda}{T_0}\right)\nabla^2\delta T + 4\lambda\partial_i\delta\omega^{i0}, \tag{25a}$$

$$0 = (\kappa+\lambda)\frac{\partial^2\delta u^i}{\partial t^2} - h_0\frac{\partial\delta u^i}{\partial t} + (\eta+\gamma)\nabla^2\delta u^i + (\zeta+\eta/2-\gamma)\partial^i\boldsymbol{\nabla}\cdot\boldsymbol{\delta u} + \frac{(\kappa-\lambda)}{T_0}\frac{\partial\nabla^i\delta T}{\partial t} - \partial^i\delta P + 4\lambda\frac{\partial\delta\omega^{i0}}{\partial t} - 4\gamma\partial_l\delta\omega^{li}, \tag{25b}$$

$$0 = \frac{\partial\delta S^{0i}}{\partial t} + 8\lambda\delta\omega^{i0} - \frac{\chi_1}{T_0}\nabla^2\omega^{i0} + 2\lambda\frac{\partial}{\partial t}\delta u^i - \frac{2\lambda}{T_0}\partial^i\delta T, \tag{25c}$$

$$0 = \frac{\partial\delta S^{ij}}{\partial t} - 2\gamma(\partial^i\delta u^j - \partial^i\delta u^j - 4\delta\omega^{ij}) - \frac{\chi_1}{T_0}\nabla^2\omega^{ij}, \tag{25d}$$

$$0 = \frac{\partial\delta n}{\partial t} + n_0\boldsymbol{\nabla}\cdot\boldsymbol{\delta u}, \tag{25e}$$

where $h_0 = \epsilon_0 + P_0$ is the enthalpy density of the initial state. By setting $\delta\mathcal{Q} = \tilde{\delta\mathcal{Q}}\exp(-\omega t + i\boldsymbol{k}\cdot\boldsymbol{x})$, one can convert the above differential equations into a set of a linear homogeneous algebraic equations. It is useful to consider the projections along the unit wave vector $\hat{\boldsymbol{k}}$ to get the longitudinal modes and projection perpendicular to $\hat{\boldsymbol{k}}$ for obtaining the transverse modes. The following set of algebraic equations are obtained for longitudinal and transverse modes denoted by subscript p and t, respectively:

$$0 = \left[-\omega + \frac{k^2(\kappa+\lambda)}{T_0\epsilon_T}\right]\delta\epsilon + \left[\frac{k^2(\kappa+\lambda)\epsilon_n}{T_0\epsilon_T}\right]\delta n + ik[\omega T_0(\kappa-\lambda)+h_0]\delta u_p + 4ik\lambda\delta\omega_{p0}, \tag{26a}$$

$$0 = [\omega h_0 + \omega^2 T_0(\kappa+\lambda) - k^2(\zeta+\eta/2)]\delta u_p + ik\left[c_s^2 ... \right]$$

$$+ik\left[\frac{\omega(\kappa-\lambda)\epsilon_n}{T_0\epsilon_T}\right]\delta n - 4\lambda\omega\delta\omega_{p0}, \tag{26b}$$

$$0 = [\omega h_0 + \omega^2 T_0(\kappa+\lambda) - k^2(\gamma+\eta)]\delta \mathrm{u}_t - 4ik\gamma T_0\delta\omega_{\mathrm{pt}} - 4\lambda T_0\omega\delta\omega_{t0}, \tag{26c}$$

$$0 = \left[8\gamma - \omega\chi_s + \frac{\chi_1}{T_0}k^2\right]\delta\omega_{\mathrm{pt}} - 4i\gamma k\delta u_t, \tag{26d}$$

$$0 = \left[8\gamma - \omega\chi_b + \frac{\chi_1}{T_0}k^2\right]\delta\omega_{\mathrm{p0}} + \frac{2\delta\mathrm{e}\lambda(ik)}{C_V T_0} - \frac{2\delta\mathrm{n}(ik)\lambda\epsilon_n}{\epsilon_T T_0} + 2\lambda\omega\delta u_p, \tag{26e}$$

$$0 = \left[8\gamma - \omega\chi_b + \frac{\chi_1}{T_0}k^2\right]\delta\omega_{\mathrm{t0}} + 2\lambda T_0\omega\delta u_t, \tag{26f}$$

$$0 = -\omega\delta n + ikn_0\delta u_p, \tag{26g}$$

where $\chi_b = \frac{\partial S^{i0}}{\partial\omega^{i0}}$ and $\chi_s = \frac{\partial S^{ij}}{\partial\omega^{ij}}$, where i and j denote spatial indices [6]. Here the subscripts p and t, respectively, describe longitudinal and transverse parts. Further, we have used $\delta T = \frac{1}{\epsilon_T}\delta\epsilon - \frac{\epsilon_n}{\epsilon_T}\delta n$, where $\epsilon_T = \frac{\partial\epsilon}{\partial T}|_n$ and $\epsilon_n = \frac{\partial\epsilon}{\partial n}|_T$ in the above equations.

Since the equations for longitudinal and transverse parts are decoupled, one can treat them separately to obtain the dispersion relations for the linear mode. For the longitudinal part,

$$\mathbb{M}\mathcal{Q}_l = 0, \tag{27}$$

where

$$\mathcal{Q}_l = \begin{pmatrix} \delta\epsilon \\ \delta\mathrm{u}_p \\ \delta\omega_{\mathrm{p0}} \\ \delta n \end{pmatrix} \tag{28}$$

and

$$\mathbb{M} = \begin{pmatrix} \frac{k^2(\kappa+\lambda)}{\epsilon_T T_0} - \omega & ikh_0 + ik\omega(\kappa-\lambda) & 4ik\lambda & -\frac{k^2\epsilon_n(\kappa+\lambda)}{\epsilon_T T_0} \\ -ik\left(c_s^2 + \frac{\omega(\kappa-\lambda)}{\epsilon_T T_0}\right) & \omega h_0 + \omega^2(\kappa+\lambda) - k^2(\zeta + \frac{4\eta}{3}) & -4\lambda\omega & -\frac{ik\omega\epsilon_n(\kappa-\lambda)}{\epsilon_T T_0} \\ -\frac{2ik\lambda}{C_V T_0} & -2\lambda\omega & \omega\chi_b - \frac{k^2\chi_1}{T_0} + 8\lambda & \frac{2ik\lambda\epsilon_n}{\epsilon_T T_0} \\ 0 & ikn_0 & 0 & -\omega \end{pmatrix}. \tag{29}$$

The nontrivial solutions are obtained by setting $\mathbb{M} = 0$ leading to the following four roots and hence four dispersion relations for the longitudinal modes:

$$\omega_{1l} = (\kappa+\lambda)\frac{n_0\epsilon_n}{\epsilon_T T_0 h_0}k^2,$$
$$\omega_{2l} = \pm i c_s k + \left[\frac{(\zeta+\frac{4}{3}\eta)}{2h_0} + \lambda\left(\frac{1}{\epsilon_T T_0} + \frac{c_s^2}{h_0}\right) - \frac{\kappa n_0 \epsilon_n}{\epsilon_T T_0 h_0}\right]k^2,$$
$$\omega_{3l} = -\frac{8\lambda}{\chi_b} + \frac{\chi_1}{\chi_b T_0}k^2,$$
$$\omega_{4l} = -\frac{h_0}{(\kappa+\lambda)}. \tag{30}$$

