Boson stars and their relatives in semiclassical gravity

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We construct boson star configurations in quantum field theory using the semiclassical gravity approximation. Restricting our attention to the static case, we show that the semiclassical Einstein-Klein-Gordon system for a *single real quantum* scalar field whose state describes the excitation of N *identical particles*, each one corresponding to a given energy level, can be reduced to the Einstein-Klein-Gordon system for N *complex classical* scalar fields. Particular consideration is given to the spherically symmetric static scenario, where energy levels are labeled by quantum numbers n, ℓ, and m. When all particles are accommodated in the ground state $n = \ell = m = 0$, one recovers the standard static boson star solutions, that can be excited if $n \neq 0$. On the other hand, for the case where all particles have fixed radial and total angular momentum numbers n and ℓ, with $\ell \neq 0$, but are homogeneously distributed with respect to their magnetic number m, one obtains the ℓ-boson stars, whereas when $\ell = m = 0$ and n takes multiple values, the multistate boson star solutions are obtained. Further generalizations of these configurations are presented, including the multi-ℓ multistate boson stars, that constitute the most general solutions to the N-particle, static, spherically symmetric, semiclassical real Einstein-Klein-Gordon system, in which the total number of particles is definite. In spite of the fact that the same spacetime configurations also appear in multifield classical theories, in semiclassical gravity, they arise naturally as the quantum fluctuations associated with the state of a single field describing a many-body system. Our results could have potential impact on direct detection experiments in the context of ultralight scalar field/fuzzy dark matter candidates.

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I. INTRODUCTION

Boson stars are exotic objects made of bosons in which the gravitational force that pulls matter together is counterbalanced by the dispersive nature of a scalar field. They were first proposed in the late 1960s by Kaup [1] and Ruffini and Bonazzola [2], and since then they have been actively studied for more than half a century [3–9]—see Refs. [10–14] for reviews on boson stars and Refs. [15–19] for references on other soliton solutions. At present, boson stars remain largely theoretical, although they have been employed to describe dark compact objects [20–25] and galactic halo cores [26–32] in models of axion [33–38] and axionlike [39–47] particles.

More pragmatically, a boson star is a regular, localized solution to the *classical* Einstein-Klein-Gordon (EKG) system. Nevertheless, nature is *quantum* at a fundamental level, and as such, these objects must also allow an interpretation in quantum field theory. The purpose of this paper is to construct boson star configurations in semiclassical gravity, to catalog their spectrum of spherically symmetric equilibrium solutions, and to compare them with those of the classical theory. Previous attempts to construct semiclassical boson stars have been carried out in Refs. [2,48–52]. See also Refs. [53–55] for an analysis of the semiclassical gravitational collapse of quantum matter and Ref. [56] for a recent study on fluid stars in semiclassical gravity.

The semiclassical theory of gravity is an effective description of gravitational phenomena that deals with gravitons at tree level and with matter fields at one loop [57,58,64–67]. The resulting equations of motion are those of quantum field theory on curved spacetimes coupled to the semiclassical Einstein equations, where the expectation value of the stress energy-momentum tensor operator acts as the source term on its right-hand side. In this article, we introduce a well-defined operational program that deals with free quantum fields acting as sources of stationary spacetimes. Our program relies on the *semiclassical self-consistent configurations* proposed in Ref. [68], and it can be summarized into three steps:

(i) Consider a stationary, globally hyperbolic background spacetime on which the free quantum fields are defined. The assumption of stationarity allows one to introduce a preferred space of "positive-norm" solutions of the matter field equations, hence a preferred vacuum state. This in turn provides a well-defined theory for the quantum fields, which we describe in terms of a Fock space representation. In particular, field operators can be written (formally) as linear combinations of creation and annihilation operators, wherein the "coefficients" are mode functions $f_I(x)$ that solve the *classical complex* field equations.

