


Consistency of chiral symmetry breaking in chiral Yang-Mills theory with adiabatic continuity

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We study the pattern of chiral symmetry breaking in the $\psi\chi\eta$ model [with the chiral fermion sector containing $\psi^{\{ij\}}$, $\chi_{[ij]}$, and η_i^A ; see Armoni and Shifman *Phys. Rev. D* **85**, 105003 (2012)] on $\mathbb{R}^3 \times S_L^1$ and derive implications forcing \mathbb{R}^4 physics. Center-symmetric vacua are stabilized by a double-trace deformation. With the center symmetry maintained at small $L(S^1) \ll \Lambda^{-1}$, i.e., at weak coupling, no phase transitions are expected in passing to large $L(S^1) \gg \Lambda^{-1}$ (here, Λ is the dynamical Yang-Mills scale). Starting with the small- L limit, we find the leading-order nonperturbative corrections in the given theory. The instanton-monopole operators induce the adjoint chiral condensate $\langle \psi^{\{ij\}} \chi_{[jk]} \rangle \neq 0$ at weak coupling, i.e., at $L(S^1) \ll \Lambda^{-1}$. Then adiabatic continuity tells us that $\langle \psi^{\{ij\}} \chi_{[jk]} \rangle \neq 0$ exists on \mathbb{R}^4 , in full accord with the prediction from Bolognesi *et al.* [*Phys. Rev. D* **97**, 094007 (2018)]. Simultaneously with $\langle \psi^{\{ij\}} \chi_{[jk]} \rangle \sim \Lambda^3 \delta_k^i$, the $SU(N_c)$ gauge symmetry is spontaneously broken at strong coupling down to its maximal Abelian subgroup.

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I. INTRODUCTION

Chiral Yang-Mills theory is an important element in understanding non-Abelian gauge dynamics. If at weak coupling (i.e., in the Standard Model) everything is transparent this cannot be said about the strong-coupling regime. In a special “hybrid” chiral model [1] (also known as the $\psi\chi\eta$ model) a pattern of the chiral symmetry breaking (χ SB) was established [2] on the basis of the ’t Hooft anomaly considerations and some additional arguments. Two possible scenarios were revealed. In this paper, we invoke Ünsal’s adiabatic continuity for additional verification of the results reported in [2]. The pattern of the chiral symmetry breaking in the $\psi\chi\eta$ model following from the adiabatic continuity perfectly coincides with one of the scenarios in [2]. We also discuss some other implications of the adiabatic continuity in the $\psi\chi\eta$ model.

The chiral Yang-Mills theories were studied at weak coupling and in various models (e.g., Maximal Attractive Channel) in the 1980s [3–5] and more recently at strong coupling; see, e.g., [1,2,6–14]. Owing to the recent discoveries of generalized global symmetries in quantum field

theory [15,16], and the corresponding ’t Hooft matching, this topic has attracted renewed attention [10,11,17,18].

We will focus on one particular subclass of strongly coupled Yang-Mills theory with a special chiral fermions matter sector.

Two simplest cases (the so-called $\psi\eta$ and $\chi\eta$ model) were previously studied in [2,6,7,10–12,17,18] from various perspectives. In particular, in [2,6], under the assumption of a nonvanishing condensate $\langle \psi\eta \rangle$ based on the traditional ’t Hooft anomaly, matching the color-flavor locking between the gauge $SU(N)$ and the global $SU(N+4)$ was observed. The χ SB pattern then takes the form

$$SU(N)_c \times SU(N+4)_f \times U(1) \rightarrow SU(N)_{cf} \times U'(1) \times SU(4)_f.$$

The infrared-bound baryon states $(\psi^{\{ij\}} \eta_i^A \eta_j^B)$ were constructed.

Then, the authors of [12] argued, on the basis of a mixed ’t Hooft anomaly, that a chirally symmetric vacuum is impossible. However, the authors of [10] (based on recasting the center of the symmetry group) raised objections concerning the result of Bolognesi *et al.* [12]. In view of the ongoing debate on the vacua of the $\psi\eta$ and $\chi\eta$ models we will leave the simplest cases aside for the time being and limit ourselves to the hybrid $\psi\chi\eta$ version in the framework of the adiabatic continuity approach.