The spin-transport coefficient λ associated with the heat conduction is seen to be contributing together with the conventional heat conduction characterized by coefficient κ. The acausal behavior seen in the NS equation can also be seen in the first equation. The parameter λ also contributes to giving instability together with the conventional heat conductivity κ. Further, it should be noted that λ and κ appear in the denominator of the unstable mode [fourth root in Eq. (30)]. In conventional first-order hydrodynamics, this kind of unstable mode was discussed in Ref. [65] and was regarded to be unphysical. Next, for finite baryon density, the sound mode mode ω_{2l} can be stable if the condition $[\frac{(\zeta+\frac{4}{3}\eta)}{2h_0} + \lambda(\frac{1}{\epsilon_T T_0} + \frac{c_s^2}{h_0}) \geq \frac{\kappa n_0 \epsilon_n}{\epsilon_T T_0 h_0}]$ is satisfied. If this condition is violated, then instability sets in as the conventional heat conduction can contribute toward increasing pressure and that may result in having an unstable mode. Interestingly, λ also contributes toward damping the sound modes described by ω_{2l}. In the absence of conventional heat conduction, i.e., $\kappa = 0$, the parameter λ can give damping of the sound wave. Finally, the third in Eq. (30) is a new mode that has no presence in conventional fluid dynamics. This mode can be unstable when $8\lambda > \frac{\chi_1}{T_0}k^2$. Here it may be noted that this mode can be made stable if one introduces a term $(\frac{S^{\alpha\beta}}{\tau_s})$ for the relaxation of $S^{\alpha\beta}$ in the left-hand side of Eq. (17), where τ_s is the spin-relaxation time. In addition, ω_{3l} can also exhibit an acausal behavior for sufficiently large values of wave vector k.

Similarly, the transverse parts in Eqs. (26a)–(26g) can be written as

$$\mathbb{M}_t \mathcal{Q}_t = 0, \tag{31}$$

where

$$\mathcal{Q}_t = \begin{pmatrix} \delta \mathrm{u}_t \\ \delta\omega_{\mathrm{pt}} \\ \delta\omega_{\mathrm{t0}} \end{pmatrix} \tag{32}$$

and

$$\mathbb{M}_t = \begin{pmatrix} \omega h_0 + \omega^2(\kappa+\lambda) - k^2(\gamma+\eta) & -4i\gamma k & -4\lambda\omega \\ -2i\gamma k & 8\gamma + \frac{k^2\chi_1}{T_0} - \omega\chi_s & 0 \\ -2\lambda\omega & 0 & \omega\chi_b - \frac{k^2\chi_1}{T_0} + 8\lambda \end{pmatrix} m. \tag{33}$$

By setting the determinant $\mathbb{M}_t = 0$, the following expressions for the dispersion relations of the transverse modes are obtained:

$$\omega_{1t} = \frac{(\gamma+\eta)}{h_0}k^2,$$
$$\omega_{2t} = -\frac{8\lambda}{\chi_b} + \frac{\chi_1}{\chi_b T_0}k^2,$$
$$\omega_{3t} = \frac{8\gamma}{\chi_s} + \frac{\chi_1}{T_0\chi_s}k^2,$$
$$\omega_{4t} = -\frac{h_0}{(\kappa+\lambda)}. \tag{34}$$

Just like the four longitudinal modes in Eq. (30), there are four transverse modes also. The modes represented by ω_{1t} and ω_{4t} have a combination of the conventional and spin-transport coefficients. It is to be noted that coefficient γ is associated with the traceless part of the anisotropic stress tensor $\phi^{\mu\nu}$ in Eq. (24), and therefore it appears together with the shear viscous coefficient η in ω_{1t}. The group velocity associated with ω_{1t} can exhibit acausal behavior. Such behavior is well known in the dispersion relation resulting from the relativistic NS equation (for example, see Ref. [64]). Those modes described by ω_{2t} and ω_{3t} are new and they have no analog in conventional hydrodynamics. The mode ω_{2t} is unstable if the condition $8\lambda > \frac{\chi_1}{T_0}k^2$ is satisfied. The expression for mode ω_{2t} is exactly similar to the longitudinal mode ω_{3l} and the instability associated with this mode can be regulated by introducing a spin-relaxation time. However, the group velocity associated with ω_{2t} can still become acausal for sufficiently high values of k. The mode ω_{3t} is stable, but it can have similar acausal behavior as ω_{2t}. However, the transport coefficient γ contributes toward giving a damping term that is independent of k. Finally, ω_{4t} gives an instability that has the same form as ω_{4l} and this mode has a counterpart in the conventional relativistic hydrodynamic theory.

Before we proceed to discuss the normal mode analysis in the nondissipative limit for the model discussed in Ref. [8], a few comments are in order. The new modes introduced by the inclusion of spin dynamics depend on the spin-transport coefficients $\gamma\lambda$ and χ_1. The instability arising due to λ can rather be controlled by introducing spin-relaxation time τ_s, provided the term with the relaxation time dominates over the term giving the instability. This point of view was also discussed in Ref. [6]. It might be possible to control the acausal behavior for modes ω_{3l}, ω_{2t}, and ω_{3t}. However, this may require an explicit calculation of the spin-transport coefficients. For example, the group velocity of mode ω_{2t} is $2\frac{\chi_1}{\chi_b T_0}k$. Therefore, even if the coefficient of k is small, the group velocity may still exceed the speed of light for large values of k. However, for the validity of the hydrodynamics, the upper limit of k is determined by its corresponding wavelength, $\lambda = 2\pi/k$ which should be larger than the mean free path of the particles. This requires the explicit calculation of spin-transport coefficients using a microscopic theory.

There are new modes in spin hydrodynamics that have no counterpart in the conventional limit; therefore, they are in no way equivalent, in general. This is a clear indication that, for a general case where vorticity can take any value and there could be other sources of spin polarization such as symmetric gradients and magnetic field, the modified conventional hydrodynamics may not be equivalent to spin hydrodynamics. This issue will be further discussed in Sec. III.