(ii) Compute the expectation value $\langle \hat{T}_{\mu\nu} \rangle$ of the stress energy-momentum tensor operator with respect to a given state in the Fock space. In order to do so, we need a regularization and renormalization prescription that removes the ill-defined ultraviolet behavior of the theory, leading to sensible finite outcomes. To achieve this, in this work, we impose normal ordering. More sophisticated approaches include, e.g., adiabatic subtraction [69] and Pauli-Villars renormalization [70–72], although we expect the differences between such methods and ours to be suppressed in the limit of large occupation numbers, as we consider in our configurations, which we also assume to be far from the Planck scale. More generally, we can also compute a statistical average by tracing $\hat{T}_{\mu\nu}$ with a density operator. This offers the interesting possibility of considering, for instance, thermal states with a given temperature.

(iii) Solve the semiclassical Einstein equations $G_{\mu\nu} = 8\pi G \langle \hat{T}_{\mu\nu} \rangle$ sourced by the expectation value (or statistical average) of the (renormalized) stress energy-momentum tensor. This step takes into account the backreaction of the quantum fields on the classical geometry.

Of course, one of the main difficulties of this approach is that the exact spacetime geometry is not known *a priori* in step *i*, and it has to be constructed in a self-consistent way together with the other two steps. For the purpose of illustration, we will restrict our attention to the case in which matter consists of a single, free, minimally coupled real scalar field, although more involved situations can also be explored, including quantum fields of higher rank. (The case of a complex scalar field is analyzed in an Appendix.) In order to simplify the analysis, we shall further concentrate on static, spherically symmetric configurations, but a generalization of our formalism to describe stationary and axisymmetric rotating objects should be possible. In particular, as we show, in the static case, the semiclassical EKG equations can be reduced to a system of self-gravitating classical complex scalar fields with harmonic time-dependency of the form $e^{-i\omega_I t}$, which leads to a nonlinear multieigenvalue problem for the frequencies ω_I. These "classical" fields arise from the mode functions $f_I(x)$ that appear in the decomposition of the field operator and represent the "wave functions" of individual particles in first quantization. Such problems were treated in the Newtonian limit long ago; see, for example, the seminal work by Lieb [73] or Ref. [74] for more recent work in this direction.

In the static case, the resulting semiclassical solutions can be interpreted as describing equilibrium self-gravitating objects made of bosons. Specifically, we construct general boson star configurations in spherical symmetry, for which the number of particles in the different energy levels is definite. These objects interpolate between standard boson stars [1–14], whose particles all lie in the lowest possible energy configuration, to more general situations where the particles are accommodated in states with higher energy and angular momentum, which include ℓ-boson stars [75–79] and multistate boson stars [49–51], as well as new configurations obtained in this article: multi-ℓ multistate boson stars. As we show, these constitute the most general solutions to the static, spherically symmetric, semiclassical real EKG system for which the total number of particles is definite (we shall refer to these configurations as N-particle systems in this paper) and encompasses the previous boson star solutions reported in the literature. A family tree of these solutions is provided in Table II, where we show how they are connected to each other. A relevant question is whether or not such configurations in which particles populate not only the ground state but also higher energy levels, are stable. For instance, it has been found that boson stars with $\ell = 0$ including only excited states are unstable [6]. However, as discussed in Refs. [49–51], a possible mechanism of stabilization is to have a suitable combination of particles in the ground and higher energy states.[1] These multistate configurations arise naturally within our semiclassical description. See Refs. [81–83] for ℓ-boson stars in the Newtonian limit and also Refs. [84,85] for related configurations which include particles in the excited states.

Before we proceed with the construction of these objects, some words are needed regarding the relation between the classical and quantum descriptions of boson stars. Classical fields emerge from quantum theories in the limit when the quantum fluctuations become negligible. This limit is

[1] See also Ref. [80] for a stabilization mechanism that includes another type of matter.

TABLE I. Minimal ingredients to construct boson stars and their relatives in the different regimes of a quantum scalar field theory in the semiclassical gravity approximation. In this paper, we concentrate on static configurations. When the quantum field is in a coherent state, the mode functions are related with the excitation of a classical field. In contrast, when the quantum field is in a N-particle state, the mode functions are related to the particle wave functions describing a many-body system.