Ünsal suggested [19,20] the adiabatic continuity method [9,17,18,21,22] to study strongly coupled gauge theories.

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See Ref. [23] for the recent review. The starting point of this method is as follows. Consider theories on the spacetime with one spatial dimension compactified, say, $\mathbb{R}^3 \times S^1_L$. Unlike compactified time dimension in thermalized theories [24,25], adiabatic continuity construction was argued to contain no phase transition in passing from the small L limit $L(S^1) \ll \Lambda^{-1}$ to large L —approaching \mathbb{R}^4 —provided the vacuum of the model at small $L(S^1)$ is center symmetric. In this case the vanishing of the Polyakov line is preserved intact at both limits. Then the model Abelianizes, i.e. all gauge bosons outside the Cartan subalgebra acquire masses and the theory becomes weakly coupled and the others are still confining according to the Polyakov criterion. Perturbative calculations turn out to be reliable. Moreover, some nonperturbative features such as instanton monopoles and bions can be explicitly computed in the quasiclassical approximation and provide nontrivial results for four-dimensional physics upon continuation to large $L(S^1) \gg \Lambda^{-1}$. In Sec. III, we will show how the instanton-monopole operators allow us to verify the χ SB pattern found in [2].

In particular, in the $\psi\chi\eta$ model, three types of monopole operators show up—only one of which is a global gauge rotation singlet participating in the low-energy effective Lagrangian. It is such an \mathcal{M}_j -type monopole that induces chiral condensates $\langle\psi\chi\rangle$. The dynamical Abelianization takes place at large $L(S^1)$ and hence on \mathbb{R}^4 by the standard lore of adiabatic continuity. This verification perfectly matches the recent study of χ SB in the $\psi\chi\eta$ model via the mixed 't Hooft anomaly [8].

In the first part of the paper, the center-symmetry stabilization is achieved by virtue of a double-trace deformation [19,21]. In the second part, we replace the double-trace deformation by additional fermions (additional with regards to the primary chiral matter), which do the same job. We find a few exceptional cases when the additional fermions can stabilize center symmetry. We note a potential disadvantage of implementing stabilization by virtue of additional adjoint fermions and appropriate flavor-twisted boundary conditions for (chiral) fermions in the fundamental representation.

The paper is organized as follows. In Sec. II, we briefly review the $\psi\chi\eta$ model and some previously conjectured χ SB patterns. In Sec. III, the technique of adiabatic continuation is applied to examine possible scenarios of χ SB with the purpose of corroborating them. In Sec. IV, the interplay between the stabilization strategy (double-trace deformation vs additional fermions) is discussed and the corresponding fermion condensates are found in two examples. Our conclusions are summarized in Sec. V.

II. THE $\psi\chi\eta$ MODEL

In this section, we review some basic elements of the $\psi\chi\eta$ model playing the main role in the following discussion. The $\psi\chi\eta$ model refers to a chiral $SU(N_c)$ gauge theory with the fermion sector consisting of a pair of

two-index fermions, one symmetric while the other antisymmetric in color indices, plus eight antifundamentals. Namely,

$$\psi^{\{ij\}}, \quad \chi_{[ij]}, \quad \text{and } \eta_i^A \text{ for } A = 1, 2, \dots, 8, \quad (1)$$

where $\psi^{\{ij\}}$ is the symmetric fermion and $\chi_{[ij]}$ is antisymmetric. The model is self-consistent and enjoys the symmetry structure [1,2,6,8]

$$G = \frac{SU(N_c) \times SU(8)_f \times U(1) \times \tilde{U}(1)}{\mathbb{Z}_{N_c} \times \mathbb{Z}_{8/N^*}} \quad (2)$$

in which N^* stands for the greatest common divisor of $N_c + 2$ and $N_c - 2$. The overlapped phase rotations of fermions between the centers and $U(1)$ groups are quotiented out.

