Three-point functions of conserved currents in 3D CFT: General formalism for arbitrary spins

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We analyze the general structure of the three-point functions involving conserved bosonic and fermionic higher-spin currents in three-dimensional conformal field theory. Using the constraints of conformal symmetry and conservation equations, we use a computational formalism to analyze the general structure of $\langle J_{s_1} J'_{s_2} J''_{s_3} \rangle$, where J_{s_1}, J'_{s_2}, and J''_{s_3} are conserved currents with spins s_1, s_2, and s_3 respectively (integer or half-integer). The calculations are completely automated for any chosen spins and are limited only by computer power. We find that the correlation function is in general fixed up to two independent "even" structures, and one "odd" structure, subject to a set of triangle inequalities. We also analyze the structure of three-point functions involving higher-spin currents and fundamental scalars and spinors.

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I. INTRODUCTION

It is widely understood that in any conformal field theory, the general structure of three-point correlation functions is determined up to finitely many parameters by conformal symmetry. However, it remains a nontrivial problem to construct explicit solutions for three-point functions for various classes of primary operators. Among the most important primary operators are conserved currents, whose scale dimension saturates the unitarity bound. The fundamental examples of conserved currents in any conformal field theory are the energy-momentum tensor and vector currents; the three-point functions of these currents were analyzed in [1,2], where a systematic approach to study correlation functions of primary operators was introduced (see also Refs. [3–12] for earlier works).

The analysis was performed in general dimensions; however, it did not consider higher-spin conserved currents, which can exist in more general conformal field theories. It also did not account for the possibility of parity-violating structures, which appear in the three-point functions of the energy-momentum tensor and vector currents in three dimensions. These structures were found in [13], where correlation functions of higher-spin conserved currents were considered, and were also found to contain parity-violating structures. Soon after, it was proven in [14] that under certain assumptions (which are, however, violated in the presence of fermionic higher-spin currents) all correlation functions involving the energy-momentum tensor and higher-spin currents are equal to those of free theories. This is an extension of the Coleman-Mandula theorem [15] to conformal field theories; it was originally proven in three dimensions and was later generalized to four- and higher-dimensional cases in [16–20].

There are also approaches to the construction of correlation functions of conserved currents which make use of embedding formalisms [21–26] (see also [27,28] for supersymmetric extensions), while others carry out the calculations in momentum space [29–38]. Results have also been obtained within the framework of the AdS/CFT correspondence (see e.g., [39–43]). The study of correlation functions of conserved currents has also been extended to superconformal field theories in diverse dimensions [44–59].

The general structure of the three-point functions of conserved higher-spin, bosonic, vector currents was proposed by Giombi *et al.* [13] in three dimensions, and further analysis was undertaken by Stanev [17,18,60] (see also [61,62]) in the four-dimensional case, and by Zhiboedov [16] in general dimensions. Despite the obvious success, the analysis in [13,16,17] appears to have some limitations. First, the results only apply to conserved currents of integer spin. Second, it is unclear how the results comprise all linearly independent structures for a given choice of spins. In particular, in [16,17], the conserved three-point functions are presented in the form of generating functions which are proposed (to the best of our understanding) without proof of the latter.

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In this paper, we develop a formalism to study the general structure of the three-point correlation function

$$\langle J_{s_1}(x_1) J'_{s_2}(x_2) J''_{s_3}(x_3) \rangle, \tag{1.1}$$

in three-dimensional conformal field theory, assuming only the constraints imposed by conformal symmetry and conservation equations. Here by J_s we denote a conserved current of spin s. Our formalism is suitable for both integer and half-integer spin. Within our approach we reproduce all known results concerning the structure of three-point functions of bosonic conserved currents and also extend the results to three-point functions involving currents of an arbitrary half-integer spin. We also apply it to correlation functions of scalar/spinor operators thus covering essentially all possible three-point functions in three-dimensional conformal field theory. Our method is exhaustive; first we construct all possible structures for the correlation function for a given set of spins s_1, s_2, and s_3, consistent with its conformal properties. We then systematically extract the linearly independent structures and then, finally, impose the conservation equations and symmetries under permutations of spacetime points. As a result we obtain the three-point function in a very explicit form which can be explicitly presented even for relatively high spins.[1] Our method can be applied for arbitrary s_1, s_2, and s_3 and is limited only by computer power. Due to these limitations we were able to carry out computations up to $s_i = 20$; however, with a sufficiently powerful computer one could probably extend our results up to $s_i \sim 50$ as in [17]. We demonstrate that in all cases with $s_i \leq 20$, including examples involving conserved *half-integer* spin currents, that the correlation function is fixed up to the following form:

$$\langle J_{s_1} J'_{s_2} J''_{s_3} \rangle = a_1 \langle J_{s_1} J'_{s_2} J''_{s_3} \rangle_{E_1} + a_2 \langle J_{s_1} J'_{s_2} J''_{s_3} \rangle_{E_2} + b \langle J_{s_1} J'_{s_2} J''_{s_3} \rangle_{O}, \tag{1.2}$$

where $\langle J_{s_1} J'_{s_2} J''_{s_3} \rangle_{E_1}$ and $\langle J_{s_1} J'_{s_2} J''_{s_3} \rangle_{E_2}$ are parity-even solutions (in the bosonic case corresponding to free bosonic and fermionic theories respectively), while $\langle J_{s_1} J'_{s_2} J''_{s_3} \rangle_{O}$ is a parity-violating (or parity-odd) solution. Parity-odd solutions are unique to three dimensions and have been shown to correspond to Chern-Simons theories interacting with parity-violating matter [63–73].[2] Further, the existence of the odd solution depends on a set of triangle inequalities:

$$s_1 \leq s_2 + s_3, \qquad s_2 \leq s_1 + s_3, \qquad s_3 \leq s_1 + s_2. \tag{1.3}$$

When the triangle inequalities are simultaneously satisfied there are two even solutions, and one odd solution. However, when any one of the above relations is not satisfied there are only two even solutions; the odd solution is incompatible with conservation equations.

The analysis quickly becomes cumbersome due to the proliferation of tensor indices; to streamline the calculations we develop a hybrid, index-free formalism which combines the approach of Osborn and Petkou [1] and a method based on contraction of tensor indices with auxiliary spinors. This method is widely used throughout the literature to construct correlation functions involving more complicated tensor operators. Our particular approach, however, describes the correlation function completely in terms of a polynomial which is a function of a single conformally covariant three-point building block, X, and the auxiliary spinor variables u, v, and w. Hence, one does not have to work with the spacetime points explicitly when imposing conservation equations. To find all solutions for the polynomial, we construct a generating function which produces an exhaustive list of all possible linearly dependent structures for a given set of spins using *Mathematica*. With the use of pattern-matching functions, we then systematically apply linear dependence relations to this set of structures to form a linearly independent ansatz for the correlation function. Once this ansatz is obtained, we impose conservation equations and any symmetries due to permutation of spacetime points. The tensor structures [related to the leading singular operator product expansion (OPE) coefficient, as in [1]] may then be read off by acting on the polynomials with appropriate partial derivatives in the auxiliary spinors. The computational approach we have developed is essentially automatic and limited only by computer power; one simply chooses the spins of the fields and the solution for the three-point function consistent with conservation and point-switch symmetries is generated.

The results of this paper are organized as follows. In Sec. II we review the essentials of the group theoretic formalism used to construct correlation functions of primary operators in three dimensions. In Sec. III we develop the formalism necessary to impose all constraints arising from conservation equations and point-switch symmetries on three-point functions. In particular, we introduce an index-free, auxiliary spinor formalism which allows us to construct a generating function for the three-point functions, and we outline the important aspects of our computational approach. Section IV is then devoted to the analysis of three-point functions involving bosonic conserved currents.

We show that we reproduce the known results previously found and proposed in [13,14,16]. In Sec. V we analyze the structure of correlation functions involving fermionic currents. We present an explicit analysis for three-point correlation functions involving combinations of a

[1] A similar analysis can also be done in the four-dimensional case and will appear elsewhere.

[2] The parity-odd terms in correlation functions involving scalars and spinors can also arise in theories without a Chern-Simons term, for example in theories with fermions in three dimensions, because $\bar{\psi}\psi$ is a parity-odd pseudoscalar; see e.g., [74,75]. We are grateful to S. Prakash for pointing this out.

"supersymmetry-like" spin-3/2 current, the energy-momentum tensor and the conserved vector current. The results are then expanded to include higher-spin conserved currents. In Sec. VI, for completeness, we perform the analysis of correlation functions involving combinations of scalars, spinors and conserved higher-spin currents. Finally, in Sec. VII, we comment on the general results in the context of superconformal field theories. The appendices are devoted to mathematical conventions, various useful identities and extra results for higher-spin conserved currents. In particular, in Appendix B we present some extra results for higher-spin three-point functions to illustrate that our method produces very explicit results even for relatively high spins.

II. CONFORMAL BUILDING BLOCKS

In this section we will review the pertinent aspects of the group theoretic formalism used to compute correlation functions of primary operators in three-dimensional conformal field theories. For a more detailed review of the formalism as applied to correlation functions of bosonic primary fields, the reader may consult [1].

A. Two-point building blocks

Consider 3D Minkowski space $\mathbb{M}^{1,2}$, parametrized by coordinates x^m, where $m = \mathbf{0}, 1, 2$ are Lorentz indices. Given two points, x_1 and x_2, we can define the covariant two-point function

$$x_{12}^m = (x_1 - x_2)^m, \qquad x_{21}^m = -x_{12}^m. \tag{2.1}$$

Next, following Osborn and Petkou [1], we introduce the conformal inversion tensor, I_{mn}, which is defined as follows:

$$I_{mn}(x) = \eta_{mn} - 2\frac{x_m x_n}{x^2}, \qquad I_{ma}(x) I^{an}(x) = \delta_m^n. \tag{2.2}$$

This object played a pivotal role in the construction of correlation functions in [1], as the full conformal group may be generated by considering Poincaré transformations supplemented by inversions. However, in the context of this work, we require an analogous operator for the spinor representation. Hence, we convert the vector two-point functions (2.1) into spinor notation using the conventions outlined in Appendix A:

$$x_{12\alpha\beta} = (\gamma^m)_{\alpha\beta} x_{12m}, \qquad x_{12}^{\alpha\beta} = (\gamma^m)^{\alpha\beta} x_{12m},$$
$$x_{12}^2 = -\frac{1}{2} x_{12}^{\alpha\beta} x_{12\alpha\beta}. \tag{2.3}$$

In this form the two-point functions possess the following useful properties:

$$x_{12\alpha\beta} = x_{12\beta\alpha}, \qquad x_{12}^{\alpha\sigma} x_{12\sigma\beta} = -x_{12}^2 \delta_\beta^\alpha. \tag{2.4}$$

Hence, we find

$$(x_{12}^{-1})^{\alpha\beta} = -\frac{x_{12}^{\alpha\beta}}{x_{12}^2}. \tag{2.5}$$

We also introduce the normalized two-point functions, denoted by $\hat{x}_{12}$,

$$\hat{x}_{12\alpha\beta} = \frac{x_{12\alpha\beta}}{(x_{12}^2)^{1/2}}, \qquad \hat{x}_{12}^{\alpha\sigma} \hat{x}_{12\sigma\beta} = -\delta_\beta^\alpha. \tag{2.6}$$

From here we can now construct an operator analogous to the conformal inversion tensor acting on the space of symmetric traceless spin-tensors of arbitrary rank. Given a two-point function x, we define the operator

$$\mathcal{I}_{\alpha(k)\beta(k)}(x) = \hat{x}_{(\alpha_1(\beta_1} \ldots \hat{x}_{\alpha_k)\beta_k)}, \tag{2.7}$$

along with its inverse

$$\mathcal{I}^{\alpha(k)\beta(k)}(x) = \hat{x}^{(\alpha_1(\beta_1} \ldots \hat{x}^{\alpha_k)\beta_k)}. \tag{2.8}$$

The spinor indices may be raised and lowered using the standard conventions as follows:

$$\mathcal{I}_{\alpha(k)}{}^{\beta(k)}(x) = \varepsilon^{\beta_1\gamma_1} \ldots \varepsilon^{\beta_k\gamma_k} \mathcal{I}_{\alpha(k)\gamma(k)}(x). \tag{2.9}$$

Now due to the property

$$\mathcal{I}_{\alpha(k)\beta(k)}(-x) = (-1)^k \mathcal{I}_{\alpha(k)\beta(k)}(x), \tag{2.10}$$

the following identity holds for products of inversion tensors:

$$\mathcal{I}_{\alpha(k)\sigma(k)}(x_{12}) \mathcal{I}^{\sigma(k)\beta(k)}(x_{21}) = \delta_{(\alpha_1}^{(\beta_1} \ldots \delta_{\alpha_k)}^{\beta_k)}. \tag{2.11}$$

The objects (2.7), (2.8) prove to be essential in the construction of correlation functions of primary operators with arbitrary spin. Indeed, the vector representation of the inversion tensor may be recovered in terms of the spinor two-point functions as follows:

$$I_{mn}(x) = -\frac{1}{2} \mathrm{Tr}(\gamma_m \hat{x} \gamma_n \hat{x}). \tag{2.12}$$

B. Three-point building blocks

Given three distinct points in Minkowski space, x_i, with $i = 1, 2, 3$, we define conformally covariant three-point functions in terms of the two-point functions as in [1]

$$X_{ij} = \frac{x_{ik}}{x_{ik}^2} - \frac{x_{jk}}{x_{jk}^2}, \qquad X_{ji} = -X_{ij}, \qquad X_{ij}^2 = \frac{x_{ij}^2}{x_{ik}^2 x_{jk}^2}, \quad (2.13)$$

where (i, j, k) is a cyclic permutation of (1, 2, 3). For example we can have

$$X_{12}^m = \frac{x_{13}^m}{x_{13}^2} - \frac{x_{23}^m}{x_{23}^2}, \qquad X_{12}^2 = \frac{x_{12}^2}{x_{13}^2 x_{23}^2}. \quad (2.14)$$

There are several useful identities involving the two- and three-point functions along with the conformal inversion tensor. For example one can prove the algebraic identities

$$I_m{}^a(x_{13}) I_{an}(x_{23}) = I_m{}^a(x_{12}) I_{an}(X_{13}),$$
$$I_{mn}(x_{23}) X_{12}^n = \frac{x_{12}^2}{x_{13}^2} X_{13m}, \quad (2.15a)$$

$$I_m{}^a(x_{23}) I_{an}(x_{13}) = I_m{}^a(x_{21}) I_{an}(X_{32}),$$
$$I_{mn}(x_{13}) X_{12}^n = \frac{x_{12}^2}{x_{23}^2} X_{32m}. \quad (2.15b)$$

The three-point functions also possess the following differential properties:

$$\partial_{(1)m} X_{12n} = \frac{1}{x_{13}^2} I_{mn}(x_{13}), \qquad \partial_{(2)m} X_{12n} = -\frac{1}{x_{23}^2} I_{mn}(x_{23}). \quad (2.16)$$

Converting to spinor notation, the three-point functions may be represented as follows:

$$X_{ij,\alpha\beta} = (\gamma_m)_{\alpha\beta} X_{ij}^m, \qquad X_{ij,\alpha\beta} = -(x_{ik}^{-1})_{\alpha\sigma} x_{ij}^{\sigma\gamma} (x_{jk}^{-1})_{\gamma\beta}. \quad (2.17)$$

These objects satisfy properties similar to the two-point functions, as in (2.4). It is also convenient to define the normalized three-point functions, $\hat{X}_{ij}$, and the inverses, (X_{ij}^{-1}),

$$\hat{X}_{ij,\alpha\beta} = \frac{X_{ij,\alpha\beta}}{(X_{ij}^2)^{1/2}}, \qquad (X_{ij}^{-1})^{\alpha\beta} = -\frac{X_{ij}^{\alpha\beta}}{X_{ij}^2}. \quad (2.18)$$

Now given an arbitrary three-point building block, X, it is useful to construct the following higher-spin inversion operator:

$$\mathcal{I}_{\alpha(k)\beta(k)}(X) = \hat{X}_{(\alpha_1(\beta_1} \ldots \hat{X}_{\alpha_k)\beta_k)}, \quad (2.19)$$

along with its inverse

$$\mathcal{I}^{\alpha(k)\beta(k)}(X) = \hat{X}^{(\alpha_1(\beta_1} \ldots \hat{X}^{\alpha_k)\beta_k)}. \quad (2.20)$$

These operators possess properties similar to the two-point higher-spin inversion operators (2.7), (2.8). There are also some useful algebraic identities relating the two- and three-point functions at various points, such as

$$\mathcal{I}_{\alpha\sigma}(x_{13}) \mathcal{I}^{\sigma\gamma}(x_{12}) \mathcal{I}_{\gamma\beta}(x_{23}) = \mathcal{I}_{\alpha\beta}(X_{12}),$$
$$\mathcal{I}^{\alpha\sigma}(x_{13}) \mathcal{I}_{\sigma\gamma}(X_{12}) \mathcal{I}^{\gamma\beta}(x_{13}) = \mathcal{I}^{\alpha\beta}(X_{32}). \quad (2.21)$$

These identities (and cyclic permutations of them) are analogous to (2.15a), (2.15b), and also admit higher-spin generalizations, for example

$$\mathcal{I}^{\alpha(k)\sigma(k)}(x_{13}) \mathcal{I}_{\sigma(k)\gamma(k)}(X_{12}) \mathcal{I}^{\gamma(k)\beta(k)}(x_{13}) = \mathcal{I}^{\alpha(k)\beta(k)}(X_{32}). \quad (2.22)$$

In addition, similar to (2.16), there are also the following identities:

$$\partial_{(1)\alpha\beta} X_{12}^{\gamma\delta} = -\frac{2}{x_{13}^2} \mathcal{I}_{(\alpha}{}^{\gamma}(x_{13}) \mathcal{I}_{\beta)}{}^{\delta}(x_{13}),$$
$$\partial_{(2)\alpha\beta} X_{12}^{\gamma\delta} = \frac{2}{x_{23}^2} \mathcal{I}_{(\alpha}{}^{\gamma}(x_{23}) \mathcal{I}_{\beta)}{}^{\delta}(x_{23}). \quad (2.23)$$

These identities allow us to account for the fact that correlation functions of primary fields obey differential constraints which can arise due to conservation equations. Indeed, given a tensor field $\mathcal{T}_{\mathcal{A}}(X)$, there are the following differential identities which arise as a consequence of (2.23):

$$\partial_{(1)\alpha\beta} \mathcal{T}_{\mathcal{A}}(X_{12}) = \frac{1}{x_{13}^2} \mathcal{I}_{\alpha}{}^{\gamma}(x_{13}) \mathcal{I}_{\beta}{}^{\delta}(x_{13}) \frac{\partial}{\partial X_{12}^{\gamma\delta}} \mathcal{T}_{\mathcal{A}}(X_{12}), \quad (2.24a)$$

$$\partial_{(2)\alpha\beta} \mathcal{T}_{\mathcal{A}}(X_{12}) = -\frac{1}{x_{23}^2} \mathcal{I}_{\alpha}{}^{\gamma}(x_{23}) \mathcal{I}_{\beta}{}^{\delta}(x_{23}) \frac{\partial}{\partial X_{12}^{\gamma\delta}} \mathcal{T}_{\mathcal{A}}(X_{12}). \quad (2.24b)$$

III. GENERAL FORMALISM FOR CORRELATION FUNCTIONS OF PRIMARY OPERATORS

In this section we develop a formalism to construct correlation functions of higher-spin primary operators in 3D conformal field theories. We utilize a hybrid method which combines auxiliary spinors with the approach of Osborn and Petkou [1].

A. Two-point functions

Let $\Phi_{\mathcal{A}}$ be a primary field with dimension Δ, where $\mathcal{A}$ denotes a collection of Lorentz spinor indices. The two-point correlation function of $\Phi_{\mathcal{A}}$ is fixed by conformal symmetry to the form

$$\langle \Phi_{\mathcal{A}}(x_1)\Phi^{\mathcal{B}}(x_2)\rangle = c\,\frac{\mathcal{I}_{\mathcal{A}}{}^{\mathcal{B}}(x_{12})}{(x_{12}^2)^{\Delta}}, \qquad (3.1)$$

where $\mathcal{I}$ is an appropriate representation of the inversion tensor and c is a constant real parameter. The denominator of the two-point function is determined by the conformal dimension of $\Phi_{\mathcal{A}}$, which guarantees that the correlation function transforms with the appropriate weight under scale transformations.

B. Three-point functions

Now concerning three-point correlation functions, let Φ, Ψ, Π be primary fields with scale dimensions Δ_1, Δ_2 and Δ_3 respectively. The three-point function may be constructed using the general ansatz

$$\langle \Phi_{\mathcal{A}_1}(x_1)\Psi_{\mathcal{A}_2}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle = \frac{\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{A}_1'}(x_{13})\mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{A}_2'}(x_{23})}{(x_{13}^2)^{\Delta_1}(x_{23}^2)^{\Delta_2}}\mathcal{H}_{\mathcal{A}_1'\mathcal{A}_2'\mathcal{A}_3}(X_{12}). \qquad (3.2)$$

The tensor $\mathcal{H}_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3}(X)$ encodes all information about the correlation function, and is related to the leading singular OPE coefficient [1]. It is highly constrained by conformal symmetry as follows:

(i) Under scale transformations of Minkowski space $x^m \mapsto x'^m = \lambda^{-2}x^m$, the three-point building blocks transform as $X^m \mapsto X'^m = \lambda^2 X^m$. As a consequence, the correlation function transforms as

$$\langle \Phi_{\mathcal{A}_1}(x_1')\Psi_{\mathcal{A}_2}(x_2')\Pi_{\mathcal{A}_3}(x_3')\rangle = (\lambda^2)^{\Delta_1+\Delta_2+\Delta_3}\langle \Phi_{\mathcal{A}_1}(x_1)\Psi_{\mathcal{A}_2}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle, \qquad (3.3)$$

which implies that $\mathcal{H}$ obeys the scaling property

$$\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(\lambda^2 X) = (\lambda^2)^{\Delta_3-\Delta_2-\Delta_1}\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X), \quad \forall\ \lambda \in \mathbb{R}\backslash\{0\}. \qquad (3.4)$$

This guarantees that the correlation function transforms correctly under scale transformations.

(ii) If any of the fields Φ, Ψ, Π obey differential equations, such as conservation laws in the case of conserved currents, then the tensor $\mathcal{H}$ is also constrained by differential equations which may be derived with the aid of identities (2.24a), (2.24b).

(iii) If any (or all) of the operators Φ, Ψ, Π coincide, the correlation function possesses symmetries under permutations of spacetime points, e.g.,

$$\langle \Phi_{\mathcal{A}_1}(x_1)\Phi_{\mathcal{A}_2}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle = (-1)^{\epsilon(\Phi)}\langle \Phi_{\mathcal{A}_2}(x_2)\Phi_{\mathcal{A}_1}(x_1)\Pi_{\mathcal{A}_3}(x_3)\rangle, \qquad (3.5)$$

where $\epsilon(\Phi)$ is the Grassmann parity of Φ. As a consequence, the tensor $\mathcal{H}$ obeys constraints which will be referred to as "point-switch identities."

The constraints above fix the functional form of $\mathcal{H}$ (and therefore the correlation function) up to finitely many independent parameters. Hence, using the general formula (3.2), the problem of computing three-point correlation functions is reduced to deriving the general structure of the tensor $\mathcal{H}$ subject to the above constraints.

1. Conserved currents

In this paper we are primarily interested in the structure of three-point correlation functions of conserved currents. In three-dimensional conformal field theory, a conserved current with spin s (integer or half-integer) is defined as a totally symmetric spin-tensor, $J_{\alpha_1\ldots\alpha_{2s}}(x) = J_{(\alpha_1\ldots\alpha_{2s})}(x)$, satisfying a conservation equation of the form:

$$(\gamma^m)^{\alpha_1\alpha_2}\partial_m J_{\alpha_1\alpha_2\ldots\alpha_{2s}} = 0. \qquad (3.6)$$

Conserved currents are primary fields, as they possesses the following infinitesimal conformal transformation properties [76]:

$$\delta J_{\alpha_1\ldots\alpha_{2s}}(x) = -\xi J_{\alpha_1\ldots\alpha_{2s}}(x) - \Delta_J\sigma(x)J_{\alpha_1\ldots\alpha_{2s}}(x) + 2s\,\omega_{(\alpha_1}{}^{\delta}(x)J_{\alpha_2\ldots\alpha_{2s})\delta}(x), \qquad (3.7)$$

where ξ is a conformal Killing vector field, and $\sigma(x)$, $\omega_{\alpha\beta}(x)$ are local parameters defined in terms of ξ, which are associated with local scale and Lorentz transformations. The dimension Δ_J is uniquely fixed by the conservation condition (3.6), as it may be shown that this condition is primary provided that $\Delta_J = s+1$. This is also the dimension of the two-point correlation function (3.1), in the case of conserved currents.