Regime	State	Boson stars	Multi-ℓ multistate boson stars
Classical field excitation	Coherent state	One complex field	N complex fields
Many-body system	N-particle state	One real/complex field	One real/complex field

manifest, for instance, in the case of coherent states, that saturate the quantum uncertainty principle and lead to field configurations in which quantum fluctuations are reduced to their minimum. However, in the real scalar field theory, coherent states are not compatible with a static spacetime geometry, as we prove later. This is not surprising, given that for a real scalar field there are no static configurations in the classical theory [86]. This can be traced back to the properties of the classical limit itself. The existence of soliton solutions relies on the presence of conserved charges [15,87] which allow localized field configurations whose energy per unit charge is less than in any other solutions, including those in which all the charge is radiated to infinity. In the classical field theory for a real scalar field, there is no such charge, which explains the absence of static solutions associated with the coherent states.

The situation is different in the quantum theory, where the conserved charge is the particle number that remains constant if the configuration is static.[2] Interestingly, despite the fact that the N-particle solutions describe many-body systems and appear as a result of quantum fluctuations (and, consequently, lie beyond the classicality of boson stars based on path integral arguments discussed in, e.g., Ref. [88]), they still have a counterpart in multifield classical theories. This is due to the relation that exists between the semiclassical EKG system describing a single real scalar field in a quantum state with a definite number of particles N and the EKG system for N complex classical scalar fields. In this way, we show that boson stars and their relatives (i.e., the multi-ℓ multistate boson stars) can be understood within our program by invoking a single real quantum scalar field without the need of postulating the existence of a fixed number of independent complex classical scalar fields, like, for example, the number $2\ell + 1$ in the construction of ℓ-boson stars as originally required in Ref. [75]. Nevertheless, the interpretation of the solutions is different in the classical and the quantum limits, and the difference can be found in the role that the mode functions $f_I(x)$ play in the different regimes of the theory. On one side, in an N-particle state, the resulting complex fields $f_I(x)$ are understood as the wave functions of the individual particles in first quantization, and they represent the many-body Hartree approximation [89] of a system of N particles that live in the mean gravitational field that they produce, where the N-particle wave function is just the product of one-particle wave functions [90]. On the other hand, if the state of the quantum field is coherent, the mode functions play the role of a classical field excitation $\sum_I[\alpha_I f_I(x) + \alpha_I^* f_I^*(x)]$, and a description in terms of particles is not appropriate in this case, in the same way that a description in terms of photons is not suitable in classical electrodynamics. Table I sketches the connection between the classical and the quantum regimes. Bearing in mind the different regimes of the theory may be relevant for potential direct detection experiments, such as those carried out in Refs. [91–94]. References [95–100] delve on the discussion of the classical and the quantum regimes of a scalar field in different cosmological and astrophysical situations.

This paper is organized as follows. In Sec. II, we review the main ingredients of quantum field theory on curved spaces and semiclassical gravity. In Sec. III. we focus on the static case, and next, in Sec. IV, we further specialize to the static spherically symmetric situation. This leads to the main theoretical result of this article, which is summarized in the semiclassical EKG system of Eqs. (40), (44a), and (44c). Remarkably, as a consequence of the semiclassical approach, the resulting system of equations includes as particular case the system for N classical complex fields. This constitutes the starting point for the subsequent analysis of this paper. Numerical solutions presenting new configurations which arise naturally in our formalism, including multi-ℓ, multistate, and multi-ℓ multistate boson stars are presented in Sec. V. Conclusions are drawn in Sec. VI, and technical aspects of our calculations are included in Appendix A. In Appendix B, we introduce the static, spherically symmetric, semiclassical complex EKG system which, in addition to the N-particle configurations, allows solutions sourced by coherent states that are not static.