The $\psi\chi\eta$ model has three $U(1)$ symmetries, with the generating currents

$$j_{(\psi)}^{\dot{\alpha}\alpha} = \bar{\psi}^{\dot{\alpha}}\psi^{\alpha}, \quad j_{(\chi)}^{\dot{\alpha}\alpha} = \bar{\chi}^{\dot{\alpha}}\chi^{\alpha}, \quad j_{(\eta)}^{\dot{\alpha}\alpha} = \bar{\eta}^{\dot{\alpha}}\eta^{\alpha}, \quad (3)$$

(no summation over the flavor index A in the definition of $j_{(\eta)}^{\dot{\alpha}\alpha}$). Each of the above currents is anomalous,

$$\partial_{\dot{\alpha}\dot{\alpha}} j^{\dot{\alpha}\alpha} = \partial_{\mu} j^{\mu} = \begin{pmatrix} N+2, & \text{for } \psi \\ N-2, & \text{for } \chi \\ 1, & \text{for } \eta \end{pmatrix} \times \frac{1}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{\mu\nu a}. \quad (4)$$

The last expression allows us to establish two nonanomalous combinations of the currents denoted by $U(1)$ and $\tilde{U}(1)$ in Eq. (2). For instance,

$$\begin{aligned} U(1): & (N-2)j_{(\psi)} - (N+2)j_{(\chi)}, \\ \tilde{U}(1): & 2j_{(\psi)} - 2j_{(\chi)} - \sum_A j_{(\eta)}. \end{aligned} \quad (5)$$

Equation (5) corresponds to the following charge assignment of fermions:

$$\begin{aligned} U(1): & Q_{\psi} = \frac{N_c - 2}{N^*}, \quad Q_{\chi} = -\frac{N_c + 2}{N^*}, \quad Q_{\eta} = 0, \\ \tilde{U}(1): & \tilde{Q}_{\psi} = 2, \quad \tilde{Q}_{\chi} = -2, \quad \tilde{Q}_{\eta} = -1. \end{aligned} \quad (6)$$

The phase rotation of fermions under the centers of $SU(N_c)$ and $SU(8)$ together with the $U(1)$'s takes the form

$$\begin{aligned} \psi & \rightarrow e^{2(N_c-2)\pi i\alpha/N^*} e^{4\pi i\beta} e^{4\pi ki/N_c} \psi, \\ \chi & \rightarrow e^{-2(N_c+2)\pi i\alpha/N^*} e^{-4\pi i\beta} e^{-4\pi ki/N_c} \chi, \\ \eta & \rightarrow e^{-2\pi i\beta} e^{-2\pi ki/N_c} e^{\pi mi/4} \eta, \end{aligned} \quad (7)$$

in which $\alpha, \beta \in (0, 1)$, $k \in \mathbb{Z} \pmod{N_c}$, and $m \in \mathbb{Z} \pmod{8}$. Here, α and β stand for the transformations under $U(1)$ and $\tilde{U}(1)$, respectively, while the integers k, m parametrize the center transformation of $SU(N_c)$ and $SU(8)$ groups. Solutions for $\alpha, \beta \neq 0$ exist, which leave (8) intact; they are parametrized by $\mathbb{Z}_{N_c} \times \mathbb{Z}_{8/N_c}$. See a more detailed discussion in [6,8].

As first noted in [2] and further analyzed in [6,8], the $\psi\chi\eta$ model *a priori* can have two different χ SB scenarios in the infrared regime. First, let us suppose both $\langle\psi\eta\rangle$ and $\langle\psi\chi\rangle$ develop nonzero vacuum expectation values (VEVs). The former condensate then implies that the color and flavor indices are entangled in such a way that the condensate is invariant under the diagonal $SU(8)$ transformation. This is the so-called color-flavor locking. As a consequence, if $N_c \geq 12$, the chiral symmetry breaks as follows:

$$\begin{aligned} & SU(N_c) \times SU(8)_f \times U(1) \times \tilde{U}(1) \\ & \rightarrow SU(8)_{\text{cf}} \times U(1)^{N_c-p+1} \times SU(p-8)_c, \end{aligned} \quad (8)$$

where we ignore the center part. At $p = 12$, the theory saturates the $SU(8)^3$ 't Hooft anomaly. On the other hand, if only $\psi\chi$ has a nonvanishing VEV (i.e., $\langle\psi\eta\rangle = 0$), we arrive at a Higgs/confinement phase.