2. Comments on differential constraints

An important aspect of this construction which requires further elaboration is that it is sensitive to the configuration of the fields in the correlation function. Indeed, depending on the exact way in which one constructs the general ansatz (3.2), it can be difficult to impose conservation equations on one of the three fields due to a lack of useful identities such as (2.24a), (2.24b). To illustrate this more clearly, consider the following example; suppose we want to determine the solution for the correlation function $\langle \Phi_{\mathcal{A}_1}(x_1)\Psi_{\mathcal{A}_2}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle$, with the ansatz

$$\langle \Phi_{\mathcal{A}_1}(x_1)\Psi_{\mathcal{A}_2}(x_2)\Pi_{\mathcal{A}_3}(x_3)\rangle = \frac{\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{A}_1'}(x_{13})\mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{A}_2'}(x_{23})}{(x_{13}^2)^{\Delta_1}(x_{23}^2)^{\Delta_2}}\mathcal{H}_{\mathcal{A}_1'\mathcal{A}_2'\mathcal{A}_3}(X_{12}). \qquad (3.8)$$

All information about this correlation function is encoded in the tensor $\mathcal{H}$; however, this particular formulation of the three-point function prevents us from imposing conservation on the field Π in a straightforward way. To rectify this issue we reformulate the ansatz with Π at the front as follows:

$$\langle \Pi_{\mathcal{A}_3}(x_3)\Psi_{\mathcal{A}_2}(x_2)\Phi_{\mathcal{A}_1}(x_1)\rangle = \frac{\mathcal{I}^{(3)}{}_{\mathcal{A}_3}{}^{\mathcal{A}'_3}(x_{31})\mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{A}'_2}(x_{21})}{(x_{31}^2)^{\Delta_3}(x_{21}^2)^{\Delta_2}} \times \tilde{\mathcal{H}}_{\mathcal{A}_1\mathcal{A}'_2\mathcal{A}'_3}(X_{32}). \tag{3.9}$$

In this case, all information about this correlation function is now encoded in the tensor $\tilde{\mathcal{H}}$, which is a completely different solution compared to $\mathcal{H}$. Conservation on Π can now be imposed by treating x_3 as the first point with the aid of identities analogous to (2.23), (2.24a), and (2.24b). We now require an equation relating the tensors $\mathcal{H}$ and $\tilde{\mathcal{H}}$, which correspond to different representations of the same correlation function. Since the two ansatz above must be equal, we obtain the following:

$$\begin{aligned}\tilde{\mathcal{H}}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X_{32}) = (x_{13}^2)^{\Delta_3-\Delta_1}\left(\frac{x_{21}^2}{x_{23}^2}\right)^{\Delta_2}\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{A}'_1}(x_{13}) \\ \times \mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{B}_2}(x_{12})\mathcal{I}^{(2)}{}_{\mathcal{B}_2}{}^{\mathcal{A}'_2}(x_{23}) \\ \times \mathcal{I}^{(3)}{}_{\mathcal{A}_3}{}^{\mathcal{A}'_3}(x_{13})\mathcal{H}_{\mathcal{A}'_1\mathcal{A}'_2\mathcal{A}'_3}(X_{12}),\end{aligned} \tag{3.10}$$

where we have absorbed any signs due to Grassmann parity into the overall normalization of $\tilde{\mathcal{H}}$. In general, this equation is impractical to work with due to the presence of both two- and three-point functions, hence, further simplification is required. At this point it is convenient to partition our solution into "even" and "odd" sectors as follows:

$$\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X) = \mathcal{H}^{(+)}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X) + \mathcal{H}^{(-)}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X), \tag{3.11}$$

where $\mathcal{H}^{(+)}$ contains all structures involving an even number of spinor metrics, $\varepsilon_{\alpha\beta}$, and $\mathcal{H}^{(-)}$ contains structures involving an odd number of spinor metrics. With this choice of convention, as a consequence of (2.21), (2.22), the following relation holds:

$$\begin{aligned}\mathcal{H}^{(\pm)}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X_{32}) = \pm(x_{13}^2X_{32}^2)^{\Delta_3-\Delta_2-\Delta_1}\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{A}'_1}(x_{13}) \\ \times \mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{A}'_2}(x_{13})\mathcal{I}^{(3)}{}_{\mathcal{A}_3}{}^{\mathcal{A}'_3}(x_{13}) \\ \times \mathcal{H}^{(\pm)}_{\mathcal{A}'_1\mathcal{A}'_2\mathcal{A}'_3}(X_{12}).\end{aligned} \tag{3.12}$$

This equation is an extension of (2.14) in [1] to spin-tensor representations, and it allows us to construct an equation relating the different representations of the correlation function. After substituting (3.12) directly into (3.10), we apply identities such as (2.21) to obtain the following relation between $\mathcal{H}$ and $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{H}}^{(\pm)}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X) = \pm(X^2)^{\Delta_1-\Delta_3}\mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{A}'_2}(X)\mathcal{H}^{(\pm)}_{\mathcal{A}_1\mathcal{A}'_2\mathcal{A}_3}(X). \tag{3.13}$$

We see here that $\mathcal{I}$ acts as an intertwining operator between the different representations of the correlation function. Once $\tilde{\mathcal{H}}$ is obtained we can then impose conservation on Π as if it were located at the "first point," using identities analogous to (2.23). It is also important to note that the even and odd sectors of the correlation function are linearly independent, and therefore may be considered separately in the constraint analysis. Another result that follows from the properties (2.19), (2.20), and (2.22) of the inversion tensor is

$$\begin{aligned}\mathcal{H}^{(\pm)}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X) = \pm\mathcal{I}^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{A}'_1}(X)\mathcal{I}^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{A}'_2}(X)\mathcal{I}^{(3)}{}_{\mathcal{A}_3}{}^{\mathcal{A}'_3}(X) \\ \times \mathcal{H}^{(\pm)}_{\mathcal{A}'_1\mathcal{A}'_2\mathcal{A}'_3}(X).\end{aligned} \tag{3.14}$$

That is, "even" structures are invariant under the action of $\mathcal{I}$, while "odd" structures are pseudo-invariant under the action of $\mathcal{I}$.

If we now consider the correlation function of three conserved primaries $J_{\alpha(I)}$, $J'_{\beta(J)}$, $J''_{\gamma(K)}$, where $I = 2s_1$, $J = 2s_2$, $K = 2s_3$, then the general ansatz is

$$\begin{aligned}&\langle J_{\alpha(I)}(x_1)J'_{\beta(J)}(x_2)J''_{\gamma(K)}(x_3)\rangle \\ &= \frac{\mathcal{I}_{\alpha(I)}{}^{\alpha'(I)}(x_{13})\mathcal{I}_{\beta(J)}{}^{\beta'(J)}(x_{23})}{(x_{13}^2)^{\Delta_1}(x_{23}^2)^{\Delta_2}}\mathcal{H}_{\alpha'(I)\beta'(J)\gamma(K)}(X_{12}),\end{aligned} \tag{3.15}$$

where $\Delta_i = s_i + 1$. The constraints on $\mathcal{H}$ are then as follows:

(i) Homogeneity:

$$\mathcal{H}_{\alpha(I)\beta(J)\gamma(K)}(\lambda^2X) = (\lambda^2)^{\Delta_3-\Delta_2-\Delta_1}\mathcal{H}_{\alpha(I)\beta(J)\gamma(K)}(X), \quad \forall\ \lambda\in\mathbb{R}\backslash\{0\}. \tag{3.16}$$

(ii) Differential constraints: After application of the identities (2.24a), (2.24b) we obtain the following constraints:

$$\text{conservation at } x_1:\ \partial_X^{\alpha_1\alpha_2}\mathcal{H}_{\alpha_1\alpha_2\alpha(I-2)\beta(J)\gamma(K)}(X) = 0, \tag{3.17a}$$

$$\text{conservation at } x_2:\ \partial_X^{\beta_1\beta_2}\mathcal{H}_{\alpha(I)\beta_1\beta_2\beta(J-2)\gamma(K)}(X) = 0, \tag{3.17b}$$

$$\text{conservation at } x_3:\ \partial_X^{\gamma_1\gamma_2}\tilde{\mathcal{H}}_{\alpha(I)\beta(J)\gamma_1\gamma_2\gamma(K-2)}(X) = 0, \tag{3.17c}$$

where

$$\tilde{\mathcal{H}}^{(\pm)}_{\alpha(I)\beta(J)\gamma(K)}(X) = \pm(X^2)^{\Delta_1-\Delta_3}\mathcal{I}_{\beta(J)}{}^{\beta'(J)}(X) \times \mathcal{H}^{(\pm)}_{\alpha(I)\beta'(J)\gamma(K)}(X). \tag{3.18}$$

(iii) Point-switch symmetries: If the fields J and J' coincide, then we obtain the following point-switch identity

$$\mathcal{H}_{\alpha(I)\beta(I)\gamma(K)}(X) = (-1)^{\epsilon(J)}\mathcal{H}_{\beta(I)\alpha(I)\gamma(K)}(-X), \tag{3.19}$$

where $\epsilon(J)$ is the Grassmann parity of J. Likewise, if the fields J and J'' coincide, then we obtain the constraint

$$\tilde{\mathcal{H}}_{\alpha(I)\beta(J)\gamma(I)}(X) = (-1)^{\epsilon(J)}\mathcal{H}_{\gamma(I)\beta(J)\alpha(I)}(-X). \tag{3.20}$$

In practice, imposing the constraints above on correlation functions involving higher-spin currents quickly becomes unwieldy using the tensor formalism, particularly due to the sheer number of possible tensor structures for a given set of spins. Hence, in the next subsections we will develop an index-free formalism to handle the computations efficiently.

3. Auxiliary spinor formalism

To study and impose constraints on correlation functions of primary fields with general spins it is often advantageous to use the formalism of auxiliary spinors to streamline the calculations. Suppose we must analyze the constraints on a general spin-tensor $\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X)$, where $\mathcal{A}_1 = \{\alpha_1, ..., \alpha_I\}$, $\mathcal{A}_2 = \{\beta_1, ..., \beta_J\}, \mathcal{A}_3 = \{\gamma_1, ..., \gamma_K\}$ represent sets of totally symmetric spinor indices associated with the fields at points x_1, x_2, and x_3 respectively. We introduce sets of commuting auxiliary spinors for each point; u at x_1, v at x_2, and w at x_3, where the spinors satisfy

$$u^2 = \varepsilon_{\alpha\beta}u^\alpha u^\beta = 0, \qquad v^2 = \varepsilon_{\alpha\beta}v^\alpha v^\beta = 0, \\ w^2 = \varepsilon_{\alpha\beta}w^\alpha w^\beta = 0. \tag{3.21}$$

Now if we define the objects

$$\mathbf{U}^{\mathcal{A}_1} \equiv \mathbf{U}^{\alpha(I)} = u^{\alpha_1}...u^{\alpha_I}, \tag{3.22a}$$

$$\mathbf{V}^{\mathcal{A}_2} \equiv \mathbf{V}^{\beta(J)} = v^{\beta_1}...v^{\beta_J}, \tag{3.22b}$$

$$\mathbf{W}^{\mathcal{A}_3} \equiv \mathbf{W}^{\gamma(K)} = w^{\gamma_1}...w^{\gamma_K}, \tag{3.22c}$$

then the generating polynomial for $\mathcal{H}$ is constructed as follows:

$$\mathcal{H}(X; u, v, w) = \mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X)\mathbf{U}^{\mathcal{A}_1}\mathbf{V}^{\mathcal{A}_2}\mathbf{W}^{\mathcal{A}_3}. \tag{3.23}$$

There is in fact a one-to-one mapping between the space of symmetric traceless spin tensors and the polynomials constructed using the above method. Indeed, the tensor $\mathcal{H}$ is extracted from the polynomial by acting on it with the following partial derivative operators:

$$\frac{\partial}{\partial \mathbf{U}^{\mathcal{A}_1}} \equiv \frac{\partial}{\partial \mathbf{U}^{\alpha(I)}} = \frac{1}{I!}\frac{\partial}{\partial u^{\alpha_1}}\cdots\frac{\partial}{\partial u^{\alpha_I}}, \tag{3.24a}$$

$$\frac{\partial}{\partial \mathbf{V}^{\mathcal{A}_2}} \equiv \frac{\partial}{\partial \mathbf{V}^{\beta(J)}} = \frac{1}{J!}\frac{\partial}{\partial v^{\beta_1}}\cdots\frac{\partial}{\partial v^{\beta_J}}, \tag{3.24b}$$

$$\frac{\partial}{\partial \mathbf{W}^{\mathcal{A}_3}} \equiv \frac{\partial}{\partial \mathbf{W}^{\gamma(K)}} = \frac{1}{K!}\frac{\partial}{\partial w^{\gamma_1}}\cdots\frac{\partial}{\partial w^{\gamma_K}}. \tag{3.24c}$$

The tensor $\mathcal{H}$ is then extracted from the polynomial as follows:

$$\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X) = \frac{\partial}{\partial \mathbf{U}^{\mathcal{A}_1}}\frac{\partial}{\partial \mathbf{V}^{\mathcal{A}_2}}\frac{\partial}{\partial \mathbf{W}^{\mathcal{A}_3}}\mathcal{H}(X; u, v, w). \tag{3.25}$$

Auxiliary vectors/spinors are widely used in the construction of correlation functions throughout the literature (see e.g., [13,16,17,23,50,61]); however, usually the entire correlator is contracted with auxiliary variables and as a result one produces a polynomial depending on all three spacetime points and the auxiliary spinors. In contrast, our approach contracts the auxiliary spinors with the tensor $\mathcal{H}_{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}(X)$, which depends on only a single variable. This is advantageous as it becomes quite straightforward to impose constraints on the correlation function (particularly conservation), since $\mathcal{H}$ does not depend on any of the spacetime points explicitly. After converting the constraints summarized in the previous subsection into the auxiliary spinor formalism, we obtain:

(i) Homogeneity:

$$\mathcal{H}(\lambda^2 X; u(I), v(J), w(K)) \\ = (\lambda^2)^{\Delta_3-\Delta_2-\Delta_1}\mathcal{H}(X; u(I), v(J), w(K)), \tag{3.26}$$

where we have used the notation $u(I)$, $v(J)$, $w(K)$ to keep track of the homogeneity of the auxiliary spinors u, v and w.

(ii) Differential constraints:

$$\text{conservation at } x_1: \ \frac{\partial}{\partial X_{\alpha\beta}}\frac{\partial}{\partial u^\alpha}\frac{\partial}{\partial u^\beta}\mathcal{H}(X; u(I), v(J), w(K)) = 0, \tag{3.27a}$$

$$\text{conservation at } x_2: \ \frac{\partial}{\partial X_{\alpha\beta}} \frac{\partial}{\partial v^\alpha} \frac{\partial}{\partial v^\beta} \mathcal{H}(X; u(I), v(J), w(K)) = 0, \tag{3.27b}$$

$$\text{conservation at } x_3: \ \frac{\partial}{\partial X_{\alpha\beta}} \frac{\partial}{\partial w^\alpha} \frac{\partial}{\partial w^\beta} \tilde{\mathcal{H}}(X; u(I), v(J), w(K)) = 0. \tag{3.27c}$$

In the auxiliary spinor formalism, $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}^{(+)} + \tilde{\mathcal{H}}^{(-)}$ is computed as follows:

$$\begin{aligned} &\tilde{\mathcal{H}}^{(\pm)}(X; u(I), v(J), w(K)) \\ &= \pm \frac{1}{J!} (X^2)^{\Delta_1 - \Delta_3} (v \hat{X} \partial_t)^J \\ &\quad \times \mathcal{H}^{(\pm)}(X; u(I), t(J), w(K)), \end{aligned} \tag{3.28}$$

where $(v\hat{X}\partial_t) = v^\alpha \hat{X}_\alpha{}^\beta \frac{\partial}{\partial t^\beta}$.

(iii) Point-switch symmetries: If the fields Φ and Ψ coincide (hence $I = J$), then we obtain the following point-switch constraint

$$\begin{aligned} &\mathcal{H}(X; u(I), v(I), w(K)) \\ &= (-1)^{\epsilon(\Phi)} \mathcal{H}(-X; v(I), u(I), w(K)), \end{aligned} \tag{3.29}$$

where, again, $\epsilon(\Phi)$ is the Grassmann parity of Φ. Similarly, if the fields Φ and Π coincide (hence $I = K$) then we obtain the constraint

$$\begin{aligned} &\tilde{\mathcal{H}}(X; u(I), v(J), w(I)) \\ &= (-1)^{\epsilon(\Phi)} \mathcal{H}(-X; w(I), v(J), u(I)). \end{aligned} \tag{3.30}$$

4. Generating function method

The approach outlined above proves to be quite tractable, computationally speaking, as the polynomial, (3.23), is now constructed out of scalar combinations of X, and the auxiliary spinors u, v and w with the appropriate homogeneity. At this point it is convenient to introduce the following "primitive" structures:

$$P_1 = \varepsilon_{\alpha\beta} v^\alpha w^\beta, \qquad P_2 = \varepsilon_{\alpha\beta} w^\alpha u^\beta, \qquad P_3 = \varepsilon_{\alpha\beta} u^\alpha v^\beta, \tag{3.31a}$$

$$Q_1 = \hat{X}_{\alpha\beta} v^\alpha w^\beta, \qquad Q_2 = \hat{X}_{\alpha\beta} w^\alpha u^\beta, \qquad Q_3 = \hat{X}_{\alpha\beta} u^\alpha v^\beta, \tag{3.31b}$$

$$Z_1 = \hat{X}_{\alpha\beta} u^\alpha u^\beta, \qquad Z_2 = \hat{X}_{\alpha\beta} v^\alpha v^\beta, \qquad Z_3 = \hat{X}_{\alpha\beta} w^\alpha w^\beta. \tag{3.31c}$$

The most general ansatz for the polynomial $\mathcal{H}$ is comprised of all possible combinations of the above structures which possess the correct homogeneity in u, v and w. In general, it is a nontrivial technical problem to come up with an exhaustive list of possible solutions for the polynomial $\mathcal{H}$ for a given set of spins. However, this problem can be simplified by introducing a generating function for the polynomial $\mathcal{H}(X; u, v, w)$:

$$\mathcal{F}(X; \Gamma) = X^\delta P_1^{k_1} P_2^{k_2} P_3^{k_3} Q_1^{l_1} Q_2^{l_2} Q_3^{l_3} Z_1^{m_1} Z_2^{m_2} Z_3^{m_3}, \tag{3.32}$$

where $\delta = \Delta_3 - \Delta_2 - \Delta_1$, and the non-negative integers, $\Gamma = \{k_i, l_i, m_i\}$, $i = 1, 2, 3$, are solutions to the following linear system:

$$k_2 + k_3 + l_2 + l_3 + 2m_1 = I, \tag{3.33a}$$

$$k_1 + k_3 + l_1 + l_3 + 2m_2 = J, \tag{3.33b}$$

$$k_1 + k_2 + l_1 + l_2 + 2m_3 = K, \tag{3.33c}$$

and $I = 2s_1$, $J = 2s_2$, $K = 2s_3$ specify the spin structure of the correlation function. These equations are obtained by comparing the homogeneity of the auxiliary spinors u, v, w in the generating function (3.32), against the index structure of the tensor $\mathcal{H}$. The solutions correspond to a linearly dependent basis of possible structures in which the polynomial $\mathcal{H}$ can be decomposed. Using *Mathematica*, it is straightforward to generate all possible solutions to (3.33) for fixed (and in some cases arbitrary) values of the spins.

Now let us assume there exists a finite number of solutions Γ_i, $i = 1, ..., N$ to (3.33) for a given choice of I, J, K. The set of solutions $\Gamma = \{\Gamma_i\}$ may be partitioned into "even" and "odd" sets Γ^+ and Γ^- respectively by counting the number of spinor metrics, $\varepsilon_{\alpha\beta}$, present in a particular solution. Since only the P_i contain $\varepsilon_{\alpha\beta}$, we define

$$\Gamma^+ = \Gamma|_{k_1+k_2+k_3 (\mathrm{mod}\ 2)=0}, \qquad \Gamma^- = \Gamma|_{k_1+k_2+k_3 (\mathrm{mod}\ 2)=1}. \tag{3.34}$$

Hence, the even solutions are those such that $k_1 + k_2 + k_3 = \text{even}$ (i.e., contains an even number of spinor metrics), while the odd solutions are those such that $k_1 + k_2 + k_3 = \text{odd}$ (contains an odd number of spinor metrics).[3] Let

[3]This convention agrees with the known result that "odd" solutions typically contain the Levi-Civita tensor, while the "even" solutions do not.

$|\Gamma^+| = N^+$ and $|\Gamma^-| = N^-$, with $N = N^+ + N^-$, then the most general ansatz for the polynomial $\mathcal{H}$ in (3.23) is as follows:

$$\mathcal{H}(X; u, v, w) = \mathcal{H}^{(+)}(X; u, v, w) + \mathcal{H}^{(-)}(X; u, v, w), \tag{3.35a}$$

$$\mathcal{H}^{(+)}(X; u, v, w) = \sum_{i=1}^{N^+} A_i \mathcal{F}(X; \Gamma_i^+),$$
$$\mathcal{H}^{(-)}(X; u, v, w) = \sum_{i=1}^{N^-} B_i \mathcal{F}(X; \Gamma_i^-), \tag{3.35b}$$

where A_i and B_i are a set of real constants. Since the even and odd sectors of the correlation function do not mix with each other, they may be considered independently.

Using the above method it is quite simple to generate all the possible structures for a given set of spins $\{s_1, s_2, s_3\}$; however, at this stage we must recall that the solutions generated using this approach are linearly dependent. To form a linearly independent set of solutions we must systematically take into account the following nonlinear relations between the primitive structures:

$$P_1 Z_1 + P_2 Q_3 + P_3 Q_2 = 0, \tag{3.36a}$$

$$P_2 Z_2 + P_1 Q_3 + P_3 Q_1 = 0, \tag{3.36b}$$

$$P_3 Z_3 + P_1 Q_2 + P_2 Q_1 = 0, \tag{3.36c}$$

$$Q_1 Z_1 - Q_2 Q_3 - P_2 P_3 = 0, \tag{3.37a}$$

$$Q_2 Z_2 - Q_1 Q_3 - P_1 P_3 = 0, \tag{3.37b}$$

$$Q_3 Z_3 - Q_1 Q_2 - P_1 P_2 = 0, \tag{3.37c}$$

$$Z_2 Z_3 + P_1^2 - Q_1^2 = 0, \tag{3.38a}$$

$$Z_1 Z_3 + P_2^2 - Q_2^2 = 0, \tag{3.38b}$$

$$Z_1 Z_2 + P_3^2 - Q_3^2 = 0, \tag{3.38c}$$

$$P_1 P_2 P_3 + P_1 Q_2 Q_3 + P_2 Q_1 Q_3 + P_3 Q_1 Q_2 = 0. \tag{3.39}$$

This appears to be an exhaustive list of relations, and similar results have been obtained in other approaches which make use of auxiliary spinors [13]. Applying the relations above to a set of linearly dependent polynomial structures is relatively straightforward to implement using *Mathematica's* built-in pattern matching capabilities.

Now that we have taken care of linear dependence, it now remains to impose conservation on all three points in addition to the various point-switch symmetries. Introducing the P, Q and Z objects proves to streamline this analysis significantly. First let us consider conservation; we define the following three differential operators:

$$D_1 = \frac{\partial}{\partial X_{\alpha\beta}} \frac{\partial}{\partial u^\alpha} \frac{\partial}{\partial u^\beta}, \qquad D_2 = \frac{\partial}{\partial X_{\alpha\beta}} \frac{\partial}{\partial v^\alpha} \frac{\partial}{\partial v^\beta},$$
$$D_3 = \frac{\partial}{\partial X_{\alpha\beta}} \frac{\partial}{\partial w^\alpha} \frac{\partial}{\partial w^\beta}. \tag{3.40}$$

To impose conservation on x_1 (for either sector) we compute

$$D_1 \mathcal{H}(X; u, v, w) = D_1 \left\{ \sum_{i=1}^{N} c_i \mathcal{F}(X; \Gamma_i) \right\}$$
$$= \sum_{i=1}^{N} c_i D_1 \mathcal{F}(X; \Gamma_i). \tag{3.41}$$

We then solve for the c_i such that the result above vanishes. It is apparent that it would be extremely useful to obtain an explicit expression for $D_1 \mathcal{F}(X; \Gamma)$, as this would allow us to impose conservation in a simple manner; this proves to be very cumbersome to carry out by hand; however it is possible to obtain an exact result computationally (which we will not present here as it is ∼200 terms long). Hence, given a particular solution $\mathcal{F}(X; \Gamma_i)$, we can compute $D_1 \mathcal{F}(X; \Gamma_i)$. The fact that $D_1 \mathcal{F}(X; \Gamma)$ can also be expressed using the primitive structures (3.31) is due to the following reasoning: let $\mathbf{P}[X(\delta); u(I), v(J), w(K)]$ represent the space of polynomials which are homogeneous degree δ in X, I in u, J in v, and K in w; any polynomial in this space can naturally be constructed in terms of the primitives (3.31). The operator D_1 may then be interpreted as follows:

$$D_1 : \mathbf{P}[X(\delta); u(I), v(J), w(K)]$$
$$\longmapsto \mathbf{P}[X(\delta - 1); u(I-2), v(J), w(K)]. \tag{3.42}$$

Hence, D_1 is a map from $\mathbf{P}[X(\delta); u(I), v(J), w(K)]$ to $\mathbf{P}[X(\delta - 1); u(I-2), v(J), w(K)]$, that is, the space of polynomials homogeneous degree $\delta - 1$ in X, $I - 2$ in u, J in v and K in w. Any polynomial in this space can naturally be constructed using the same primitives defined in (3.31). Analogous results also apply for $D_2 \mathcal{F}(X; \Gamma)$.