Our conventions are as follows. We use the mostly plus signature convention for the spacetime metric, $(-,+,+,+)$, and to simplify the notation, we work in terms of natural

[2]As we show in Appendix A, if the spacetime is static, the particle number operator commutes with the Hamiltonian of the system, which is the generator of the time translations.

units for which $\hbar = c = 1$. Numerical results are obtained using Planck units, where in addition we set $G = 1$.

II. THEORETICAL FRAMEWORK

The quantization of a free field on a curved, globally hyperbolic spacetime is well understood. This program was initiated by Parker in the late 1960s and developed further by Fulling, Ford and Wald, among others (see, e.g., Refs. [59–62] for relevant textbooks and references to the aforementioned original work). If in addition the quantum fields act as a source of the spacetime metric, the semiclassical theory of gravity [57,58] provides an effective description that combines the quantum nature of matter with the classical behavior that gravity exhibits at macroscopic scales.[3] In this section, we review the main ingredients of this construction.

A. Quantum spin-0 fields in curved spaces

For the following, we consider a globally hyperbolic spacetime $(\mathcal{M}, ds^2)$ which is foliated by three-dimensional Cauchy hypersurfaces Σ_t. In terms of the standard $3+1$ decomposition, the spacetime metric is written as

$$ds^2 = -(\alpha^2 - \beta_j\beta^j)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j. \quad (1)$$

Here, $\alpha(x)$ is the lapse function, $\beta^j(x)$ is the shift vector, and $\gamma_{ij}(x)$ is the induced metric on Σ_t, with $x = (t, \vec{x})$ denoting a generic point in the spacetime manifold. Latin indices $i, j, k, \ldots$ take natural values in the range from 1 to 3 and are raised and lowered with the three-metric γ_{ij}, e.g., $\beta_i = \gamma_{ij}\beta^j$.

At the classical level, a real free massive scalar field satisfies the Klein-Gordon equation

$$(\Box - m_0{}^2)\phi = 0, \quad (2)$$

where $\Box := g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the curved d'Alembertian operator in four dimensions, $g^{\mu\nu}$ is the inverse of the spacetime metric, and ∇_μ is the covariant derivative with respect to this metric. The parameter m_0, which we assume to be positive, denotes the inverse Compton length of the field (that plays the role of the rest mass of the particles in the quantum theory), and for simplicity, a minimal coupling with gravity has been considered.

For the quantization of the field ϕ, one extends the space of real classical solutions to the space of complex-valued classical solutions of Eq. (2). Let us call this space X in the following. Given two such solutions $\phi_1, \phi_2 \in X$, one introduces the four-current vector field (with $\nabla^\mu := g^{\mu\nu}\nabla_\nu$)

$$j^\mu(\phi_1, \phi_2) := -i[\phi_1(\nabla^\mu\phi_2^*) - (\nabla^\mu\phi_1)\phi_2^*], \quad (3)$$

which, by virtue of Eq. (2), is divergence-free ($\nabla_\mu j^\mu = 0$) and satisfies the symmetries $j^\mu(\phi_1, \phi_2) = [j^\mu(\phi_2, \phi_1)]^* = -j^\mu(\phi_2^*, \phi_1^*)$. Here and in the following, $\phi_2^*(x)$ denotes the complex conjugate of $\phi_2(x)$. The four-current (3) gives rise to an inner product on X, defined as

$$(\phi_1, \phi_2) := \int_{\Sigma_t} j^\mu n_\mu d\gamma = -i\int_{\Sigma_t}[\phi_1(\pounds_n\phi_2^*) - (\pounds_n\phi_1)\phi_2^*]d\gamma. \quad (4)$$