Once the adjoint condensate $\langle\psi^{ij}\chi_{jk}\rangle$ develops, the pattern of χ SB takes the form

$$\begin{aligned} & SU(N_c) \times SU(8)_f \times U(1) \times \tilde{U}(1) \\ & \rightarrow \prod_{l=1}^{N_c-1} U(1)_l \times SU(8)_f \times \tilde{U}(1). \end{aligned} \quad (9)$$

The fundamental fermions stay massless and weakly coupled with the massless dual photons in the infrared regime [2].

III. ADIABATIC CONTINUITY ANALYSIS

In the following discussion, we will show that the above scenario is compatible with the prediction from the adiabatic continuity of the deformed $\psi\chi\eta$ model on $\mathbb{R}^3 \times S_L^1$. To proceed, let us consider the $\psi\chi\eta$ model with a double-trace deformation [9,21,22], say,

$$S = \frac{1}{g^2} \int_{\mathbb{R}^3 \times S_L^1} d^4x \left\{ \frac{1}{2} \text{Tr} F^2 + \sum_{\mathcal{R}} i\bar{\Psi}^{\mathcal{R}} \mathcal{D}\Psi^{\mathcal{R}} \right\}, \quad (10)$$

in which \mathcal{R} runs over the relevant irreducible representations of chiral fermions in the theory. Along S_L^1 , all fermions (as well as bosons) obey the periodic boundary condition. $S_{\text{double trace}}$ here is fine-tuned in such a way that

the effective potential is stabilized¹ at the center-symmetric point [26]

$$\{La_j\}_{j=1}^{N_c} = \left\{ \frac{N_c-1}{N_c}\pi, \frac{N_c-3}{N_c}\pi, \dots, -\frac{N_c-1}{N_c}\pi \right\}. \quad (11)$$

In the small- L regime, the gauge symmetry $SU(N_c)$ is broken down to $[U(1)]^{N_c-1}$ at the center-symmetric point and the off-diagonal gauge bosons as well as fermions acquire three-dimensional masses. In particular, the mass terms of two-index fermions ψ^{ij} and χ_{ij} are

$$m_{ij} = \frac{2\pi}{LN_c} (N_c + 1 - i - j), \quad (12)$$

implying ψ and χ remain massless in perturbation theory if $i + j = N_c + 1$. For later convenience, let us denote

$$\psi^j := \psi^{\{j, N_c+1-j\}} \quad \text{and} \quad \chi_j := \chi_{[j, N_c+1-j]}. \quad (13)$$

Before directly getting into the chiral symmetry breaking in transition from the small radius to the four-dimensional limit, we first consider the model at small $L(S^1)$, at weak coupling. It is straightforward to obtain the perturbation theory by integrating the modes along the S_L^1 direction. On the other hand, we emphasize that the building blocks to describe no-perturbative dynamics of the $\psi\chi\eta$ model on $\mathbb{R}^3 \times S_L^1$ are the monopole-instanton operators. In $SU(N_c)$ gauge theory, we deal with $(N_c - 1)$ Bogomol'nyi-Prasad-Sommerfield monopoles and one Kaluza-Klein monopole [27–29]. The fermion zero mode distribution can be found from the index theorem [30,31]. Namely,

$$\begin{aligned} \mathcal{M}_j &= e^{-S_0 + i\alpha_j \sigma} (\psi^j + \psi^{j+1}) (\chi_j + \chi_{j+1}), \\ \mathcal{M}_{N_c-1} &= e^{-S_0 + i\alpha_{N_c-1} \sigma} (\psi^1 + \psi^2)^2 \eta^{A_1} \dots \eta^{A_8} \epsilon_{A_1 A_2 \dots A_8}, \\ \mathcal{M}_{N_c} &= e^{-S_0 + i\alpha_{N_c} \sigma} (\psi)^2 \end{aligned} \quad (14)$$

for $j = 1, 2, \dots, N_c - 2$. Note that σ is the dual photon field in three dimensions, and $\{\alpha_1, \alpha_2, \dots, \alpha_{N_c}\}$ are affine roots of the $su(N_c)$ algebra. In Eq. (14), we will keep only massless fermions for the zero modes as the leading-order contribution. To see which of these monopole operators generate the relevant condensates, first recall that under the global $[U(1)]^{N_c-1}$ gauge rotations,

$$\begin{aligned} \psi^{\{ab\}} &\rightarrow e^{i(\mathbf{H}_{aa} + \mathbf{H}_{bb})\xi} \psi^{\{ab\}}, \\ \chi_{[ab]} &\rightarrow e^{-i(\mathbf{H}_{aa} + \mathbf{H}_{bb})\xi} \chi_{[ab]}, \\ \eta_a &\rightarrow e^{-i\mathbf{H}_{aa}\xi} \eta_a, \end{aligned} \quad (15)$$