However, to impose conservation on x_3 we must first obtain an explicit expression for $\tilde{\mathcal{H}}$ in terms of $\mathcal{H}$, that is, we must compute (e.g., for the even sector)

$$\tilde{\mathcal{H}}(X; u(I), v(J), w(K)) = \frac{1}{J!} (X^2)^{\Delta_1 - \Delta_3} (v \hat{X} \partial_t)^J$$
$$\times \mathcal{H}(X; u(I), t(J), w(K)). \tag{3.43}$$

Recalling the fact that any solution for $\mathcal{H}$ can be written in the form of the generating function $\mathcal{F}(X;\Gamma)$, we compute

$$\begin{aligned}\tilde{\mathcal{F}}(X;\Gamma) &= \frac{1}{J!}(X^2)^{\Delta_1-\Delta_3}(v\hat{X}\partial_t)^J\mathcal{F}(X;\Gamma)\\ &= \frac{1}{J!}(X^2)^{\Delta_1-\Delta_3}(v\hat{X}\partial_t)^J\{X^\delta P_1^{k_1}P_2^{k_2}P_3^{k_3}Q_1^{l_1}Q_2^{l_2}Q_3^{l_3}Z_1^{m_1}Z_2^{m_2}Z_3^{m_3}\}\\ &= \frac{1}{J!}X^{\Delta_1-\Delta_2-\Delta_3}P_2^{k_2}Q_2^{l_2}Z_1^{m_1}Z_3^{m_3}(v\hat{X}\partial_t)^J\{P_1^{k_1}P_3^{k_3}Q_1^{l_1}Q_3^{l_3}Z_2^{m_2}\}.\end{aligned} \tag{3.44}$$

Since P_1, P_3, Q_1, Q_3, and Z_2 are the only objects with t dependence, if we make use of the fact that $k_1+k_3+l_1+l_3+2m_2=J$, in addition to the identities

$$(v\hat{X}\partial_t)P_1 = -Q_1, \qquad (v\hat{X}\partial_t)P_3 = Q_3, \tag{3.45a}$$

$$(v\hat{X}\partial_t)Q_1 = -P_1, \qquad (v\hat{X}\partial_t)Q_3 = P_3, \tag{3.45b}$$

$$(v\hat{X}\partial_t)^2 Z_2 = 2Z_2, \tag{3.45c}$$

then it may be shown that

$$\begin{aligned}\tilde{\mathcal{F}}(X;\Gamma) &= X^{\tilde{\delta}}(-Q_1)^{k_1}P_2^{k_2}Q_3^{k_3}(-P_1)^{l_1}Q_2^{l_2}P_3^{l_3}Z_1^{m_1}Z_2^{m_2}Z_3^{m_3},\\ &= (-1)^{k_1+l_1}X^{\tilde{\delta}}P_1^{l_1}P_2^{k_2}P_3^{l_3}Q_1^{k_1}Q_2^{l_2}Q_3^{k_3}Z_1^{m_1}Z_2^{m_2}Z_3^{m_3},\end{aligned} \tag{3.46}$$

where $\tilde{\delta} = \Delta_1 - \Delta_2 - \Delta_3$. Hence we arrive at the following result:

$$\tilde{\mathcal{F}}(X;\Gamma) = (-1)^{k_1+l_1}\mathcal{F}(X;\Gamma)|_{\delta\to\tilde{\delta},k_1\leftrightarrow l_1,k_3\leftrightarrow l_3}. \tag{3.47}$$

Therefore the computation of $\tilde{\mathcal{H}}$ is actually quite straightforward: we take each term in the ansatz for $\mathcal{H}$ and make appropriate swaps of the primitive structures. This also simplifies imposing conservation at x_3, as we can now use the same generating function that we used for conservation at x_1 and x_2 as follows:

$$D_3\tilde{\mathcal{F}}(X;\Gamma) = (-1)^{k_1+l_1}D_3\mathcal{F}(X;\Gamma)|_{\delta\to\tilde{\delta},k_1\leftrightarrow l_1,k_3\leftrightarrow l_3}. \tag{3.48}$$

Now that we have exact expressions for $D_i\mathcal{F}(X;\Gamma)$, it remains to find out how point-switch symmetries act on the primitive structures. For permutation of spacetime points x_1 and x_2, we have $X\to -X$, $u\leftrightarrow v$. This results in the following replacement rules for the basis objects (3.31):

$$P_1\to -P_2, \qquad P_2\to -P_1, \qquad P_3\to -P_3, \tag{3.49a}$$

$$Q_1\to -Q_2, \qquad Q_2\to -Q_1, \qquad Q_3\to -Q_3, \tag{3.49b}$$

$$Z_1\to -Z_2, \qquad Z_2\to -Z_1, \qquad Z_3\to -Z_3. \tag{3.49c}$$

Likewise, for permutation of spacetime points x_1 and x_3 we have $X\to -X$, $u\leftrightarrow w$, resulting in the following replacements:

$$P_1\to -P_3, \qquad P_2\to -P_2, \qquad P_3\to -P_1, \tag{3.50a}$$

$$Q_1\to -Q_3, \qquad Q_2\to -Q_2, \qquad Q_3\to -Q_1, \tag{3.50b}$$

$$Z_1\to -Z_3, \qquad Z_2\to -Z_2, \qquad Z_3\to -Z_1. \tag{3.50c}$$

We have now developed all the formalism necessary to analyze the structure of three-point correlation functions in 3D CFT. In the remaining sections of this paper we will analyze the three-point functions of conserved higher-spin currents (for both integer and half-integer spin) using the following method:

(1) We construct all possible (linearly dependent) structures for $\mathcal{H}(X;u,v,w)$ for a given set of spins, which is governed by the solutions to (3.33). The solutions are sorted into even and odd sectors and analyzed separately.
(2) In each sector, we apply an algorithm to the set of dependent structures which systematically reduces it to a linearly independent set through repeated application of the identities (3.36a), (3.37a), (3.38a), and (3.39). This is sufficient to form the most general linearly independent ansatz.
(3) Using the method outlined in Sec. III B 4, we impose the conservation equations (3.27) on each sector.
(4) Once the general form of the polynomial $\mathcal{H}(X;u,v,w)$ (associated with the conserved three-point function $\langle J_{s_1}J'_{s_2}J''_{s_3}\rangle$) is obtained for a given set of spins $\{s_1,s_2,s_3\}$, we then impose any symmetries under permutation of spacetime points, that is, (3.29) and (3.30) (if applicable). In certain cases, imposing these constraints can eliminate the remaining structures.

Due to computational limitations such as CPU clock speed and available RAM, we could carry out this explicit analysis up to $s_i = 20$; however, with more optimization of the code and sufficient computational resources this approach should hold for arbitrary spins. Since there are an enormous number of possible three-point functions with $s_i \leq 20$, we present the final results for $\mathcal{H}(X;u,v,w)$ for some particularly interesting examples, as the solutions and coefficient constraints become cumbersome to present beyond low spin cases. We are primarily interested in counting the number of independent tensor structures after imposing all the constraints.

The results in the next sections are organized as follows: in Sec. IV we analyze the correlation functions involving bosonic conserved currents, commenting on some of the general features. Many of these results are known in the literature [13,16]; however, they have not been derived explicitly using this construction based on the conformal inversion tensor. In addition, within the framework of the generating function methods used in [16,17], it is unclear how the generating functions in these works are derived and how they produce an exhaustive list of independent structures. It is in this regard that our analysis is very explicit, as we find all possible structures for a given set of spins and systematically apply linear dependence relations to them. In Sec. V we analyze the mixed three-point functions involving fermionic conserved currents; these results are new and are naturally of interest within the context of superconformal field theories. Finally, in Sec. VI, we analyze correlation functions involving combinations of higher-spin currents and fundamental scalars/spinors. We stress that our analysis is based only on symmetries and conservation equations and does not take into account any other features of local field theory. The results are completely analytic and we present explicit formula for $\mathcal{H}(X; u, v, w)$ in all cases; the results are copied directly from the *Mathematica* code.

IV. CORRELATION FUNCTIONS INVOLVING BOSONIC CURRENTS

Three-point correlation functions of conserved bosonic currents have been extensively studied in 3D CFT. In particular, it has been shown that the general structure of the three-point correlation function $\langle J_{s_1} J'_{s_2} J''_{s_3}\rangle$ is fixed up to the following form [13,14,16]:

$$\langle J_{s_1} J'_{s_2} J''_{s_3}\rangle = a_1 \langle J_{s_1} J'_{s_2} J''_{s_3}\rangle_B + a_2 \langle J_{s_1} J'_{s_2} J''_{s_3}\rangle_F + b \langle J_{s_1} J'_{s_2} J''_{s_3}\rangle_{\text{odd}}. \tag{4.1}$$

The solutions $\langle J_{s_1} J'_{s_2} J''_{s_3}\rangle_B$, $\langle J_{s_1} J'_{s_2} J''_{s_3}\rangle_F$ are generated by theories of a free-boson and free-fermion respectively, while the "odd" structure, $\langle J_{s_1} J'_{s_2} J''_{s_3}\rangle_{\text{odd}}$, is not generated by a free CFT; instead it is generated by a Chern-Simons theory interacting with parity-violating matter [63–73]. Furthermore, the existence of the odd solution depends on the following set of triangle inequalities:

$$s_1 \leq s_2 + s_3, \qquad s_2 \leq s_1 + s_3, \qquad s_3 \leq s_1 + s_2. \tag{4.2}$$

When the triangle inequalities are simultaneously satisfied, there are two even solutions and one odd solution; however, if any of the inequalities above are not satisfied then the odd solution is incompatible with current conservation.[4] Further, if any of the J, J', J'' coincide (i.e., in cases where the currents are unique and have the same spin), then the resulting point-switch symmetries can kill off the remaining structures. Our comments on the general results for three-point functions of bosonic currents are summarized below:

(i) When the triangle inequalities are simultaneously satisfied, each polynomial structure in the solution for all three-point functions can be written as a product of at most 5 of the P_i, Q_i, with the Z_i completely eliminated.

(ii) For the three-point functions $\langle J_{s_1} J'_{s_1} J''_{s_2}\rangle$, for arbitrary integer s_1 and s_2: when the triangle inequalities are satisfied there are two even solutions and one odd solution, otherwise there are only two even solutions. After imposing $J = J'$ the solutions exist only when s_2 is an even integer. Note that for $s_1 > s_2$ the triangle inequalities are always satisfied.

(iii) For the three-point functions $\langle J_s J_s J_s\rangle$, with s an integer, there are two even solutions and one odd solution; however they exist only for s even. For s odd the solutions survive only if the currents carry a flavor index associated with a non-Abelian symmetry group.

Another observation is that the triangle inequalities can be encoded in a discriminant, σ, which we define as follows:

$$\sigma(s_1, s_2, s_3) = q_1 q_2 q_3, \qquad q_i = s_i - s_j - s_k - 1, \tag{4.3}$$

where (i, j, k) is a cyclic permutation of (1, 2, 3). For $\sigma(s_1, s_2, s_3) < 0$, there are two even solutions and one odd solution, while for $\sigma(s_1, s_2, s_3) \geq 0$ there are only two even solutions. The origin of this discriminant equation is actually quite simple within the framework of this formalism: recall that the correlation function can be encoded in a tensor $\mathcal{H}$, which is a function of a single three-point covariant, X. There are three different (but equivalent) representations of a given correlation function, call them $\mathcal{H}^{(i)}$, where the superscript i denotes which point we set to act as the "third point" in the ansatz (3.2). As shown in Sec. III B 2, the representations are related by the intertwining operator $\mathcal{I}$, with each $\mathcal{H}^{(i)}$ being homogeneous degree q_i. After exhaustive analysis of the three-point functions with $s_i \leq 20$, a clear pattern emerges: the odd structure survives if and only if $\forall\, i, q_i < 0$. In other words, each $\mathcal{H}^{(i)}$ must be a rational function of X with homogeneity $q_i < 0$. The discriminant (4.3) simply encodes information about whether the $\mathcal{H}^{(i)}$ are simultaneously of negative homogeneity.

[4] Existence and uniqueness of the parity-odd solution (inside and outside triangle inequalities) has been proven in the "lightlike" limit in [69]. Similar arguments can be made to show there are only two forms for the parity-even solutions; these are sketched [14].

In the next subsections we analyze the structure of three-point functions involving conserved bosonic currents. As a test of our approach we begin with an analysis of correlation functions involving low-spin currents such as the energy-momentum tensor and vector current.

A. Energy-momentum tensor and vector current correlators

The conserved currents which are fundamental in any conformal field theory are the conserved vector current, V_m, and the symmetric, traceless energy-momentum tensor, T_{mn}. The vector current has scale dimension $\Delta_V = 2$ and satisfies $\partial^m V_m = 0$, while the energy-momentum tensor has scale dimension $\Delta_T = 3$ and satisfies the conservation equation $\partial^m T_{mn} = 0$. Converting to spinor notation we have

$$V_{\alpha_1\alpha_2}(x) = (\gamma^m)_{\alpha_1\alpha_2} V_m(x),$$
$$T_{\alpha_1\alpha_2\alpha_3\alpha_4}(x) = (\gamma^m)_{(\alpha_1\alpha_2}(\gamma^n)_{\alpha_3\alpha_4)} T_{mn}(x). \tag{4.4}$$

These objects possess fundamental information associated with internal and spacetime symmetries, hence, analysis of their three-point functions is of great importance. The general structure of correlation functions involving these fields has been widely studied throughout the literature of conformal field theory; here we present the solutions for them using our formalism. The possible three-point functions involving the conserved vector current and the energy-momentum tensor are

$$\langle V_{\alpha(2)}(x_1) V_{\beta(2)}(x_2) V_{\gamma(2)}(x_3)\rangle, \qquad \langle V_{\alpha(2)}(x_1) V_{\beta(2)}(x_2) T_{\gamma(4)}(x_3)\rangle, \tag{4.5}$$

$$\langle T_{\alpha(4)}(x_1) T_{\beta(4)}(x_2) V_{\gamma(2)}(x_3)\rangle, \qquad \langle T_{\alpha(4)}(x_1) T_{\beta(4)}(x_2) T_{\gamma(4)}(x_3)\rangle. \tag{4.6}$$

In all cases, we note that the triangle inequalities (4.2) are simultaneously satisfied, hence, we expect that each of these correlation functions should possess a parity-odd solution after imposing conservation on all three points. The analysis of these three-point functions is quite simple using our computational approach. Let us first consider $\langle VVV\rangle$; within the framework of our formalism we study the three-point function $\langle J_1 J'_1 J''_1\rangle$.

1. Correlation function $\langle J_1 J'_1 J''_1\rangle$

The general ansatz for this correlation function, according to (3.15), is

$$\langle J_{\alpha(2)}(x_1) J'_{\beta(2)}(x_2) J''_{\gamma(2)}(x_3)\rangle = \frac{\mathcal{I}_{\alpha(2)}{}^{\alpha'(2)}(x_{13}) \mathcal{I}_{\beta(2)}{}^{\beta'(2)}(x_{23})}{(x_{13}^2)^2 (x_{23}^2)^2} \times \mathcal{H}_{\alpha'(2)\beta'(2)\gamma(2)}(X_{12}). \tag{4.7}$$

Using the formalism outlined in Sec. III B, all information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X; u(2), v(2), w(2)) = \mathcal{H}_{\alpha(2)\beta(2)\gamma(2)}(X) \mathbf{U}^{\alpha(2)} \mathbf{V}^{\beta(2)} \mathbf{W}^{\gamma(2)}, \tag{4.8}$$

where here and in all examples that follow we make the replacement $X_{12} \to X$ where X is some representative three-point building block which has no explicit dependence on the spacetime points. This may be done without loss of generality as the conservation equations (3.27), and algebraic constraints on $\mathcal{H}$ depend only on X. Using *Mathematica* we solve (3.33) for the chosen spins and substitute each solution into the generating function (3.32). This provides us with the following list of (linearly dependent) polynomial structures in the even and odd sectors respectively:

$$\text{even: } \{Z_1Z_2Z_3, Q_3^2Z_3, Q_2^2Z_2, Q_1Q_2Q_3, Q_1^2Z_1, P_3^2Z_3, P_2P_3Q_1, P_2^2Z_2, P_1P_3Q_2, P_1P_2Q_3, P_1^2Z_1\}, \tag{4.9a}$$

$$\text{odd: } \{P_3Q_3Z_3, P_3Q_1Q_2, P_2Q_2Z_2, P_2Q_1Q_3, P_1Q_2Q_3, P_1Q_1Z_1, P_1P_2P_3\}. \tag{4.9b}$$

Next, we systematically apply the linear dependence relations (3.36a) to these lists, reducing them to the following sets of linearly independent structures:

$$\text{even: } \{P_2P_3Q_1, P_1P_3Q_2, P_1P_2Q_3, Q_1Q_2Q_3\}, \tag{4.10a}$$

$$\text{odd: } \{P_3Q_1Q_2, P_2Q_1Q_3, P_1Q_2Q_3\}. \tag{4.10b}$$

Note that application of the linear-dependence relations eliminates all terms involving Z_i in this case. Next we construct an ansatz out of the linearly independent structures, see (3.35), where $\mathcal{H}_i^{(\pm)}$ denotes a structure at position "i" in the even/odd list respectively. After imposing conservation on all three points using the methods outlined in Sec. III B 4, we obtain the following relations between the coefficients:

$$\text{even}: \{A_1 \to A_1, A_2 \to A_1, A_3 \to A_3, A_4 \to A_1\}, \quad (4.11a)$$

$$\text{odd}: \{B_1 \to B_1, B_2 \to 0, B_3 \to 0\}. \quad (4.11b)$$

Hence, the final solutions for the even and odd sectors are

$$\text{even}: \frac{A_3}{X^2} P_1 P_2 Q_3 + \frac{A_1}{X^2}(P_2 P_3 Q_1 + P_1 P_3 Q_2 + Q_2 Q_3 Q_1), \quad (4.12a)$$

$$\text{odd}: \frac{B_1 P_3 Q_1 Q_2}{X^2}. \quad (4.12b)$$

After imposing symmetries under permutation of space-time points, e.g., $J = J' = J''$, the remaining structures vanish unless the currents possess a flavor index associated with a non-Abelian symmetry group, in which case all three structures survive. The next example to consider is the mixed correlator $\langle VVT \rangle$. To study this case we may examine the correlation function $\langle J_1 J'_1 J''_2 \rangle$.

2. Correlation function $\langle J_1 J'_1 J''_2 \rangle$

Using the general formula, the ansatz for this three-point function reads

$$\langle J_{\alpha(2)}(x_1) J'_{\beta(2)}(x_2) J''_{\gamma(4)}(x_3) \rangle = \frac{\mathcal{I}_{\alpha(2)}{}^{\alpha'(2)}(x_{13}) \mathcal{I}_{\beta(2)}{}^{\beta'(2)}(x_{23})}{(x_{13}^2)^2 (x_{23}^2)^2} \mathcal{H}_{\alpha'(2)\beta'(2)\gamma(4)}(X_{12}). \quad (4.13)$$

Using the formalism outlined in Sec. III B, all information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X; u(2), v(2), w(4)) = \mathcal{H}_{\alpha(2)\beta(2)\gamma(4)}(X) \mathbf{U}^{\alpha(2)} \mathbf{V}^{\beta(2)} \mathbf{W}^{\gamma(4)}. \quad (4.14)$$

After solving (3.33), we find the following linearly dependent polynomial structures in the even and odd sectors respectively:

$$\text{even}: \{Z_1 Z_2 Z_3^2, Q_3^2 Z_3^2, Q_2^2 Z_2 Z_3, Q_1 Q_2 Q_3 Z_3, Q_1^2 Z_1 Z_3, Q_1^2 Q_2^2, P_3^2 Z_3^2, P_2 P_3 Q_1 Z_3, P_2^2 Z_2 Z_3, P_2^2 Q_1^2, P_1 P_3 Q_2 Z_3, P_1 P_2 Q_3 Z_3, P_1 P_2 Q_1 Q_2, P_1^2 Z_1 Z_3, P_1^2 Q_2^2, P_1^2 P_2^2\}, \quad (4.15a)$$

$$\text{odd}: \{P_3 Q_3 Z_3^2, P_3 Q_1 Q_2 Z_3, P_2 Q_2 Z_2 Z_3, P_2 Q_1 Q_3 Z_3, P_2 Q_1^2 Q_2, P_1 Q_2 Q_3 Z_3, P_1 Q_1 Z_1 Z_3, P_1 Q_1 Q_2^2, P_1 P_2 P_3 Z_3, P_1 P_2^2 Q_1, P_1^2 P_2 Q_2\}. \quad (4.15b)$$

Next we systematically apply the linear dependence relations (3.36a) to these lists, reducing them to the following linearly independent structures:

$$\text{even}: \{P_2^2 Q_1^2, P_1 P_2 Q_1 Q_2, P_1^2 Q_2^2, P_1^2 P_2^2, Q_1^2 Q_2^2\}, \quad (4.16a)$$

$$\text{odd}: \{P_2 Q_1^2 Q_2, P_1 Q_1 Q_2^2, P_1 P_2^2 Q_1, P_1^2 P_2 Q_2\}. \quad (4.16b)$$

After constructing an appropriate ansatz for each sector, we then impose conservation on all three points using the methods outlined in Sec. III B 4 and obtain the following relations between the coefficients:

$$\text{even}: \left\{A_1 \to A_1, A_2 \to A_2, A_3 \to A_1, A_4 \to A_2 - 3A_1, A_5 \to -\frac{3A_1}{5}\right\}, \quad (4.17a)$$

$$\text{odd}: \{B_1 \to B_1, B_2 \to B_1, B_3 \to -B_1, B_4 \to -B_1\}. \quad (4.17b)$$

Hence, the final solutions for the even and odd sectors are

$$\text{even}: \frac{A_2}{X}(P_1 P_2 Q_1 Q_2 + P_1^2 P_2^2) + \frac{A_1}{X}\left(P_2^2 Q_1^2 + P_1^2 Q_2^2 - 3P_1^2 P_2^2 - \frac{3}{5} Q_1^2 Q_2^2\right), \quad (4.18a)$$

$$\text{odd}: \frac{B_1}{X}(-P_2 P_1^2 Q_2 + P_1 Q_1 Q_2^2 - P_2^2 P_1 Q_1 + P_2 Q_1^2 Q_2). \quad (4.18b)$$

All structures survive after setting $J = J'$. Hence, this correlation function is fixed up to two independent even structures, and one odd structure.

Since the number of tensor structures rapidly increases with spin, we will skip the technical details for the other correlation functions and present only the final results after imposing conservation. For $\langle TTV\rangle$ we may consider the correlation function $\langle J_2 J'_2 J''_1\rangle$, for which we find the solution:

$$\text{even:}\ \frac{A_2}{X^4}\left(\frac{5}{3}P_1P_2Q_3^3 + P_2P_3Q_1Q_3^2 + P_1P_3Q_2Q_3^2 + P_3^2Q_1Q_2Q_3\right) + \frac{A_1}{X^4}\left(P_2P_3^3Q_1 + P_1P_3^3Q_2 - 6P_2P_3Q_1Q_3^2 - 6P_1P_3Q_2Q_3^2 - \frac{23}{3}P_1P_2Q_3^3 - \frac{7}{3}Q_1Q_2Q_3^3\right), \tag{4.19a}$$

$$\text{odd:}\ \frac{B_1}{X^4}(P_3^3Q_1Q_2 - P_2P_3^2Q_1Q_3 - P_1P_3^2Q_2Q_3 - 3P_3Q_1Q_2Q_3^2). \tag{4.19b}$$

In this case, all structures vanish after setting $J = J'$ and imposing the required symmetries under the exchange of x_1 and x_2. Hence, the correlation function $\langle TTV\rangle$ vanishes in any CFT. Finally, to study $\langle TTT\rangle$ we can analyze the correlation function $\langle J_2 J'_2 J''_2\rangle$, which has the solution:

$$\text{even:}\ \frac{A_3}{X^3}\left(P_2^2Q_3^2Q_1^2 + P_2P_3Q_2Q_3Q_1^2 + P_1P_2Q_2Q_3^2Q_1 + P_1P_3Q_2^2Q_3Q_1 - \frac{9}{5}P_1^2P_2^2Q_3^2 + P_1^2Q_2^2Q_3^2\right) + \frac{A_1}{X^3}\left(P_2^2P_3^2Q_1^2 - \frac{30}{7}P_3^2Q_2^2Q_1^2 + P_1^2P_3^2Q_2^2 + \frac{3}{5}P_1^2P_2^2Q_3^2 + Q_2^2Q_3^2Q_1^2\right), \tag{4.20a}$$

$$\text{odd:}\ \frac{B_1}{X^3}(P_2P_3^2Q_2Q_1^2 + P_3Q_2^2Q_3Q_1^2 + P_1P_3^2Q_2^2Q_1). \tag{4.20b}$$

In all cases, it is clear that the general solutions are determined up to two independent even structures, and one odd structure. These results are consistent with [13] in terms of the number of independent polynomial structures; however it is difficult to make a direct comparison.

