Dipole-dipole scattering amplitude in the CGC approach

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(Received 23 September 2022; accepted 20 February 2023; published 9 March 2023)

In this paper we propose recurrence relations for the dipole densities in QCD, which allows us to find these densities from the solution to the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation. We resolve these relations in the diffusion approximation for the BFKL kernel. Based on this solution, we found the sum of large Pomeron loops. This sum generates the scattering amplitude that decreases at large values of rapidity Y. It turns out that such behavior of the scattering amplitudes is an artifact of diffusion approximation. This approximation leads to the unitarization without saturation both in deep inelastic scattering and in dipole-dipole interaction at high energies.

DOI: 10.1103/PhysRevD.107.054012

I. INTRODUCTION

The main ideas of the color glass condensate/saturation approach (see Ref. [1] for a review), including the saturation of the dipole density and the new dimensional scale (Q_s), which increases with energy, have become the common language for discussing the high energy scattering in QCD. However, in spite of intensive work [2–36], we have several problems that have not been solved. One of the principle problems is summing Pomeron loops, which without solving we cannot consider the dilute-dilute and dense-dense parton densities collisions. As has been recently shown [35,36], even the Balitsky-Kovchegov (BK) equation that governs the dilute-dense parton density scattering (deep inelastic scattering of electron with proton) has to be modified due to contributions of Pomeron loops.

In this paper, we attempt to sum Pomeron loops for dipole-dipole scattering amplitude at high energies. This attempt is based on the experience with the simple, but exactly solvable, two-dimensional models [10,37–51], which we will discuss in the next section. From these models, we learned that the scattering amplitude at high energies is determined by the sum of large Pomeron loops. Actually, the formalism for summing large Pomeron loops in QCD has been developed [10,52–56]. In this paper, we propose new recurrence relations for the parton densities in QCD, which allows us to find all parton densities from the solution of the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation. We resolve these recurrence relations in the diffusion approximation for the BFKL kernel and suggest the explicit form for the scattering amplitude. We believe that we have completed the approach that was started in Refs. [55,56].

II. POMERON CALCULUS IN ZERO TRANSVERSE DIMENSION: A RECAP

The simple toy model, the Pomeron calculus in zero transverse dimension, is a respectable tool and a well-known training ground for the interaction at high energies [10,11,37–51]. Because of the simplicity of these models, we are able to formulate and solve the Reggeon field theory for the interacting Pomerons. This theory satisfies both the s and t channel unitarity constraints and includes the emission of the dipoles as well as the saturation effects in the corresponding parton cascades. The simple toy model also gives examples of theories that have the probabilistic interpretation for the scattering amplitude in letter and spirit of the partonic approach.

In Ref. [10] the simple probabilistic formula for the S matrix is suggested,

$$S(Y) = \sum_{n,m} e^{-m\,n\,\gamma} P_n^{\mathrm{BFKL}}(Y_0) P_m^{\mathrm{BFKL}}(Y - Y_0), \qquad (1)$$

where γ is the scattering amplitude of two dipoles and $P_n^{\mathrm{BFKL}}(Y)$ is the probability distribution in the BFKL cascade. For $P_n(Y)$ we have equations in the following form for the zero transverse dimension:

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$$\frac{dP_n^{\rm BFKL}(Y)}{dY} = -\Delta n P_n^{\rm BFKL}(Y) + \Delta(n-1) P_{n-1}^{\rm BFKL}(Y), \qquad (2)$$

with the solution

$$P_n^{BK}(Y) = \frac{1}{N(Y)-1}\left(1-\frac{1}{N(Y)}\right)^n \xrightarrow{Y\gg 1} \frac{1}{N(Y)}\exp\left(-\frac{n}{N(Y)}\right), \qquad (3)$$

where $N(Y)$ is the first factorial moment or multiplicity of dipoles: $N(Y) = e^{\Delta Y}$. Generally speaking, Eq. (1) does depend on the reference frame (on the value of Y_0) and, as has been discussed in Refs. [10,43,50,51], we need to change Eq. (2) to obtain the Pomeron calculus that satisfies both t and s channel unitarity. However, at large values of $Y - Y_0$ and Y_0, Eq. (1) leads to the scattering amplitude that does not depend on the value of Y_0 [51]. Indeed,

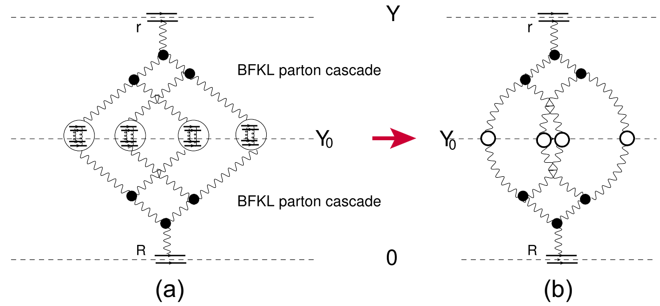

FIG. 1. Summing large Pomeron loops. The wavy lines denote the BFKL Pomeron exchanges in (a) and, in (b), the Pomeron in the two-dimensional Pomeron calculus with Green's function $G_{\mathbb{P}}(Y) = \exp(\Delta Y)$. The black circles stand for the triple Pomeron vertices in both figures, which are equal to Δ in (b), while the white circles denote the amplitude γ. In (a) we show the dipole-dipole scattering amplitude in the Born approximation of perturbative QCD in the circles.

$$S(Y) = \sum_{n,m} e^{-m\,n\,\gamma} P_n^{\rm BFKL}(Y_0) P_m^{\rm BFKL}(Y-Y_0) = \sum_{k=0}^{\infty}\frac{(-\gamma)^k}{k!}\underbrace{\left(\sum_{n=0}^{\infty} n^k P_n(Y_0)\right)}_{c_k(Y_0)}\underbrace{\left(\sum_{m=0}^{\infty} m^k P_m(Y-Y_0)\right)}_{c_k(Y-Y_0)}$$
$$= \sum_{k=0}^{\infty}(-\gamma N(Y_0)N(Y-Y_0))^k k! = \sum_{k=0}^{\infty}(-\gamma N(Y))^k k! = -\frac{e^{\frac{1}{\gamma N(Y)}}\mathrm{Ei}(-\frac{1}{N(Y)\gamma})}{\gamma N(Y)}. \qquad (4)$$

In Eq. (4) we used that (1) $N(Y-Y_0)N(Y_0) = N(Y)$, and (2) the factorial moments of the distribution of Eq. (3) has the following form:

$$M_k(Y) = k! N(Y)(N(Y)-1)^{k-1} \xrightarrow{N(Y)\gg 1} c_k(Y) = k! N^k(Y). \qquad (5)$$

One can see that $S(Y)$, being a function of $N(Y)$, does not depend on the reference frame. It turns out that this S matrix at large $Y - Y_0$ and Y_0 coincides with the one calculated in the unitarity toy model [10,50,51] theory, which is independent of a reference frame at any value of Y_0.

It is easy to see that the S matrix of Eq. (1) sums the large BFKL Pomeron loops shown in Fig. 1. Indeed, the contribution of large Pomeron loops can be written in the following form [9,52,54–56]:

$$S(Y) = \sum_n \frac{(-\gamma)^n}{n!} M_n(Y-Y_0) M_n(Y_0) \xrightarrow{Y-Y_0, Y_0\gg 1} \sum_n (-\gamma)^n n! (N(Y-Y_0)N(Y_0))^n$$
$$= \sum_n (-\gamma)^n n! (G_{\mathbb{P}}(Y-Y_0)G_{\mathbb{P}}(Y_0))^n = -\frac{e^{\frac{1}{\gamma N(Y)}}\mathrm{Ei}(-\frac{1}{N(Y)\gamma})}{\gamma N(Y)}, \qquad (6)$$

where M_n are the factorial moments that we replace by $M_n(Y) = n! N^n(Y)$ at large Y for distribution of Eq. (3). In Eq. (6), $N(Y) = N(Y-Y_0)N(Y_0)$. The advantage of this derivation is that it can be easily generalized to the QCD case, which is the main goal of this paper.