C. Instability and the heat flux

The term Du^ν appearing in the expression for heat flux can be replaced by using Eq. (25b) in favor of the spatial gradient of pressure and other terms first order in the derivative. Thus, if we keep only the first-order term in the heat flux with no time derivative of the fluid velocity, then it gets a correction from the first-order dissipation. We use Eqs. (15) and (21) to find the following form of heat fluxes:

$$h^\mu = \kappa\left[\frac{1}{P+\epsilon} - \frac{1}{P+\epsilon-\mu n}\right]\Delta^{\mu\nu}\partial_\nu P + \kappa\frac{1}{P+\epsilon-\mu n}P_n\Delta^{\mu\nu}\partial_\nu n + \mathcal{O}(\partial^2), \tag{35}$$

and

$$q^\mu = -\lambda\left[\frac{1}{P+\epsilon-\mu n} + \frac{1}{P+\epsilon}\right]\Delta^{\mu\nu}\partial_\nu P + \lambda\frac{1}{P+\epsilon-\mu n}P_n\Delta^{\mu\nu}\partial_\nu n + 4\lambda\omega^{\mu\nu}u_\nu + \mathcal{O}(\partial^2), \tag{36}$$

where $P_n = \frac{\partial P}{\partial n}|_T$. We have used $\partial^\mu T = \frac{1}{P_T}\partial^\mu P - \frac{P_n}{P_T}\partial^\mu n$, where $P_T = \frac{\partial P}{\partial T}|_n$ and $P_n = \frac{\partial P}{\partial n}|_T$. For the baryon free case, i.e., for $n = 0$ and $P_n = 0$, we have $h^\mu = 0 + \mathcal{O}(\partial^2)$ and $q^\mu = -2\lambda\frac{1}{P+\epsilon}\partial^\mu P + \mathcal{O}(\partial^2)$. This is the situation considered in Ref. [6]. We consider the general case with nonzero baryon density. In such a situation, the linearized equations become

$$0 = \frac{\partial\delta\epsilon}{\partial t} + h_0\boldsymbol{\nabla}\cdot\boldsymbol{\delta u} + c_s^2\left(\frac{\kappa-\lambda}{h_0} - \frac{\kappa+\lambda}{h_0-\mu_0 n_0}\right)\nabla^2\delta\epsilon + \frac{(\kappa+\lambda)P_n}{h_0-\mu_0 n_0}\nabla^2\delta n + 4\lambda\partial_i\delta\omega^{i0}, \tag{37a}$$

$$\begin{aligned}0 = &-h_0\boldsymbol{\nabla}\cdot\boldsymbol{\delta u} - h_0\frac{\partial\delta u^i}{\partial t} + (\eta+\gamma)\nabla^2\delta u^i + (\zeta+\eta/2-\gamma)\partial^i\boldsymbol{\nabla}\cdot\boldsymbol{\delta u}\\ &- c_s^2\left(\frac{\kappa+\lambda}{h_0} - \frac{\kappa-\lambda}{h_0-\mu_0 n_0}\right)\frac{\partial}{\partial t}\partial^i\delta\epsilon - \frac{(\kappa-\lambda)P_n}{h_0-\mu_0 n_0}\frac{\partial}{\partial t}\partial^i\delta n - \partial^i\delta P + 4\lambda\frac{\partial\delta\omega^{i0}}{\partial t} - 4\gamma\partial_l\delta\omega^{li},\end{aligned} \tag{37b}$$

$$0 = \frac{\partial\delta S^{0i}}{\partial t} + 8\lambda\delta\omega^{i0} - \frac{\chi_1}{T_0}\nabla^2\omega^{i0} - 2\lambda c_s^2\left(\frac{1}{h_0} + \frac{1}{h_0-\mu_0 n_0}\right)\partial^i\delta\epsilon + 2\lambda\frac{1P_n}{h_0-\mu_0 n_0}\partial^i\delta n, \tag{37c}$$

$$0 = \frac{\partial\delta S^{ij}}{\partial t} - 2\gamma(\partial^i\delta u^j - \partial^i\delta u^j - 4\delta\omega^{ij}) - \frac{\chi_1}{T_0}\nabla^2\omega^{ij}, \tag{37d}$$

$$0 = \frac{\partial\delta n}{\partial t} + n_0\boldsymbol{\nabla}\cdot\boldsymbol{\delta u}. \tag{37e}$$

In the above equations, putting perturbations as $\delta\mathcal{Q} = \delta\tilde{\mathcal{Q}}\exp(-\omega t + i\boldsymbol{k}\cdot\boldsymbol{x})$, we get

$$0 = \delta\mathrm{e}\left(-\left(k^2c_s^2\left(\frac{\kappa-\lambda}{h_0} - \frac{\kappa+\lambda}{h_0-\mu_0 n_0}\right)\right) - \omega\right) - \frac{\delta\mathrm{n}k^2(\kappa+\lambda)P_n}{h_0-\mu_0 n_0} + \delta\mathrm{u}_p(ikh_0) + 4i\lambda k\delta\omega_{\mathrm{p}0}, \tag{38a}$$

$$0 = -i\delta\mathrm{e}kc_s^2\left(1 - \omega\left(\frac{\kappa+\lambda}{h_0} - \frac{\kappa-\lambda}{h_0-\mu_0 n_0}\right)\right) + \delta\mathrm{u}_p\left(\omega h_0 - k^2\left(\zeta + \frac{4\eta}{3}\right)\right) + \frac{\delta\mathrm{n}(ik)\omega(\kappa-\lambda)P_n}{h_0-\mu_0 n_0} - 4\lambda\omega\delta\omega_{\mathrm{p}0}, \tag{38b}$$

$$0 = \delta \mathrm{u}_t(\omega h_0 - k^2(\gamma+\eta)) - 4i\gamma k \delta\omega_{\mathrm{pt}} - 4\lambda\omega\delta\omega_{\mathrm{t0}}, \tag{38c}$$

$$0 = -\delta\omega_{\mathrm{p0}}\left(-\omega\chi_b + \frac{k^2\chi_1}{T_0} - 8\lambda\right) - 2\delta \mathrm{e}(ik)\lambda c_s^2\left(\frac{1}{h_0-\mu_0 n_0} + \frac{1}{h_0}\right) + \frac{2\delta \mathrm{n}(ik)\lambda P_n}{h_0 - \mu_0 n_0}, \tag{38d}$$

$$0 = -\delta\omega_{\mathrm{t0}}\left(-\omega\chi_b + \frac{k^2\chi_1}{T_0} - 8\lambda\right), \tag{38e}$$

$$0 = \delta\omega_{\mathrm{pt}}\left(8\gamma + \frac{k^2\chi_1}{T_0} - \omega\chi_s\right) - 2i\gamma k\delta \mathrm{u}_t, \tag{38f}$$

$$0 = -\omega\delta n + ikn_0\delta u_p. \tag{38g}$$

Following the same procedure as earlier, we get the dispersion relations that are linear in transport coefficients for longitudinal and transverse modes. The longitudinal modes read

$$\begin{aligned}
\omega_{1l} &= (\kappa+\lambda)\frac{n_0\epsilon_n}{\epsilon_T T_0 h_0}k^2, \\
\omega_{2l} &= \pm i c_s k + \left[\frac{(\zeta+\frac{4}{3}\eta)}{2h_0} + \frac{\lambda}{\epsilon_0 - \mu_0 n_0 + P_0}\left(2c_s^2 - \frac{n_0(\mu_0 c_s^2 + P_n)}{h_0}\right)\right]k^2, \\
\omega_{3l} &= -\frac{8\lambda}{\chi_b} + \frac{\chi_1}{\chi_b T_0}k^2,
\end{aligned} \tag{39}$$

and we have for the transverse modes

$$\begin{aligned}
\omega_{1t} &= -\frac{8\lambda}{\chi_b} + \frac{\chi_1}{\chi_b T_0}k^2, \\
\omega_{2t} &= \frac{(\gamma+\eta)}{h_0}k^2, \\
\omega_{3t} &= \frac{8\gamma}{\chi_s} + \frac{\chi_1}{T_0\chi_s}k^2.
\end{aligned} \tag{40}$$