In this equation, $(n_\mu) = (-\alpha, 0, 0, 0)$ is the future-directed timelike unit normal covector field to the Cauchy hypersurfaces Σ_t, $\pounds_n\phi_2 = n^\mu\nabla_\mu\phi_2$ refers to the Lie derivative of ϕ_2 with respect to the corresponding vector field $n = g^{\mu\nu}n_\mu\partial_\nu$, and $d\gamma = \sqrt{\det(\gamma_{ij})}d^3x$ denotes the volume element on this hypersurface. As long as the space X is restricted to those solutions of Eq. (2) which decay sufficiently fast at spatial infinity, the inner product (4) does not depend on the choice of the Cauchy hypersurface. Note also that by construction the inner product (4) is linear in its first argument and inherits the symmetries of the four-current, such that $(\phi_1, \phi_2) = (\phi_2, \phi_1)^* = -(\phi_2^*, \phi_1^*)$. However, it fails to be positive definite. Indeed, for $\phi_1 = \phi_2 =: \phi \in X$,

$$(\phi, \phi) = 2\mathrm{Im}\int_{\Sigma_t}\phi(\pounds_n\phi^*)d\gamma \quad (5)$$

may assume any real (positive or negative) value, since the restrictions of the functions ϕ and $\pounds_n\phi$ on Σ_t represent the Cauchy data for Eq. (2), which is free.

At the quantum level, the scalar field and its conjugate momentum $\pi(x) := \sqrt{\det(\gamma_{ij})}\pounds_n\phi(x)$ are promoted to self-adjoint field operators $\hat{\phi}(x)$ and $\hat{\pi}(x) = \sqrt{\det(\gamma_{ij})}\pounds_n\hat{\phi}(x)$ acting on an abstract Hilbert space $\mathscr{H}$. These operators satisfy the standard equal time commutation relations

$$[\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}),$$
$$[\hat{\phi}(t, \vec{x}), \hat{\phi}(t, \vec{y})] = [\hat{\pi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = 0, \quad (6)$$

for all $(t, \vec{x}), (t, \vec{y}) \in \Sigma_t$ and $\hat{\phi}(x)$ satisfying the Klein-Gordon equation (2). Note that we follow a canonical quantization scheme, and we are working in the Heisenberg representation, where the evolution is codified in the operators and the state vectors remain independent of time. Instead of the field operator $\hat{\phi}(x)$, which is really an operator-valued distribution on $\mathcal{M}$, it is sometimes convenient to work with its "smeared-out" versions, given by the operators

$$\hat{a}(f) := (\hat{\phi}, f), \quad f \in X, \quad (7)$$

[3] The regime of applicability of semiclassical gravity is an open question, due mainly to the fact that we do not have access to a complete, satisfactory theory of quantum gravity. See the discussion in Ref. [101], Sec. II A (and references therein), for a critical examination of the origin of the semiclassical equations.

with $(\cdot,\cdot)$ the same inner product as in Eq. (4), such that the definition is again independent of the choice of the Cauchy surface and $\hat{a}(f)$ is constant in time. Because $\hat{\phi} = \hat{\phi}^\dagger$ is self-adjoint, it follows that $\hat{a}^\dagger(f) = -(\hat{\phi}, f^*) = -\hat{a}(f^*)$ and the commutation relations (6) imply

$$[\hat{a}(f), \hat{a}(g)^\dagger] = (g, f), \qquad [\hat{a}(f), \hat{a}(g)] = -(g^*, f), \tag{8}$$

for all $f, g \in X$.

For the following, we make the important assumption that the space of complex-valued, classical solutions X of the Klein-Gordon equation (2) can be decomposed in the form [102]

$$X = X_+ \oplus X_+^*, \tag{9}$$

with the subspace X_+ consisting of solutions with positive norm [that is, $(f, f) > 0$ for all $f \in X_+$ with $f \neq 0$] and its complex conjugate X_+^* being orthogonal to it, such that $(f, g^*) = 0$ for all $f, g \in X_+$.[4] For a detailed discussion on the validity and uniqueness of this decomposition, we refer the reader to Ref. [103]. For a generic spacetime manifold $(\mathcal{M}, ds^2)$, it does not seem clear if a split of this kind exists and is unique; however, for static or stationary spacetimes, i.e., those admitting a globally defined timelike Killing vector field, the decomposition of X exists, and, moreover, the "energy requirement" of Ref. [103] selects a preferred one. Under these assumptions, the vacuum state (which in general depends on the choice of the decomposition) can be characterized as the state $|0\rangle \in \mathscr{H}$ for which $\langle 0|0\rangle = 1$ and $\hat{a}(f)|0\rangle = 0$ for all $f \in X_+$ in the preferred decomposition. The particular case for which $(\mathcal{M}, ds^2)$ is *static* will be reviewed in the next section. In this case, the natural choice for X_+ satisfying the energy requirement can be constructed directly from the space of "positive-frequency" solutions of the Klein-Gordon equation (2).