The factorial moments will play an essential role in our approach. Bearing this in mind, we wish to write the equation for them in the simple BFKL cascade of Eq. (2). The solution of Eq. (3) is easy to obtain introducing the generating function

$$Z(Y, u) = \sum_{n=1}^{\infty} P_n(Y) u^n. \qquad (7)$$

One can see that

$$P_n(Y) = \frac{1}{n!}\frac{\partial Z(Y,u)}{\partial u^n}\Big|_{u=0};$$
$$M_n(Y) = \sum_n (n(n-1)\dots(n-k+1))P_n(Y) = \frac{\partial Z(Y,u)}{\partial u^n}\Big|_{u=1}. \tag{8}$$

From Eq. (2), the equation for Z takes the form

$$\frac{\partial Z(Y,u)}{\partial Y} = -\Delta u(1-u)\frac{\partial Z(Y,u)}{\partial u}. \tag{9}$$

Taking n derivatives from Eq. (9) and substituting $u = 1$, we obtain the following equation for $M_n(Y)$:

$$\frac{\partial M_n(Y)}{\partial Y} = \Delta n M_n(Y) + \Delta n(n-1)M_{n-1}(Y). \tag{10}$$

Equation (10) has a more elegant form for $\rho_n(Y) = M_n(Y)/n!$,

$$\frac{\partial \rho_n(Y)}{\partial Y} = \Delta n \rho_n(Y) + \Delta(n-1)\rho_{n-1}(Y). \tag{11}$$

For sum of the large Pomeron loops, we have the following formula, using $\rho_n(Y)$:

$$S(Y) = \sum_n (-\gamma)^n n! \rho_n(Y - Y_0)\rho_n(Y_0). \tag{12}$$

Equation (12) has been generalized to the QCD case in Refs. [52,54] and we will use it in our approach.

Equation (9) for the generating function Z can be rewritten as the nonlinear equation for Z in the form

$$\frac{\partial Z(Y,u)}{\partial \Delta Y} = -Z(Y,u)(1 - Z(Y,u)). \tag{13}$$

Bearing in mind that $\rho_n(Y) = \frac{1}{n!}\frac{\partial Z(Y,u)}{\partial u^n}|_{u=1}$, we can obtain the equation for ρ_n differentiating Eq. (13) and putting $u = 1$,

$$\frac{\partial \rho_n(Y)}{\partial \Delta Y} = \rho_n(Y) + \sum_{k=1}^{n-1}\rho_{n-k}(Y)\rho_k(Y). \tag{14}$$

Subtracting this equation from Eq. (11), we obtain the following recurrence relation for ρ_n:

$$(n-1)\rho_n(Y) = \sum_{k=1}^{n-1}\rho_{n-k}(Y)\rho_k(Y) - (n-1)\rho_{n-1}(Y). \tag{15}$$

The solution to Eq. (15) has the form

$$\rho_n(Y) = \rho_1(Y)(\rho_1(Y) - 1)^{n-1}. \tag{16}$$

Summarizing what we have obtained in this section for the simple models of Reggeon field theory, we conclude that (i) the scattering amplitude at high energies can be calculated from Eq. (12) using the parton densities $\rho_n(Y)$; (ii) these parton densities satisfy the two evolution equations of Eqs. (11) and (14); and (iii) these two equations lead to the recurrence relation for ρ_n [see Eq. (15)]. The main goal of this paper is to generalize these ingredients to the case of QCD and obtain the QCD scattering amplitude at high energies.

III. SUMMING LARGE POMERON LOOPS IN QCD

As it has been mentioned, our main goal is to sum large Pomeron loops in QCD to obtain the scattering amplitude. Our approach includes two stages. First, we need to generalize Eq. (12) to the case of QCD. Actually, this problem has been solved in Refs. [1,9–11,52,54] and we are going to discuss it here. Second, we need to get the evolution equations for the parton densities and find their solutions. This problem has been partly solved in Ref. [54], but in this section we will find the second evolution equation and suggest the recurrence relations for ρ_n. Finally, we need to find the solution for the parton densities and this topic is the main one of this section.

A. BFKL parton cascade: Evaluation equations and recurrence relations for the parton densities

The simple Eq. (2) has been generalized to the QCD case in Refs. [1,9,54] and has the following form:

$$\frac{\partial P_n(Y,\boldsymbol{r},\boldsymbol{b};\{\boldsymbol{r}_i,\boldsymbol{b}_i\})}{\partial Y} = -\sum_{i=1}^{n}\omega_G(r_i)P_n(Y,\boldsymbol{r},\boldsymbol{b};\{\boldsymbol{r}_i,\boldsymbol{b}_i\}) + \bar{\alpha}_S\sum_{i=1}^{n-1}\frac{(\boldsymbol{r}_i+\boldsymbol{r}_n)^2}{(2\pi)r_i^2 r_n^2} \times P_{n-1}(Y,\boldsymbol{r},\boldsymbol{b};\{\boldsymbol{r}_j,\boldsymbol{b}_j,\boldsymbol{r}_i+\boldsymbol{r}_n,\boldsymbol{b}_{in}\}),$$

where $P_n(Y,r,b;\{r_i,b_i\})$ is the probability to have n dipoles of size r_i, at impact parameter b_i and at rapidity Y. $\boldsymbol{b}_{in}$ in Eq. (17) is equal to $\boldsymbol{b}_{in} = \boldsymbol{b}_i + \frac{1}{2}\boldsymbol{r}_i = \boldsymbol{b}_n - \frac{1}{2}\boldsymbol{r}_i$.

Equation (17) is a typical cascade equation in which the first term describes the reduction of the probability to find n dipoles, due to the possibility that one of n dipoles can decay into two dipoles of arbitrary sizes, while the second term describes the growth due to the splitting of $(n-1)$ dipoles into n dipoles. We introduce the generating functional [9]

$$Z(Y,\boldsymbol{r},\boldsymbol{b};[u_i]) = \sum_{n=1}^{\infty}\int P_n(Y,\boldsymbol{r},\boldsymbol{b};\{\boldsymbol{r}_i\boldsymbol{b}_i\}) \times \prod_{i=1}^{n} u(\boldsymbol{r}_i\boldsymbol{b}_i)d^2r_i d^2b_i, \tag{17}$$

where $u(\boldsymbol{r}_i\boldsymbol{b}_i) \equiv u_i$ is an arbitrary function. The initial and boundary conditions for Eq. (17) take the following form for the functional Z:

$$Z(Y=0,\boldsymbol{r},\boldsymbol{b};[u_i]) = u(\boldsymbol{r},\boldsymbol{b}), \tag{18a}$$

$$Z(Y,r,[u_i=1]) = 1. \tag{18b}$$

Multiplying both parts of Eq. (17) by $\prod_{i=1}^{n} u(\boldsymbol{r}_i\boldsymbol{b}_i)$ and integrating over r_i and b_i, we obtain the following linear functional equation [54]:

$$\frac{\partial Z(Y,\boldsymbol{r},\boldsymbol{b};[u_i])}{\partial Y} = \int d^2r' K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r})\left(-u(r,b) + u\left(\boldsymbol{r}',\boldsymbol{b}+\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}')\right)u\left(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}\boldsymbol{r}'\right)\right)\frac{\delta Z}{\delta u(r,b)}, \tag{19a}$$

$$K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r}) = \frac{\bar{\alpha}_S}{2\pi}\frac{r^2}{r'^2(\boldsymbol{r}-\boldsymbol{r}')^2}; \qquad \omega_G(r) = \int d^2r' K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r}). \tag{19b}$$

The n-dipole densities $\rho_n(r_1,b_1,\ldots,r_n,b_n)$ are defined as follows [54]:

$$\rho_n(r_1,b_1\ldots,r_n,b_n;Y-Y_0) = \frac{1}{n!}\prod_{i=1}^{n}\frac{\delta}{\delta u_i}Z(Y-Y_0;[u])|_{u=1}. \tag{20}$$

Taking nth functional derivatives from Eq. (19a) and substituting $u_1=1$, we obtain for ρ_n [54]

$$\frac{\partial\rho_n(\{r_i,b_i\})}{\partial Y} = -\sum_{i=1}^{n}\omega_G(r_i)\rho_n(\{r_i,b_i\}) + 2\sum_{i=1}^{n}\int\frac{d^2r'}{2\pi}\frac{r'^2}{r_i^2(\boldsymbol{r}_i-\boldsymbol{r}')^2}\rho_n(\ldots r',b_i-r'/2\ldots) + \bar{\alpha}_S\frac{(\boldsymbol{r}_i+\boldsymbol{r}_n)^2}{r_i^2r_n^2}\sum_{i=1}^{n-1}\rho_{n-1}(\ldots(\boldsymbol{r}_i+\boldsymbol{r}_n),b_{in}\ldots). \tag{21}$$