We find that there are no unstable modes of the form $\omega = -\frac{h_0}{(\kappa+\lambda)}$ [see the last equation in Eqs. (30)]. The appearance of this mode can be understood from the part $(\kappa+\lambda)\frac{\partial^2 \delta u^i}{\partial t^2} - h_0\frac{\partial \delta u^i}{\partial t}$ of Eq. (25b), which gives $(\omega h_0 + \omega^2 T_0(\kappa+\lambda))$ in the coefficient of δu_p in Eq. (26b). $(\omega h_0 + \omega^2 T_0(\kappa+\lambda)) = 0$ gives $\omega = -\frac{h_0}{\kappa+\lambda}$. The term $(\kappa+\lambda)\frac{\partial^2\delta u^i}{\partial t^2}$ originates in the equation through the expression of heat flux in Eq. (24), where already a time derivative of velocity appears on the right-hand side. The unstable mode is found to disappear if the time derivative in the expression of heat fluxes [Eq. (24)] is replaced by terms up to first order in gradients in hydrodynamic fields by using Eq. (15). This unstable mode is there without spin field [62]; here the presence of spin adds to that through its contribution to heat flux through q^μ (or λ). So the source of instability found by Lindblom and Hiscock [62] is due to the presence of a second-order correction entering in first-order hydrodynamics through the expression of heat flux, which contains time variation of fluid velocity (Du^μ). The second-order effect in heat fluxes comes through Du^μ because of its dependence on gradients of the dissipative fluxes through the velocity equations, gradients of the first-order dissipative fluxes being second order. Since the instability is related to the expression of heat flux, it can be removed by redefining heat fluxes, as already shown for the spinless case in Ref. [63]. However, there may be unstable modes due to the presence of the spin polarization, $\omega_{1t} = -\frac{8\lambda}{\chi_b} + \frac{\chi_1}{\chi_b T_0}k^2$. At first order, the term due to the spin dissipation is dropped by considering it second order. Now if the spin potential is first order itself, then $\omega_{1t} = -\frac{8\lambda}{\chi_b}$ is always unstable when the contribution to the heat flux from the spin potential is nonzero at first order in the gradients of other hydrodynamic fields. Of course, this mode is unstable only if $\chi_b > 0$, i.e., if the direction of spin potential is along the spin polarization. The sign dependence of χ_b on charges, helicity, and chirality of particles will then enable the separation of the contribution from opposite charges. However, if the spin potential gets a contribution from the zeroth order, then even for species that give an unstable contribution, the modes with $|k| < 2\sqrt{\frac{2\lambda T_0}{\chi_1}}$ are stable. It is to be noted from the expression of heat fluxes q^μ [Eq. (24)] in Eqs. (38c)–(38e) that this unstable linear mode vanishes when (i) the spin potential satisfies $\omega^{\mu\nu}u_\mu = 0$ and/or (ii) $\lambda = 0$. i.e., when the contribution to the heat flux from the spin potential vanishes at first order in the gradients of other hydrodynamic fields [66].

Now, the question of which form of the heat fluxes are to be used in the first-order theory to avoid instability developed at $\omega = -\frac{h_0}{(\kappa+\lambda)}$ arises. One may argue that replacement of the time derivative of the velocity by first-order spatial derivative of the hydrodynamic field by using Eq. (15) in heat flux is a good remedy for it. We note that the form of heat flux that contains the time derivative of the fluid velocity comes from the positivity of four divergences of the entropy current, and it contains corrections from the first order of the dissipative fluxes through Eq. (15). Though the contribution to the entropy is first order in dissipative fluxes [Eq. (23)], the dissipative fluxes are not restricted by this condition that it is to be first order in the gradients of hydrodynamic fields due to the presence of the time derivative of the fluid velocity in the form of fluxes. The timescale of growth of this instability is $t \sim \omega^{-1} \sim \frac{(\kappa+\lambda)}{h_0}$. So for smaller κ and λ, this may be very short, which means that, in a very short time, the contribution from the second order would grow to lead to instability. So the truncation of higher-order effects in heat fluxes may not be applicable in that situation. For the general situation, this demands a consistent second-order theory. The instability of first-order theory is tamed only when the contribution of dissipation on the time variation of fluid velocity (or acceleration) is negligible compared to what it gets from the pressure gradients.

Another important issue to note is that in two situations the contribution of conductivities in the dissipation of sound modes is different. However, in both cases, the dissipation of sound gets a contribution from the new transport coefficient (λ) due to the spin polarization. For certain values of n_0, this contribution may lead to growth also and the condition for the growth is different for a different form of heat fluxes. There is always a contribution from spin polarization in the transverse modes through γ. So the spin polarization affects the dissipation in the system.

III. EQUIVALENCE OF SPIN HYDRODYNAMICS WITH SECOND-ORDER THEORY IN NONDISSIPATIVE LIMIT

Next, we consider the stability analysis of the spin hydrodynamics in the dissipationless limit discussed in Ref. [8]. As we have discussed before, in Ref. [8] it was shown that the inclusion of spin variable in the relativistic hydrodynamical framework in the nondissipative limit is equivalent to the conventional hydrodynamics with the second-order corrections. The dissipationless limit requires that entropy current s^μ satisfies $\partial_\mu s^\mu = 0$. In the previous section, we have seen how the new dissipative fluxes arise due to the inclusion of spin variable and how they contribute to some of the known problems related to the relativistic Navier-Stokes theory [62,64]. Thus, it would be interesting to check if similar issues still persist in the nondissipative limit or not.