B. Higher-spin correlators

In this subsection we obtain explicit results for three-point functions involving higher-spin currents. We present the final results after imposing conservation on all three points.

1. Correlation function $\langle J_1 J'_1 J''_3\rangle$: $\sigma = 0$

$$\text{Even:}\ A_1\left(P_2^2Q_1^2Z_3 + P_1^2Q_2^2Z_3 - 5P_1^2P_2^2Z_3 - \frac{3}{7}Q_1^2Q_2^2Z_3\right) + A_2(P_1P_2Q_1Q_2Z_3 + P_1^2P_2^2Z_3), \tag{4.21a}$$

$$\text{odd:}\ 0. \tag{4.21b}$$

This is an instance in which one of the triangle inequalities is not satisfied, and we can see here that the odd solution vanishes. The even structures vanish after imposing $J = J'$.

2. Correlation function $\langle J_1 J'_1 J''_4\rangle$: $\sigma > 0$

$$\text{Even:}\ A_1\left(P_2^2Q_1^2Z_3^2 + P_1^2Q_2^2Z_3^2 - 7P_1^2P_2^2Z_3^2 - \frac{1}{3}Q_1^2Q_2^2Z_3^2\right) + XA_2(P_1P_2Q_1Q_2Z_3^2 + P_1^2P_2^2Z_3^2), \tag{4.22a}$$

$$\text{odd:}\ 0. \tag{4.22b}$$

3. Correlation function $\langle J_1 J'_2 J''_3\rangle$: $\sigma < 0$

$$\text{Even:}\ \frac{A_3}{X}\left(-\frac{7}{3}P_2P_1^3Q_1Q_2 + P_2^2P_1^2Q_1^2 + P_2P_1Q_1^3Q_2 - \frac{7}{3}P_2^2P_1^4\right) + \frac{A_1}{X}\left(\frac{21}{5}P_1^4Q_2^2 - \frac{112}{15}P_2P_1^3Q_1Q_2 + 14P_2^2P_1^2Q_1^2 - \frac{42}{5}P_1^2Q_1^2Q_2^2 - \frac{7}{5}P_2^2Q_1^4 - \frac{301}{15}P_2^2P_1^4 + Q_1^4Q_2^2\right), \tag{4.23a}$$

$$\text{odd}:\ \frac{B_1}{X}(2P_2P_1^4Q_2-4P_1^3Q_1Q_2^2+2P_2^2P_1^3Q_1-6P_2P_1^2Q_1^2Q_2-2P_2^2P_1Q_1^3+2P_1Q_1^3Q_2^2+P_2Q_1^4Q_2). \tag{4.23b}$$

4. Correlation function $\langle J_2J_2'J_3''\rangle$: $\sigma<0$

$$\begin{aligned}\text{Even}:\ &\frac{A_6}{X^2}\left(\frac{47}{15}P_2^3P_1^3Q_3-\frac{7}{5}P_2P_1^3Q_2^2Q_3-\frac{7}{5}P_2^3P_1Q_1^2Q_3+\frac{4}{5}P_2^2P_1^2Q_1Q_2Q_3+P_2P_1Q_1^2Q_2^2Q_3\right)\\&+\frac{A_1}{X^2}\left(-\frac{9}{5}P_3P_1^3Q_2^3-\frac{276}{5}P_2^3P_1^3Q_3+\frac{117}{5}P_2P_1^3Q_2^2Q_3+9P_1^2Q_1Q_2^3Q_3-\frac{81}{5}P_2^2P_1^2Q_1Q_2Q_3\right.\\&\left.+15P_3P_1Q_1^2Q_2^3+\frac{117}{5}P_2^3P_1Q_1^2Q_3-\frac{9}{5}P_2^3P_3Q_1^3+15P_2P_3Q_1^3Q_2^2+9P_2^2Q_1^3Q_2Q_3+Q_1^3Q_2^3Q_3\right),\end{aligned} \tag{4.24a}$$

$$\begin{aligned}\text{odd}:\ &\frac{B_1}{X^2}\left(P_3Q_2^3Q_1^3-\frac{2}{5}P_2^2P_3Q_2Q_1^3-\frac{2}{15}P_2^3Q_3Q_1^3+\frac{4}{5}P_2Q_2^2Q_3Q_1^3+\frac{4}{5}P_1Q_2^3Q_3Q_1^2-\frac{2}{5}P_1P_2^2Q_2Q_3Q_1^2\right.\\&\left.-\frac{2}{5}P_1^2P_3Q_2^3Q_1-\frac{2}{5}P_1^2P_2Q_2^2Q_3Q_1-\frac{2}{15}P_1^3Q_2^3Q_3\right).\end{aligned} \tag{4.24b}$$

5. Correlation function $\langle J_2J_2'J_4''\rangle$: $\sigma<0$

$$\begin{aligned}\text{Even}:\ &\frac{A_4}{X}\left(-\frac{9}{5}P_2^2P_1^4Q_2^2-\frac{9}{5}P_2P_1^3Q_1Q_2^3+\frac{29}{5}P_2^3P_1^3Q_1Q_2-\frac{9}{5}P_2^4P_1^2Q_1^2+\frac{9}{5}P_2^2P_1^2Q_1^2Q_2^2+P_2P_1Q_1^3Q_2^3\right.\\&\left.-\frac{9}{5}P_2^3P_1Q_1^3Q_2+5P_2^4P_1^4\right)+\frac{A_1}{X}\left(\frac{99}{35}P_1^4Q_2^4-\frac{1278}{35}P_2^2P_1^4Q_2^2-\frac{288}{35}P_2P_1^3Q_1Q_2^3+\frac{384}{5}P_2^3P_1^3Q_1Q_2-\frac{54}{7}P_1^2Q_1^2Q_2^4\right.\\&\left.-\frac{1278}{35}P_2^4P_1^2Q_1^2+\frac{324}{5}P_2^2P_1^2Q_1^2Q_2^2-\frac{288}{35}P_2^3P_1Q_1^3Q_2+\frac{99}{35}P_2^4Q_1^4-\frac{54}{7}P_2^2Q_1^4Q_2^2+81P_2^4P_1^4+Q_1^4Q_2^4\right),\end{aligned} \tag{4.25a}$$

$$\begin{aligned}\text{odd}:\ &\frac{B_1}{X}\left(\frac{5}{3}P_2P_1^4Q_2^3-P_2^3P_1^4Q_2-\frac{5}{3}P_1^3Q_1Q_2^4+6P_2^2P_1^3Q_1Q_2^2-P_2^4P_1^3Q_1-6P_2P_1^2Q_1^2Q_2^3+6P_2^3P_1^2Q_1^2Q_2\right.\\&\left.+P_1Q_1^3Q_2^4+\frac{5}{3}P_2^4P_1Q_1^3-6P_2^2P_1Q_1^3Q_2^2+P_2Q_1^4Q_2^3-\frac{5}{3}P_2^3Q_1^4Q_2\right).\end{aligned} \tag{4.25b}$$

6. Correlation function $\langle J_2J_2'J_5''\rangle$: $\sigma=0$

$$\begin{aligned}\text{Even}:\ &A_2\left(P_2^2P_1^4Q_2^2Z_3+P_2P_1^3Q_1Q_2^3Z_3-\frac{41}{11}P_2^3P_1^3Q_1Q_2Z_3+P_2^4P_1^2Q_1^2Z_3-\frac{9}{11}P_2^2P_1^2Q_1^2Q_2^2Z_3-\frac{5}{11}P_2P_1Q_1^3Q_2^3Z_3\right.\\&\left.+P_2^3P_1Q_1^3Q_2Z_3-\frac{37}{11}P_2^4P_1^4Z_3\right)+A_1\left(P_1^4Q_2^4Z_3-14P_2^2P_1^4Q_2^2Z_3+\frac{224}{11}P_2^3P_1^3Q_1Q_2Z_3-\frac{30}{13}P_1^2Q_1^2Q_2^4Z_3\right.\\&-14P_2^4P_1^2Q_1^2Z_3+\frac{252}{11}P_2^2P_1^2Q_1^2Q_2^2Z_3-\frac{160}{143}P_2P_1Q_1^3Q_2^3Z_3+P_2^4Q_1^4Z_3-\frac{30}{13}P_2^2Q_1^4Q_2^2Z_3\\&\left.+\frac{343}{11}P_2^4P_1^4Z_3+\frac{35}{143}Q_1^4Q_2^4Z_3\right),\end{aligned} \tag{4.26a}$$

$$\text{odd}:\ 0. \tag{4.26b}$$

7. Correlation function $\langle J_3 J'_3 J''_3 \rangle$: $\sigma < 0$

$$
\begin{aligned}
\text{Even: } & \frac{A_2}{X^4}\Big(\frac{23}{10}P_2^2Q_2Q_3^3Q_1^3+\frac{33}{10}P_2P_3Q_2^2Q_3^2Q_1^3-\frac{27}{70}P_2^3P_3Q_3^2Q_1^3+P_3^2Q_2^3Q_3Q_1^3-\frac{27}{70}P_2^2P_3^2Q_2Q_3Q_1^3-\frac{67}{35}P_1P_2^3Q_3^3Q_1^2 \\
&+\frac{349}{70}P_1P_2Q_2^2Q_3^3Q_1^2+\frac{33}{10}P_1P_3Q_2^3Q_3^2Q_1^2+\frac{23}{10}P_1^2Q_2^3Q_3^3Q_1-\frac{67}{35}P_1^2P_2^2Q_2Q_3^3Q_1-\frac{27}{70}P_1^2P_3^2Q_2^3Q_3Q_1 \\
&+\frac{3191P_1^3P_2^3Q_3^3}{1470}-\frac{67}{35}P_1^3P_2Q_2^2Q_3^3-\frac{27}{70}P_1^3P_3Q_2^3Q_3^2\Big)+\frac{A_1}{X^4}\Big(P_2^3P_3^3Q_1^3+\frac{159}{11}P_2^2Q_2Q_3^3Q_1^3-\frac{105}{11}P_2P_3^3Q_2^2Q_1^3 \\
&+24P_2P_3Q_2^2Q_3^2Q_1^3-\frac{24}{11}P_2^3P_3Q_3^2Q_1^3-\frac{24}{11}P_2^2P_3^2Q_2Q_3Q_1^3-\frac{105}{11}P_1P_3^3Q_2^3Q_1^2-\frac{135}{11}P_1P_2^3Q_3^3Q_1^2 \\
&+\frac{342}{11}P_1P_2Q_2^2Q_3^3Q_1^2+24P_1P_3Q_2^3Q_3^2Q_1^2+\frac{159}{11}P_1^2Q_2^3Q_3^3Q_1-\frac{135}{11}P_1^2P_2^2Q_2Q_3^3Q_1-\frac{24}{11}P_1^2P_3^2Q_2^3Q_3Q_1 \\
&+P_1^3P_3^3Q_2^3+\frac{1062}{77}P_1^3P_2^3Q_3^3-\frac{135}{11}P_1^3P_2Q_2^2Q_3^3-\frac{24}{11}P_1^3P_3Q_2^3Q_3^2+Q_2^3Q_3^3Q_1^3\Big), \qquad (4.27a)
\end{aligned}
$$

$$
\text{odd: } 0. \qquad (4.27b)
$$

8. Correlation function $\langle J_4 J'_4 J''_4 \rangle$: $\sigma < 0$

$$
\begin{aligned}
\text{Even: } & \frac{A_2}{X^5}\Big(-\frac{451P_2^4Q_3^4Q_1^4}{1428}+\frac{10229P_2^2Q_2^2Q_3^4Q_1^4}{4284}+\frac{14513P_2P_3Q_2^3Q_3^3Q_1^4}{4284}-\frac{407}{612}P_2^3P_3Q_2Q_3^3Q_1^4+P_3^2Q_2^4Q_3^2Q_1^4 \\
&+\frac{143P_2^4P_3^2Q_3^2Q_1^4}{4284}-\frac{2189P_2^2P_3^2Q_2^2Q_3^2Q_1^4}{4284}-\frac{11}{68}P_2P_3^3Q_2^3Q_3Q_1^4+\frac{143P_2^3P_3^3Q_2Q_3Q_1^4}{4284}+\frac{1195}{252}P_1P_2Q_2^3Q_3^4Q_1^3 \\
&-\frac{2165P_1P_2^3Q_2Q_3^4Q_1^3}{1428}+\frac{14513P_1P_3Q_2^4Q_3^3Q_1^3}{4284}-\frac{11}{68}P_1P_3^3Q_2^4Q_3Q_1^3+\frac{421}{476}P_1^2P_2^4Q_3^4Q_1^2+\frac{10229P_1^2Q_2^4Q_3^4Q_1^2}{4284} \\
&-\frac{1293}{476}P_1^2P_2^2Q_2^2Q_3^4Q_1^2-\frac{2189P_1^2P_3^2Q_2^4Q_3^2Q_1^2}{4284}-\frac{2165P_1^3P_2Q_2^3Q_3^4Q_1}{1428}+\frac{421}{476}P_1^3P_2^3Q_2Q_3^4Q_1-\frac{407}{612}P_1^3P_3Q_2^4Q_3^3Q_1 \\
&+\frac{143P_1^3P_3^3Q_2^4Q_3Q_1}{4284}-\frac{9337P_1^4P_2^4Q_3^4}{12852}-\frac{451P_1^4Q_2^4Q_3^4}{1428}+\frac{421}{476}P_1^4P_2^2Q_2^2Q_3^4+\frac{143P_1^4P_3^2Q_2^4Q_3^2}{4284}\Big) \\
&+\frac{A_1}{X^5}\Big(P_2^4P_3^4Q_1^4+\frac{462}{13}P_3^4Q_2^4Q_1^4-\frac{563}{85}P_2^4Q_3^4Q_1^4+Q_2^4Q_3^4Q_1^4+\frac{877}{17}P_2^2Q_2^2Q_3^4Q_1^4+\frac{5813}{85}P_2P_3Q_2^3Q_3^3Q_1^4 \\
&-\frac{1209}{85}P_2^3P_3Q_2Q_3^3Q_1^4-\frac{84}{5}P_2^2P_3^4Q_2^2Q_1^4+\frac{83}{85}P_2^4P_3^2Q_3^2Q_1^4-\frac{499}{85}P_2^2P_3^2Q_2^2Q_3^2Q_1^4+\frac{147}{85}P_2P_3^3Q_2^3Q_3Q_1^4 \\
&+\frac{83}{85}P_2^3P_3^3Q_2Q_3Q_1^4+\frac{511}{5}P_1P_2Q_2^3Q_3^4Q_1^3-\frac{2727}{85}P_1P_2^3Q_2Q_3^4Q_1^3+\frac{5813}{85}P_1P_3Q_2^4Q_3^3Q_1^3+\frac{147}{85}P_1P_3^3Q_2^4Q_3Q_1^3 \\
&-\frac{84}{5}P_1^2P_3^4Q_2^4Q_1^2+\frac{1601}{85}P_1^2P_2^4Q_3^4Q_1^2+\frac{877}{17}P_1^2Q_2^4Q_3^4Q_1^2-\frac{4891}{85}P_1^2P_2^2Q_2^2Q_3^4Q_1^2-\frac{499}{85}P_1^2P_3^2Q_2^4Q_3^2Q_1^2 \\
&-\frac{2727}{85}P_1^3P_2Q_2^3Q_3^4Q_1+\frac{1601}{85}P_1^3P_2^3Q_2Q_3^4Q_1-\frac{1209}{85}P_1^3P_3Q_2^4Q_3^3Q_1+\frac{83}{85}P_1^3P_3^3Q_2^4Q_3Q_1+P_1^4P_3^4Q_2^4 \\
&-\frac{3938}{255}P_1^4P_2^4Q_3^4-\frac{563}{85}P_1^4Q_2^4Q_3^4+\frac{1601}{85}P_1^4P_2^2Q_2^2Q_3^4+\frac{83}{85}P_1^4P_3^2Q_2^4Q_3^2\Big), \qquad (4.28a)
\end{aligned}
$$

$$
\begin{aligned}
\text{odd: } & \frac{B_1}{X^5}\Big(-5P_2P_3^4Q_2^3Q_1^4+P_3Q_2^4Q_3^3Q_1^4+\frac{10}{3}P_2P_3^2Q_2^3Q_3^2Q_1^4+P_2^3P_3^4Q_2Q_1^4-\frac{5}{3}P_3^3Q_2^4Q_3Q_1^4-5P_1P_3^4Q_2^4Q_1^3 \\
&+\frac{10}{3}P_1P_3^2Q_2^4Q_3^2Q_1^3+P_1^3P_3^4Q_2^4Q_1\Big). \qquad (4.28b)
\end{aligned}
$$

The solutions quickly become cumbersome to present beyond these cases; however, our method effectively produces explicit results for any chosen spins within the confines of our computational limitations. We have explicitly tested our approach up to $s_i = 20$, and we present additional results in Appendix B.

V. MIXED CORRELATORS INVOLVING FERMIONIC CURRENTS

In this section we will evaluate three-point functions involving conserved fermionic currents. To the best of our knowledge, these correlation functions have not been studied in much detail in the literature, particularly in three dimensions. The most important example of a fermionic conserved current is the supersymmetry current, $Q_{m,\alpha}$, which is prevalent in $\mathcal{N}$-extended superconformal field theories. Such a field is primary with dimension $\Delta_Q = 5/2$, and satisfies the conservation equation $\partial^m Q_{m,\alpha} = 0$. In spinor notation, we have

$$Q_{\alpha(3)}(x) = (\gamma^m)_{(\alpha_1\alpha_2} Q_{m,\alpha_3)}(x). \tag{5.1}$$

Recall that in three-dimensional superconformal field theory, the supersymmetry current and the energy-momentum tensor are contained in the supercurrent multiplet, $\mathbf{J}_{\alpha(3)}(z)$, where $z^{\mathcal{A}} = (x^a, \theta^\alpha)$ is a point in $\mathcal{N} = 1$ superspace. The supersymmetry current, $Q_{\alpha(3)}$, and the energy-momentum tensor, $T_{\alpha(4)}$, are extracted through bar-projection as follows:

$$\begin{aligned} Q_{\alpha(3)}(x) &= \mathbf{J}_{\alpha(3)}(z)|_{\theta=0}, \\ T_{\alpha(4)}(x) &= D_{(\alpha_1}\mathbf{J}_{\alpha_2\alpha_3\alpha_4)}(z)|_{\theta=0}, \end{aligned} \tag{5.2}$$

where

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}(\gamma^m)_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^m} \tag{5.3}$$

is the standard spinor-covariant derivative [76]. Likewise, the conserved vector current, $V_{\alpha(2)}$, is contained within the flavor current multiplet, $\mathbf{L}_\alpha(z)$, and is extracted as follows:

$$V_{\alpha(2)}(x) = D_{(\alpha_1}\mathbf{L}_{\alpha_2)}(z)|_{\theta=0}. \tag{5.4}$$

Since the supersymmetry current is a conserved current associated with supersymmetry transformations, it is interesting to study the correlation functions involving the supersymmetry current, the vector current and the energy-momentum tensor. The two three-point functions involving Q, V and T which are of interest in $\mathcal{N} = 1$ superconformal field theories are

$$\begin{aligned} &\langle Q_{\alpha(3)}(x_1) Q_{\beta(3)}(x_2) V_{\gamma(2)}(x_3)\rangle, \\ &\langle Q_{\alpha(3)}(x_1) Q_{\beta(3)}(x_2) T_{\gamma(4)}(x_3)\rangle. \end{aligned} \tag{5.5}$$

These correlation functions are naturally contained in the following supersymmetric three-point functions:

$$\begin{aligned} &\langle \mathbf{J}_{\alpha(3)}(z_1)\mathbf{J}_{\beta(3)}(z_2)\mathbf{L}_\gamma(z_3)\rangle, \\ &\langle \mathbf{J}_{\alpha(3)}(z_1)\mathbf{J}_{\beta(3)}(z_2)\mathbf{J}_{\gamma(3)}(z_3)\rangle. \end{aligned} \tag{5.6}$$

In three dimensions, $\langle \mathbf{JJL}\rangle$ vanishes, while $\langle \mathbf{JJJ}\rangle$ is fixed up to a single tensor structure [50,51,54]. Therefore, since the component correlators (5.5) are obtained by bar-projecting the supersymmetric correlation functions, we find $\langle QQV\rangle = 0$, while $\langle QQT\rangle$ is fixed up to a single tensor structure.

However, in this paper we do not assume supersymmetry. Our goal is to find the most general structure of the correlation functions consistent with only conformal symmetry. Hence, in the next subsection we will evaluate the correlation functions

$$\begin{aligned} &\langle \tilde{Q}_{\alpha(3)}(x_1)\tilde{Q}'_{\beta(3)}(x_2) V_{\gamma(2)}(x_3)\rangle, \\ &\langle \tilde{Q}_{\alpha(3)}(x_1)\tilde{Q}'_{\beta(3)}(x_2) T_{\gamma(4)}(x_3)\rangle, \end{aligned} \tag{5.7}$$

where in this case $\tilde{Q}_{\alpha(3)}$ and $\tilde{Q}'_{\beta(3)}$ now denote "supersymmetry-like" currents; that is, they possess identical properties to supersymmetry currents but are not necessarily equal to them. It is of interest to us to examine the number of independent tensor structures contained within these three-point functions to see if they are consistent with the supersymmetric results. A similar analysis was recently carried out in four dimensions [62], where it was found that the number of independent structures is, in general, inconsistent with supersymmetry.

In the next subsections we use our formalism to constrain the general form of correlation functions involving supersymmetry-like currents consistent with conservation equations and point-switch symmetries. This is followed by an analysis of correlation functions involving conserved fermionic higher-spin currents. Our comments on the results for correlation functions involving fermionic symmetry currents for general spins are summarized below:

(i) In general, the structure of the three-point function $\langle J_{s_1} J'_{s_2} J''_{s_3}\rangle$, for arbitrary half-integer s_1, s_2 and integer s_3, adheres to the triangle inequalities (4.2), the same as the bosonic case. That is, if the triangle inequalities are satisfied we obtain two even structures and one odd structure. Otherwise, there are just two even structures.

(ii) For the three-point functions $\langle J_{s_1} J'_{s_1} J''_{s_2}\rangle$, for arbitrary half-integer s_1 and integer s_2: when the triangle inequalities are satisfied there are two even solutions and one odd solution, otherwise there are only two even solutions. After imposing $J = J'$ the solutions exist only when s_2 is an even integer. Note that for $s_1 > s_2$ the triangle inequalities are always satisfied.

A. Spin-3/2 current correlators

In this subsection we present an explicit analysis of the general structure of the correlation functions involving $\tilde{Q}$, V and T that are compatible with the constraints of conformal symmetry and conservation equations. Let us first consider $\langle \tilde{Q}\tilde{Q}' V\rangle$, for which we may analyze the general structure of the correlation function $\langle J_{3/2} J'_{3/2} J''_1\rangle$.