To proceed, it is convenient to work with an orthonormal set of basis functions $f_1, f_2, \ldots \in X_+$, such that $(f_I, f_J) = \delta_{IJ}$, which are usually refereed to as the *mode functions*, and to introduce the corresponding creation and annihilation operators

$$\hat{a}_I \coloneqq \hat{a}(f_I), \qquad \hat{a}_I^\dagger \coloneqq \hat{a}^\dagger(f_I), \tag{10}$$

which, by virtue of Eq. (8) and the decomposition (9), fulfill the commutation relations

$$[\hat{a}_I, \hat{a}_J^\dagger] = \delta_{IJ}, \qquad [\hat{a}_I, \hat{a}_J] = 0. \tag{11}$$

In terms of the operators $\hat{a}_I$ and $\hat{a}_I^\dagger$, the field operator $\hat{\phi}(x)$ can be decomposed as

$$\hat{\phi}(x) = \sum_I [\hat{a}_I f_I(x) + \hat{a}_I^\dagger f_I^*(x)], \tag{12}$$

which allows one to disentangle the field properties, codified in the spacetime functions $f_I(x)$ and $f_I^*(x)$, from the time-independent quantum operators $\hat{a}_I$ and $\hat{a}_I^\dagger$. Notice that this decomposition is not unique, and any choice of the orthonormal set of basis functions works equally well.

The Hilbert space $\mathscr{H}$ can now be constructed (*à la* Fock) by successive applications of creation operators on the vacuum state. A generic element in the base of the Fock construction can be written in the form

$$|N_1, N_2, \ldots\rangle = \frac{(\hat{a}_1^\dagger)^{N_1}}{\sqrt{N_1!}} \frac{(\hat{a}_2^\dagger)^{N_2}}{\sqrt{N_2!}} \ldots |0\rangle, \tag{13}$$

with $N_1, N_2, \ldots$ non-negative integer numbers such that $\sum_I N_I$ is finite. Using the commutation relations (11), one easily shows that the basis vectors (13) are normalized and mutually orthogonal and that

$$\hat{a}_K^\dagger |N_1, \ldots N_K, \ldots\rangle = \sqrt{N_K + 1} |N_1, \ldots, N_K + 1, \ldots\rangle, \tag{14a}$$

$$\hat{a}_K |N_1, \ldots, N_K, \ldots\rangle = \sqrt{N_K} |N_1, \ldots, N_K - 1, \ldots\rangle. \tag{14b}$$

Furthermore, the states (13) are eigenvectors of the particle number operator $\hat{N}_K \coloneqq \hat{a}_K^\dagger \hat{a}_K$, such that $\hat{N}_K |N_1, \ldots, N_K, \ldots\rangle = N_K |N_1, \ldots, N_K, \ldots\rangle$, with N_K representing the number of particles in the Kth mode, and thus the states (13) describe a system of $N = \sum_I N_I$ identical quantum particles, with N_1 of them in the one-particle state corresponding to mode 1, N_2 of them in the state corresponding to mode 2, and so on. Note that the expectation value of the field operator $\hat{\phi}(x)$ vanishes when evaluated on a state with a definite number of particles, i.e., $\langle N_1, N_2, \ldots | \hat{\phi}(x) | N_1, N_2, \ldots\rangle = 0$, although this does not imply that the expectation value of the stress energy-momentum tensor also vanishes, as we will see later. Because the creation operators $\hat{a}_I^\dagger$ commute with each other, these states are totally symmetric, and thus the particles satisfy the Bose-Einstein statistics and describe bosons.