In the following, first, we discuss how the structure of the equivalent second-order theory in Ref. [8] can resemble the spin hydrodynamics in its pseudogauge transformed form. For the symmetric Belinfante-Rosenfeld EMT with pseudogauge transformation with the choice of gauge to be $S^{\alpha\mu\nu} = \Sigma^{\alpha\mu\nu}$ [35,57,69], we have

$$
\begin{aligned}
T^{\mu\nu} &= \Theta^{\mu\nu} + \frac{1}{2}\partial_\alpha(S^{\alpha\mu\nu} - S^{\mu\alpha\nu} - S^{\nu\alpha\mu}) \\
&= \Theta^{\mu\nu} + \frac{1}{2}\partial_\alpha(\Sigma^{\alpha\mu\nu} - \Sigma^{\mu\alpha\nu} - \Sigma^{\nu\alpha\mu}) \\
&= \frac{1}{2}(\Theta^{\mu\nu} + \Theta^{\nu\mu}) - \frac{1}{2}\partial_\alpha(\Sigma^{\mu\alpha\nu} + \Sigma^{\nu\alpha\mu}) \\
&= eu^\mu u^\nu + \mathrm{P}\Delta^{\mu\nu} + \Pi\Delta^{\mu\nu} + h^\mu u^\nu + u^\mu h^\nu + \pi^{\mu\nu} - \frac{1}{2}\partial_\alpha(u^\mu S^{\alpha\nu} + u^\nu S^{\alpha\mu}) - \frac{1}{2}\partial_\alpha(\Delta\Sigma^{\mu\alpha\nu} + \Delta\Sigma^{\nu\alpha\mu}) \\
&= eu^\mu u^\nu + \mathrm{P}\Delta^{\mu\nu} + \Pi\Delta^{\mu\nu} + h^\mu u^\nu + u^\mu h^\nu + \pi^{\mu\nu} - \frac{1}{2}(\partial_\alpha u^\mu)S^{\alpha\nu} - \frac{1}{2}(\partial_\alpha u^\nu)S^{\alpha\mu} \\
&\quad - \frac{1}{2}(u^\mu\partial_\alpha S^{\alpha\nu} + u^\nu\partial_\alpha S^{\alpha\mu}) - \frac{1}{2}\partial_\alpha(\Delta\Sigma^{\mu\alpha\nu} + \Delta\Sigma^{\nu\alpha\mu}) \\
&= eu^\mu u^\nu + \mathrm{P}\Delta^{\mu\nu} + \left(\Pi - \frac{1}{6}\Delta_{\lambda\rho}\partial_\alpha(\Delta\Sigma^{\lambda\alpha\rho} + \Delta\Sigma^{\rho\alpha\lambda})\right)\Delta^{\mu\nu} + \left(h^\mu - \frac{1}{2}\partial_\alpha S^{\alpha\mu}\right)u^\nu \\
&\quad + u^\mu\left(h^\nu - \frac{1}{2}\partial_\alpha S^{\alpha\nu}\right) + \pi^{\mu\nu} - \frac{1}{2}\Delta^{\mu\nu}_{\lambda\rho}\partial_\alpha(\Delta\Sigma^{\lambda\alpha\rho} + \Delta\Sigma^{\rho\alpha\lambda}) - \frac{1}{2}(\partial_\alpha u^\mu)S^{\alpha\nu} - \frac{1}{2}(\partial_\alpha u^\nu)S^{\alpha\mu}.
\end{aligned}
\tag{41}
$$

In the nondissipative limit, the dissipative tensor related to viscosity $\Pi\Delta^{\mu\nu}$, $\pi^{\mu\nu}$, and spin $\Delta\Sigma^{\alpha\mu\nu}$ are zero, while the heat flux h^μ still has a nondissipative contribution due to the vorticity driven thermal Hall effect [8]. Here we have used $\Sigma^{\alpha\mu\nu} = u^\alpha S^{\mu\nu}$. The term $\frac{1}{2}(\partial_\alpha u^\mu)S^{\alpha\nu} + \frac{1}{2}(\partial_\alpha u^\nu)S^{\alpha\mu}$ can be decomposed as a combination that contains $\Delta^{\mu\nu}S^{\lambda\rho}\omega_{\lambda\rho}$ and $S^\mu_\lambda\omega_{\lambda\nu}$. From the above equation, it is clear that if $S^{\alpha\nu}$ is connected to the vorticity as $S^{\alpha\nu} = \chi\omega^{\alpha\nu}$ [8], the EMT looks like that of a second-order theory that contains a second-order derivative in the expansion of the EMT in field gradients.

The above pseudogauge transformation makes the spin tensor disappear from the total angular momentum since the transformed spin tensor is $\tilde{\Sigma}^{\alpha\mu\nu} = \Sigma^{\alpha\mu\nu} - S^{\alpha\mu\nu}$. It is easy to check that $\partial_\mu T^{\mu\nu} = 0$, using the identities [7]

$$\begin{aligned} &\partial_\mu\partial_\alpha(\Sigma^{\alpha\mu\nu} - \Sigma^{\mu\alpha\nu} - \Sigma^{\nu\alpha\mu}) = 0, \\ \text{or}\quad &\partial_\mu\partial_\alpha(u^\alpha S^{\mu\nu} + u^\nu S^{\nu\alpha} + u^\nu S^{\mu\alpha}) = 0, \end{aligned} \tag{42}$$

as $S^{\mu\nu}$'s are antisymmetric in its indices.

A. Structure of the equivalent second-order theory

If the vorticity is the predominant gradient in the system, where other dissipative gradients that are responsible for the transport are very small, for highly rotating fluid, with the vorticity $\omega_{\mu\nu} = \frac{1}{2}(\Delta^\alpha_\mu\partial_\alpha u_\nu - \Delta^\alpha_\nu\partial_\alpha u_\mu)$, the symmetric energy-momentum tensor and the conserved charge current of a parity-even plasma is written as [8]

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \Delta T^{\mu\nu}, \tag{43}$$

$$\Delta T^{\mu\nu} = a_0\Delta^{\mu\nu}\omega^{\lambda\rho}\omega_{\lambda\rho} + a_1\omega^\mu_\lambda\omega_{\lambda\nu}, \tag{44}$$

$$J^\nu = nu^\nu + \Delta J^\nu, \tag{45}$$

$$\Delta J^\mu = c_1\Delta^\mu_\rho\partial_\nu\omega^{\nu\rho} + c_2\omega^{\mu\nu}\partial_\nu\beta, \tag{46}$$

where a_0, a_1, c_1, and c_2 are second-order transport coefficients. For ideal evolution ($\partial_\mu s^\mu = 0$), these transport coefficients are related [8]. The assumption behind the structure of the theory is that the vorticity is the dominating scale over other gradients in the theory. In certain cases, this can be a physical situation, since for a uniform rotation, the vorticity can have arbitrarily high values without entropy generation in the system. However, in general, the local vorticity can have a wide range of values and the gradient appearing through the vorticity can be larger with significant entropy production. So the assumption of the above theory is rather valid for a specific situation of high rotation with a lower gradient appearing in the vorticity. The scales are as follows: for the vorticity $\omega^{\mu\nu} \sim \delta_\omega$, with symmetric gradient $\theta^{\mu\nu} = \frac{1}{2}(\Delta^\alpha_\mu\partial_\alpha u_\nu + \Delta^\alpha_\nu\partial_\alpha u_\mu) \sim \partial^\perp_\mu\alpha \sim \delta$, $\partial^\perp_\mu\beta \sim \delta'$ and spatial derivative of $\omega^{\mu\nu}$, β brings extra δ' such that $\partial^\perp_\mu\omega^{\mu\nu} \sim \delta'\delta_\omega$, $\partial^\perp_\mu\partial^\perp_\nu\beta \sim \delta'^2$, whereas for spatial derivative of $\theta^{\mu\nu}$ and α, extra δ appear: $\partial^\perp_\mu\theta^{\mu\nu} \sim \partial^\perp_\mu\partial^\perp_\nu\alpha \sim \delta^2$, where $\alpha = \mu/T$ and $\partial^\perp_\mu = \Delta^\rho_\mu\partial_\rho$. The assumption for the above theory in terms of these scales is given by

$$\delta'^2 \ll \delta \ll \delta_\omega\delta' \ll \delta^2_\omega \ll \delta' \ll \delta_\omega \ll 1. \tag{47}$$