Introducing

$$\bar{\rho}_n(\boldsymbol{r},\boldsymbol{b};\{r_i,b_i\}) = \prod_{i=1}^{n} r_i^2\rho_n(\{r_i,b_i\}), \tag{22}$$

we reduce Eq. (21) to the following form:

$$\frac{\partial\bar{\rho}_n(\{\boldsymbol{r}_i,\boldsymbol{b}_i\})}{\partial Y} = \sum_{i=1}^{n}\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}_i-\boldsymbol{r}'|\boldsymbol{r}_i)\{\bar{\rho}_n(\{\boldsymbol{r}_j,\boldsymbol{b}_j\},\boldsymbol{r}',\boldsymbol{b}_i-(\boldsymbol{r}_i-\boldsymbol{r}')/2) + \bar{\rho}_n(\{\boldsymbol{r}_j,\boldsymbol{b}_j\},\boldsymbol{r}_i-\boldsymbol{r}',\boldsymbol{b}_i-\boldsymbol{r}'/2) - \bar{\rho}_n(\{\boldsymbol{r}_i,\boldsymbol{b}_i\})\} + \bar{\alpha}_S\sum_{i=1}^{n-1}\bar{\rho}_{n-1}(\ldots(\boldsymbol{r}_i+\boldsymbol{r}_n),b_{in}\ldots). \tag{23}$$

Equation (19a) can be rewritten as the nonlinear equation for Z [9,54],

$$\frac{\partial Z(Y,\boldsymbol{r},\boldsymbol{b};[u_i])}{\partial Y} = \int d^2r' K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r})\left(-Z(Y,\boldsymbol{r},\boldsymbol{b};[u_i]) + Z\left(\boldsymbol{r}',\boldsymbol{b}+\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}');[u_i]\right)Z\left(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}\boldsymbol{r}';[u_i]\right)\right). \tag{24}$$

Using the definition of Eq. (20) and differentiating Eq. (24) [see Eq. (20)], we obtain a new equation for ρ_n[1]:

$$\frac{\partial\rho_n(\boldsymbol{r},\boldsymbol{b};\{\boldsymbol{r}_i,\boldsymbol{b}_i\})}{\partial Y} = \bar{\alpha}_S\int d^2r' K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r}) \times\left\{\left(\rho_n\left(\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}');\{\boldsymbol{r}_i,\boldsymbol{b}_i\}\right) + \rho_n\left(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}\boldsymbol{r}';\{\boldsymbol{r}_i,\boldsymbol{b}_i\}\right) - \rho_n(\boldsymbol{r},\boldsymbol{b};\{\boldsymbol{r}_i,\boldsymbol{b}_i\})\right) + \bar{\alpha}_S\sum_{k=1}^{n-1}\rho_{n-k}\left(\boldsymbol{r}',\boldsymbol{b}+\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}';\{\boldsymbol{r}_i,\boldsymbol{b}_i\}\right)\rho_k\left(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}\boldsymbol{r}';\{\boldsymbol{r}_i,\boldsymbol{b}_i\}\right)\right\}. \tag{25}$$

[1]The analogous equation for the factorial moments of multiplicity distribution has been derived in Refs. [57,58].

For $\bar{\rho}_n$, it has the same form. This equation together with Eq. (23) leads to the recurrence relation for $\bar{\rho}_n$, which has the form

$$\int d^2r' K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r})\left\{\left(\rho_n\left(\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}');\{\boldsymbol{r}_i,\boldsymbol{b}_i\}\right)+\rho_n\left(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}\boldsymbol{r}';\{\boldsymbol{r}_i,\boldsymbol{b}_i\}\right)-\rho_n(\boldsymbol{r},\boldsymbol{b};\{\boldsymbol{r}_i,\boldsymbol{b}_i\})\right)\right.$$
$$\left.+\sum_{k=1}^{n-1}\rho_{n-k}\left(\boldsymbol{r}',\boldsymbol{b}+\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}';\{\boldsymbol{r}_i,\boldsymbol{b}_i\}\right)\rho_k\left(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}\boldsymbol{r}';\{\boldsymbol{r}_i,\boldsymbol{b}_i\}\right)\right\}=\sum_{i=1}^{n}\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}_i-\boldsymbol{r}'|\boldsymbol{r}_i)$$
$$\times\{\bar{\rho}_n(\boldsymbol{r},\{\boldsymbol{r}_j,\boldsymbol{b}_j\},\boldsymbol{r}',\boldsymbol{b}_i-(\boldsymbol{r}_i-\boldsymbol{r}')/2)+\bar{\rho}_n(\boldsymbol{r},\{\boldsymbol{r}_j,\boldsymbol{b}_j\},\boldsymbol{r}_i-\boldsymbol{r}',\boldsymbol{b}_i-\boldsymbol{r}'/2)-\bar{\rho}_n(\boldsymbol{r},\{\boldsymbol{r}_i,\boldsymbol{b}_i\})\}$$
$$+\sum_{i=1}^{n-1}\bar{\rho}_{n-1}(\boldsymbol{r},\ldots(\boldsymbol{r}_i+\boldsymbol{r}_n),b_{in}\ldots). \tag{26}$$

One can see that just from the general form of Eq. (23) the leading energy behavior stems from the inhomogeneous term of this equation and we expect that $\bar{\rho}_n\propto\sum_k^{n-1}\bar{\rho}_{n-k}\bar{\rho}_k$.

B. Main formula

The scattering amplitude shown in Fig. 1(a) can be written in the following way [54–56]:

$$A(Y,r,R;\boldsymbol{b})=\sum_{n=1}^{\infty}(-1)^{n+1}n!\int\prod\frac{d^2r_i}{r_i^4}\frac{d^2r'_i}{r'^4_i}d^2b'_i\int d^2\delta b_i\gamma^{BA}(r_1,r'_i,\boldsymbol{b}_i-\boldsymbol{b}'_i\equiv\delta\boldsymbol{b}_i)\bar{\rho}(Y-Y_0,\{\boldsymbol{r}_i,\boldsymbol{b}_i\})\bar{\rho}(Y_0,\{\boldsymbol{r}'_i,\boldsymbol{b}'_i\}), \tag{27}$$

where γ^{BA} is the scattering amplitude of two dipoles in the Born approximation of perturbative QCD. Considering $Y-Y_0\gg1$ and $Y_0\gg1$, one can see that the typical $\ln b_i^2\sim\sqrt{Y-Y_0}$ and $\ln b'^2_i\sim\sqrt{Y_0}$ are large, while in γ^{BA}, $\delta b_i\sim r_1,r'_i$. Hence, we can neglect the contribution of δb_i in $\bar{\rho}$. Therefore, in Eq. (27) enters $\int d^2\delta b_i\gamma^{BA}(r_1,r'_i,\boldsymbol{b}_i-\boldsymbol{b}'_i\equiv\delta\boldsymbol{b}_i)$, which can be written as [9]

$$\sigma^{BA}(\boldsymbol{r}_i,\boldsymbol{r}'_i)=\int d^2\delta b_i\gamma^{BA}(r_1,r'_i,\boldsymbol{b}_i-\boldsymbol{b}'_i\equiv\delta\boldsymbol{b}_i)=4\pi\bar{\alpha}_S^2\int\frac{dl}{l^3}(1-J_0(lr_i))(1-J_0(lr'_i)). \tag{28}$$

In Eq. (27), $\boldsymbol{b}_i=\boldsymbol{b}-\boldsymbol{b}'_i$.