¹The effective potential $V_{\psi+\chi} \sim \sum_p (\text{tr}_F \Omega^p)^2$ develops a local minimum at the center-symmetric point, and the double-trace term is only used to stabilize V_η generated by fundamental fermions as in [9].

where \mathbf{H} is the elements of the Cartan subalgebra of $su(N_c)$. Then, to construct a global-rotation invariant monopole operator, we have to integrate out all $U(1)$ phases,

$$e^{-S_0 + i\alpha_j \sigma} \cdot \int \psi^j \chi_j + e^{i[\mathbf{H}_{j+1, j+1} + \mathbf{H}_{N_c - j, N_c - j} - (j \rightarrow j-1)] \xi} \psi^{j+1} \chi_j + (j \leftrightarrow j+1) d\xi, \quad (16)$$

which results in

$$\mathcal{M}_j = e^{-S_0 + i\alpha_j \sigma} (\psi^j \chi_j + \psi^{j+1} \chi_{j+1}). \quad (17)$$

All other operators vanish. This is different from the observations in [9,17,18]. In our model, a certain kind of monopole operator (17) dressed with chiral fermions forms a global gauge rotation singlet, while in the aforementioned publications, monopole operators with chiral zero modes do not survive and only act as a building block for higher-order operators. In fact, this is not so surprising because $\psi^{ik} \chi_{kj}$ transforms as an adjoint boson, and we saw in many examples [22,26,32] that the instanton-monopole operators with the adjoint zero modes do play a role in the nonperturbative dynamics of the theory.

The existence of the \mathcal{M}_j -type monopole operators then implies the low-energy (large-distance) effective Lagrangian of the form

$$\begin{aligned} \mathcal{L}_{\text{dual}} = & \frac{g_3^2}{32\pi^2} (\partial_\mu \sigma)^2 + \bar{\Psi} i \partial \Psi \\ & + \left(\sum_j e^{-S_0 + i\alpha_j \sigma} (\psi^j \chi_j + \psi^{j+1} \chi_{j+1}) + \text{H.c.} \right) \\ & + \dots, \end{aligned} \quad (18)$$

where the former two terms come from the perturbation theory, while the dots stand for the higher-order non-perturbative contributions such as bions and instantons.

Now, let us probe the pattern of χ SB by taking into account the nonperturbative contributions mentioned above. In light of (18), there exists a nonvanishing chiral condensate

$$\langle \psi^{ik} \chi_{kj} \rangle \sim \langle \psi^l \chi_l \rangle \delta_j^i = c_j \Lambda^3 e^{4i\alpha/N^*} \delta_j^i \quad (19)$$

with

$$\sum_{j=1}^{N_c} c_j = 0,$$

where Λ is the strong scale. This is exactly the same as what was argued in [2,8] and therefore dynamical Abelianization (9) ensues. Note that due to the shift symmetry,

$$\begin{aligned} \alpha_j \sigma & \rightarrow \alpha_j \sigma + \frac{4\alpha}{N^*}, \\ \alpha_{N_c-1} \sigma & \rightarrow \alpha_{N_c-1} \sigma - \frac{2(N_c-2)\alpha}{N^*} - 4\beta, \\ \alpha_{N_c} \sigma & \rightarrow \alpha_{N_c} \sigma - \frac{2(N_c-2)\alpha}{N^*} + 4\beta, \end{aligned} \quad (20)$$

the bion excitations can show up in the spectrum and generate the dual photon masses. To be more precise, the magnetic bions are neutral under the shift symmetry and, according to Eqs. (14) and (20), the bion operators can be explicitly written in the form

$$\mathcal{B}_{ij} \sim e^{-2S_0} e^{i(\alpha_i - \alpha_j) \sigma}, \quad (21)$$

where j can be $i+1$, $N_c - i$, $N_c - 1 - i$, or $N_c + 1 - i$. Note that i, j are both smaller than $N_c - 1$ since it is the pairing among $\mathcal{M}_1, \dots, \mathcal{M}_{N_c-2}$ as implied by the shift symmetry. Thus, $\sigma_1, \dots, \sigma_{N_c-2}$ become massive while σ_{N_c-1} remains massless. Those massive gauge bosons then result in the theory being confining by virtue of the Polyakov mechanism [33].