The energy-momentum conservation equation ($\partial_\mu T^{\mu\nu} = 0$) and $\partial_\mu J^\mu = 0$ can be written as

$$D\epsilon + (\epsilon + P)\theta + a_0\theta(\omega^{\lambda\rho}\omega_{\lambda\rho}) - a_1\omega^\mu_\lambda u_\nu\partial_\mu\omega_{\lambda\nu} = 0, \tag{48}$$

$$\begin{aligned} &(\epsilon + P)Du^\alpha + \Delta^{\alpha\mu}\partial_\mu P + a_0(Du^\alpha)\omega^{\lambda\rho}\omega_{\lambda\rho} \\ &\quad + a_0\Delta^{\alpha\mu}\partial_\mu(\omega^{\lambda\rho}\omega_{\lambda\rho}) + a_1\partial^\alpha_\nu(\omega^\mu_\lambda\omega_{\lambda\nu}) = 0, \end{aligned} \tag{49}$$

$$n\theta + Dn + \partial_\mu\Delta J^\mu = 0. \tag{50}$$

If we linearize the theory around a static equilibrium, where the background quantities are independent of space-time, as considered in Sec. II, then the contribution from the second-order terms vanishes in the linearized form and, consequently, we have

$$\frac{\partial}{\partial t}\delta\epsilon + (h_0)\delta\theta = 0, \tag{51}$$

$$(h_0)\frac{\partial}{\partial t}u^\alpha + \Delta^{\alpha\mu}\partial_\mu\delta P = 0, \tag{52}$$

$$n_0\delta\theta + \frac{\partial}{\partial t}\delta n = 0. \tag{53}$$

If we consider the perturbation of the form $\delta Q = \tilde{\delta Q}\exp(-\omega t + i\boldsymbol{k}\cdot\boldsymbol{x})$, then these lead to ideal and stable propagation of perturbations with only longitudinal propagating modes, $\omega_{2l} = \pm ic_s k$. This supports only sound waves and the transport coefficients introduced for the ideal (nondissipative) hydrodynamics do not contribute to the linear modes for the given choice of the background with no vorticity. Here we note that if the background has finite vorticity, then the new transport coefficients in this dissipationless limit may contribute to the dispersion relation. Now let us investigate whether the first-order spin hydrodynamics as discussed in Ref. [6] gives the same dispersion in this order of scaling. If we put the same order of scaling as in Eq. (47), with the spin chemical potential tensor being the vorticity and it is the dominating order, then the spin hydrodynamics also has no dissipation and we have only $\omega_{2l} = \pm ic_s k$, since then all the dissipative fluxes are absent at that order, and the structure of the EMT of ideal spin hydrodynamics becomes $\Theta^{\mu\nu} = \epsilon u^\mu u^\nu + \mathrm{P}\Delta^{\mu\nu}$ without any contribution from vorticity at all. Thus the first-order spin hydrodynamics becomes ideal for the scheme of ordering mentioned in Eq. (47) which does not carry problems of causality and stability.

However, if the spin chemical potential, though being of the same order as the vorticity, is not identical to it (which is the case in a general situation, since the symmetric shear and the magnetic field can also be the cause of spin polarization), then the surviving dissipative fluxes from Eq. (24) are $q^\mu \equiv 4\lambda T\omega^{\mu\nu}u_\nu$ and $\phi^{\mu\nu} = -2\gamma[\frac{1}{2}(\Delta^{\mu\alpha}\partial_\alpha u^\nu - \Delta^{\nu\alpha}\partial_\alpha u^\mu) - \Delta^\mu_\rho\Delta^\nu_\lambda\omega^{\rho\lambda}]$. In that case, the linear analysis around the static background gives the longitudinal modes linear in transport coefficients, $\omega_{1li} = \pm ikc_s$ and $\omega_{2li} = \frac{8\gamma}{\chi_b}$ and transverse modes linear in transport coefficients, $\omega_{1ti} = \frac{8\gamma}{\chi_b}$ and $\omega_{2ti} = \frac{\pm\sqrt{(8\gamma\epsilon_0+\gamma k^2\chi_s+8\gamma P_0)^2-32\gamma^2k^2(-\epsilon_0\chi_s-P_0\chi_s)}+8\gamma\epsilon_0+\gamma k^2\chi_s+8\gamma P_0}{2(\epsilon_0\chi_s+P_0\chi_s)}$, which give acausal diffusion. In such situations, these modes have no counterpart in the conventional equivalent hydrodynamics. Here we would like to note that it is possible that the hierarchy described by Eq. (47) may not be satisfied in a more general situation. For example, when the Reynolds number is not very large, it is likely that the dissipative fluxes (related to the spin degree of freedom also) will play a dominant role. The inclusion of such dissipative fluxes may lead to the unphysical behavior that we have already discussed above. Further, it is not clear in this situation how the equivalence between the conventional second-order fluid theory and the spin hydrodynamics can be established. Another instance when the hierarchy is not respected is $\delta_\omega \ll \delta'$. In this case, too, the conventional second-order fluid dynamics and the spin hydrodynamics in the ideal limit may not be equivalent.

However, it is important to note that, when the above hierarchy [Eq. (47)] is respected, in the dissipationless limit, the spin hydrodynamics is equivalent to the conventional fluid theory with the second-order corrections as established in Ref. [8]. This equivalence allows one to have the convenience of choosing from either of the models of equivalent hydrodynamics. So far, we have considered the background fluid state without any vorticity. Since the second-order corrections in the equivalent conventional theory are dependent on vorticity, it is of interest to consider a linear stability analysis with the background having nonzero vorticity. In the following, we consider such a case. Such analysis also will help to understand whether, in this prescription of scales, the hydrodynamics will always be causal and stable or not. In the following, we investigate the dispersion structure of the spin hydrodynamics with ideal evolution as given in Ref. [8].

B. Nondissipative evolution in a uniformly rotating background

The ideal counterpart of the spin hydrodynamic energy-momentum tensor can be written as in Ref. [8],

$$\Theta^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} + h^\nu u^\mu + h^\mu u^\nu - \frac{1}{2}\partial_\alpha\Sigma^{\alpha\mu\nu},$$

$$\text{with}\quad h^\mu = \frac{\chi}{2\beta}\omega^{\mu\nu}\partial_\nu\beta,$$

$$\text{and}\quad \Sigma^{\alpha\mu\nu} = S^{\mu\nu}u^\alpha. \tag{54}$$

The h^ν given above vanishes at first order, for static background. To have nonzero h^ν at first order, we consider a rotating background with background equilibrium fluid velocity profile,

$$u_0^\mu = (-1, 0, 0, v_z),$$
$$v_z = \frac{v_0}{L}(y - x), \tag{55}$$

and we consider $\frac{v_0}{L}$ to be very small (such that v_z can be treated in first-order perturbation).