C. Solutions for $\bar{\rho}_n(Y,\{\boldsymbol{r}_i,\boldsymbol{b}_i\})$

1. $\bar{\rho}_1(Y,\boldsymbol{r},\boldsymbol{r}_1,\boldsymbol{b}_1)$

From Eq. (23) for $\bar{\rho}_1$, we have the linear equation

$$\frac{\partial\bar{\rho}_1(Y;r_1,b)}{\bar{\alpha}_S\partial Y}=-\omega_G(r_1)\bar{\rho}_1(Y;r_1,b)+2\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r})\bar{\rho}_1(Y,r',b). \tag{29}$$

The physical meaning of ρ_1 is clear from Eq. (20): it is the mean number of dipoles with size r_1 in the partonic wave function of the projectile or target. It is proven in Ref. [3] that the eigenfunction of the BFKL equation has the following form:

$$\phi_\gamma(\boldsymbol{r},\boldsymbol{r}_1,\boldsymbol{b}_1)=\left(\frac{r^2r_1^2}{(\boldsymbol{b}_1+\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}_1))^2(\boldsymbol{b}_1-\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}_1))^2}\right)^\gamma\xrightarrow{b_1\gg r,r_1}\left(\frac{r^2r_1^2}{b_1^4}\right)^\gamma\equiv e^{\gamma\xi} \tag{30}$$

for any kernel that satisfies the conformal symmetry. In Eq. (30), r is the size of the initial dipole at $Y=0$, while r_1 is the size of the dipole with rapidity Y. As has been discussed in the previous section, the typical b_i in $\bar{\rho}_n$ in Eq. (27) is large. Hence, we can use the variable ξ from Eq. (30).

For the kernel of the leading-order BFKL equation [see Eq. (19b)], the eigenvalues take the form

$$\omega(\bar{\alpha}_S,\gamma)=\bar{\alpha}_S\chi(\gamma)=\bar{\alpha}_S(2\psi(1)-\psi(\gamma)-\psi(1-\gamma))\xrightarrow{\gamma\to\frac{1}{2}}\underbrace{4\ln2\bar{\alpha}_S}_{\Delta_{\rm BFKL}}+\underbrace{14\zeta(3)\bar{\alpha}_S}_{D}\left(\gamma-\frac{1}{2}\right)^2, \tag{31}$$

where $\psi(z)$ is the Euler psi function $\psi(z) = d\ln\Gamma(z)/dz$, $\bar{\alpha}_S = N_c\alpha_S/\pi$, where N_c is the number of colors. The general solution to Eq. (29) takes the form

$$\bar{\rho}_1(Y;r_1,b) = \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\omega}{2\pi i}\frac{1}{\omega-\omega(\bar{\alpha}_S,\gamma)}e^{\omega Y+\gamma\xi}\phi_{in}(\gamma) = \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}e^{\omega(\bar{\alpha}_S,\gamma)Y+\gamma\xi}\phi_{in}(\gamma). \tag{32}$$

Function $\phi_{in}(\gamma)$ has been found from the initial conditions at $Y = 0$ [3] (see also Refs. [4,9]): $\phi_{in} = i\nu/\pi$, where $\gamma = \frac{1}{2} + i\nu$. For large Y we can estimate the integral over γ using the steepest descent method. The equation for the saddle point has the form

$$\frac{d\omega(\bar{\alpha}_S,\gamma)}{d\gamma}Y|_{\gamma=\gamma_{\rm SP}} + \xi = 0. \tag{33}$$

The solution to Eq. (33) gives $\gamma_{\rm SP} = \frac{1}{2} - \frac{\xi}{2DY}\xrightarrow{Y\gg\xi}\frac{1}{2}$.

Plugging Eq. (33) in Eq. (32) and using $\phi_{in} = i\nu_{\rm SP}/\pi$, we obtain

$$\bar{\rho}_1^{\rm d.a.}(Y;r,r_1,b) = \frac{2}{(DY)^{3/2}}\xi e^{\Delta_{\rm BFKL}Y-\frac{\xi^2}{4DY}}e^{\frac{1}{2}\xi} = \frac{2}{(DY)^{3/2}}\frac{rr_1}{b_1^2}\ln\left(\frac{b_1^4}{r^2r_1^2}\right)\times\exp\left(\Delta_{\rm BFKL}Y - \frac{\ln^2(\frac{r^2r_1^2}{b_1^4})}{4DY}\right), \tag{34}$$

where d.a. denotes diffusion approximation for the BFKL kernel in the vicinity of $\gamma = \frac{1}{2}$ [see Eq. (19b)], which has been used in deriving Eq. (34). It is instructive to note that for $\bar{\rho}_1(Y;r_1,b)$, Eq. (25) has the same form as Eq. (23) and the same solution as Eq. (34). We will use below $\bar{\rho}_n$ in the momentum representation, viz.

$$\bar{\rho}_n(Y,\boldsymbol{k}_T,\boldsymbol{b};Y,\{\boldsymbol{r}_i,\boldsymbol{b}_i\}) = \int d^2r\, e^{-i\boldsymbol{k}_T\cdot\boldsymbol{r}}\frac{\bar{\rho}_n(Y,\boldsymbol{r},\boldsymbol{b};\{\boldsymbol{r}_i,\boldsymbol{b}_i\})}{r^2}. \tag{35}$$

It turns out that in the vicinity of $\gamma_i = \frac{1}{2}$ we can obtain the momentum representation of $\bar{\rho}_n$ (and vice versa) using the simple substitute $r_i \rightleftarrows k_T^i/2$ [see Ref. [59] formula 6.561 (14)]. Hence,

$$G_{\mathbb{P}}(Y;k_T,r_1,b) \equiv \bar{\rho}_1^{\rm d.a.}(Y;k_T,r_1,b) = \frac{2}{(DY)^{3/2}}\xi' e^{\Delta_{\rm BFKL}Y-\frac{\xi'^2}{4DY}}e^{\frac{1}{2}\xi'}, \tag{36}$$

with $\xi' = \ln(\frac{4r_1^2}{k_T^2b_1^4})$. In conclusion, we see that $\bar{\rho}_1$ is described by the exchange of the BFKL Pomeron, which in diffusion approximation has the form of Eq. (36).

2. $\bar{\rho}_2(Y,\boldsymbol{r},\boldsymbol{r}_1,\boldsymbol{b}_1,\boldsymbol{r}_2,\boldsymbol{b}_2)$

For $\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)$, Eq. (26) can be rewritten as follows:

$$\begin{aligned}&\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}_1-\boldsymbol{r}'|\boldsymbol{r}_1)\{\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}',b_1,\boldsymbol{r}_2,b_2)+\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1-\boldsymbol{r}',b_1,\boldsymbol{r}_2,b_2)-\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)\}\\&+\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}_2-\boldsymbol{r}'|\boldsymbol{r}_2)\{\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}',b_2)+\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2-\boldsymbol{r}',b_2)-\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)\}\\&+\rho_1(Y,\boldsymbol{r};\boldsymbol{r}_1+\boldsymbol{r}_2,b)\\&=\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r})\left\{\left(\bar{\rho}_2\left(Y,\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}');\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2\right)+\bar{\rho}_2\left(Y,\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}\boldsymbol{r}';\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2\right)\right.\right.\\&\left.\left.-\bar{\rho}_2(Y,\boldsymbol{r},\boldsymbol{b};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)\right)+\bar{\rho}_1(Y,\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1)\bar{\rho}_1(Y,\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2)\right\}.\end{aligned} \tag{37}$$

In Eq. (20) we neglect the shifts in the impact parameters due to the sizes of dipoles, since in Eq. (27) all b_i are much larger then r_i.

The simplest solution to Eq. (20) we obtain in diffusion approximation for the BFKL kernel [see Eq. (31)]. In this approximation

$$\begin{aligned}&\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}_1-\boldsymbol{r}'|\boldsymbol{r}_1)\{\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}',b_1,\boldsymbol{r}_2,b_2)+\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2-\boldsymbol{r}',b_2)-\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)\}\\&=\left(\Delta_{\rm BFKL}+D\frac{\partial^2}{\partial\xi_1^2}\right)\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)\equiv L^{\rm d.a.}\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)\end{aligned}$$

for all r_i and r in Eq. (20).