We can also imagine a situation that the gauge symmetry $[U(1)]^{N_c-1}$ is spontaneously broken. That is, we lift the constraints on the invariance of the instanton monopoles under the global gauge rotations. In this case, all instanton-monopole vertices in Eq. (14) participate in nonperturbative dynamics. The additional condensate is then

$$\langle (\psi^1 + \psi^2)^2 \eta^{A_1} \dots \eta^{A_8} \varepsilon_{A_1 A_2 \dots A_8} \rangle \neq 0 \quad (22)$$

coming from \mathcal{M}_{N_c-1} . Note that because the massless fermions (ψ^1 and ψ^{N_c}) do not couple to \mathcal{M}_{N_c} , it cannot be included in the leading contribution in the sense that massive fermions decouple in the weak-coupling region. Lastly, in the presence of the \mathcal{M}_j instanton monopoles within this framework, the suggested symmetry breaking pattern is identical to (9) by the same argument. This is one of the scenarios first advocated in [2].

IV. CENTER STABILIZATION WITH ADJOINT FERMIONS

In this section, we will abandon the idea of the double-trace deformation and discuss the stabilization problem in some models with chiral fermions in mixed representations—including adjoint fermions—and how these mixed ensembles affect the prediction for χ SB based on adiabatic continuity. Adding adjoint fermions is intended to stabilize center-symmetric vacua in the effective potential. As has been mentioned in footnote 1, there is no need to stabilize the ψ - χ pair.

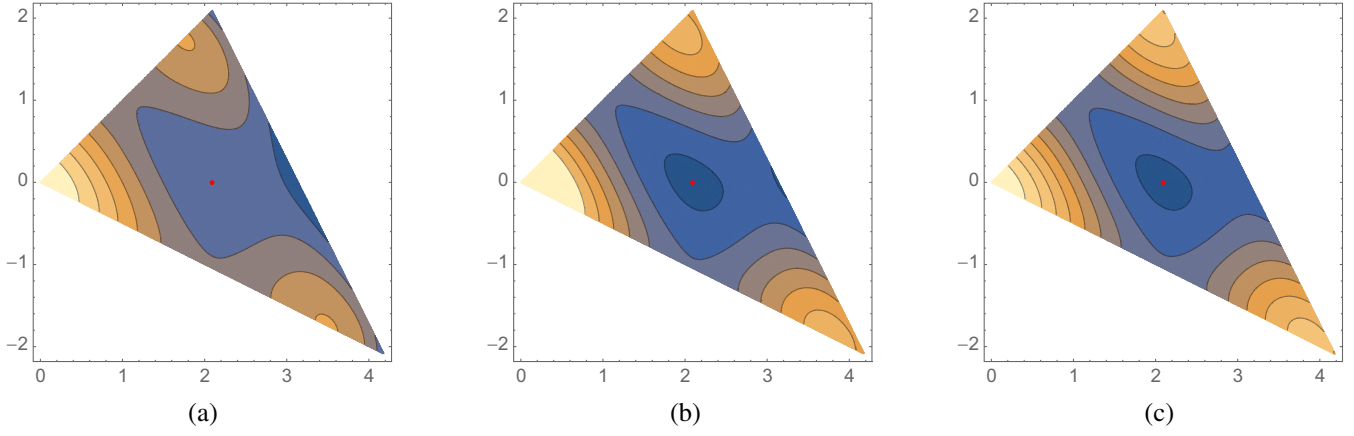


FIG. 1. The effective potentials of one, two, and three adjoint fermions added in the Weyl chamber. The horizontal and vertical axes represent two components of the background holonomy, respectively. The red dot illustrates the center-symmetric point. (a) One extra adjoint fermion, (b) two extra adjoint fermions, and (c) three extra adjoint fermions.