Then, with $S^{\mu\nu} = \chi\omega^{\mu\nu}$, the linearized conservation equations become

$$0 = D_0\delta\epsilon + h_0\nabla\cdot\delta\boldsymbol{u} + \frac{\chi v_0}{2L}(\partial_t\partial_z)(\delta u^y - \delta u^x) + v_z\frac{\chi}{4}\partial_t(\partial_x^2 + \partial_y^2 + \partial_z^2)\delta u^z, \tag{56}$$

$$0 = h_0 D_0\delta u^i + v_z\delta^{zi}\frac{\partial}{\partial t}\delta P + \partial^i\delta P + \frac{\chi}{2}(\partial_t^2)\delta\omega^{0i} + \frac{\chi}{2}(\partial_t\partial_l)\delta\omega^{li} + \frac{\chi}{2}\omega_0^{li}\partial_l\nabla\cdot\delta\boldsymbol{u} - \delta^{iz}\left(\frac{v_0}{L}\right)(\delta h^x - \delta h^y) - \frac{\partial}{\partial t}\delta h^i, \tag{57}$$

where $\delta\omega^{0i} = -\frac{1}{2}v_z\partial_z\delta u^i - \frac{1}{2}\delta^{iz}\delta u^l\partial_l v_z$ and $\delta\omega^{ij} = \frac{1}{2}(\partial^i\delta u^j - \partial^j\delta u^i) + \frac{v_z}{2}(\delta^{iz}\partial_t\delta u^j - \delta^{jz}\partial_t\delta u^i)$.

We have in ω-k space, with $\delta Q = \delta\tilde{Q}e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$, where Q stands for hydrodynamic fields (it is to be noted that before this we considered the perturbations to be of the form $\delta Q = \delta\tilde{Q}e^{-\omega t + \boldsymbol{k}\cdot\boldsymbol{x}}$. So from here onward, the real part of ω would correspond to (oscillatory or) wave mode,

$$\begin{aligned}0 = {}& \delta\epsilon\left(\frac{k_z(v_0\chi\omega)}{4LT_0\epsilon_T} + ic_s^2k_x\right) - \frac{\delta\mathrm{n}(k_z(v_0\chi\omega\epsilon_n))}{4LT_0\epsilon_T} + \delta\mathrm{u}_x\left\{\frac{1}{4}\chi\left(-\frac{v_0k_xk_z}{L} + i\omega(k_y^2 + k_z^2)\right)\right.\\ & \left. - ih_0(\omega - k_zv_z)\right\} - \frac{1}{4}\delta\mathrm{u}_y\left(\chi\left(\frac{v_0k_yk_z}{L} + i\omega k_xk_y\right)\right) - \frac{1}{4}\chi\delta\mathrm{u}_z\left(\frac{v_0k_z^2}{L} + i\omega k_xk_z\right),\end{aligned}$$

$$
\begin{aligned}
0 &= \delta\epsilon\left(-\frac{k_z(v_0\chi\omega)}{4LT_0\epsilon_T} + ic_s^2k_y\right) + \frac{\delta \mathrm{n}(k_z(v_0\chi\omega\epsilon_n))}{4LT_0\epsilon_T} + \delta \mathrm{u}_y\left\{\frac{1}{4}\chi\left(\frac{v_0k_yk_z}{L} + i\omega(k_x^2+k_z^2)\right)\right. \\
&\quad \left. - ih_0(\omega - k_zv_z)\right\} - \frac{1}{4}\delta \mathrm{u}_x\left(\chi\left(-\frac{v_0k_xk_z}{L} + i\omega k_xk_y\right)\right) - \frac{1}{4}\chi\delta \mathrm{u}_z\left(-\frac{v_0k_z^2}{L} + i\omega k_yk_z\right), \\
0 &= -i\delta\epsilon c_s^2(\omega v_z - k_z) + \delta \mathrm{u}_z\left(-ih_0(\omega - k_zv_z) - \frac{k_z(v_0\chi)(k_x-k_z)}{4L} + \frac{1}{2}i\chi\omega^2k_zv_z + \frac{1}{4}i\chi\omega k_x^2\right) \\
&\quad + \delta \mathrm{u}_x\left(-\frac{k_x(v_0\chi)(k_x-k_y)}{4L} + \frac{1}{4}i\chi\omega^2k_xv_z + \frac{1}{2}i\chi\omega k_xk_z - \frac{\omega^2(v_0\chi)}{2L}\right) \\
&\quad + \delta \mathrm{u}_y\left(-\frac{k_y(v_0\chi)(k_x-k_y)}{4L} + \frac{1}{4}i\chi\omega^2k_yv_z + \frac{1}{2}i\chi\omega k_yk_z + \frac{\omega^2(v_0\chi)}{2L}\right), \\
0 &= \delta \mathrm{u}_x\left(-\frac{v_0\chi\omega k_z}{2L} + (e_0+p_0)(ik_x)\right) + \delta \mathrm{u}_y\left(\frac{v_0\chi\omega k_z}{2L} + (e_0+p_0)(ik_y)\right) \\
&\quad + \delta \mathrm{u}_z\left((e_0+p_0)(ik_z) + \frac{1}{4}(ik)k\chi\omega v_z\right) - i\delta \mathrm{e}(\omega - k_zv_z), \\
0 &= n_0(ik_j)\delta \mathrm{u}^j - i\delta \mathrm{n}(\omega - k_zv_z),
\end{aligned} \tag{58}
$$

where $\epsilon_n = \frac{\partial\epsilon}{\partial n}|_T$. We have used $\partial^\mu T = \frac{1}{\epsilon_T}\partial^\mu e - \frac{\epsilon_n}{\epsilon_T}\partial^\mu n$, where $\epsilon_T = \frac{\partial\epsilon}{\partial T}|_n$. In the following, we consider $n_0 = 0$ and $\epsilon_n = 0$. If we consider only the perturbation that propagates in the z direction, then $k_x = k_y = 0$, and from the above equations, for energy perturbation, we get

$$
0 = \delta\epsilon\left[2ic_s^2(k_z - \omega v_z) + \frac{2(\omega - k_zv_z)(-4ih_0(\omega - k_zv_z) + \frac{v_0\chi k_z^2}{L} + 2i\chi\omega^2k_zv_z)}{k(4\epsilon_0 + k\chi\omega v_z + 4P_0)}\right]. \tag{59}
$$