We resolve the recurrence relation of Eq. (20) by neglecting all contributions of the order of $\frac{\xi_i^2}{4DY^2}$ in kernels K, replacing them by Δ_{BFKL} [see Eq. (33)]. Indeed, in this case

$$\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)=\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\left\{\int d^2r'K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r})\bar{\rho}_1(Y,\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1)\rho_1(Y,\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2)-\bar{\rho}_1(Y,\boldsymbol{r};\boldsymbol{r}_1+\boldsymbol{r}_2,b)\right\}. \quad (38)$$

Equation (38) can be rewritten in more economic form going to the momentum representation [see Eq. (35)],

$$\bar{\rho}_2(Y,\boldsymbol{k}_T;\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)=\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\{\bar{\rho}_1(Y,\boldsymbol{k}_T,\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1)\rho_1(Y,\boldsymbol{k}_T,\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2)-\bar{\rho}_1(Y,\boldsymbol{k}_T;\boldsymbol{r}_1+\boldsymbol{r}_2,b)\}, \quad (39)$$

which leads to the following estimates in the diffusion approximation:

$$\begin{aligned}\bar{\rho}_2(Y,\xi_1',\xi_2')&=\left(\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\right)\{\bar{\rho}_1^{\mathrm{d.a.}}(Y;\xi_1')\bar{\rho}_1^{\mathrm{d.a.}}(Y;\xi_2')-\bar{\rho}_1^{\mathrm{d.a.}}(Y;\xi_{12}')\}\\ &\xrightarrow{Y\gg1}\left(\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\right)\bar{\rho}_1^{\mathrm{d.a.}}(Y;\xi_1')\bar{\rho}_1^{\mathrm{d.a.}}(Y;\xi_2')=\left(\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\right)G_{\mathbb{P}}(Y;\xi_1')G_{\mathbb{P}}(Y;\xi_2'), \end{aligned}\quad (40)$$

where ξ_i' is defined in Eq. (36). ξ_{ik}' is the same as ξ_i' where r_i is replaced by $|\boldsymbol{r}_i+\boldsymbol{r}_k|$. For large values of Y, Eq. (40) has a very simple meaning shown in Fig. 2: it stems from the simple "fan" diagram after integration over Y'. Note that in momentum representation the triple Pomeron vertex is equal to $\bar{\alpha}_S$.

3. $\bar{\boldsymbol{\rho}}_3(\boldsymbol{Y},\boldsymbol{r},\boldsymbol{r}_1,\boldsymbol{b}_1,\boldsymbol{r}_2,\boldsymbol{b}_2,\boldsymbol{r}_3,\boldsymbol{b}_2)$

From Eq. (26) we have the following equation for $\bar{\rho}_3(Y,k_T,r_1,b_1,r_2,b_2,r_3,b_2)$:

$$\begin{aligned}&\int d^2r'K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r})\Big\{\Big(\bar{\rho}_3\Big(Y,\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}');\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2,\boldsymbol{r}_3,b_3\Big)+\bar{\rho}_3\Big(Y,\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b}-\frac{1}{2}\boldsymbol{r}';\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2,\boldsymbol{r}_3,b_3\Big)\\
&-\bar{\rho}_3(Y,\boldsymbol{r},\boldsymbol{b};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2)\Big)+\bar{\rho}_1(Y,\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1)\bar{\rho}_2(Y,\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2,\boldsymbol{r}_3,b_3)\\
&+\bar{\rho}_2(Y,\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1,\boldsymbol{r}_3,b_3)\bar{\rho}_1(Y,\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2)\Big\}=\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}_1-\boldsymbol{r}'|\boldsymbol{r}_1)\\
&\times\{\bar{\rho}_3(Y,\boldsymbol{k}_T;\boldsymbol{r}',b_1,\boldsymbol{r}_2,b_2,\boldsymbol{r}_3,b_3)+\bar{\rho}_3(Y,\boldsymbol{k}_T;\boldsymbol{r}_1-\boldsymbol{r}',b_1,\boldsymbol{r}_2,b_2,\boldsymbol{r}_3,b_3)-\bar{\rho}_3(Y,\boldsymbol{k}_T,\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2,\boldsymbol{r}_3,b_3)\}\\
&+\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}_2-\boldsymbol{r}'|\boldsymbol{r}_2)\\
&\times\{\bar{\rho}_3(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}',b_2,\boldsymbol{r}_3,b_3)+\bar{\rho}_3(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2-\boldsymbol{r}',b_2,\boldsymbol{r}_3,b_3)-\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2,\boldsymbol{r}_3,b_3)\}\\
&+\int\frac{d^2r'}{2\pi}K(\boldsymbol{r}',\boldsymbol{r}_3-\boldsymbol{r}'|\boldsymbol{r}_3)\\
&\times\{\bar{\rho}_3(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2,\boldsymbol{r}',b_3)+\bar{\rho}_3(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2,\boldsymbol{r}_3-\boldsymbol{r}',b_3)-\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2,b_2,\boldsymbol{r}_3,b_3)\}\\
&+\{\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1+\boldsymbol{r}_3,b_1,\boldsymbol{r}_2,b_2)+\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2+\boldsymbol{r}_3,b_2)\}.\end{aligned}\quad (41)$$

Replacing $\int d^2r'K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r})\{\ldots\}=\Delta_{\mathrm{BFKL}}\bar{\rho}_3(Y,\boldsymbol{r};\boldsymbol{r}_1,\boldsymbol{b}_1,\boldsymbol{r}_2,\boldsymbol{b}_2,\boldsymbol{r}_3,\boldsymbol{b}_3)$, we obtain that

$$\begin{aligned}\bar{\rho}_3(Y,\boldsymbol{r};\boldsymbol{r}_1,\boldsymbol{b}_1,\boldsymbol{r}_2,\boldsymbol{b}_2,\boldsymbol{r}_3,\boldsymbol{b}_3)=\frac{1}{2}\Big\{\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}&\int d^2r'K(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r}) &(42)\\
&\times\{\bar{\rho}_2(Y,\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1,\boldsymbol{r}_3,\boldsymbol{b}_2)\bar{\rho}_1(Y,\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2)\\
&+\bar{\rho}_1(Y,\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1)\bar{\rho}_2(Y,\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2,\boldsymbol{r}_3,\boldsymbol{b}_3)\}\\
&-\{\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1+\boldsymbol{r}_3,b_1,\boldsymbol{r}_2,b_2)+\bar{\rho}_2(Y,\boldsymbol{r};\boldsymbol{r}_1,b_1,\boldsymbol{r}_2+\boldsymbol{r}_3,b_2)\}\Big\}. &(43)\end{aligned}$$

This equation has a simplified form in the momentum representation [see Eq. (35)]

$$\bar{\rho}_3(Y,\boldsymbol{k}_T;\boldsymbol{r}_1,\boldsymbol{b}_1,\boldsymbol{r}_2,\boldsymbol{b}_2,\boldsymbol{r}_3,\boldsymbol{b}_3) = \frac{1}{2}\left(\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\right)\{\bar{\rho}_2(Y,\boldsymbol{k}_T,\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1,\boldsymbol{r}_3,\boldsymbol{b}_2)\bar{\rho}_1(Y,\boldsymbol{k}_T,\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2)$$
$$+\bar{\rho}_1(Y,\boldsymbol{k}_T,\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1)\bar{\rho}_2(Y,\boldsymbol{k}_T,\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2,\boldsymbol{r}_3,\boldsymbol{b}_2) - (\bar{\rho}_2(Y,\boldsymbol{k}_T;\boldsymbol{r}_1+\boldsymbol{r}_3,b_1,\boldsymbol{r}_2,b_2)$$
$$+\bar{\rho}_2(Y,\boldsymbol{k}_T;\boldsymbol{r}_1,b_1,\boldsymbol{r}_2+\boldsymbol{r}_3,b_2))\}. \tag{44}$$

For the toy model in which $\bar{\rho}_n$ do not depend on the dipole sizes, Eq. (44) leads to $\bar{\rho}_3 = \bar{\rho}_1(\bar{\rho}_1-1)^2$, if we assume that $\Delta_{\mathrm{BFKL}} = \bar{\alpha}_S$. One can see that the calculations started to be cumbersome, but for our approach the most important conclusion is that the main contribution, which is proportional to $e^{3\Delta_{\mathrm{BFKL}}Y}$, has a very simple form,

$$\bar{\rho}_3(Y,\boldsymbol{k}_T;\boldsymbol{r}_1,\boldsymbol{b}_1,\boldsymbol{r}_2,\boldsymbol{b}_2,\boldsymbol{r}_3,\boldsymbol{b}_3) = \left(\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\right)^2\bar{\rho}_1(Y,\boldsymbol{k}_T,\boldsymbol{b};\boldsymbol{r}_1,\boldsymbol{b}_1)\bar{\rho}_1(Y,\boldsymbol{k}_T,\boldsymbol{b};\boldsymbol{r}_2,\boldsymbol{b}_2)\bar{\rho}_1(Y,\boldsymbol{r},\boldsymbol{b};\boldsymbol{r}_3,\boldsymbol{b}_3) + \mathcal{O}(e^{2\Delta_{\mathrm{BFKL}}Y})\}$$
$$\times\left(\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\right)^2\bar{\rho}_1^{\mathrm{d.a}}(Y;\xi'_1)\bar{\rho}_1^{\mathrm{d.a.}}(Y;\xi'_2)\bar{\rho}_1^{\mathrm{d.a.}}(Y;\xi'_3) = \left(\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\right)^2 G_{\mathbb{P}}(Y;\xi'_1)G_{\mathbb{P}}(Y;\xi'_2)G_{\mathbb{P}}(Y;\xi'_3). \tag{45}$$