An arbitrary (pure) state in the Hilbert space $\mathscr{H}$ can be expressed as a linear combination of the elements in the Fock construction,

$$|\psi\rangle = \sum_{N_1, N_2, \ldots = 0}^{\infty} (C_{N_1 N_2 \ldots}) |N_1, N_2, \ldots\rangle, \tag{15}$$

with $C_{N_1 N_2 \ldots}$ arbitrary complex numbers such that $\sum_{N_1, N_2, \ldots = 0}^{\infty} |C_{N_1 N_2 \ldots}|^2 = 1$. A case of particular interest consists of the coherent states, defined as those elements of $\mathscr{H}$ that saturate the quantum uncertainty principle and most closely resemble a classical field excitation; see, e.g., p. 97

[4]Given the properties of the inner product, the elements of X_+^* have negative norm; however, this will not be relevant for what follows.

of Ref. [104] for a brief description of the coherent states in the context of the single-particle quantum harmonic oscillator and Refs. [105,106] for comprehensive reviews. In the context of a field theory, they are usually referred as Glauber states [107] and are defined as the eigenstates of the (non-Hermitian) annihilation operators $\hat{a}_I$,

$$\hat{a}_K|\alpha_1, \alpha_2, \ldots\rangle = \alpha_K|\alpha_1, \alpha_2, \ldots\rangle, \tag{16}$$

with α_K being in general complex numbers. Note that, contrary to what happens for the states with a definite number of particles, the expectation value of the scalar field does not vanish when evaluated on a coherent state, where we obtain $\langle\alpha_1, \alpha_2, \ldots|\hat{\phi}(x)|\alpha_1, \alpha_2, \ldots\rangle = \sum_I[\alpha_I f_I(x) + \alpha_I^* f_I^*(x)]$, which is a solution to the classical Klein-Gordon equation (2). This is not surprising and actually is a consequence of Ehrenfest's theorem and the fact that we are dealing with a linear theory, so the expectation value of the field operator $\hat{\phi}(x)$ always satisfies the classical equations of motion.

B. Semiclassical gravity

So far, we have ignored the backreaction of the quantum fields, and we have assumed that the spacetime background is given *a priori*. However, according to general relativity, the spacetime metric is determined dynamically by the distribution of matter through Einstein's field equations, for which a (classical) stress energy-momentum tensor is required. One possibility to address this problem is to follow an effective field theory approach where, starting from the generating functional $Z[J, T^{\mu\nu}]$ (with J and $T^{\mu\nu}$ external sources of ϕ and $g_{\mu\nu}$), one expands the effective action $\Gamma[\phi, g]$ at tree level in gravitons and one loop in matter fields [64–67] (see also Ref. [108] for a derivation of the quantum corrected equations of motion of the metric in terms of an analysis of graviton fluctuations). The resulting theory is known as semiclassical gravity, where, in addition to the quantum field theory summarized in Sec. II A, one enforces Einstein's equations sourced by the expectation value of the stress energy-momentum tensor [57,58],

$$G_{\mu\nu} = 8\pi G\langle\hat{T}_{\mu\nu}\rangle. \tag{17}$$

Here, $G_{\mu\nu}$ is the Einstein tensor, and $\langle\hat{T}_{\mu\nu}\rangle = \langle\psi|\hat{T}_{\mu\nu}|\psi\rangle$ denotes the expectation value of the stress energy-momentum operator when evaluated on an arbitrary state $|\psi\rangle$ in $\mathscr{H}$. Further details on how to solve this problem based on the notion of semiclassical self-consistent configurations introduced in Ref. [68] will be given below. For recent rigorous results on the initial-value problem for semiclassical gravity, see, for instance, Refs. [109–112].