What could possibly help in this situation is to introduce the (flavor) twisted-boundary condition (FTBC) [34–36].² While in Sec. III, all fermions obey the periodic boundary condition

$$\psi(x^\mu, x^3 + L) = \psi(x^\mu, x^3), \quad (23)$$

now we switch on a flavor twist for fundamentals to construct an additional flavor-dependent phase.³ Instead of Eq. (23), now we require

$$\eta_k^A(x^\mu, x^3 + L) = \pm \omega^A \eta_k^A(x^\mu, x^3) \quad \text{with} \quad \omega^n = 1. \quad (24)$$

This then alters the fundamental fermion contribution to the effective potential

$$V_{\text{eff}} \sim \sum_{p=1}^{\infty} \frac{1}{p^4} \text{tr}_F \Omega^{2p} \rightarrow \sum_{p=1}^{\infty} \frac{1}{n^3 p^4} \text{tr}_F \Omega^{2np}, \quad (25)$$

so that more models fall into the (local) minimum at the center-symmetric point as extra adjoint fermions are added. In the following we provide two examples with FTBC fundamental fermions and point out the shortcoming of this boundary condition in determining the schemes of χ SB.

First, let us consider $SU(3)$ Yang-Mills theory with the following fermion sector: ψ^{ij} , χ^{ij} , and η_i^A for $A = 1, 2, \dots, 6$ where the fundamentals η_k^A satisfy the boundary condition

²For recent developments on the implication of χ SB in QCD with fundamental or adjoint fermions through introduction of twisted boundary conditions, see Refs. [37,38].

³One may want to consider the most general FTBC, $\eta^A(x_3 + L) = U^{AB} \eta_B(x_3)$ with $U^{AB} \in SU(N_f)$. Yet, the twist matrix can always be diagonalized by making a flavor rotation.

$$\eta_k^A(x^\mu, x^3 + L) = e^{\frac{2\pi Ai}{3}} \eta_k^A(x^\mu, x^3). \quad (26)$$

The effective potential with the varying number of the adjoint fermions is demonstrated in Fig. 1. The center symmetry is achieved at least at the local minimum and as long as the number of adjoint fermions is greater than one, it is in fact a global minimum. Next, the instanton-monopole operators of the theory under consideration take the form

$$\begin{aligned} \mathcal{M}_1 &= e^{-S_0} e^{i\alpha_1 \cdot \sigma} \chi \psi \eta^3 \eta^6, & \mathcal{M}_2 &= e^{-S_0} e^{i\alpha_2 \cdot \sigma} (\psi)^2 \eta^1 \eta^4, \\ \mathcal{M}_3 &= e^{-S_0} e^{i\alpha_3 \cdot \sigma} (\psi)^2 \eta^2 \eta^5. \end{aligned} \quad (27)$$

Four-fermion zero mode vertices appear for each monopole. The corresponding lowest-order condensate then should be a four-fermion composite so that it cannot provide further information as to the next-to-leading-order contribution because the minimal condensates argued in [6] consist of only two fermions.

Another class of center-stabilized anomaly-free theories is $SU(N_c)$ gauge theories with one $\psi^{\{ij\}}$, a number of two-index antisymmetric $\tilde{\chi}^{[ij]}$ and $\chi_{[ij]}$ fields, and pairs of fundamental fermions satisfying periodic and antiperiodic boundary conditions. There is only a finite number of combinations for this kind of model with stabilization at the center-symmetric point, namely $N_c = 4, 6, 8, 10, 14, 16, 20, 24, 36, 52$. The idea is to cancel the χ contribution driving the effective potential away from the center-symmetric point by pairs of periodic and antiperiodic fermions, namely,

$$-\frac{k-1}{2} (\text{tr}_F \Omega^{2p}) + \frac{n_f/2}{2^3} (\text{tr}_F \Omega^{2p}) = 0, \quad (28)$$

where k is the total number of $\tilde{\chi}^{ij}$ and χ_{ij} . The distribution of the fermion zero modes on the monopole operators can be sketched as follows:

$$\begin{aligned}
\mathcal{M}_j &= e^{-S_0} e^{i\alpha_j \sigma} \psi \tilde{\chi}^{k_1} \chi^{k_2}, \\
\mathcal{M}_{\frac{N_c}{2}} &= e^{-S_0} e^{i\alpha_{N_c/2} \sigma} \psi^2 \eta_1 \eta_2 \cdots \eta_{n_f}, \\
\mathcal{M}_{N_c} &= e^{-S_0} e^{i\alpha_{N_c} \sigma} \psi^2.
\end{aligned} \tag{29}$$

In the same manner as in the beginning of this section, the related condensates contain multifermion composites and turn out to be irrelevant in verifying the pattern of χ SB.