We see again that at large Y $\bar{\rho}_3$ is described by the fan diagram of the same type as in Fig. 2, but with three Pomerons, in which two integrations of the positions of two triple Pomeron vertices lead to factor $(1/\Delta_{\mathrm{BFKL}})^2$, while $\bar{\alpha}_S^2$ appears due to the value of the triple Pomeron vertex equal to $\bar{\alpha}_S$ and we have two vertices in these diagrams.

4. $\bar{\rho}_n(Y,\boldsymbol{r},\{\boldsymbol{r}_i,\boldsymbol{b}_i\})$ at high energies

Having solutions for $\bar{\rho}_2$ and $\bar{\rho}_3$, one can see that the leading term of the solution to Eq. (26), which behaves as $\exp(n\Delta_{\mathrm{BFKL}}Y)$ at high energies, has the form

$$\bar{\rho}_n(Y,\{\xi'_i\}) = \left(\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\right)^{n-1}\underbrace{\prod_{i=1}^{n}\bar{\rho}_1^{\mathrm{d.a.}}(Y;\xi'_i)}_{\sim\exp(n\Delta_{\mathrm{BFKL}}Y)}. \tag{46}$$

We can check by the direct substitution in Eq. (26) that $\bar{\rho}_n$ is equal to

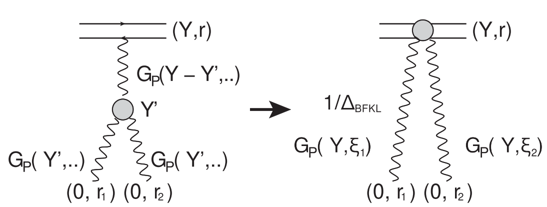

FIG. 2. The graphic form of Eq. (40). The wavy lines describe the BFKL Pomerons. The shaded circle corresponds to the triple Pomeron vertex. Factor $1/\Delta_{\mathrm{BFKL}}$ stems from integration over Y' in the triple Pomeron diagram.

$$\bar{\rho}_n(Y,\{\xi_i\}) = \left(\frac{\bar{\alpha}_S}{\Delta_{\mathrm{BFKL}}}\right)^{n-1}\left\{\underbrace{\prod_{i=1}^{n}\bar{\rho}_1^{\mathrm{d.a.}}(Y;\xi_i)}_{\sim\exp(n\Delta_{\mathrm{BFKL}})}\right.$$
$$\left.-\frac{1}{n-1}\underbrace{\sum_{i=1}^{n-1}\bar{\rho}_{n-1}^{\mathrm{d.a.}}(\{\xi_j,\xi_{i,n}\})}_{\sim\exp((n-1)\Delta_{\mathrm{BFKL}})}\right\}. \tag{47}$$

Accuracy of this solution is on the order of $n(\xi_i^2/(4DY))^2$ and we will show below that the typical value of n does not increase with Y.

IV. SCATTERING AMPLITUDE

A. The BFKL Pomeron exchange

From Eq. (27) the contribution of the single BFKL Pomeron exchange to the scattering amplitude is equal to the following expression [9]:

$$A^{\mathrm{BFKL}}(Y,\boldsymbol{k}_T,\boldsymbol{k}'_T;\boldsymbol{b})$$
$$= \int\frac{d^2r_1}{r_2^4}\frac{d^2r'_1}{r_1'^4}d^2b'_1\bar{\rho}(Y-Y_0,\boldsymbol{k}_T;\boldsymbol{r}_1,\boldsymbol{b}-\boldsymbol{b}'_1)\sigma^{BA}(\boldsymbol{r}_1,\boldsymbol{r}'_1)$$
$$\times\bar{\rho}(Y_0,\boldsymbol{k}'_T;\boldsymbol{r}'_1,\boldsymbol{b}'_1), \tag{48}$$

where $\boldsymbol{k}'_T$ appears as the momentum variable corresponding the R dependence of the scattering amplitude.

Equation (48) has been estimated (see Refs. [9,11]), however, for completeness of presentation, we briefly outline here the main points of these estimates. Plugging Eqs. (28) and (32) into Eq. (48), we can first integrate over r_1 and r'_1, obtaining the contribution $\propto l^{2+2i\nu+2i\nu'}$, with $\gamma_r = \frac{1}{2}+i\nu$ and $\gamma'_{r'} = \frac{1}{2}+i\nu'$. Integration over l leads to the pole $1/(\nu+\nu')$, for which the contribution leads to the independence of the amplitude with the value of Y_0. The integral over $d^2b'_1$ has the form

$$\int d^2 b'_1 \frac{1}{((\boldsymbol{b}-\boldsymbol{b}'_1)^2 b'^2_1)^{1+i\nu}} = \frac{2}{i\nu}\frac{1}{b^{2(1+i\nu)}}. \tag{49}$$

In Eq. (49), the main contributions stem from two kinematic regions: $b'_1 \ll b$ and $|\boldsymbol{b}-\boldsymbol{b}'_1| \ll b$. Finally, we obtain the result of Ref. [9],

$$A^{\mathrm{BFKL}}(Y,\boldsymbol{k}_T,\boldsymbol{k}'_T;\boldsymbol{b}) = \frac{8\bar{\alpha}_S^2\pi}{(DY)^{3/2}}\frac{4}{k_T k'_T b^2}\ln\left(\frac{k_T k'_T b^2}{4}\right)\exp\left(\Delta_{\mathrm{BFKL}}Y - \frac{\ln^2(\frac{16}{b^4 k_T^2 k'^2_T})}{4DY}\right) = \frac{8\pi\bar{\alpha}_S^2}{(DY)^{3/2}}\xi e^{-\xi}\exp\left(\Delta_{\mathrm{BFKL}}Y - \frac{\xi^2}{4DY}\right), \tag{50}$$

where $\xi = \ln(\frac{k_T k'_T b^2}{4})$.

B. Scattering amplitude from the main formula in the momentum representation

Plugging Eq. (50) into our master equation [see Eq. (27)], one can see that the scattering amplitude takes the following form:

$$\begin{aligned} A(Y,\boldsymbol{k}_T,\boldsymbol{k}'_T;\boldsymbol{b}) &= \sum_{n=1}^{\infty}(-1)^{n+1}n!\left(\frac{\bar{\alpha}_S^2}{\Delta^2_{\mathrm{BFKL}}}\right)^{n-1}(A^{\mathrm{BFKL}}(Y,k_T,k'_T;\boldsymbol{b}))^n \\ &= \left(\frac{\Delta^2_{\mathrm{BFKL}}}{\bar{\alpha}_S^2}\right)\left\{1 - e^{\frac{1}{\kappa}}\Gamma\left(0,\frac{1}{\kappa}\right)\Big/\kappa\right\} \to \begin{cases} (\frac{\Delta^2_{\mathrm{BFKL}}}{\bar{\alpha}_S^2})\kappa = A^{\mathrm{BFKL}}(Y,r,R;\boldsymbol{b}) & \text{for } \kappa \ll 1; \\ (\frac{\Delta^2_{\mathrm{BFKL}}}{\bar{\alpha}_S^2})(1 + \frac{-\ln\kappa + C_E}{\kappa}) & \text{for } \kappa \gg 1, \end{cases} \end{aligned} \tag{51}$$

where $\kappa = (\frac{\bar{\alpha}_S^2}{\Delta^2_{\mathrm{BFKL}}})A^{\mathrm{BFKL}}(Y,\boldsymbol{k}_T,\boldsymbol{k}'_T;\boldsymbol{b})$. One can see that at low energies (small values of Y) the scattering amplitude reproduces the exchange of one BFKL Pomeron. At high energies (at $Y \gg 1$), the amplitude approaches the constant value $(\frac{\Delta_{\mathrm{BFKL}}}{\bar{\alpha}_S})$. Since this amplitude is in the momentum representation, the unitarity limit for it is $\frac{1}{2}\ln k_T^2 \frac{1}{2}\ln k'^2_T$, but not the unity.