1. Correlation function $\langle J_{3/2} J'_{3/2} J''_1\rangle$

Using the general formula, the ansatz for this three-point function reads

$$\langle J_{\alpha(3)}(x_1) J'_{\beta(3)}(x_2) J''_{\gamma(2)}(x_3)\rangle = \frac{\mathcal{I}_{\alpha(3)}{}^{\alpha'(3)}(x_{13}) \mathcal{I}_{\beta(3)}{}^{\beta'(3)}(x_{23})}{(x_{13}^2)^{5/2}(x_{23}^2)^{5/2}} \mathcal{H}_{\alpha'(3)\beta'(3)\gamma(2)}(X_{12}). \tag{5.8}$$

Using the formalism outlined in Sec. III B, all information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X; u(3), v(3), w(2)) = \mathcal{H}_{\alpha(3)\beta(3)\gamma(2)}(X) \mathbf{U}^{\alpha(3)} \mathbf{V}^{\beta(3)} \mathbf{W}^{\gamma(2)}. \tag{5.9}$$

After solving (3.33), we find the following linearly dependent polynomial structures in the even and odd sectors respectively:

$$\begin{aligned} \text{even:}\ \{&Q_3Z_1Z_2Z_3, Q_3^3Z_3, Q_2^2Q_3Z_2, Q_1Q_2Z_1Z_2, Q_1Q_2Q_3^2, Q_1^2Q_3Z_1, P_3^2Q_3Z_3, P_3^2Q_1Q_2, P_2P_3Q_2Z_2, P_2P_3Q_1Q_3, P_2^2Q_3Z_2,\\ &P_1P_3Q_2Q_3, P_1P_3Q_1Z_1, P_1P_2Z_1Z_2, P_1P_2Q_3^2, P_1P_2P_3^2, P_1^2Q_3Z_1\}, \end{aligned} \tag{5.10a}$$

$$\begin{aligned} \text{odd:}\ \{&P_3Z_1Z_2Z_3, P_3Q_3^2Z_3, P_3Q_2^2Z_2, P_3Q_1Q_2Q_3, P_3Q_1^2Z_1, P_3^3Z_3, P_2Q_2Q_3Z_2, P_2Q_1Z_1Z_2, P_2Q_1Q_3^2, P_2P_3^2Q_1, P_2^2P_3Z_2,\\ &P_1Q_2Z_1Z_2, P_1Q_2Q_3^2, P_1Q_1Q_3Z_1, P_1P_3^2Q_2, P_1P_2P_3Q_3, P_1^2P_3Z_1\}. \end{aligned} \tag{5.10b}$$

Next we systematically apply the linear dependence relations (3.36a) to these lists, reducing them to the following linearly independent structures:

$$\text{even:}\ \{P_3^2Q_1Q_2, P_2P_3Q_1Q_3, P_1P_3Q_2Q_3, P_1P_2Q_3^2, Q_1Q_2Q_3^2\}, \tag{5.11a}$$

$$\text{odd:}\ \{P_2P_3^2Q_1, P_1P_3^2Q_2, P_3Q_1Q_2Q_3, P_2Q_1Q_3^2, P_1Q_2Q_3^2\}. \tag{5.11b}$$

After constructing an appropriate ansatz for each sector, we obtain the following relations between the coefficients after imposing conservation on all three points:

$$\text{even:}\ \left\{A_1 \to A_1, A_2 \to A_2, A_3 \to A_2, A_4 \to \frac{10A_1}{9} + \frac{17A_2}{9}, A_5 \to \frac{5A_2}{9} - \frac{5A_1}{9}\right\}, \tag{5.12a}$$

$$\text{odd:}\ \{B_1 \to B_1, B_2 \to B_1, B_3 \to 3B_1, B_4 \to 0, B_5 \to 0\}. \tag{5.12b}$$

The final solutions for the even and odd sectors are

$$\text{even:}\ \frac{A_2}{X^3}\left(\frac{17}{9}P_1P_2Q_3^2 + P_2P_3Q_1Q_3 + P_1P_3Q_2Q_3 + \frac{5}{9}Q_1Q_2Q_3^2\right) + \frac{A_1}{X^3}\left(P_3^2Q_1Q_2 + \frac{10}{9}P_1P_2Q_3^2 - \frac{5}{9}Q_1Q_2Q_3^2\right), \tag{5.13a}$$

$$\text{odd:}\ \frac{B_1}{X^3}(P_2P_3^2Q_1 + P_1P_3^2Q_2 + 3P_3Q_1Q_2Q_3). \tag{5.13b}$$

Hence, we see that the correlation function $\langle J_{3/2} J'_{3/2} J''_1\rangle$, and therefore $\langle \tilde{Q}\tilde{Q}' V\rangle$ is fixed up to two even structures and one odd structure. It may be shown that all structures vanish for $J = J'$ as they do not possess the correct symmetry under permutation of points x_1 and x_2; therefore, we find that the correlation function $\langle \tilde{Q}\tilde{Q} V\rangle$ vanishes.

Next we will analyze the general structure of $\langle \tilde{Q}\tilde{Q}T\rangle$, which is associated with the correlation function $\langle J_{3/2}J'_{3/2}J''_{2}\rangle$ using the ansatz (3.15).

2. Correlation function $\langle J_{3/2}J'_{3/2}J''_{2}\rangle$

According to the general formula (3.15), the ansatz for this three-point function is

$$\langle J_{\alpha(3)}(x_1)J'_{\beta(3)}(x_2)J''_{\gamma(4)}(x_3)\rangle = \frac{\mathcal{I}_{\alpha(3)}{}^{\alpha'(3)}(x_{13})\mathcal{I}_{\beta(3)}{}^{\beta'(3)}(x_{23})}{(x_{13}^2)^{5/2}(x_{23}^2)^{5/2}}\mathcal{H}_{\alpha'(3)\beta'(3)\gamma(4)}(X_{12}). \tag{5.14}$$

Using the formalism outlined in Sec. III B, all information about this correlation function is encoded in the following polynomial:

$$\mathcal{H}(X;u(3),v(3),w(4)) = \mathcal{H}_{\alpha(3)\beta(3)\gamma(4)}(X)\mathbf{U}^{\alpha(3)}\mathbf{V}^{\beta(3)}\mathbf{W}^{\gamma(4)}. \tag{5.15}$$

After solving (3.33), we find the following linearly dependent polynomial structures in the even and odd sectors respectively:

$$\begin{aligned}\text{even}:\ \{&Q_3Z_1Z_2Z_3^2, Q_3^3Z_3^2, Q_2^2Q_3Z_2Z_3, Q_1Q_2Z_1Z_2Z_3, Q_1Q_2Q_3^2Z_3, Q_1Q_2^3Z_2, Q_1^2Q_3Z_1Z_3, Q_1^2Q_2^2Q_3, Q_1^3Q_2Z_1, P_3^2Q_3Z_3^2,\\ &P_3^2Q_1Q_2Z_3, P_2P_3Q_2Z_2Z_3, P_2P_3Q_1Q_3Z_3, P_2P_3Q_1^2Q_2, P_2^2Q_3Z_2Z_3, P_2^2Q_1Q_2Z_2, P_2^2Q_1^2Q_3, P_1P_3Q_2Q_3Z_3,\\ &P_1P_3Q_1Z_1Z_3, P_1P_3Q_1Q_2^2, P_1P_2Z_1Z_2Z_3, P_1P_2Q_3^2Z_3, P_1P_2Q_2^2Z_2, P_1P_2Q_1Q_2Q_3, P_1P_2Q_1^2Z_1, P_1P_2P_3^2Z_3,\\ &P_1P_2^2P_3Q_1, P_1P_2^3Z_2, P_1^2Q_3Z_1Z_3, P_1^2Q_2^2Q_3, P_1^2Q_1Q_2Z_1, P_1^2P_2P_3Q_2, P_1^2P_2^2Q_3, P_1^3P_2Z_1\},\end{aligned} \tag{5.16a}$$

$$\begin{aligned}\text{odd}:\ \{&P_3Z_1Z_2Z_3^2, P_3Q_3^2Z_3^2, P_3Q_2^2Z_2Z_3, P_3Q_1Q_2Q_3Z_3, P_3Q_1^2Z_1Z_3, P_3Q_1^2Q_2^2, P_3^3Z_3^2, P_2Q_2Q_3Z_2Z_3, P_2Q_1Z_1Z_2Z_3,\\ &P_2Q_1Q_3^2Z_3, P_2Q_1Q_2^2Z_2, P_2Q_1^2Q_2Q_3, P_2Q_1^3Z_1, P_2P_3^2Q_1Z_3, P_2^2P_3Z_2Z_3, P_2^2P_3Q_1^2, P_2^3Q_1Z_2, P_1Q_2Z_1Z_2Z_3,\\ &P_1Q_2Q_3^2Z_3, P_1Q_2^3Z_2, P_1Q_1Q_3Z_1Z_3, P_1Q_1Q_2^2Q_3, P_1Q_1^2Q_2Z_1, P_1P_3^2Q_2Z_3, P_1P_2P_3Q_1Q_2, P_1P_2^2Q_2Z_2,\\ &P_1P_2^2Q_1Q_3, P_1^2P_3Z_1Z_3, P_1^2P_3Q_2^2, P_1^2P_2Q_2Q_3, P_1^2P_2Q_1Z_1, P_1^2P_2^2P_3, P_1^3Q_2Z_1, P_1P_2P_3Q_3Z_3\}.\end{aligned} \tag{5.16b}$$

Next we systematically apply the linear dependence relations (3.36a) to these lists, reducing them to the following linearly independent structures:

$$\text{even}:\ \{P_2P_3Q_1^2Q_2, P_1P_3Q_1Q_2^2, P_2^2Q_1^2Q_3, P_1P_2Q_1Q_2Q_3, P_1^2Q_2^2Q_3, P_1^2P_2^2Q_3, Q_1^2Q_2^2Q_3\}, \tag{5.17a}$$

$$\text{odd}:\ \{P_2^2P_3Q_1^2, P_3Q_1^2Q_2^2, P_2Q_1^2Q_2Q_3, P_1Q_1Q_2^2Q_3, P_1^2P_3Q_2^2, P_1P_2^2Q_1Q_3, P_1^2P_2Q_2Q_3\}. \tag{5.17b}$$

After constructing an appropriate ansatz for each sector, we obtain the following relations between the coefficients after imposing conservation on all three points:

$$\text{even}:\ \left\{A_1\to A_1, A_2\to A_1, A_3\to A_3, A_4\to \frac{7A_1}{15}-\frac{4A_3}{15}, A_5\to A_3, A_6\to\frac{14A_1}{15}-\frac{53A_3}{15}, A_7\to\frac{A_1}{3}-\frac{A_3}{3}\right\}, \tag{5.18a}$$

$$\text{odd}:\ \{B_1\to B_1, B_2\to -4B_1, B_3\to -2B_1, B_4\to -2B_1, B_5\to B_1, B_6\to 0, B_7\to 0\}. \tag{5.18b}$$

Therefore the final solutions for the even and odd sectors are

$$\begin{aligned}\text{even}:\ &\frac{A_3}{X^2}\left(-\frac{53}{15}P_1^2P_2^2Q_3+P_2^2Q_1^2Q_3+P_1^2Q_2^2Q_3-\frac{4}{15}P_1P_2Q_1Q_2Q_3-\frac{1}{3}Q_1^2Q_2^2Q_3\right)\\ &+\frac{A_1}{X^2}\left(\frac{14}{15}P_1^2P_2^2Q_3+P_3P_2Q_1^2Q_2+\frac{7}{15}P_1P_2Q_1Q_2Q_3+P_1P_3Q_1Q_2^2+\frac{1}{3}Q_1^2Q_2^2Q_3\right),\end{aligned} \tag{5.19a}$$

$$\text{odd}:\ \frac{B_1}{X^2}(-4P_3Q_2^2Q_1^2+P_2^2P_3Q_1^2-2P_2Q_2Q_3Q_1^2-2P_1Q_2^2Q_3Q_1+P_1^2P_3Q_2^2). \tag{5.19b}$$

Hence, we note that the three-point function $\langle J_{3/2} J'_{3/2} J''_{2} \rangle$, and therefore $\langle \tilde{Q} \tilde{Q}' T \rangle$ is fixed up to two independent "even" structures and one "odd" structure. In this case, both structures survive after imposing the symmetry under the exchange of x_1 and x_2, that is, when $J = J'$. Hence, the correlation function $\langle \tilde{Q} \tilde{Q} T \rangle$ is also fixed up to two even structures and a single odd structure.

B. Higher-spin correlators

In this subsection we compile some results for three-point correlation functions involving fermionic higher-spin currents. We present only the final results after imposing conservation on all three points.

1. Correlation function $\langle J_{3/2} J'_{3/2} J''_{3} \rangle$: $\sigma < 0$

$$\begin{aligned}\text{Even:}\ &\frac{A_3}{X}\left(-\frac{7}{9}P_2P_1^3Q_2^2+\frac{4}{3}P_2^2P_1^2Q_1Q_2-\frac{7}{9}P_2^3P_1Q_1^2+P_2P_1Q_1^2Q_2^2+\frac{59}{27}P_2^3P_1^3\right)\\&+\frac{A_1}{X}\left(-\frac{14}{5}P_2P_1^3Q_2^2-\frac{21}{5}P_1^2Q_1Q_2^3+21P_2^2P_1^2Q_1Q_2-\frac{14}{5}P_2^3P_1Q_1^2-\frac{21}{5}P_2^2Q_1^3Q_2+\frac{56}{3}P_2^3P_1^3+Q_1^3Q_2^3\right),\end{aligned} \tag{5.20a}$$

$$\text{odd:}\ \frac{B_1}{X}\left(-\frac{2}{3}P_1^3Q_2^3+P_2^2P_1^3Q_2-3P_2P_1^2Q_1Q_2^2+P_2^3P_1^2Q_1+P_1Q_1^2Q_2^3-3P_2^2P_1Q_1^2Q_2-\frac{2}{3}P_2^3Q_1^3+P_2Q_1^3Q_2^2\right). \tag{5.20b}$$

Imposing the symmetry under permutation of spacetimes points x_1 and x_2, i.e., when $J = J'$, requires that the three-point function must vanish.

2. Correlation function $\langle J_{3/2} J'_{3/2} J''_{4} \rangle$: $\sigma = 0$

$$\begin{aligned}\text{Even:}\ &A_2\left(-P_2P_1^3Q_2^2Z_3+\frac{4}{3}P_2^2P_1^2Q_1Q_2Z_3-P_2^3P_1Q_1^2Z_3+P_2P_1Q_1^2Q_2^2Z_3+\frac{29}{9}P_2^3P_1^3Z_3\right)\\&+A_1\left(\frac{2}{3}P_2P_1^3Q_2^2Z_3+P_1^2Q_1Q_2^3Z_3-7P_2^2P_1^2Q_1Q_2Z_3+\frac{2}{3}P_2^3P_1Q_1^2Z_3+P_2^2Q_1^3Q_2Z_3-\frac{56}{9}P_2^3P_1^3Z_3-\frac{5}{27}Q_1^3Q_2^3Z_3\right),\end{aligned} \tag{5.21a}$$

$$\text{odd:}\ 0. \tag{5.21b}$$

This correlation function is compatible with the symmetry under permutation of spacetimes points x_1 and x_2.

3. Correlation function $\langle J_{3/2} J'_{3/2} J''_{5} \rangle$: $\sigma > 0$

$$\begin{aligned}\text{Even:}\ &XA_2\left(-\frac{11}{9}P_2P_1^3Q_2^2Z_3^2+\frac{4}{3}P_2^2P_1^2Q_1Q_2Z_3^2-\frac{11}{9}P_2^3P_1Q_1^2Z_3^2+P_2P_1Q_1^2Q_2^2Z_3^2+\frac{41}{9}P_2^3P_1^3Z_3^2\right)\\&+XA_1\left(\frac{2}{3}P_2P_1^3Q_2^2Z_3^2+P_1^2Q_1Q_2^3Z_3^2-9P_2^2P_1^2Q_1Q_2Z_3^2+\frac{2}{3}P_2^3P_1Q_1^2Z_3^2+P_2^2Q_1^3Q_2Z_3^2-8P_2^3P_1^3Z_3^2-\frac{5}{33}Q_1^3Q_2^3Z_3^2\right),\end{aligned} \tag{5.22a}$$

$$\text{odd:}\ 0. \tag{5.22b}$$

4. Correlation function $\langle J_{5/2} J'_{3/2} J''_{1} \rangle$: $\sigma < 0$

$$\begin{aligned}\text{Even:}\ &\frac{A_3}{X^4}\left(-\frac{1}{5}P_2P_3^3Q_2-\frac{1}{5}P_2^2P_3^2Q_3+P_2P_3Q_2Q_3^2+P_2^2Q_3^3\right)\\&+\frac{A_1}{X^4}\left(-\frac{12}{35}P_2P_3^3Q_2+\frac{9}{5}P_2^2P_3^2Q_3-\frac{9}{7}P_3^2Q_2^2Q_3-3P_2^2Q_3^3+Q_2^2Q_3^3\right)\end{aligned} \tag{5.23a}$$

$$\text{odd:}\ \frac{B_1}{X^4}\left(-\frac{2}{9}P_3^3Q_2^2+\frac{4}{3}P_2P_3^2Q_2Q_3-\frac{2}{3}P_2^2P_3Q_3^2+P_3Q_2^2Q_3^2-\frac{2}{3}P_2Q_2Q_3^3+\frac{1}{3}P_2^2P_3^3\right). \tag{5.23b}$$

5. Correlation function $\langle J_{5/2} J'_{3/2} J''_2 \rangle$: $\sigma < 0$

$$\text{Even:}\ \frac{A_2}{X^3}\Big(-\frac{29}{5}P_1P_2^3Q_3^2-\frac{7}{5}P_3P_2^3Q_1Q_3+\frac{11}{5}P_2^2Q_1Q_2Q_3^2-\frac{7}{5}P_3^2P_2^2Q_1Q_2+\frac{26}{5}P_1P_2Q_2^2Q_3^2+\frac{16}{5}P_3P_2Q_1Q_2^2Q_3 + P_3^2Q_1Q_2^3+P_1P_3Q_2^3Q_3\Big)+\frac{A_1}{X^3}\Big(-\frac{115}{7}P_1P_2^3Q_3^2-5P_3P_2^3Q_1Q_3+\frac{40}{7}P_2^2Q_1Q_2Q_3^2+\frac{106}{7}P_1P_2Q_2^2Q_3^2 +\frac{90}{7}P_3P_2Q_1Q_2^2Q_3+\frac{15}{7}P_1P_3Q_2^3Q_3+Q_1Q_2^3Q_3^2\Big), \tag{5.24a}$$

$$\text{odd:}\ \frac{B_1}{X^3}\Big(-\frac{2}{3}P_3^2P_2^3Q_1-\frac{4}{3}P_3P_2^2Q_1Q_2Q_3+2P_3^2P_2Q_1Q_2^2-\frac{2}{3}P_2Q_1Q_2^2Q_3^2+\frac{1}{3}P_1P_3^2Q_2^3+P_3Q_1Q_2^3Q_3\Big). \tag{5.24b}$$

6. Correlation function $\langle J_{5/2} J'_{3/2} J''_3 \rangle$: $\sigma < 0$

$$\text{Even:}\ \frac{A_2}{X^2}\Big(\frac{107}{50}P_1^2P_2^4Q_3-\frac{27}{50}P_2^4Q_1^2Q_3-\frac{27}{50}P_3P_2^3Q_1^2Q_2+\frac{13}{50}P_1P_2^3Q_1Q_2Q_3-\frac{59}{35}P_1^2P_2^2Q_2^2Q_3+P_2^2Q_1^2Q_2^2Q_3+P_3P_2Q_1^2Q_2^3 +\frac{23}{35}P_1P_2Q_1Q_2^3Q_3+\frac{3}{10}P_1P_3Q_1Q_2^4+\frac{3}{10}P_1^2Q_2^4Q_3\Big)+\frac{A_1}{X^2}\Big(-43P_1^2P_2^4Q_3+9P_2^4Q_1^2Q_3-8P_1P_2^3Q_1Q_2Q_3 +\frac{226}{7}P_1^2P_2^2Q_2^2Q_3-10P_2^2Q_1^2Q_2^2Q_3+\frac{8}{7}P_1P_2Q_1Q_2^3Q_3-3P_1^2Q_2^4Q_3+Q_1^2Q_2^4Q_3\Big), \tag{5.25a}$$

$$\text{odd:}\ \frac{B_1}{X^2}\Big(\frac{2}{9}P_3P_2^4Q_1^2-\frac{4}{9}P_2^3Q_1^2Q_2Q_3-\frac{4}{3}P_3P_2^2Q_1^2Q_2^2-\frac{2}{3}P_1P_2^2Q_1Q_2^2Q_3-\frac{2}{9}P_1^2P_2Q_2^3Q_3+\frac{2}{3}P_2Q_1^2Q_2^3Q_3 +P_3Q_1^2Q_2^4-\frac{4}{45}P_1^2P_3Q_2^4+\frac{4}{5}P_1Q_1Q_2^4Q_3\Big). \tag{5.25b}$$

7. Correlation function $\langle J_{5/2} J'_{3/2} J''_4 \rangle$: $\sigma < 0$

$$\text{Even:}\ \frac{A_4}{X}\Big(\frac{33}{25}P_1P_2^5Q_1^2-\frac{24}{5}P_1^2P_2^4Q_1Q_2+\frac{58}{15}P_1^3P_2^3Q_2^2-\frac{18}{5}P_1P_2^3Q_1^2Q_2^2+\frac{8}{5}P_1^2P_2^2Q_1Q_2^3-\frac{3}{5}P_1^3P_2Q_2^4+P_1P_2Q_1^2Q_2^4 -\frac{27}{5}P_1^3P_2^5\Big)+\frac{A_1}{X}\Big(\frac{396}{35}P_1P_2^5Q_1^2+\frac{99}{7}P_2^4Q_1^3Q_2-\frac{639}{7}P_1^2P_2^4Q_1Q_2+48P_1^3P_2^3Q_2^2-\frac{108}{7}P_1P_2^3Q_1^2Q_2^2 -\frac{90}{7}P_2^2Q_1^3Q_2^3+54P_1^2P_2^2Q_1Q_2^3-\frac{18}{7}P_1^3P_2Q_2^4-\frac{27}{7}P_1^2Q_1Q_2^5-\frac{558}{7}P_1^3P_2^5+Q_1^3Q_2^5\Big), \tag{5.26a}$$

$$\text{odd:}\ \frac{B_1}{X}\Big(P_2^5Q_1^3-P_1^2P_2^5Q_1-P_1^3P_2^4Q_2+5P_1P_2^4Q_1^2Q_2-\frac{10}{3}P_2^3Q_1^3Q_2^2+6P_1^2P_2^3Q_1Q_2^2+2P_1^3P_2^2Q_2^3-6P_1P_2^2Q_1^2Q_2^3 +P_2Q_1^3Q_2^4-3P_1^2P_2Q_1Q_2^4-\frac{1}{3}P_1^3Q_2^5+\frac{3}{5}P_1Q_1^2Q_2^5\Big). \tag{5.26b}$$

8. Correlation function $\langle J_{5/2} J'_{3/2} J''_5 \rangle$: $\sigma = 0$

$$\begin{aligned}\text{Even: } & A_2\Big(-\frac{83}{140}P_1P_2^5Q_1^2Z_3+\frac{1}{7}P_2^4Q_1^3Q_2Z_3+P_1^2P_2^4Q_1Q_2Z_3-\frac{3}{2}P_1^3P_2^3Q_2^2Z_3+\frac{3}{2}P_1P_2^3Q_1^2Q_2^2Z_3-\frac{10}{91}P_2^2Q_1^3Q_2^3Z_3\\ &+\frac{1}{4}P_1^3P_2Q_2^4Z_3-\frac{135}{364}P_1P_2Q_1^2Q_2^4Z_3-\frac{3}{91}P_1^2Q_1Q_2^5Z_3+\frac{41}{20}P_1^3P_2^5Z_3+\frac{1}{143}Q_1^3Q_2^5Z_3\Big)+A_4\Big(-\frac{55}{56}P_1P_2^5Q_1^2Z_3\\ &+\frac{11}{21}P_2^4Q_1^3Q_2Z_3+\frac{11}{4}P_1P_2^3Q_1^2Q_2^2Z_3-\frac{110}{273}P_2^2Q_1^3Q_2^3Z_3+P_1^2P_2^2Q_1Q_2^3Z_3+\frac{11}{24}P_1^3P_2Q_2^4Z_3-\frac{535}{728}P_1P_2Q_1^2Q_2^4Z_3\\ &-\frac{11}{91}P_1^2Q_1Q_2^5Z_3+\frac{55}{24}P_1^3P_2^5Z_3+\frac{1}{39}Q_1^3Q_2^5Z_3-\frac{25}{12}P_1^3P_2^3Q_2^2Z_3\Big),\end{aligned} \tag{5.27a}$$

$$\text{odd: } 0. \tag{5.27b}$$

9. Correlation function $\langle J_{5/2} J'_{5/2} J''_1 \rangle$: $\sigma < 0$

$$\begin{aligned}\text{Even: } & \frac{A_2}{X^5}\Big(-\frac{1}{10}P_3^4Q_1Q_2-\frac{1}{10}P_2P_3^3Q_1Q_3-\frac{1}{10}P_1P_3^3Q_2Q_3+P_3^2Q_1Q_2Q_3^2+P_2P_3Q_1Q_3^3+P_1P_3Q_2Q_3^3+\frac{7}{6}P_1P_2Q_3^4\Big)\\ &+\frac{A_1}{X^5}\Big(\frac{4}{21}P_3^4Q_1Q_2-P_2P_3^3Q_1Q_3-P_1P_3^3Q_2Q_3+\frac{10}{3}P_2P_3Q_1Q_3^3+\frac{10}{3}P_1P_3Q_2Q_3^3+\frac{29}{9}P_1P_2Q_3^4+Q_1Q_2Q_3^4\Big),\end{aligned} \tag{5.28a}$$

$$\text{odd: } \frac{B_1}{X^5}\Big(-\frac{1}{10}P_2P_3^4Q_1-\frac{1}{10}P_1P_3^4Q_2-\frac{2}{3}P_3^3Q_1Q_2Q_3+\frac{1}{3}P_2P_3^2Q_1Q_3^2+\frac{1}{3}P_1P_3^2Q_2Q_3^2+P_3Q_1Q_2Q_3^3\Big). \tag{5.28b}$$