In the above examples, the theories we have identified all have an exact center-symmetric vacuum. In fact, the condition that the effective potential has to stabilize at the center-symmetric point can be further relaxed as long as the Abelianization of the vacua (of a theory) persists. This then opens up more classes of theories whose chiral symmetry breaking can be verified via the adiabatic continuity. At the moment, however, a systematic search is hardly possible in an analytic (or even a numerical) way because of the vagueness of the condition.

Summarizing, imposing the flavor-twisted boundary conditions allows a number of theories to be stabilized at the center-symmetric point. This circumstance makes possible the method of adiabatic continuity. However, the downside is forcing the fundamental fermion zero modes to be evenly distributed among the monopole operators. The emerging condensates then consist of a larger number of fermions than needed for the $\psi\chi$ condensate (in the leading approximation). Therefore, the FTBC-based line of reasoning in the present form is useless for corroborating the pattern of the χ SB which was argued for in Refs. [2,6].

V. COMMENTS AND CONCLUSIONS

In this work, we start from the UV symmetry group in the $\psi\chi\eta$ model and consider possible ways of χ SB on \mathbb{R}^4 . After a brief introduction, we discuss dynamics, especially nonperturbative aspects, in the $\psi\chi\eta$ model on $\mathbb{R}^3 \times S^1_L$ at small L . We limit ourselves to the leading approximation (instanton monopoles). The center-symmetric vacuum is guaranteed by a double-trace deformation. The dynamical Abelianization is achieved. Then, the corresponding instanton-monopole operators are found to induce the chiral condensate $\langle\psi\chi\rangle$. Adiabatic continuity propagates this

result to \mathbb{R}^4 . This is a successful and desired part of our work. With $\langle\psi^{\{ij\}}\chi_{[jk]}\rangle \sim \Lambda^3 \delta_k^i$ the $SU(N_c)$ gauge symmetry is spontaneously broken down to its maximal torus $[U(1)]^{N_c-1}$. This is our target—verification of the prediction [2,6] in Sec. III.

The strategy of abandoning the double-trace deformation in favor of adding extra adjoint fermions and flavor twisting of boundary conditions for the fundamental fields η for stabilization at small L in essence failed. Although stabilization can be achieved in a limited number of cases, no useful information can be obtained in this way about the condensate of interest.

In conclusion it is worth adding a comment about the planar equivalence between $\mathcal{N} = 1$ Yang-Mills theory (without matter) and the $\psi\chi\eta$ model. The latter model was designed [1] in the context of the concept of planar equivalence in the common sectors which, in the case at hand, covers all correlators of purely gluon operators.

At small $L(S^1)$, the $\psi\chi\eta$ model is characterized by Polyakov confinement due to bions [see Eq. (21)]. If the adiabatic continuity is correct, we expect confinement at large $L(S^1)$ [on \mathbb{R}^4 in the limit $L(S^1) \rightarrow \infty$]. However, because of the fact that $\langle\psi^{\{ij\}}\chi_{[jk]}\rangle \sim \Lambda^3 \delta_k^i$, dynamically this confinement is somewhat different from that in $\mathcal{N} = 1$ Yang-Mills theory on \mathbb{R}^4 . One can call it the Higgs/confinement phase. The vacuum structure is also different: While in $\mathcal{N} = 1$ Yang-Mills theory we have N vacua marked by the gluino order parameter, $\langle\lambda\lambda\rangle \sim \Lambda^3 \exp(i\frac{2\pi k}{N})$, in the $\psi\chi\eta$ model the order parameter (19) continuously rotates under the action of the anomaly-free current $\frac{1}{N^*}[(N-2)j_{(\psi)} - (N+2)j_{(\chi)}]$; see Eqs. (5) and (6). This implies a continuous $U(1)$ vacuum manifold which in turn implies the existence of a massless particle.

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