The scattering amplitude of Eq. (51) does not generate a correct behavior of the cross section, which increases as a power of the energy, resulting from the powerlike behavior of the scattering amplitude at large impact parameters. From Eq. (51) one can see that the corrections at large values of Y show the increase with the growth of b, demonstrating that in the scattering amplitude we have even more severe problems with the Froissart theorem [60] than for the BK evolution [61–64].

In all our estimates, we assume that $\sum_1^n \frac{\xi'_i}{2D,Y} \ll 1$. Therefore, we need to estimate the typical values of n in this sum. From Eq. (51), we can find the average value of n,

$$\begin{aligned} \bar{n} &= \frac{d\ln A(Y,r,R;\boldsymbol{b})}{d\ln A^{\mathrm{BFKL}}} \\ &= \left(\frac{\mu e^{1/\kappa}\Gamma(0,\frac{1}{\kappa})}{\kappa^2} - \frac{\mu}{\kappa} + \frac{\mu e^{1/\kappa}\Gamma(0,\frac{1}{\kappa})}{\kappa}\right)\Big/\left(1 - \frac{\mu e^{1/\kappa}\Gamma(0,\frac{1}{\kappa})}{\kappa}\right), \end{aligned} \tag{52}$$

where $\kappa = (\frac{\bar{\alpha}_S^2}{\Delta^2_{\mathrm{BFKL}}})A^{\mathrm{BFKL}}(Y,r,R;\boldsymbol{b})$ and $\mu = \Delta^2_{\mathrm{BFKL}}/\bar{\alpha}_S^2$. One can see that $\bar{n}$ decreases at large values of Y. Therefore, our assumption $\sum_1^n \frac{\xi'_i}{2D,Y} \ll 1$ looks plausible.

One can also see that, at fixed b, the scattering amplitude approaches the constant value of $\Delta_{\mathrm{BFKL}}/\bar{\alpha}_S$ as follows:

$$\begin{aligned} &A(Y,\boldsymbol{k}_T,\boldsymbol{k}'_T;\boldsymbol{b}) = 1 - \mathrm{Const}\,\ln(A^{\mathrm{BFKL}}(Y,r,R;\boldsymbol{b}))/ \\ &A^{\mathrm{BFKL}}(Y,\boldsymbol{k}_T,\boldsymbol{k}'_T;\boldsymbol{b}) = 1 - \mathcal{C}(\xi',Y)(e^{\xi'}Q_s^2(Y))^{-\bar{\gamma}}, \end{aligned} \tag{53}$$

where the saturation momentum $Q_s^2(Y) = \exp(2DY(1 + \frac{\Delta_{\mathrm{BFKL}}}{D}))$, $\bar{\gamma} = \frac{1}{2} + \frac{\Delta_{\mathrm{BFKL}}}{D}$, and function $\mathcal{C}$ is a smooth function of ξ and Y.

It is instructive to note that at first sight such approach is in contradiction both with the BK nonlinear evolution equation [65] and with estimates for the scattering dipole-dipole amplitude in Ref. [53]. However, we will show in the next section that it is not the case, considering the solution to the BK nonlinear equation in the diffusion approximation for the BFKL kernel.

C. Solution to BK equation with the diffusion kernel at high energies

The surprising result is that the amplitude in the momentum representation turns out to be constant at high energies and it does not show the geometric scaling behavior. In this section, we show that both these features are the artifact of the simplified BFKL kernel that we have used. As have been discussed, we used the diffusion approximation of Eq. (31) (at $\gamma \to \frac{1}{2}$) for the BFKL kernel. The BK equation in the momentum representation (k_T) takes the following form:

$$\frac{\partial \tilde{N}(k_\perp, b; Y)}{\partial Y} = \bar{\alpha}_S \left\{ \chi\left(-\frac{\partial}{\partial \tilde{\xi}}\right) \tilde{N}(k_\perp, b; Y) - \tilde{N}^2(k_\perp, b; Y) \right\}, \tag{54}$$

with $\tilde{\xi} = \ln k_T^2$ and

$$\bar{\alpha}_S \chi\left(-\frac{\partial}{\partial \tilde{\xi}}\right) = \Delta_{\mathrm{BFKL}} + D \frac{\partial^2}{\partial \tilde{\xi}^2} \tag{55}$$

[see Eq. (31) at $\gamma \to \frac{1}{2}$].

The asymptotic solution to Eq. (54) has a simple form: $\tilde{N}^{\mathrm{asymp}}(k_\perp) = \frac{\Delta_{\mathrm{BFKL}}}{\bar{\alpha}_S}$. Plugging in Eq. (54) $\tilde{N}(k_\perp, b; Y) = \tilde{N}^{\mathrm{asymp}}(k_\perp) - \Delta\tilde{N}(k_\perp, b; Y)$ and considering $\Delta\tilde{N}(k_\perp, b; Y) \ll 1$, we obtain the following linear equation for $\Delta\tilde{N}(k_\perp, b; Y)$:

$$\frac{\partial \Delta\tilde{N}(k_\perp, b; Y)}{\partial Y} = -\Delta_{\mathrm{BFKL}} \Delta\tilde{N}(k_\perp, b; Y) + D \frac{\partial^2}{\partial \tilde{\xi}^2} \Delta\tilde{N}(k_\perp, b; Y). \tag{56}$$

Equation (56) has the same form as the linear BFKL equation but with the negative intercept. Hence, the solution to Eq. (56) can be obtain from Eq. (36),

$$\Delta\tilde{N}(k_\perp, b; Y) = \frac{2}{(DY)^{3/2}} \xi' e^{-\Delta_{\mathrm{BFKL}} Y - \frac{\xi'^2}{4DY}} e^{\frac{1}{2}\xi'}. \tag{57}$$

Therefore, one can see that (i) the solution at high energies does not show the geometric scaling behavior that was predicted in Ref. [65], and (ii) at large Y it decreases as $\exp(-\mathrm{Const}Y)$ instead of $\exp(-\mathrm{Const}Y^2)$ [65]. Note that we use the same procedure to Eq. (54) as was developed in Ref. [65] to the general kernel of the BFKL equation. On the other hand, applying the approach of Ref. [53] to the scattering amplitude taking into account Eq. (57), we obtain Eq. (51) for $\kappa \gg 1$.

Concluding, we state that the scattering amplitude of Eq. (51) gives more microscopic insight in the structure of the scattering amplitude and reproduces both approaches of Refs. [52,65]. It is worth mentioning that Eq. (54), being the Fisher-Kolmogorov-Petrovskii-Piskunov (F-KPP) equation [66,67], shows the geometrical scaling behavior in the preasymptotic region in the vicinity of the saturation scale. Equation (57) also indicates that our assumption $\xi/Y \ll 1$ is not substantial for the main features of the amplitude behavior.

D. Dipole-dipole scattering amplitude at high energies

The dipole-dipole scattering amplitude takes the following form:

$$A(Y, r, R; \boldsymbol{b}) = r^2 R^2 \int k_T dk_T J_0(k_T r) \int k'_T dk'_T J_0(k'_T R) \quad A(Y, \boldsymbol{k}_T, \boldsymbol{k}'_T; \boldsymbol{b}). \tag{58}$$

We use the Mellin transform for $J_0(k_T r)$ (see formula 6.8.1 of Ref. [68]),

$$J_0(k_T r) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} (k_T r)^{-\gamma} \frac{2^{\gamma-1}\Gamma(\frac{1}{2}\gamma)}{\Gamma(1 - \frac{1}{2}\gamma)}. \tag{59}$$

Note that in Eq. (59) we take $\frac{3}{2} > \epsilon > 1$.