10. Correlation function $\langle J_{5/2} J'_{5/2} J''_2 \rangle$: $\sigma < 0$

$$\begin{aligned}\text{Even: } & \frac{A_2}{X^4}\Big(-\frac{7}{22}P_1P_3^3Q_1Q_2^2-\frac{7}{22}P_2P_3^3Q_1^2Q_2-\frac{7}{22}P_2^2P_3^2Q_1^2Q_3-\frac{7}{22}P_1^2P_3^2Q_2^2Q_3+P_3^2Q_1^2Q_2^2Q_3+\frac{40}{11}P_1P_3Q_1Q_2^2Q_3^2\\ &+\frac{40}{11}P_2P_3Q_1^2Q_2Q_3^2-\frac{75}{22}P_1^2P_2^2Q_3^3+\frac{29}{11}P_2^2Q_1^2Q_3^3+\frac{29}{11}P_1^2Q_2^2Q_3^3+\frac{81}{22}P_1P_2Q_1Q_2Q_3^3\Big)+\frac{A_1}{X^4}\Big(-\frac{125}{33}P_1P_3^3Q_1Q_2^2\\ &-\frac{125}{33}P_2P_3^3Q_1^2Q_2-\frac{100}{99}P_2^2P_3^2Q_1^2Q_3-\frac{100}{99}P_1^2P_3^2Q_2^2Q_3+\frac{1850}{99}P_1P_3Q_1Q_2^2Q_3^2+\frac{1850}{99}P_2P_3Q_1^2Q_2Q_3^2\\ &-\frac{184}{11}P_1^2P_2^2Q_3^3+\frac{1300}{99}P_2^2Q_1^2Q_3^3+\frac{1300}{99}P_1^2Q_2^2Q_3^3+\frac{601}{33}P_1P_2Q_1Q_2Q_3^3+Q_1^2Q_2^2Q_3^3\Big),\end{aligned} \tag{5.29a}$$

$$\text{odd: } \frac{B_1}{X^4}\Big(\frac{1}{5}P_2^2P_3^3Q_1^2+\frac{1}{5}P_1^2P_3^3Q_2^2-\frac{9}{10}P_3^3Q_1^2Q_2^2+\frac{3}{5}P_1P_3^2Q_1Q_2^2Q_3+\frac{3}{5}P_2P_3^2Q_1^2Q_2Q_3+P_3Q_1^2Q_2^2Q_3^2\Big). \tag{5.29b}$$

11. Correlation function $\langle J_{5/2} J'_{5/2} J''_3 \rangle$: $\sigma < 0$

$$\begin{aligned}\text{Even}:\ &\frac{A_2}{X^3}\left(\frac{2253}{490}P_2^3P_1^3Q_3^2-\frac{107}{35}P_2P_1^3Q_2^2Q_3^2-\frac{27}{70}P_3P_1^3Q_2^3Q_3-\frac{27}{70}P_3^2P_1^2Q_1Q_2^3+\frac{23}{10}P_1^2Q_1Q_2^3Q_3^2-\frac{373}{245}P_2^2P_1^2Q_1Q_2Q_3^2\right.\\&-\frac{107}{35}P_2^3P_1Q_1^2Q_3^2+\frac{397}{70}P_2P_1Q_1^2Q_2^2Q_3^2+\frac{33}{10}P_3P_1Q_1^2Q_2^3Q_3+P_3^2Q_1^3Q_2^3+\frac{23}{10}P_2^2Q_1^3Q_2Q_3^2-\frac{27}{70}P_2^2P_3^2Q_1^3Q_2\\&\left.-\frac{27}{70}P_2^3P_3Q_1^3Q_3+\frac{33}{10}P_2P_3Q_1^3Q_2^2Q_3\right)+\frac{A_1}{X^3}\left(\frac{1501}{21}P_2^3P_1^3Q_3^2-\frac{430}{9}P_2P_1^3Q_2^2Q_3^2-5P_3P_1^3Q_2^3Q_3\right.\\&+\frac{305}{9}P_1^2Q_1Q_2^3Q_3^2-\frac{1511}{63}P_2^2P_1^2Q_1Q_2Q_3^2-\frac{430}{9}P_2^3P_1Q_1^2Q_3^2+\frac{757}{9}P_2P_1Q_1^2Q_2^2Q_3^2+\frac{125}{3}P_3P_1Q_1^2Q_2^3Q_3\\&\left.+\frac{305}{9}P_2^2Q_1^3Q_2Q_3^2-5P_2^3P_3Q_1^3Q_3+\frac{125}{3}P_2P_3Q_1^3Q_2^2Q_3+Q_1^3Q_2^3Q_3^2\right),\end{aligned}\tag{5.30a}$$

$$\begin{aligned}\text{odd}:\ &\frac{B_1}{X^3}\left(\frac{1}{3}P_2^3P_3^2Q_1^3-3P_2P_3^2Q_2^2Q_1^3+2P_2Q_2^2Q_3^2Q_1^3+P_3Q_2^3Q_3Q_1^3-P_2^2P_3Q_2Q_3Q_1^3-3P_1P_3^2Q_2^3Q_1^2\right.\\&\left.+2P_1Q_2^3Q_3^2Q_1^2-P_1^2P_3Q_2^3Q_3Q_1+\frac{1}{3}P_1^3P_3^2Q_2^3\right).\end{aligned}\tag{5.30b}$$

12. Correlation function $\langle J_{5/2} J'_{5/2} J''_4 \rangle$: $\sigma < 0$

$$\begin{aligned}\text{Even}:\ &\frac{A_6}{X^2}\left(-\frac{1367}{441}P_2^4P_1^4Q_3-\frac{11}{49}P_1^4Q_2^4Q_3+\frac{90}{49}P_2^2P_1^4Q_2^2Q_3-\frac{11}{49}P_3P_1^3Q_1Q_2^4-\frac{10}{49}P_2P_1^3Q_1Q_2^3Q_3-\frac{211}{147}P_2^3P_1^3Q_1Q_2Q_3\right.\\&+P_1^2Q_1^2Q_2^4Q_3+\frac{90}{49}P_2^4P_1^2Q_1^2Q_3-\frac{123}{49}P_2^2P_1^2Q_1^2Q_2^2Q_3+P_3P_1Q_1^3Q_2^4+\frac{101}{63}P_2P_1Q_1^3Q_2^3Q_3-\frac{10}{49}P_2^3P_1Q_1^3Q_2Q_3\\&\left.+P_2P_3Q_1^4Q_2^3-\frac{11}{49}P_2^3P_3Q_1^4Q_2-\frac{11}{49}P_2^4Q_1^4Q_3+P_2^2Q_1^4Q_2^2Q_3\right)+\frac{A_1}{X^2}\left(\frac{10943}{63}P_2^4P_1^4Q_3+\frac{55}{7}P_1^4Q_2^4Q_3\right.\\&-\frac{710}{7}P_2^2P_1^4Q_2^2Q_3-\frac{80}{7}P_2P_1^3Q_1Q_2^3Q_3+\frac{1744}{21}P_2^3P_1^3Q_1Q_2Q_3-10P_1^2Q_1^2Q_2^4Q_3-\frac{710}{7}P_2^4P_1^2Q_1^2Q_3\\&\left.+\frac{732}{7}P_2^2P_1^2Q_1^2Q_2^2Q_3+\frac{16}{9}P_2P_1Q_1^3Q_2^3Q_3-\frac{80}{7}P_2^3P_1Q_1^3Q_2Q_3+\frac{55}{7}P_2^4Q_1^4Q_3-10P_2^2Q_1^4Q_2^2Q_3+Q_1^4Q_2^4Q_3\right),\end{aligned}\tag{5.31a}$$

$$\begin{aligned}\text{odd}:\ &\frac{B_1}{X^2}\left(\frac{1}{28}P_3P_1^4Q_2^4+\frac{5}{84}P_2P_1^4Q_2^3Q_3-\frac{9}{28}P_1^3Q_1Q_2^4Q_3+\frac{5}{28}P_2^2P_1^3Q_1Q_2^2Q_3-\frac{1}{2}P_3P_1^2Q_1^2Q_2^4-\frac{25}{28}P_2P_1^2Q_1^2Q_2^3Q_3\right.\\&+\frac{5}{28}P_2^3P_1^2Q_1^2Q_2Q_3+\frac{13}{14}P_1Q_1^3Q_2^4Q_3+\frac{5}{84}P_2^4P_1Q_1^3Q_3-\frac{25}{28}P_2^2P_1Q_1^3Q_2^2Q_3+\frac{1}{28}P_2^4P_3Q_1^4+P_3Q_1^4Q_2^4\\&\left.-\frac{1}{2}P_2^2P_3Q_1^4Q_2^2+\frac{13}{14}P_2Q_1^4Q_2^3Q_3-\frac{9}{28}P_2^3Q_1^4Q_2Q_3\right).\end{aligned}\tag{5.31b}$$

13. Correlation function $\langle J_{5/2} J'_{5/2} J''_{5} \rangle$: $\sigma < 0$

$$\begin{aligned}\text{Even}:\ \frac{A_4}{X}\bigg(&\frac{143}{175}P_2P_1^5Q_2^4-\frac{1254}{175}P_2^3P_1^5Q_2^2-\frac{176}{35}P_2^2P_1^4Q_1Q_2^3+\frac{592}{35}P_2^4P_1^4Q_1Q_2-\frac{22}{7}P_2P_1^3Q_1^2Q_2^4-\frac{1254}{175}P_2^5P_1^3Q_1^2\\&+\frac{492}{35}P_2^3P_1^3Q_1^2Q_2^2+\frac{16}{7}P_2^2P_1^2Q_1^3Q_2^3-\frac{176}{35}P_2^4P_1^2Q_1^3Q_2+\frac{143}{175}P_2^5P_1Q_1^4+P_2P_1Q_1^4Q_2^4-\frac{22}{7}P_2^3P_1Q_1^4Q_2^2\\&+\frac{11563}{875}P_2^5P_1^5\bigg)+\frac{A_1}{X}\bigg(\frac{572}{63}P_2P_1^5Q_2^4-\frac{10516}{63}P_2^3P_1^5Q_2^2+\frac{715}{63}P_1^4Q_1Q_2^5-\frac{11770}{63}P_2^2P_1^4Q_1Q_2^3+\frac{3135}{7}P_2^4P_1^4Q_1Q_2\\&-\frac{1100}{63}P_2P_1^3Q_1^2Q_2^4-\frac{10516}{63}P_2^5P_1^3Q_1^2+\frac{1760}{7}P_2^3P_1^3Q_1^2Q_2^2-\frac{110}{9}P_1^2Q_1^3Q_2^5+\frac{1100}{7}P_2^2P_1^2Q_1^3Q_2^3-\frac{11770}{63}P_2^4P_1^2Q_1^3Q_2\\&+\frac{572}{63}P_2^5P_1Q_1^4-\frac{1100}{63}P_2^3P_1Q_1^4Q_2^2-\frac{110}{9}P_2^2Q_1^5Q_2^3+\frac{715}{63}P_2^4Q_1^5Q_2+\frac{11528}{35}P_2^5P_1^5+Q_1^5Q_2^5\bigg),\end{aligned}\tag{5.32a}$$

$$\begin{aligned}\text{odd}:\ \frac{B_1}{X}\bigg(&\frac{7}{10}P_1^5Q_2^5-3P_2^2P_1^5Q_2^3+P_2^4P_1^5Q_2+\frac{15}{2}P_2P_1^4Q_1Q_2^4-10P_2^3P_1^4Q_1Q_2^2+P_2^5P_1^4Q_1-3P_1^3Q_1^2Q_2^5+20P_2^2P_1^3Q_1^2Q_2^3\\&-10P_2^4P_1^3Q_1^2Q_2-10P_2P_1^2Q_1^3Q_2^4-3P_2^5P_1^2Q_1^3+20P_2^3P_1^2Q_1^3Q_2^2+P_1Q_1^4Q_2^5-10P_2^2P_1Q_1^4Q_2^3+\frac{15}{2}P_2^4P_1Q_1^4Q_2\\&+\frac{7}{10}P_2^5Q_1^5+P_2Q_1^5Q_2^4-3P_2^3Q_1^5Q_2^2\bigg).\end{aligned}\tag{5.32b}$$

14. Correlation function $\langle J_{5/2} J'_{5/2} J''_{6} \rangle$: $\sigma = 0$

$$\begin{aligned}\text{Even}:\ A_2\bigg(&-\frac{3}{10}P_2P_1^5Q_2^4Z_3+\frac{43}{15}P_2^3P_1^5Q_2^2Z_3+\frac{8}{5}P_2^2P_1^4Q_1Q_2^3Z_3-\frac{248}{39}P_2^4P_1^4Q_1Q_2Z_3+P_2P_1^3Q_1^2Q_2^4Z_3+\frac{43}{15}P_2^5P_1^3Q_1^2Z_3\\&-\frac{334}{65}P_2^3P_1^3Q_1^2Q_2^2Z_3-\frac{8}{13}P_2^2P_1^2Q_1^3Q_2^3Z_3+\frac{8}{5}P_2^4P_1^2Q_1^3Q_2Z_3-\frac{3}{10}P_2^5P_1Q_1^4Z_3-\frac{7}{26}P_2P_1Q_1^4Q_2^4Z_3+P_2^3P_1Q_1^4Q_2^2Z_3\\&-\frac{3703}{650}P_2^5P_1^5Z_3\bigg)+A_1\bigg(\frac{2}{5}P_2P_1^5Q_2^4Z_3-\frac{72}{5}P_2^3P_1^5Q_2^2Z_3+P_1^4Q_1Q_2^5Z_3-18P_2^2P_1^4Q_1Q_2^3Z_3+\frac{573}{13}P_2^4P_1^4Q_1Q_2Z_3\\&-\frac{72}{5}P_2^5P_1^3Q_1^2Z_3+\frac{216}{13}P_2^3P_1^3Q_1^2Q_2^2Z_3-\frac{14}{15}P_1^2Q_1^3Q_2^5Z_3+\frac{180}{13}P_2^2P_1^2Q_1^3Q_2^3Z_3-18P_2^4P_1^2Q_1^3Q_2Z_3+\frac{2}{5}P_2^5P_1Q_1^4Z_3\\&-\frac{14}{39}P_2P_1Q_1^4Q_2^4Z_3-\frac{14}{15}P_2^2Q_1^5Q_2^3Z_3+P_2^4Q_1^5Q_2Z_3+\frac{10878}{325}P_2^5P_1^5Z_3+\frac{21}{325}Q_1^5Q_2^5Z_3\bigg),\end{aligned}\tag{5.33a}$$

$$\text{odd}:\ 0.\tag{5.33b}$$

Additional results for three-point functions involving fermionic higher-spin currents are contained in Appendix B.

VI. CORRELATORS INVOLVING SCALARS AND SPINORS

In this section, for completeness, we analyze some of the important three-point correlation functions involving scalar and spinor fields. The results are interesting because the correlation functions can contain parity-odd solutions, with their existence depending on both triangle inequalities and the scale dimensions of the scalars/spinors. The correlation functions are analyzed using the same methods as in the previous sections; the full classification of the results is presented below:

(i) The three-point function $\langle \psi \psi' O \rangle$, where ψ, ψ' are fundamental fermions and O is a fundamental scalar, is fixed up to one even structure and one odd structure. All structures remain after imposing $\psi = \psi'$.

(ii) The three-point function $\langle O O' J_s \rangle$, where O, O' are fundamental scalars with dimension δ, δ' respectively: for $\delta = \delta'$, there is a single even solution compatible with conservation which survives after imposing $O = O'$ only for even s. For $\delta \neq \delta'$ there are no solutions.

(iii) The three-point function $\langle \psi\psi' J_s \rangle$, where ψ, ψ are fundamental fermions with dimension δ, δ' respectively: when $s = 1$, the triangle inequalities are satisfied, and for $\delta = \delta'$ there are two even solutions and one odd solution. For $\delta \neq \delta'$ there is one even solution and one odd solution. In both cases, the three-point function vanishes after imposing $\psi = \psi'$. For $s > 1$, the triangle inequalities are not satisfied, and for $\delta = \delta'$ there are two even solutions which survive after imposing $\psi = \psi'$ provided that s is even. For $\delta \neq \delta'$ there are no solutions for general s.

(iv) The three-point function $\langle \psi J_s O \rangle$, for half-integer $s \geq 3/2$, where ψ is a fundamental fermion with dimension δ, and O is a scalar with dimension δ': the triangle inequalities are not satisfied for any s, and in general there are no solutions after imposing conservation for arbitrary δ, δ'. However, there are two special cases; for $\delta = 3/2$ there is an even solution for $\delta' = 1$ and an odd solution for $\delta' = 2$.

(v) For three-point functions of the form $\langle J_{s_1} J'_{s_2} O \rangle$, where O is a scalar field with dimension δ, s_1, and s_2 must be simultaneously integer/half-integer for there to be a solution. For $s_1 > s_2$, the triangle inequalities are not satisfied and there is no solution for general δ; however, there is an even solution for $\delta = 1$, and an odd solution for $\delta = 2$. For $s_1 = s_2$, the triangle inequalities are satisfied and there exists an even and odd solution for general δ. The solutions also survive after imposing the symmetry $J = J'$.

(vi) For three-point functions of the form $\langle \psi J_{s_1} J'_{s_2} \rangle$, for half-integer $s_1 \geq 3/2$ and integer s_2, where ψ is a fundamental fermion with dimension δ: for $\delta = 3/2$ there are two even solutions and one odd solution provided that the triangle inequalities are satisfied, otherwise, there are only two even solutions. In addition, for $\delta = 5/2$ there is one even solution and one odd solution when the triangle inequalities are satisfied, otherwise, there is a single odd solution. For general δ, an even and odd solution exists if the triangle inequalities are satisfied, otherwise, there are no solutions.

In the next subsections we present explicit solutions for some of the above cases.

A. Low-spin correlators

1. Correlation function $\langle \psi\psi' O \rangle$

For $\Delta_\psi = \delta_1$, $\Delta_{\psi'} = \delta'_1$, $\Delta_O = \delta_2$, there is always one even and one odd solution,

$$\text{even: } A_1 Q_3 X^{-\delta'_1 - \delta_1 + \delta_2}, \tag{6.1a}$$

$$\text{odd: } B_1 P_3 X^{-\delta'_1 - \delta_1 + \delta_2}. \tag{6.1b}$$

2. Correlation function $\langle O O' J_1 \rangle$

An even solution exists for $\Delta_O = \Delta_{O'} = \delta$:

$$\text{even: } A_1 Z_3 X^{2-2\delta}, \tag{6.2a}$$

$$\text{odd: } 0. \tag{6.2b}$$

The solution vanishes upon imposing the symmetry between x_1 and x_2, i.e., when the fields O, O' coincide.

3. Correlation function $\langle O O' J_2 \rangle$

An even solution exists for $\Delta_O = \Delta_{O'} = \delta$:

$$\text{even: } A_1 Z_3^2 X^{3-2\delta}, \tag{6.3a}$$

$$\text{odd: } 0. \tag{6.3b}$$

In general, for $\langle O O' J_s \rangle$ there is always an even solution; however, it only survives the $O = O'$ point-switch symmetry for even s.

4. Correlation function $\langle \psi\psi' J_1 \rangle$

$$\text{Even: } X^{2-2\delta}(A_2 P_1 P_2 + A_1 Q_1 Q_2), \tag{6.4a}$$

$$\text{odd: } B_1 (P_2 Q_1 + P_1 Q_2) X^{2-2\delta}. \tag{6.4b}$$

In this case the triangle inequalities are satisfied and there are two even solutions and one odd solution. All structures vanish after imposing $\psi = \psi'$.

5. Correlation function $\langle \psi\psi' J_2 \rangle$

$$\text{Even: } X^{3-2\delta}(A_1 P_1 P_2 Z_3 + A_2 Q_1 Q_2 Z_3), \tag{6.5a}$$

$$\text{odd: } 0. \tag{6.5b}$$

In this case the triangle inequalities are not satisfied and there are two even solutions. All structures survive after imposing $\psi = \psi'$.

6. Correlation function $\langle J_1 J'_1 O \rangle$

$$\text{Even: } A_1 X^{\delta-4}\left(\frac{\delta P_3^2}{\delta - 4} + Q_3^2\right), \tag{6.6a}$$

$$\text{odd: } B_1 P_3 Q_3 X^{\delta-4}. \tag{6.6b}$$

In this case the triangle inequalities are satisfied and there is one even solution and one odd solution. The structures survive after imposing $J = J'$.

7. Correlation function $\langle J_2 J_1' O\rangle$

In this case there is a single even solution for $\Delta_O = 1$:

$$\text{even}: \frac{A_1(P_3^2 Z_1 - 5Q_3^2 Z_1)}{X^4}, \tag{6.7a}$$

$$\text{odd}: 0. \tag{6.7b}$$

There is also a single odd solution for $\Delta_O = 2$:

$$\text{even}: 0, \tag{6.8a}$$

$$\text{odd}: \frac{B_1 P_3 Q_3 Z_1}{X^3}. \tag{6.8b}$$

However, there are no solutions for arbitrary Δ_O. The same results were found in [13].

8. Correlation function $\langle J_{3/2} J_{3/2}' O\rangle$

$$\text{Even}: A_1 X^{\delta-5}\left(\frac{3\delta P_3^2 Q_3}{\delta-6} + Q_3^3\right), \tag{6.9a}$$

$$\text{odd}: B_1 X^{\delta-5}\left(\frac{(\delta+1)P_3^3}{3(\delta-5)} + P_3 Q_3^2\right). \tag{6.9b}$$

9. Correlation function $\langle J_2 J_2' O\rangle$

$$\text{Even}: \frac{1}{6}A_2 X^{\delta-6}\left(\frac{(\delta+2)P_3^4}{\delta-6} + 6P_3^2 Q_3^2 + \frac{(\delta-8)Q_3^4}{\delta}\right), \tag{6.10a}$$

$$\text{odd}: B_1 X^{\delta-6}\left(\frac{(\delta+1)P_3^3 Q_3}{\delta-7} + P_3 Q_3^3\right). \tag{6.10b}$$

10. Correlation function $\langle \psi J_{3/2} O\rangle$

For $\Delta_\psi = 3/2$, there is an even solution for $\Delta_O = 1$:

$$\text{even}: \frac{A_1 Q_3 Z_2}{X^3}, \tag{6.11a}$$

$$\text{odd}: 0. \tag{6.11b}$$

There is also an odd solution for $\Delta_O = 2$:

$$\text{even}: 0, \tag{6.12a}$$

$$\text{odd}: \frac{B_1 P_3 Z_2}{X^2}. \tag{6.12b}$$

11. Correlation function $\langle \psi J_{3/2} J_1'\rangle$

In this case there is an even and odd solution for general Δ_ψ:

$$\text{even}: A_1 X^{-\delta-\frac{1}{2}}\left(\frac{2(9-2\delta)}{2\delta+3}P_1 P_3 Q_1 + \frac{(2\delta-9)}{2\delta+3}P_1^2 Q_3 + Q_1^3\right), \tag{6.13a}$$

$$\text{odd}: B_1 X^{-\delta-\frac{1}{2}}\left(\frac{(2\delta-11)P_3 P_1^2}{2\delta+1} - 2P_1 Q_1 Q_3 + P_3 Q_1^2\right). \tag{6.13b}$$

There is also an additional even solution for $\Delta_\psi = 3/2$:

$$\text{even}: \frac{A_2}{X^2}(P_1^2 Q_3 + P_3 P_1 Q_1) + \frac{A_1}{X^2}(Q_1^2 Q_3 - 3P_1^2 Q_3), \tag{6.14a}$$

$$\text{odd}: \frac{B_1}{X^2}(-2P_1 Q_1 Q_3 + P_3 Q_1^2 - 2P_3 P_1^2). \tag{6.14b}$$

12. Correlation function $\langle \psi J_{3/2} J_2'\rangle$

In this case there is one even and one odd solution for general Δ_ψ:

$$\text{even}: A_1 X^{\frac{1}{2}-\delta}\left(\frac{(13-2\delta)P_2 P_1^3}{2\delta+3} + \frac{3(2\delta-13)P_1^2 Q_1 Q_2}{2\delta+3} - 3P_2 P_1 Q_1^2 + Q_1^3 Q_2\right), \tag{6.15a}$$

$$\text{odd}: B_1 \frac{X^{\frac{1}{2}-\delta}}{(2\delta+1)(2\delta+5)}((2\delta+1)P_2 Q_1((6\delta-33)P_1^2 + (2\delta+5)Q_1^2) - (2\delta-11)P_1 Q_2((2\delta-15)P_1^2 + 3(2\delta+1)Q_1^2)). \tag{6.15b}$$

However, there is an additional even solution for $\Delta_\psi = 3/2$:

$$\text{even}: \frac{A_2}{X}\left(P_1 P_2 Q_1^2 - \frac{5}{3}P_1^3 P_2\right) + \frac{A_1}{X}\left(-5P_1^2 Q_1 Q_2 - \frac{10}{3}P_2 P_1^3 + Q_1^3 Q_2\right), \tag{6.16a}$$

$$\text{odd}: \frac{B_1}{X}(-3P_1^3 Q_2 - 3P_2 P_1^2 Q_1 + 3P_1 Q_1^2 Q_2 + P_2 Q_1^3). \tag{6.16b}$$

B. Higher-spin correlators

In this subsection we provide some more examples of three-point correlation functions involving combinations of scalars, spinors, and higher-spin conserved currents.