In the case of a nonrotating static background, $v_z = v_0 = 0$, then the above equation has solution $\omega = \pm c_sk$. This is the same as that of equivalent conventional hydrodynamics of Ref. [8]. However, for small rotation and small v_0, we get

$$
\begin{aligned}
\omega_1 &= \pm c_sk_z - \frac{k_zv_z\{(c_s^2-2) - 3/4\chi_0c_s^2k_z^2\}}{2} - \frac{iv_0\chi_0k_z^2}{8L}, \\
\omega_2 &= \frac{2}{\chi_0k_zv_z},
\end{aligned} \tag{60}
$$

where $\chi_0 = \frac{\chi}{h_0}$. So, from the first two terms of the above dispersion relation for ω_1, it is evident that in the presence of rotation of the background the propagation speed gets modified due to the presence of spin polarization arising from the vorticity (through nonzero χ) with $|\frac{d\mathrm{Re}(\omega_1)}{dk}| = \frac{9\chi c_s^2k_z^2v_z}{8h_0} - \frac{1}{2}c_s^2v_z \pm c_s + v_z$. This means that for $k_z > \frac{2\sqrt{c_s^2v_z - 2(\pm c_s) - 2v_z + 2}}{3\sqrt{\chi_0}c_s\sqrt{v_z}}$, $|\frac{d\mathrm{Re}(\omega_1)}{dk}| > 1$, i.e., the sound propagation becomes acausal. The third term tells about the decay of the mode, though we have taken ideal evolution as in Ref. [12], and this term may lead to instability for a background rotation with negative v_0. However, this decay through the diffusion is acausal due to k_z^2 dependence of this term. This means that in the nondissipative limit the prescribed spin hydrodynamics of Ref. [12] may lead to acausal and unstable propagation. However, that implies that the equivalent second-order theory may lead to acausality and instability for rotating background. The second mode is a wave mode whose propagation speed is inversely proportional to χ, that is, to vorticity to spin conversion strength, and also reduces with increasing rotation. The speed of this mode is higher for lower k_z, which means such modes with longer wavelengths propagate faster. This mode is there even in the absence of the sound mode.

Apart from these modes, there are other modes. Taking the sum of the first two equations of the set of equations given in Eq. (58), we get

$$
\begin{aligned}
0 = \frac{1}{4}i(\delta \mathrm{u}_x + \delta \mathrm{u}_y)(&4\epsilon_0k_zv_z - 4\epsilon_0\omega + 4P_0k_zv_z \\
&+ \chi\omega k_z^2 - 4P_0\omega).
\end{aligned} \tag{61}
$$

This gives the wave mode other than the sound as

$$\omega = \frac{4k_z v_z}{4 - \chi_0 k_z^2}. \tag{62}$$

However, if we keep $k_x = k_y$ [which follows from $\delta h^z = 0$ and $\delta h^0 = 0$, where δh^μ is the perturbation to h^μ appearing in Eq. (54)] and make the perturbation of energy and z component of velocity zero, then from the first two equations we get

$$\begin{aligned} 0 = (\delta \mathrm{u}_x - \delta \mathrm{u}_y)(&-4\epsilon_0(\omega - k_z v_z) + 4P_0 k_z v_z \\ &+ \chi\omega k_z^2 - 4P_0\omega)\{(-4\epsilon_0(\omega - k_z v_z) + 4P_0 k_z v_z \\ &+ 2\chi\omega k_x^2 + \chi\omega k_z^2 - 4P_0\omega\}. \end{aligned} \tag{63}$$

This gives two modes,

$$\begin{aligned} \omega_1 &= \frac{4v_z k_z}{4 - \chi_0 k_z^2}, \\ \omega_2 &= \frac{4v_z k_z}{4 - 2\chi_0 k_x^2 - \chi k_z^2}. \end{aligned} \tag{64}$$

These modes are wavelike modes and vanish when there is no rotation of background ($v_z = 0$). So for a nonrotating homogeneous-static background, the conventional hydrodynamics of Ref. [8], its equivalent ideal spin hydrodynamics has only sound modes. However, in the case of constant uniform rotation, the spin hydrodynamics may become unstable and acausal. So this equivalence, in general, makes the conventional second-order theory unusable, in the sense that it corresponds to an acausal form of the spin hydrodynamics.

IV. SUMMARY AND DISCUSSIONS

In the present work, we have carried out a linear mode analysis for the two different sets of equations of the relativistic spin hydrodynamics to study the issues related to stability and causality. For the case of dissipative spin hydrodynamics, it is found that the inclusion of spin dynamics introduces new modes and instability to the hydrodynamics. In this case, the spin hydrodynamics seem to have similar kinds of pathologies as reported in the literature of the relativistic NS equation [62]. We have investigated the origin of the kind of instability in the theory discussed in Ref. [62] and the origin is found to be in the form of the heat fluxes. The spin dissipative dynamics is characterized by three transport coefficients: (i) γ (associated with the shear stress), (ii) λ (associated with heat conduction), and (iii) χ_1 (associated with the spin dynamics). In the absence of regular dissipation ($\zeta = \eta = \kappa = 0$), the first two longitudinal modes described by Eq. (30) exhibit acausal behavior, as $|\frac{d\omega_{1,2l}}{dk}|$ can exceed the speed of light. Similar behavior can be seen in the regular relativistic NS equation also [see Eq. (30) with ζ, η and $\kappa = 0$]. The third mode [in Eq. (30)] is a new mode, which is conditionally unstable and it can also have acausal behavior. The fourth mode [in Eq. (30)] is purely an unstable mode and it has a counterpart in the relativistic NS equation [see the last mode in Eq. (30) with ζ, η and $\kappa = 0$]. The transverse modes described by Eq. (34) also exhibit acausality and instability. In Eq. (34), the second and third equations are the new modes arising due to the spin dynamics. In this case also the transport coefficient λ can drive the instability under certain conditions. It is evident that the presence of spin polarization affects the hydrodynamic responses through new coefficients in spin hydrodynamics.

We also studied the stability of the dissipationless spin dynamics described in Ref. [8]. In this case, the linear mode analysis was performed for the following two backgrounds, i.e., when the fluid is (i) static and (ii) having constant vorticity. In the first case, it is shown that the fluid supports only the sound waves. In the second case, the background velocity is in the z direction with constant vorticity in x and y directions. In this case, it is possible to study the normal Fourier modes in the z direction. The normal modes for this case are described by Eq. (60). Here, the first equation may give an instability for $v_0 < 0$, but the reason for the instability can be attributed to the source of the free energy provided by the finite flow velocity of the background. The flow velocity can also alter the sound speed. There is an equivalent second-order dissipationless conventional hydrodynamical theory as reported [8]. The underlying pseudogauge transformation may give a similar kind of dispersion relation described by Eq. (60). These issues make the conventional second-order theory in Ref. [8] and its equivalent spin hydrodynamics inadequate to describe the hydrodynamics with the spin for a general situation.

Thus, we have analyzed acausal behavior and unphysical instability arising in the relativistic spin hydrodynamics. We believe that our linear analysis shows that relativistic spin hydrodynamics faces similar issues faced by the relativistic NS equation, but the spin dynamics brings in new complexities. This points toward the need for causal and stable theories, with the spin density as an independent hydrodynamic field, which are free from acausality and instability to describe the spin dynamics of spin-polarized fluid.

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