Introducing a new variable $k_T k'_t = \zeta$, we can rewrite Eq. (58) in the form

$$A(Y, r, R; \boldsymbol{b}) = r^2 R^2 \int \zeta d\zeta \frac{dk_T}{k_T} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} (k_T r)^{-\gamma} \frac{2^{\gamma-1}\Gamma(\frac{1}{2}\gamma)}{\Gamma(1 - \frac{1}{2}\gamma)} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} \left(\frac{\zeta}{k_T} R\right)^{-\gamma'} \frac{2^{\gamma'-1}\Gamma(\frac{1}{2}\gamma')}{\Gamma(1 - \frac{1}{2}\gamma')} A(Y, \boldsymbol{k}_T, \boldsymbol{k}'_T; \boldsymbol{b}). \tag{60}$$

Noting that $A(Y, \boldsymbol{k}_T, \boldsymbol{k}'_T; \boldsymbol{b})$ depends only on ζ, we can integrate Eq. (60) over k_T and γ', which results in

$$A(Y, r, R; \boldsymbol{b}) = r^2 R^2 \int \zeta d\zeta \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} (\zeta R r)^{-\gamma} \frac{4^{\gamma-1}\Gamma^2(\frac{1}{2}\gamma)}{\Gamma^2(1 - \frac{1}{2}\gamma)} A(Y, \boldsymbol{k}_T, \boldsymbol{k}'_T; \boldsymbol{b}). \tag{61}$$

To take the integral over ζ, we simplify the expression for $A^{\mathrm{BFKL}}(Y, \boldsymbol{k}_T, \boldsymbol{k}'_T; \boldsymbol{b})$ of Eq. (50) considering $\gamma_{\mathrm{SP}} = \frac{1}{2}$ and reducing it to the following expression:

$$A^{\mathrm{BFKL}}(Y,\boldsymbol{k}_T,\boldsymbol{k}'_T;\boldsymbol{b}) = \frac{8\pi}{(DY)^{3/2}}\frac{4}{k_T k'_T b^2}\ln\left(\frac{k_T k'_T b^2}{4}\right)\exp\left(\Delta_{\mathrm{BFKL}}Y - \frac{\ln^2\left(\frac{16}{b^4 k_T^2 k'^2_T}\right)}{4DY}\right)$$
$$= \frac{1}{k_T k'_T} N(k_T \to 2/r; k'_T \to 2/R) = \frac{N}{\zeta}, \tag{62}$$

where we use Eq. (36). The new variable $\zeta = k_T k'_T$.

Using the integral representation for the incomplete gamma function [see formula 8.353(3) in Ref. [59]] and Eq. (62), we obtain

$$A(Y,r,R;\boldsymbol{b}) = \mu r^2 R^2 \int \zeta d\zeta \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}(\zeta R r)^{-\gamma}\frac{4^{\gamma-1}\Gamma^2(\frac{1}{2}\gamma)}{\Gamma^2(1-\frac{1}{2}\gamma)}\int_0^\infty d\tau e^{-\tau}\frac{\frac{\tau N}{\mu\zeta}}{1+\frac{\tau N}{\mu\zeta}}$$
$$= \mu r^2 R^2 \int \zeta d\zeta \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}(R r)^{-\gamma}\frac{4^{\gamma-1}\Gamma^2(\frac{1}{2}\gamma)}{\Gamma^2(1-\frac{1}{2}\gamma)}\int_0^\infty d\tau e^{-\tau}\left\{-\frac{\pi}{\sin(\pi\gamma)}(\tau N/\mu)^{(2-\gamma)}\right\}$$
$$= \mu \int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{d\gamma}{2\pi i}\frac{4^{\gamma-1}\Gamma^2(\frac{1}{2}\gamma)}{\Gamma^2(1-\frac{1}{2}\gamma)}\left(-\frac{\pi}{\sin(\pi\gamma)}\right)(rRN/\mu)^{2-\gamma}. \tag{63}$$

Closing the contour of integration over γ on negative γ, we obtain the scattering amplitude as the sum of $(rRN = A^{\mathrm{BFKL}}(Y,\xi_r))^n$, where $\xi_r = \ln(\frac{b^2}{rR})$. If we close the contour on the singularities for positive $\gamma > 1$, we have the asymptotic series, which determines the behavior of the scattering amplitude at high energies (at large values of Y). One can see that the integrant has no singularities at $\gamma = 2$ and the first pole appears at $\gamma = 3$, which leads to

$$A(Y,r,R;\boldsymbol{b}) = \mu^2\frac{\ln(A^{\mathrm{BFKL}}(Y,\xi_r)/\mu)}{A^{\mathrm{BFKL}}(Y,\xi_r)}, \tag{64}$$

where $\mu = \Delta_{\mathrm{BFKL}}/\bar{\alpha}_S$.

Therefore, we found that the scattering amplitude decreases at large values of Y (at high energies).

V. CONCLUSIONS

In this paper, we have three results. First, we derive the recurrence relations for dipole densities (ρ_n) in QCD for the BFKL parton cascade. These relations allow us to find the dipole densities from the solution to the BFKL equation for ρ_1. Note, that Eq. (25) for the energy evolution of the parton densities is also new. Second, using the diffusion approximation for the BFKL kernel, we resolve these recurrence relations and find the leading terms in $\rho_n \propto e^{n\Delta_{\mathrm{BFKL}}Y}$. It is worth mentioning that these relations are suited for the numerical estimate of the dipole densities, opening a new way for the numerical simulation of the scattering amplitudes using Eq. (27).

Third, for the first time, we sum analytically the large Pomeron loops in QCD using these solutions. As a result of this summation, we obtain the dipole-dipole scattering amplitude. Surprisingly, it turns out that this amplitude decreases at large values of Y. We believe that such a behavior of the scattering amplitude follows from the simplified kernel of the BFKL equation in the diffusion approximation, as has been demonstrated in Sec. IV C. The physics origin of such behavior is that the diffusion BFKL kernel does not lead to the saturation both in the BK equation and in dipole-dipole amplitude. In other words, the first attempt to sum analytically BFKL Pomeron loops in QCD leads to the scattering amplitude that satisfies both t and s channel unitarity without saturation. Hence, the sum of Pomeron loops gives the typical contribution to the S matrix at high energies, which turns out to be larger than the rare fluctuations discussed in Ref. [53] and which will lead to the main contribution to the scattering amplitude for more realistic approximation for the BFKL kernel. The fact that the diffusion approximation to the BFKL kernel is so deficient turns out to be a great surprise to us, especially because this approximation, which leads to the F-KPP equation, has been widely used to describe the deep inelastic scattering (see Ref. [1] for review). On the other hand, such a result is not new for the Pomeron calculus (see Ref. [69] and references therein). It should be emphasized that shortcomings of the diffusion approximation force us to look at numerical estimates of Refs. [10,11] with a grain of salt, since the diffusion approximation was used in these papers. We have to believe that these estimates have been made in the preasymptotic region where the diffusion approximation generates the geometric scaling behavior of the scattering amplitude.

Certainly, summing the Pomeron loops for more a realistic approximation for the BFKL kernel that leads to the saturation will be our first problem to solve in the future. We wish also to note that, even in the present form, the sum of Pomeron loops can be useful in discussion of the multiplicity distribution of the produced gluons.

We believe that our reader can take the following results from this paper. First, it is the new evolution equation for dipole densities [see Eq. (25) and the recurrence relation between them; see Eq. (26)]. They are derived for the general BFKL kernel. The recurrence relations are well suited for the numerical estimates of the scattering amplitudes. Second, it is the solution of Eq. (47), which has been found for the BFKL kernel in diffusion approximation. However, one can use these solutions only in the vicinity of the saturation scale, where they reproduce the geometric scaling behavior [66,67]. The third is the unexpected result that the diffusion approximation cannot describe the high energy asymptotic behavior both for the BK equation and for dipole-dipole scattering. The failure of the diffusion approximation is surprising and instructive since most of experts bear in mind the diffusion approximation when discussing the BFKL Pomeron contribution.

ACKNOWLEDGMENTS

We thank C. Contreras, A. Kovner, M. Lublinsky, and R. Meneses for stimulating and encouraging discussions on the subject of this paper. This research was supported by ANID PIA/APOYO AFB180002 (Chile), Fondecyt Grant No. 1180118 (Chile), and the Tel Aviv University encouragement Grant No. 5731.

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