1. Correlation function $\langle J_{5/2} J'_{5/2} O \rangle$

$$\text{Even:}\ A_2 X^{\delta-7}\left(\frac{(\delta+2)P_3^4 Q_3}{2(\delta-8)} + P_3^2 Q_3^3 + \frac{(\delta-10)Q_3^5}{10\delta}\right), \tag{6.17a}$$

$$\text{odd:}\ B_1 X^{\delta-7}\left(\frac{(\delta+1)(\delta+3)P_3^5}{5(\delta-9)(\delta-7)} + \frac{2(\delta+1)P_3^3 Q_3^2}{\delta-9} + P_3 Q_3^4\right). \tag{6.17b}$$

2. Correlation function $\langle J_{7/2} J'_{7/2} O \rangle$

$$\text{Even:}\ A_2 X^{\delta-9}\left(\frac{5(\delta+2)P_3^4 Q_3^3}{3(\delta-12)} + \frac{(\delta+2)(\delta+4)P_3^6 Q_3}{3(\delta-12)(\delta-10)} + P_3^2 Q_3^5 + \frac{(\delta-14)Q_3^7}{21\delta}\right), \tag{6.18a}$$

$$\text{odd:}\ B_2 X^{\delta-9}\left(\frac{(\delta+3)(\delta+5)P_3^7}{35(\delta-11)(\delta-9)} + \frac{3(\delta+3)P_3^5 Q_3^2}{5(\delta-11)} + \frac{(\delta-13)P_3 Q_3^6}{5(\delta+1)} + P_3^3 Q_3^4\right). \tag{6.18b}$$

3. Correlation function $\langle J_5 J'_5 O \rangle$

$$\text{Even:}\ \frac{1}{45} A_2 X^{\delta-12}\left(\frac{(\delta+2)(\delta+4)(\delta+6)(\delta+8)P_3^{10}}{(\delta-18)(\delta-16)(\delta-14)(\delta-12)} + \frac{45(\delta+2)(\delta+4)(\delta+6)P_3^8 Q_3^2}{(\delta-18)(\delta-16)(\delta-14)} + \frac{210(\delta+2)(\delta+4)P_3^6 Q_3^4}{(\delta-18)(\delta-16)}\right.$$
$$\left. + \frac{210(\delta+2)P_3^4 Q_3^6}{\delta-18} + 45 P_3^2 Q_3^8 + \frac{(\delta-20)Q_3^{10}}{\delta}\right), \tag{6.19a}$$

$$\text{odd:}\ B_2 X^{\delta-12}\left(\frac{(\delta+3)(\delta+5)(\delta+7)P_3^9 Q_3}{12(\delta-17)(\delta-15)(\delta-13)} + \frac{(\delta+3)(\delta+5)P_3^7 Q_3^3}{(\delta-17)(\delta-15)} + \frac{21(\delta+3)P_3^5 Q_3^5}{10(\delta-17)} + \frac{(\delta-19)P_3 Q_3^9}{12(\delta+1)} + P_3^3 Q_3^7\right). \tag{6.19b}$$

4. Correlation function $\langle J_{9/2} J'_{7/2} O \rangle$

In this case the triangle inequalities are not satisfied, and there is an even solution for $\Delta_O = 1$:

$$\text{even:}\ \frac{A_1}{X^9}\left(\frac{143}{5} P_3^2 Q_3^5 Z_1 - 11 P_3^4 Q_3^3 Z_1 + P_3^6 Q_3 Z_1 - \frac{143}{7} Q_3^7 Z_1\right), \tag{6.20a}$$

$$\text{odd:}\ 0. \tag{6.20b}$$

There is also an odd solution for $\Delta_O = 2$:

$$\text{even:}\ 0, \tag{6.21a}$$

$$\text{odd:}\ \frac{B_1}{X^8}\left(-27 P_3^5 Q_3^2 Z_1 + 99 P_3^3 Q_3^4 Z_1 - \frac{429}{5} P_3 Q_3^6 Z_1 + P_3^7 Z_1\right). \tag{6.21b}$$

5. Correlation function $\langle \psi J_{7/2} J'_4 \rangle$

In this case the triangle inequalities are satisfied. For $\Delta_\psi = 3/2$, we see there are two even solutions and one odd solution:

$$\text{even:}\ \frac{A_2}{X}\left(\frac{99}{5} P_2 P_1^5 Q_1^2 - 9 P_2 P_1^3 Q_1^4 + P_2 P_1 Q_1^6 - \frac{429}{35} P_2 P_1^7\right) + \frac{A_1}{X}\left(-\frac{429}{5} P_1^6 Q_1 Q_2 + \frac{396}{5} P_2 P_1^5 Q_1^2 + 99 P_1^4 Q_1^3 Q_2\right.$$
$$\left. - 18 P_2 P_1^3 Q_1^4 - 27 P_1^2 Q_1^5 Q_2 - \frac{2574}{35} P_2 P_1^7 + Q_1^7 Q_2\right), \tag{6.22a}$$

$$\text{odd:}\ \frac{B_1}{X}\bigg(-35P_1^7Q_2-35P_2P_1^6Q_1+105P_1^5Q_1^2Q_2 + \frac{175}{3}P_2P_1^4Q_1^3 \tag{6.22b}$$

$$-\frac{175}{3}P_1^3Q_1^4Q_2-21P_2P_1^2Q_1^5+7P_1Q_1^6Q_2+P_2Q_1^7\bigg). \tag{6.22c}$$

VII. DISCUSSION

The purpose of this paper is to develop a formalism to determine the general form of the three-point correlation functions of conserved currents with arbitrary spins in three-dimensional conformal field theory. Our method gives explicit results and is limited only by computer power. We managed to find solutions for spins up to $s_i = 20$, but the pattern of the number of independent structures is very clear and allows us to conclude that it holds in general. We demonstrate that in all cases where the triangle inequalities are simultaneously satisfied, there are two even solutions and one odd solution for $\langle J_{s_1} J'_{s_2} J''_{s_3}\rangle$, otherwise, there are only two even solutions. Although the results for three-point functions involving bosonic currents have been proposed previously [13,14,16], we believe that our analysis stands on its own merit, as our method for imposing conservation on all three points is very explicit and analytic at every step of the computations. In addition, we construct a discriminant equation which governs the existence of the odd structure, and we extend the scope of our analysis to include correlation functions of conserved fermionic currents. Another benefit of our approach is that it can be directly generalized to four- and higher-dimensional conformal field theories as well as to (extended) superconformal field theories in diverse dimensions. We intend to explore these ideas in future works.

Finally, let us remark on three-point functions of fermionic, spin-3/2 currents. These currents naturally appear as supersymmetry currents in (extended) superconformal field theories, hence, it is also interesting to understand the general structure of correlation functions of spin-3/2 currents when supersymmetry is not manifest. In a superconformal theory the correlation functions (5.5) are contained within the following supersymmetric three-point functions:

$$\langle \mathbf{J}_{\alpha(3)}(z_1)\mathbf{J}_{\beta(3)}(z_2)\mathbf{L}_{\gamma}(z_3)\rangle, \qquad \langle \mathbf{J}_{\alpha(3)}(z_1)\mathbf{J}_{\beta(3)}(z_2)\mathbf{J}_{\gamma(3)}(z_3)\rangle. \tag{7.1}$$

In three dimensions, $\langle \mathbf{JJL}\rangle$ vanishes, while $\langle \mathbf{JJJ}\rangle$ is fixed up to a single parity-even tensor structure [50,51,54]. Hence, it appears that supersymmetry imposes additional restrictions on the general structure of three-point correlation functions. It then follows that the general form of the correlation functions (5.7) is inconsistent with supersymmetry in the sense that they are fixed up to more tensor structures than (7.1). Similar phenomenon was also found in four-dimensional (super)conformal field theories in [55,62].

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APPENDIX A: 3D CONVENTIONS AND NOTATION

For the Minkowski metric we use the "mostly plus" convention: $\eta_{mn} = \mathrm{diag}(-1,1,1)$. Spinor indices are then raised and lowered with the $\mathrm{SL}(2,\mathbb{R})$ invariant antisymmetric ε-tensor

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha{}^\beta, \tag{A1}$$

$$\phi_\alpha = \varepsilon_{\alpha\beta}\phi^\beta, \qquad \phi^\alpha = \varepsilon^{\alpha\beta}\phi_\beta. \tag{A2}$$

The γ-matrices are chosen to be real, and are expressed in terms of the Pauli matrices σ as follows:

$$(\gamma_0)_\alpha{}^\beta = -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\gamma_1)_\alpha{}^\beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A3a}$$

$$(\gamma_2)_\alpha{}^\beta = -\sigma_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \tag{A3b}$$

$$(\gamma_m)_{\alpha\beta} = \varepsilon_{\beta\delta}(\gamma_m)_\alpha{}^\delta, \qquad (\gamma_m)^{\alpha\beta} = \varepsilon^{\alpha\delta}(\gamma_m)_\delta{}^\beta. \tag{A4}$$

The γ-matrices are traceless and symmetric

$$(\gamma_m)^\alpha{}_\alpha = 0, \qquad (\gamma_m)_{\alpha\beta} = (\gamma_m)_{\beta\alpha}, \tag{A5}$$

and also satisfy the Clifford algebra

$$\gamma_m\gamma_n + \gamma_n\gamma_m = 2\eta_{mn}. \tag{A6}$$

For products of γ-matrices we make use of the identities

$$(\gamma_m)_\alpha{}^\rho(\gamma_n)_\rho{}^\beta = \eta_{mn}\delta_\alpha{}^\beta + \epsilon_{mnp}(\gamma^p)_\alpha{}^\beta, \tag{A7a}$$

$$(\gamma_m)_\alpha{}^\rho(\gamma_n)_\rho{}^\sigma(\gamma_p)_\sigma{}^\beta = \eta_{mn}(\gamma_p)_\alpha{}^\beta - \eta_{mp}(\gamma_n)_\alpha{}^\beta + \eta_{np}(\gamma_m)_\alpha{}^\beta + \epsilon_{mnp}\delta_\alpha{}^\beta, \tag{A7b}$$

where we have introduced the 3D Levi-Civita tensor ϵ, with $\epsilon^{012} = -\epsilon_{012} = 1$. We also have the orthogonality and completeness relations for the γ-matrices

$$\begin{aligned}(\gamma^m)_{\alpha\beta}(\gamma_m)^{\rho\sigma} &= -\delta_\alpha{}^\rho\delta_\beta{}^\sigma - \delta_\alpha{}^\sigma\delta_\beta{}^\rho,\\ (\gamma_m)_{\alpha\beta}(\gamma_n)^{\alpha\beta} &= -2\eta_{mn}.\end{aligned} \tag{A8}$$

Finally, the γ-matrices are used to swap from vector to spinor indices. For example, given some three-vector x_m, it may equivalently be expressed in terms of a symmetric second-rank spinor $x_{\alpha\beta}$ as follows:

$$x^{\alpha\beta} = (\gamma^m)^{\alpha\beta}x_m, \qquad x_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}x_{\alpha\beta}, \tag{A9}$$

$$\det(x_{\alpha\beta}) = \frac{1}{2}x^{\alpha\beta}x_{\alpha\beta} = -x^m x_m = -x^2. \tag{A10}$$

The same conventions are also adopted for the spacetime partial derivatives ∂_m

$$\partial_{\alpha\beta} = (\gamma^m)^{\alpha\beta}\partial_m, \qquad \partial_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}\partial_{\alpha\beta}, \tag{A11}$$

$$\partial_m x^n = \delta_m^n, \qquad \partial_{\alpha\beta}x^{\rho\sigma} = -\delta_\alpha{}^\rho\delta_\beta{}^\sigma - \delta_\alpha{}^\sigma\delta_\beta{}^\rho, \tag{A12}$$

$$\xi^m\partial_m = -\frac{1}{2}\xi^{\alpha\beta}\partial_{\alpha\beta}. \tag{A13}$$

APPENDIX B: MORE EXAMPLES OF HIGHER-SPIN CORRELATORS

In this appendix we provide further examples of three-point functions of higher-spin currents using our formalism.

1. Correlation function $\langle J_5 J_5' J_5'' \rangle$

$$\begin{aligned}
\text{Even:}\ \frac{A_3}{X^6}\Bigg(&-\frac{38177}{546}P_2^2Q_2^3Q_3^5Q_1^5 + \frac{39002P_2^4Q_2Q_3^5Q_1^5}{3003} - \frac{715865P_2P_3Q_2^4Q_3^4Q_1^5}{7098} + \frac{151915P_2^3P_3Q_2^2Q_3^4Q_1^5}{6006} - \frac{250}{231}P_2^5P_3Q_3^4Q_1^5\\
&- \frac{36594P_3^2Q_2^5Q_3^3Q_1^5}{1183} + \frac{35065P_2^2P_3^2Q_2^3Q_3^3Q_1^5}{2002} - \frac{4475P_2^4P_3^2Q_2Q_3^3Q_1^5}{2002} + \frac{565}{91}P_2P_3^3Q_2^4Q_3^2Q_1^5 + \frac{425P_2^5P_3^3Q_3^2Q_1^5}{6006}\\
&- \frac{10225P_2^3P_3^3Q_2^2Q_3^2Q_1^5}{6006} + P_3^4Q_2^5Q_3Q_1^5 - \frac{50}{91}P_2^2P_3^4Q_2^3Q_3Q_1^5 + \frac{425P_2^4P_3^4Q_2Q_3Q_1^5}{6006} - \frac{5417P_1P_2^5Q_3^5Q_1^4}{1001}\\
&- \frac{840319P_1P_2Q_2^4Q_3^5Q_1^4}{6006} + \frac{282400P_1P_2^3Q_2^2Q_3^5Q_1^4}{5577} - \frac{715865P_1P_3Q_2^5Q_3^4Q_1^4}{7098} + \frac{565}{91}P_1P_3^3Q_2^5Q_3^2Q_1^4\\
&- \frac{38177}{546}P_1^2Q_2^5Q_3^5Q_1^3 + \frac{965766P_1^2P_2^2Q_2^3Q_3^5Q_1^3}{13013} - \frac{829960P_1^2P_2^4Q_2Q_3^5Q_1^3}{39039} + \frac{35065P_1^2P_3^2Q_2^5Q_3^3Q_1^3}{2002} - \frac{50}{91}P_1^2P_3^4Q_2^5Q_3Q_1^3\\
&+ \frac{407434P_1^3P_2^5Q_3^5Q_1^2}{39039} + \frac{282400P_1^3P_2Q_2^4Q_3^5Q_1^2}{5577} - \frac{206951P_1^3P_2^3Q_2^2Q_3^5Q_1^2}{5577} + \frac{151915P_1^3P_3Q_2^5Q_3^4Q_1^2}{6006}\\
&- \frac{10225P_1^3P_3^3Q_2^5Q_3^2Q_1^2}{6006} + \frac{39002P_1^4Q_2^5Q_3^5Q_1}{3003} - \frac{829960P_1^4P_2^2Q_2^3Q_3^5Q_1}{39039} + \frac{407434P_1^4P_2^4Q_2Q_3^5Q_1}{39039}\\
&- \frac{4475P_1^4P_3^2Q_2^5Q_3^3Q_1}{2002} + \frac{425P_1^4P_3^4Q_2^5Q_3Q_1}{6006} - \frac{9627521P_1^5P_2^5Q_3^5}{1431430} - \frac{5417P_1^5P_2Q_2^4Q_3^5}{1001} + \frac{407434P_1^5P_2^3Q_2^2Q_3^5}{39039}\\
&- \frac{250}{231}P_1^5P_3Q_2^5Q_3^4 + \frac{425P_1^5P_3^3Q_2^5Q_3^2}{6006}\Bigg) + \frac{A_1}{X^6}\Bigg(P_2^5P_3^5Q_1^5 + Q_2^5Q_3^5Q_1^5 - \frac{530}{323}P_2^2Q_2^3Q_3^5Q_1^5 - \frac{40}{323}P_2^4Q_2Q_3^5Q_1^5\\
&+ \frac{30030}{323}P_2P_3^5Q_2^4Q_1^5 + \frac{77755P_2P_3Q_2^4Q_3^4Q_1^5}{4199} + \frac{325}{323}P_2^3P_3Q_2^2Q_3^4Q_1^5 + \frac{30}{323}P_2^5P_3Q_3^4Q_1^5 - \frac{24750P_3^2Q_2^5Q_3^3Q_1^5}{4199}\\
&- \frac{1780}{323}P_2^2P_3^2Q_2^3Q_3^3Q_1^5 - \frac{25}{323}P_2^4P_3^2Q_2Q_3^3Q_1^5 - \frac{495}{19}P_2^3P_3^5Q_2^2Q_1^5 - \frac{32175}{323}P_2P_3^3Q_2^4Q_3^2Q_1^5 + \frac{5}{19}P_2^5P_3^3Q_3^2Q_1^5\\
&+ \frac{2420}{323}P_2^3P_3^3Q_2^2Q_3^2Q_1^5 + \frac{2475}{323}P_2^2P_3^4Q_2^3Q_3Q_1^5 + \frac{5}{19}P_2^4P_3^4Q_2Q_3Q_1^5 + \frac{30030}{323}P_1P_3^5Q_2^5Q_1^4 - \frac{10}{323}P_1P_2^5Q_3^5Q_1^4
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1145}{323}P_1P_2Q_2^4Q_3^5Q_1^4-\frac{1675P_1P_2^3Q_2^2Q_3^5Q_1^4}{4199}+\frac{77755P_1P_3Q_2^5Q_3^4Q_1^4}{4199}-\frac{32175}{323}P_1P_3^3Q_2^5Q_3^2Q_1^4-\frac{530}{323}P_1^2Q_2^5Q_3^5Q_1^3\\
&-\frac{1920P_1^2P_2^2Q_2^3Q_3^5Q_1^3}{4199}-\frac{225P_1^2P_2^4Q_2Q_3^5Q_1^3}{4199}-\frac{1780}{323}P_1^2P_3^2Q_2^5Q_3^3Q_1^3+\frac{2475}{323}P_1^2P_3^4Q_2^5Q_3Q_1^3-\frac{495}{19}P_1^3P_3^5Q_2^5Q_1^2\\
&-\frac{35P_1^3P_2^5Q_3^5Q_1^2}{4199}-\frac{1675P_1^3P_2Q_2^4Q_3^5Q_1^2}{4199}-\frac{320P_1^3P_2^3Q_2^2Q_3^5Q_1^2}{4199}+\frac{325}{323}P_1^3P_3Q_2^5Q_3^4Q_1^2+\frac{2420}{323}P_1^3P_3^3Q_2^5Q_3^2Q_1^2\\
&-\frac{40}{323}P_1^4Q_2^5Q_3^5Q_1-\frac{225P_1^4P_2^2Q_2^3Q_3^5Q_1}{4199}-\frac{35P_1^4P_2^4Q_2Q_3^5Q_1}{4199}-\frac{25}{323}P_1^4P_3^2Q_2^5Q_3^3Q_1+\frac{5}{19}P_1^4P_3^4Q_2^5Q_3Q_1\\
&+P_1^5P_3^5Q_2^5-\frac{63P_1^5P_2^5Q_3^5}{46189}-\frac{10}{323}P_1^5P_2Q_2^4Q_3^5-\frac{35P_1^5P_2^3Q_2^2Q_3^5}{4199}+\frac{30}{323}P_1^5P_3Q_2^5Q_3^4+\frac{5}{19}P_1^5P_3^3Q_2^5Q_3^2\bigg),
\end{aligned}
\tag{B1a}
$$

$$
\begin{aligned}
\text{odd:}\ \frac{B_1}{X^6}\bigg(&\frac{147}{10}P_3^5Q_2^5Q_1^5+P_3Q_2^5Q_3^4Q_1^5-8P_2^2P_3^5Q_2^3Q_1^5-5P_2P_3^2Q_2^4Q_3^3Q_1^5-13P_3^3Q_2^5Q_3^2Q_1^5+5P_2^2P_3^3Q_2^3Q_3^2Q_1^5\\
&+P_2^4P_3^5Q_2Q_1^5+5P_2P_3^4Q_2^4Q_3Q_1^5-5P_1P_3^2Q_2^5Q_3^3Q_1^4+5P_1P_3^4Q_2^5Q_3Q_1^4-8P_1^2P_3^5Q_2^5Q_1^3\\
&+5P_1^2P_3^3Q_2^5Q_3^2Q_1^3+P_1^4P_3^5Q_2^5Q_1\bigg).
\end{aligned}
\tag{B1b}
$$

2. Correlation function $\langle J_6 J'_6 J''_6 \rangle$

$$
\begin{aligned}
\text{Even:}\ \frac{A_3}{X^7}\bigg(&-\frac{1060375P_2^6Q_3^6Q_1^6}{1876446}-\frac{314643203P_2^2Q_2^4Q_3^6Q_1^6}{6254820}+\frac{14547479P_2^4Q_2^2Q_3^6Q_1^6}{1250964}-\frac{457205623P_2P_3Q_2^5Q_3^5Q_1^6}{6254820}\\
&+\frac{44053663P_2^3P_3Q_2^3Q_3^5Q_1^6}{2084940}-\frac{63988P_2^5P_3Q_2Q_3^5Q_1^6}{34749}-\frac{49847P_3^2Q_2^6Q_3^4Q_1^6}{2187}+\frac{86002289P_2^2P_3^2Q_2^4Q_3^4Q_1^6}{6254820}\\
&+\frac{91409P_2^6P_3^2Q_3^4Q_1^6}{1250964}-\frac{1279777P_2^4P_3^2Q_2^2Q_3^4Q_1^6}{521235}+\frac{15307P_2P_3^3Q_2^5Q_3^3Q_1^6}{2916}-\frac{7882543P_2^3P_3^3Q_2^3Q_3^3Q_1^6}{4691115}\\
&+\frac{84949P_2^5P_3^3Q_2Q_3^3Q_1^6}{568620}+P_3^4Q_2^6Q_3^2Q_1^6-\frac{2261P_2^6P_3^4Q_3^2Q_1^6}{694980}-\frac{442}{729}P_2^2P_3^4Q_2^4Q_3^2Q_1^6+\frac{236113P_2^4P_3^4Q_2^2Q_3^2Q_1^6}{2084940}\\
&-\frac{17}{162}P_2P_3^5Q_2^5Q_3Q_1^6+\frac{323P_2^3P_3^5Q_2^3Q_3Q_1^6}{8748}-\frac{2261P_2^5P_3^5Q_2Q_3Q_1^6}{694980}-\frac{629266057P_1P_2Q_2^5Q_3^6Q_1^5}{6254820}\\
&+\frac{79440343P_1P_2^3Q_2^3Q_3^6Q_1^5}{2084940}-\frac{126235P_1P_2^5Q_2Q_3^6Q_1^5}{23166}-\frac{457205623P_1P_3Q_2^6Q_3^5Q_1^5}{6254820}+\frac{15307P_1P_3^3Q_2^6Q_3^3Q_1^5}{2916}\\
&-\frac{17}{162}P_1P_3^5Q_2^6Q_3Q_1^5+\frac{1173985P_1^2P_2^6Q_3^6Q_1^4}{625482}-\frac{314643203P_1^2Q_2^6Q_3^6Q_1^4}{6254820}+\frac{331624313P_1^2P_2^2Q_2^4Q_3^6Q_1^4}{6254820}\\
&-\frac{36961603P_1^2P_2^4Q_2^2Q_3^6Q_1^4}{2084940}+\frac{86002289P_1^2P_3^2Q_2^6Q_3^4Q_1^4}{6254820}-\frac{442}{729}P_1^2P_3^4Q_2^6Q_3^2Q_1^4+\frac{79440343P_1^3P_2Q_2^5Q_3^6Q_1^3}{2084940}\\
&-\frac{117852976P_1^3P_2^3Q_2^3Q_3^6Q_1^3}{4691115}+\frac{40698149P_1^3P_2^5Q_2Q_3^6Q_1^3}{6254820}+\frac{44053663P_1^3P_3Q_2^6Q_3^5Q_1^3}{2084940}-\frac{7882543P_1^3P_3^3Q_2^6Q_3^3Q_1^3}{4691115}\\
&+\frac{323P_1^3P_3^5Q_2^6Q_3Q_1^3}{8748}-\frac{1913161P_1^4P_2^6Q_3^6Q_1^2}{694980}+\frac{14547479P_1^4Q_2^6Q_3^6Q_1^2}{1250964}-\frac{36961603P_1^4P_2^2Q_2^4Q_3^6Q_1^2}{2084940}\\
&+\frac{5804704P_1^4P_2^4Q_2^2Q_3^6Q_1^2}{521235}-\frac{1279777P_1^4P_3^2Q_2^6Q_3^4Q_1^2}{521235}+\frac{236113P_1^4P_3^4Q_2^6Q_3^2Q_1^2}{2084940}-\frac{126235P_1^5P_2Q_2^5Q_3^6Q_1}{23166}\\
&+\frac{40698149P_1^5P_2^3Q_2^3Q_3^6Q_1}{6254820}-\frac{1913161P_1^5P_2^5Q_2Q_3^6Q_1}{694980}-\frac{63988P_1^5P_3Q_2^6Q_3^5Q_1}{34749}+\frac{84949P_1^5P_3^3Q_2^6Q_3^3Q_1}{568620}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2261P_1^5P_3^5Q_2^6Q_3Q_1}{694980}+\frac{39567647P_1^6P_2^6Q_3^6}{27104220}-\frac{1060375P_1^6Q_2^6Q_3^6}{1876446}+\frac{1173985P_1^6P_2^2Q_2^4Q_3^6}{625482}-\frac{1913161P_1^6P_2^4Q_2^2Q_3^6}{694980}\\
&+\frac{91409P_1^6P_3^2Q_2^6Q_3^4}{1250964}-\frac{2261P_1^6P_3^4Q_2^6Q_3^2}{694980}\Bigg)+\frac{A_1}{X^7}\Bigg(P_2^6P_3^6Q_1^6-\frac{145860}{437}P_3^6Q_2^6Q_1^6+\frac{27662225P_2^6Q_3^6Q_1^6}{117369}+Q_2^6Q_3^6Q_1^6\\
&+\frac{234542267P_2^2Q_2^4Q_3^6Q_1^6}{11178}-\frac{54202255P_2^4Q_2^2Q_3^6Q_1^6}{11178}+\frac{340136899P_2P_3Q_2^5Q_3^5Q_1^6}{11178}-\frac{229861729P_2^3P_3Q_2^3Q_3^5Q_1^6}{26082}\\
&+\frac{3337690P_2^5P_3Q_2Q_3^5Q_1^6}{4347}+\frac{32175}{161}P_2^2P_3^6Q_2^4Q_1^6+\frac{52588822P_3^2Q_2^6Q_3^4Q_1^6}{5589}-\frac{446004101P_2^2P_3^2Q_2^4Q_3^4Q_1^6}{78246}\\
&-\frac{2376985P_2^6P_3^2Q_3^4Q_1^6}{78246}+\frac{13370582P_2^4P_3^2Q_2^2Q_3^4Q_1^6}{13041}-\frac{50490583P_2P_3^3Q_2^5Q_3^3Q_1^6}{26082}+\frac{81332834P_2^3P_3^3Q_2^3Q_3^3Q_1^6}{117369}\\
&-\frac{4880537P_2^5P_3^3Q_2Q_3^3Q_1^6}{78246}-\frac{858}{23}P_2^4P_3^6Q_2^2Q_1^6+\frac{2009P_2^6P_3^4Q_3^2Q_1^6}{1242}+\frac{707564P_2^2P_3^4Q_2^4Q_3^2Q_1^6}{13041}-\frac{958117P_2^4P_3^4Q_2^2Q_3^2Q_1^6}{26082}\\
&-\frac{210925P_2P_3^5Q_2^5Q_3Q_1^6}{1449}-\frac{370799P_2^3P_3^5Q_2^3Q_3Q_1^6}{78246}+\frac{2009P_2^5P_3^5Q_2Q_3Q_1^6}{1242}+\frac{469066453P_1P_2Q_2^5Q_3^6Q_1^5}{11178}\\
&-\frac{414392809P_1P_2^3Q_2^3Q_3^6Q_1^5}{26082}+\frac{3292775P_1P_2^5Q_2Q_3^6Q_1^5}{1449}+\frac{340136899P_1P_3Q_2^6Q_3^5Q_1^5}{11178}-\frac{50490583P_1P_3^3Q_2^6Q_3^3Q_1^5}{26082}\\
&-\frac{210925P_1P_3^5Q_2^6Q_3Q_1^5}{1449}+\frac{32175}{161}P_1^2P_3^6Q_2^6Q_1^4-\frac{30621350P_1^2P_2^6Q_3^6Q_1^4}{39123}+\frac{234542267P_1^2Q_2^6Q_3^6Q_1^4}{11178}\\
&-\frac{1729902269P_1^2P_2^2Q_2^4Q_3^6Q_1^4}{78246}+\frac{192821989P_1^2P_2^4Q_2^2Q_3^6Q_1^4}{26082}-\frac{446004101P_1^2P_3^2Q_2^6Q_3^4Q_1^4}{78246}+\frac{707564P_1^2P_3^4Q_2^6Q_3^2Q_1^4}{13041}\\
&-\frac{414392809P_1^3P_2Q_2^5Q_3^6Q_1^3}{26082}+\frac{1229630576P_1^3P_2^3Q_2^3Q_3^6Q_1^3}{117369}-\frac{212309387P_1^3P_2^5Q_2Q_3^6Q_1^3}{78246}\\
&-\frac{229861729P_1^3P_3Q_2^6Q_3^5Q_1^3}{26082}+\frac{81332834P_1^3P_3^3Q_2^6Q_3^3Q_1^3}{117369}-\frac{370799P_1^3P_3^5Q_2^6Q_3Q_1^3}{78246}-\frac{858}{23}P_1^4P_3^6Q_2^6Q_1^2\\
&+\frac{9980443P_1^4P_2^6Q_3^6Q_1^2}{8694}-\frac{54202255P_1^4Q_2^6Q_3^6Q_1^2}{11178}+\frac{192821989P_1^4P_2^2Q_2^4Q_3^6Q_1^2}{26082}-\frac{60562679P_1^4P_2^4Q_2^2Q_3^6Q_1^2}{13041}\\
&+\frac{13370582P_1^4P_3^2Q_2^6Q_3^4Q_1^2}{13041}-\frac{958117P_1^4P_3^4Q_2^6Q_3^2Q_1^2}{26082}+\frac{3292775P_1^5P_2Q_2^5Q_3^6Q_1}{1449}-\frac{212309387P_1^5P_2^3Q_2^3Q_3^6Q_1}{78246}\\
&+\frac{9980443P_1^5P_2^5Q_2Q_3^6Q_1}{8694}+\frac{3337690P_1^5P_3Q_2^6Q_3^5Q_1}{4347}-\frac{4880537P_1^5P_3^3Q_2^6Q_3^3Q_1}{78246}+\frac{2009P_1^5P_3^5Q_2^6Q_3Q_1}{1242}\\
&+P_1^6P_3^6Q_2^6-\frac{29487593P_1^6P_2^6Q_3^6}{48438}+\frac{27662225P_1^6Q_2^6Q_3^6}{117369}-\frac{30621350P_1^6P_2^2Q_2^4Q_3^6}{39123}+\frac{9980443P_1^6P_2^4Q_2^2Q_3^6}{8694}\\
&-\frac{2376985P_1^6P_3^2Q_2^6Q_3^4}{78246}+\frac{2009P_1^6P_3^4Q_2^6Q_3^2}{1242}\Bigg),
\end{aligned}
\tag{B2a}
$$

$$
\begin{aligned}
\text{odd}:\ \frac{B_1}{X^7}\Bigg(&\frac{168}{5}P_2P_3^6Q_2^5Q_1^6+P_3Q_2^6Q_3^5Q_1^6+21P_2P_3^2Q_2^5Q_3^4Q_1^6-\frac{35}{3}P_2^3P_3^6Q_2^3Q_1^6+\frac{28}{3}P_3^3Q_2^6Q_3^3Q_1^6-7P_2^2P_3^3Q_2^4Q_3^3Q_1^6\\
&-\frac{259}{5}P_2P_3^4Q_2^5Q_3^2Q_1^6+7P_2^3P_3^4Q_2^3Q_3^2Q_1^6+P_2^5P_3^6Q_2Q_1^6-\frac{56}{5}P_3^5Q_2^6Q_3Q_1^6+7P_2^2P_3^5Q_2^4Q_3Q_1^6+\frac{168}{5}P_1P_3^6Q_2^6Q_1^5\\
&+21P_1P_3^2Q_2^6Q_3^4Q_1^5-\frac{259}{5}P_1P_3^4Q_2^6Q_3^2Q_1^5-7P_1^2P_3^3Q_2^6Q_3^3Q_1^4+7P_1^2P_3^5Q_2^6Q_3Q_1^4-\frac{35}{3}P_1^3P_3^6Q_2^6Q_1^3\\
&+7P_1^3P_3^4Q_2^6Q_3^2Q_1^3+P_1^5P_3^6Q_2^6Q_1\Bigg).
\end{aligned}
\tag{B2b}
$$

3. Correlation function $\langle J_{7/2} J'_{7/2} J''_6 \rangle$

$$
\begin{aligned}
\text{Even:}\ & \frac{A_8}{X^2}\Big(-\frac{301799 P_2^6 Q_3 P_1^6}{84942}+\frac{85 Q_2^6 Q_3 P_1^6}{2178}-\frac{1435 P_2^2 Q_2^4 Q_3 P_1^6}{2178}+\frac{6503 P_2^4 Q_2^2 Q_3 P_1^6}{2178}+\frac{85 P_3 Q_1 Q_2^6 P_1^5}{2178}+\frac{175 P_2 Q_1 Q_2^5 Q_3 P_1^5}{2178} \\
&+\frac{1099}{726} P_2^3 Q_1 Q_2^3 Q_3 P_1^5-\frac{96511 P_2^5 Q_1 Q_2 Q_3 P_1^5}{28314}-\frac{25}{66} Q_1^2 Q_2^6 Q_3 P_1^4+\frac{2114 P_2^2 Q_1^2 Q_2^4 Q_3 P_1^4}{1089}+\frac{6503 P_2^6 Q_1^2 Q_3 P_1^4}{2178} \\
&-\frac{68389 P_2^4 Q_1^2 Q_2^2 Q_3 P_1^4}{14157}-\frac{25}{66} P_3 Q_1^3 Q_2^6 P_1^3-\frac{126}{121} P_2 Q_1^3 Q_2^5 Q_3 P_1^3-\frac{11833 P_2^3 Q_1^3 Q_2^3 Q_3 P_1^3}{42471}+\frac{1099}{726} P_2^5 Q_1^3 Q_2 Q_3 P_1^3 \\
&+Q_1^4 Q_2^6 Q_3 P_1^2-\frac{1435 P_2^6 Q_1^4 Q_3 P_1^2}{2178}-\frac{20101 P_2^2 Q_1^4 Q_2^4 Q_3 P_1^2}{9438}+\frac{2114 P_2^4 Q_1^4 Q_2^2 Q_3 P_1^2}{1089}+P_3 Q_1^5 Q_2^6 P_1 \\
&+\frac{1667}{858} P_2 Q_1^5 Q_2^5 Q_3 P_1-\frac{126}{121} P_2^3 Q_1^5 Q_2^3 Q_3 P_1+\frac{175 P_2^5 Q_1^5 Q_2 Q_3 P_1}{2178}+P_2 P_3 Q_1^6 Q_2^5-\frac{25}{66} P_2^3 P_3 Q_1^6 Q_2^3 \\
&+\frac{85 P_2^5 P_3 Q_1^6 Q_2}{2178}+\frac{85 P_2^6 Q_1^6 Q_3}{2178}+P_2^2 Q_1^6 Q_2^4 Q_3-\frac{25}{66} P_2^4 Q_1^6 Q_2^2 Q_3\Big)+\frac{A_1}{X^2}\Big(\frac{1207229}{429} P_2^6 Q_3 P_1^6-\frac{595}{33} Q_2^6 Q_3 P_1^6 \\
&+\frac{5565}{11} P_2^2 Q_2^4 Q_3 P_1^6-\frac{25991}{11} P_2^4 Q_2^2 Q_3 P_1^6+\frac{700}{11} P_2 Q_1 Q_2^5 Q_3 P_1^5-\frac{13944}{11} P_2^3 Q_1 Q_2^3 Q_3 P_1^5+\frac{386572}{143} P_2^5 Q_1 Q_2 Q_3 P_1^5 \\
&+\frac{525}{11} Q_1^2 Q_2^6 Q_3 P_1^4-\frac{12481}{11} P_2^2 Q_1^2 Q_2^4 Q_3 P_1^4-\frac{25991}{11} P_2^6 Q_1^2 Q_3 P_1^4+\frac{532031}{143} P_2^4 Q_1^2 Q_2^2 Q_3 P_1^4-\frac{504}{11} P_2 Q_1^3 Q_2^5 Q_3 P_1^3 \\
&+\frac{342032}{429} P_2^3 Q_1^3 Q_2^3 Q_3 P_1^3-\frac{13944}{11} P_2^5 Q_1^3 Q_2 Q_3 P_1^3-21 Q_1^4 Q_2^6 Q_3 P_1^2+\frac{5565}{11} P_2^6 Q_1^4 Q_3 P_1^2+\frac{64431}{143} P_2^2 Q_1^4 Q_2^4 Q_3 P_1^2 \\
&-\frac{12481}{11} P_2^4 Q_1^4 Q_2^2 Q_3 P_1^2+\frac{36}{13} P_2 Q_1^5 Q_2^5 Q_3 P_1-\frac{504}{11} P_2^3 Q_1^5 Q_2^3 Q_3 P_1+\frac{700}{11} P_2^5 Q_1^5 Q_2 Q_3 P_1-\frac{595}{33} P_2^6 Q_1^6 Q_3 \\
&+Q_1^6 Q_2^6 Q_3-21 P_2^2 Q_1^6 Q_2^4 Q_3+\frac{525}{11} P_2^4 Q_1^6 Q_2^2 Q_3\Big),
\end{aligned}
\tag{B3a}
$$

$$
\begin{aligned}
\text{odd:}\ & \frac{B_1}{X^2}\Big(-\frac{1}{220} P_3 P_1^6 Q_2^6-\frac{7}{330} P_2 P_1^6 Q_2^5 Q_3+\frac{7}{792} P_2^3 P_1^6 Q_2^3 Q_3+\frac{1}{11} P_1^5 Q_1 Q_2^6 Q_3-\frac{35}{264} P_2^2 P_1^5 Q_1 Q_2^4 Q_3 \\
&+\frac{7}{264} P_2^4 P_1^5 Q_1 Q_2^2 Q_3+\frac{2}{15} P_3 P_1^4 Q_1^2 Q_2^6+\frac{7}{15} P_2 P_1^4 Q_1^2 Q_2^5 Q_3-\frac{7}{24} P_2^3 P_1^4 Q_1^2 Q_2^3 Q_3+\frac{7}{264} P_2^5 P_1^4 Q_1^2 Q_2 Q_3 \\
&-\frac{115}{198} P_1^3 Q_1^3 Q_2^6 Q_3+\frac{21}{22} P_2^2 P_1^3 Q_1^3 Q_2^4 Q_3+\frac{7}{792} P_2^6 P_1^3 Q_1^3 Q_3-\frac{7}{24} P_2^4 P_1^3 Q_1^3 Q_2^2 Q_3-\frac{5}{8} P_3 P_1^2 Q_1^4 Q_2^6 \\
&-\frac{49}{33} P_2 P_1^2 Q_1^4 Q_2^5 Q_3+\frac{21}{22} P_2^3 P_1^2 Q_1^4 Q_2^3 Q_3-\frac{35}{264} P_2^5 P_1^2 Q_1^4 Q_2 Q_3+\frac{131}{132} P_1 Q_1^5 Q_2^6 Q_3-\frac{7}{330} P_2^6 P_1 Q_1^5 Q_3 \\
&-\frac{49}{33} P_2^2 P_1 Q_1^5 Q_2^4 Q_3+\frac{7}{15} P_2^4 P_1 Q_1^5 Q_2^2 Q_3-\frac{1}{220} P_2^6 P_3 Q_1^6+P_3 Q_1^6 Q_2^6-\frac{5}{8} P_2^2 P_3 Q_1^6 Q_2^4+\frac{2}{15} P_2^4 P_3 Q_1^6 Q_2^2 \\
&+\frac{131}{132} P_2 Q_1^6 Q_2^5 Q_3-\frac{115}{198} P_2^3 Q_1^6 Q_2^3 Q_3+\frac{1}{11} P_2^5 Q_1^6 Q_2 Q_3\Big).
\end{aligned}
\tag{B3b}
$$

4. Correlation function $\langle J_{9/2} J'_{9/2} J''_6 \rangle$

$$
\begin{aligned}
\text{Even:}\ \frac{A_4}{X^4}\Bigg(&\frac{5251373P_2^6Q_3^3P_1^6}{368082}-\frac{13022Q_2^6Q_3^3P_1^6}{14157}+\frac{32501P_2^2Q_2^4Q_3^3P_1^6}{4719}-\frac{139294P_2^4Q_2^2Q_3^3P_1^6}{7865}+\frac{323P_3^2Q_2^6Q_3P_1^6}{9438}\\
&+\frac{323P_3^3Q_1Q_2^6P_1^5}{9438}-\frac{91}{11}P_2Q_1Q_2^5Q_3^3P_1^5+\frac{92858P_2^3Q_1Q_2^3Q_3^3P_1^5}{7865}-\frac{1810849P_2^5Q_1Q_2Q_3^3P_1^5}{613470}-\frac{2601P_3Q_1Q_2^6Q_3^2P_1^5}{1573}\\
&+\frac{119731Q_1^2Q_2^6Q_3^3P_1^4}{9438}-\frac{250872P_2^2Q_1^2Q_2^4Q_3^3P_1^4}{7865}-\frac{139294P_2^6Q_1^2Q_3^3P_1^4}{7865}+\frac{11317369P_2^4Q_1^2Q_2^2Q_3^3P_1^4}{306735}\\
&-\frac{3451P_3^2Q_1^2Q_2^6Q_3P_1^4}{3146}-\frac{85}{234}P_3^3Q_1^3Q_2^6P_1^3+\frac{26072}{605}P_2Q_1^3Q_2^5Q_3^3P_1^3-\frac{3261463P_2^3Q_1^3Q_2^3Q_3^3P_1^3}{83655}+\frac{92858P_2^5Q_1^3Q_2Q_3^3P_1^3}{7865}\\
&+\frac{54511P_3Q_1^3Q_2^6Q_3^2P_1^3}{3146}-\frac{3355}{78}Q_1^4Q_2^6Q_3^3P_1^2+\frac{32501P_2^6Q_1^4Q_3^3P_1^2}{4719}+\frac{267967P_2^2Q_1^4Q_2^4Q_3^3P_1^2}{4290}-\frac{250872P_2^4Q_1^4Q_2^2Q_3^3P_1^2}{7865}\\
&+\frac{220}{39}P_3^2Q_1^4Q_2^6Q_3P_1^2+P_3^3Q_1^5Q_2^6P_1-\frac{73729}{858}P_2Q_1^5Q_2^5Q_3^3P_1+\frac{26072}{605}P_2^3Q_1^5Q_2^3Q_3^3P_1-\frac{91}{11}P_2^5Q_1^5Q_2Q_3^3P_1\\
&-\frac{20363}{390}P_3Q_1^5Q_2^6Q_3^2P_1+P_2P_3^3Q_1^6Q_2^5-\frac{85}{234}P_2^3P_3^3Q_1^6Q_2^3-\frac{13022P_2^6Q_1^6Q_3^3}{14157}-\frac{3355}{78}P_2^2Q_1^6Q_2^4Q_3^3\\
&+\frac{119731P_2^4Q_1^6Q_2^2Q_3^3}{9438}-\frac{20363}{390}P_2P_3Q_1^6Q_2^5Q_3^2+\frac{54511P_2^3P_3Q_1^6Q_2^3Q_3^2}{3146}-\frac{2601P_2^5P_3Q_1^6Q_2Q_3^2}{1573}+\frac{323P_2^5P_3^3Q_1^6Q_2}{9438}\\
&+\frac{323P_2^6P_3^2Q_1^6Q_3}{9438}-\frac{46}{5}P_3^2Q_1^6Q_2^6Q_3+\frac{220}{39}P_2^2P_3^2Q_1^6Q_2^4Q_3-\frac{3451P_2^4P_3^2Q_1^6Q_2^2Q_3}{3146}\Bigg)+\frac{A_1}{X^4}\Bigg(\frac{21005503P_2^6Q_3^3P_1^6}{2431}\\
&-\frac{6108}{11}Q_2^6Q_3^3P_1^6+\frac{780066}{187}P_2^2Q_2^4Q_3^3P_1^6-\frac{10029132}{935}P_2^4Q_2^2Q_3^3P_1^6+\frac{171}{11}P_3^2Q_2^6Q_3P_1^6-\frac{935172}{187}P_2Q_1Q_2^5Q_3^3P_1^5\\
&+\frac{6686424}{935}P_2^3Q_1Q_2^3Q_3^3P_1^5-\frac{21729462P_2^5Q_1Q_2Q_3^3P_1^5}{12155}-\frac{10530}{11}P_3Q_1Q_2^6Q_3^2P_1^5+7515Q_1^2Q_2^6Q_3^3P_1^4\\
&-\frac{1640196}{85}P_2^2Q_1^2Q_2^4Q_3^3P_1^4-\frac{10029132}{935}P_2^6Q_1^2Q_3^3P_1^4+\frac{271628769P_2^4Q_1^2Q_2^2Q_3^3P_1^4}{12155}-486P_3^2Q_1^2Q_2^6Q_3P_1^4\\
&+\frac{2179008}{85}P_2Q_1^3Q_2^5Q_3^3P_1^3-\frac{26059876P_2^3Q_1^3Q_2^3Q_3^3P_1^3}{1105}+\frac{6686424}{935}P_2^5Q_1^3Q_2Q_3^3P_1^3+\frac{166212}{17}P_3Q_1^3Q_2^6Q_3^2P_1^3\\
&-\frac{415674}{17}Q_1^4Q_2^6Q_3^3P_1^2+\frac{780066}{187}P_2^6Q_1^4Q_3^3P_1^2+\frac{3165261}{85}P_2^2Q_1^4Q_2^4Q_3^3P_1^2-\frac{1640196}{85}P_2^4Q_1^4Q_2^2Q_3^3P_1^2\\
&+\frac{40095}{17}P_3^2Q_1^4Q_2^6Q_3P_1^2-48846P_2Q_1^5Q_2^5Q_3^3P_1+\frac{2179008}{85}P_2^3Q_1^5Q_2^3Q_3^3P_1-\frac{935172}{187}P_2^5Q_1^5Q_2Q_3^3P_1\\
&-\frac{2400354}{85}P_3Q_1^5Q_2^6Q_3^2P_1-\frac{6108}{11}P_2^6Q_1^6Q_3^3+Q_1^6Q_2^6Q_3^3-\frac{415674}{17}P_2^2Q_1^6Q_2^4Q_3^3+7515P_2^4Q_1^6Q_2^2Q_3^3\\
&-\frac{2400354}{85}P_2P_3Q_1^6Q_2^5Q_3^2+\frac{166212}{17}P_2^3P_3Q_1^6Q_2^3Q_3^2-\frac{10530}{11}P_2^5P_3Q_1^6Q_2Q_3^2+\frac{171}{11}P_2^6P_3^2Q_1^6Q_3\\
&-\frac{324324}{85}P_3^2Q_1^6Q_2^6Q_3+\frac{40095}{17}P_2^2P_3^2Q_1^6Q_2^4Q_3-486P_2^4P_3^2Q_1^6Q_2^2Q_3\Bigg),
\end{aligned}
\tag{B4a}
$$

$$
\begin{aligned}
\text{odd:}\ \frac{B_1}{X^4}\Big(&-\frac{49}{222}P_3^3Q_2^6Q_1^6+\frac{5}{37}P_2^2P_3^3Q_2^4Q_1^6+\frac{P_2^6P_3^3Q_1^6}{1221}+\frac{406}{407}P_2Q_2^5Q_3^3Q_1^6-\frac{245P_2^3Q_2^3Q_3^3Q_1^6}{1221}+\frac{18P_2^5Q_2Q_3^3Q_1^6}{2035}\\
&-\frac{1}{37}P_2^4P_3^3Q_2^2Q_1^6+P_3Q_2^6Q_3^2Q_1^6-\frac{21}{37}P_2^2P_3Q_2^4Q_3^2Q_1^6+\frac{13}{185}P_2^4P_3Q_2^2Q_3^2Q_1^6-\frac{3P_2^6P_3Q_3^2Q_1^6}{2035}-\frac{21}{37}P_2P_3^2Q_2^5Q_3Q_1^6\\
&+\frac{6}{37}P_2^3P_3^2Q_2^3Q_3Q_1^6-\frac{28P_2^5P_3^2Q_2Q_3Q_1^6}{2035}+\frac{406}{407}P_1Q_2^6Q_3^3Q_1^5-\frac{189}{407}P_1P_2^2Q_2^4Q_3^3Q_1^5+\frac{63P_1P_2^4Q_2^2Q_3^3Q_1^5}{2035}\\
&-\frac{21}{37}P_1P_3^2Q_2^6Q_3Q_1^5+\frac{5}{37}P_1^2P_3^3Q_2^6Q_1^4-\frac{189}{407}P_1^2P_2Q_2^5Q_3^3Q_1^4+\frac{21}{407}P_1^2P_2^3Q_2^3Q_3^3Q_1^4-\frac{21}{37}P_1^2P_3Q_2^6Q_3^2Q_1^4\\
&-\frac{245P_1^3Q_2^6Q_3^3Q_1^3}{1221}+\frac{21}{407}P_1^3P_2^2Q_2^4Q_3^3Q_1^3+\frac{6}{37}P_1^3P_3^2Q_2^6Q_3Q_1^3-\frac{1}{37}P_1^4P_3^3Q_2^6Q_1^2+\frac{63P_1^4P_2Q_2^5Q_3^3Q_1^2}{2035}\\
&+\frac{13}{185}P_1^4P_3Q_2^6Q_3^2Q_1^2+\frac{18P_1^5Q_2^6Q_3^3Q_1}{2035}-\frac{28P_1^5P_3^2Q_2^6Q_3Q_1}{2035}+\frac{P_1^6P_3^3Q_2^6}{1221}-\frac{3P_1^6P_3Q_2^6Q_3^2}{2035}\Big).
\end{aligned}
\tag{B4b}
$$

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