For the case of a real free massive scalar field, the operator associated with the stress energy-momentum tensor takes the form

$$\hat{T}_{\mu\nu} = (\nabla_\mu\hat{\phi})(\nabla_\nu\hat{\phi}) - \frac{1}{2}g_{\mu\nu}[(\nabla_\alpha\hat{\phi})(\nabla^\alpha\hat{\phi}) + m_0{}^2\hat{\phi}\,\hat{\phi}]. \tag{18}$$

It is important to notice that this quantity is quadratic in field operators and that it contains products of these operators evaluated at the same spacetime point. Since $\hat{\phi}$ is a distribution, these products are not mathematically well defined, and this problem manifests itself as divergences when computing the right-hand side of Eq. (17). Some regularization and renormalization prescription is needed in order to subtract the ill-defined ultraviolet behavior from the expectation value of higher order operators, providing sensible finite results. On the one hand, this requires the introduction of counterterms into the effective action, in such a way that the divergences that appear in the free theory are absorbed into the cosmological constant, Newton's gravitational constant, and the coupling constants accompanying quadratic curvature scalars such as R^2 and $R_{\mu\nu}R^{\mu\nu}$ [58] (which are expected to be suppressed in the low energy regime and we do not include here).[5] On the other hand, this also leads to a finite contribution to the expectation value of the stress energy-momentum tensor originating from the structure of the vacuum itself, that, even if interesting in its own right, will not be explored in more detail in the present paper (this contribution is expected to be suppressed for large occupation numbers, and this is what we assume in the following.) In practice, this corresponds to assuming normal (Wick) ordering and writing, e.g., $:\hat{a}_I\hat{a}_I^\dagger: = \hat{a}_I^\dagger\hat{a}_I$ in our expressions, moving all the creation operators to the left.

Introducing the field decomposition (12) in terms of the creation and annihilation operators into the expression for the stress energy-momentum tensor (18), one obtains

$$\hat{T}_{\mu\nu} = \frac{1}{2}\sum_{I,J}[\hat{a}_I\hat{a}_J T_{\mu\nu}(f_I, f_J) + \hat{a}_I^\dagger\hat{a}_J T_{\mu\nu}(f_I^*, f_J) + \text{H.c.}]. \tag{19}$$

As usual, H.c. stands for Hermitian conjugation, and to abbreviate the notation, we have defined

$$T_{\mu\nu}(f_I, f_J) \coloneqq (\nabla_\mu f_I)(\nabla_\nu f_J) + (\nabla_\nu f_I)(\nabla_\mu f_J) - g_{\mu\nu}[(\nabla_\alpha f_I)(\nabla^\alpha f_J) + m_0{}^2 f_I f_J], \tag{20}$$

such that $T_{\mu\nu}(f_I, f_I^*)$ is the stress energy-momentum tensor corresponding to a classical complex scalar field of amplitude $f_I(x)$. With the normal order we have imposed, Eq. (19) provides sensible results. In particular, for coherent states such as (16), one has $\langle\hat{a}_I\hat{a}_J\rangle = \alpha_I\alpha_J$ and $\langle\hat{a}_I^\dagger\hat{a}_J\rangle = \alpha_I^*\alpha_J$, and the expectation value of the stress

[5]The observed value of the cosmological constant is so small that its relevance at local scales is negligible, and for that reason, we will not include this term in our analysis either.

energy-momentum tensor operator takes the same form as its classical counterpart with $\phi_{\rm cl}(x) = \langle\hat{\phi}(x)\rangle = \sum_I[\alpha_I f_I(x) + \alpha_I^* f_I^*(x)]$, that is,

$$\langle\alpha_1,\alpha_2,\ldots|\hat{T}_{\mu\nu}|\alpha_1,\alpha_2,\ldots\rangle = \frac{1}{2}T_{\mu\nu}(\phi_{\rm cl},\phi_{\rm cl}), \quad (21)$$

where the factor 1/2 on the right hand side of Eq. (21) is due to the difference in the definition of the stress energy-momentum tensor of a real and a complex field [cf. Eqs. (19) and (B2)]. For a state with a definite number of particles of the form (13) we have, however, $\langle\hat{a}_I\hat{a}_J\rangle = 0$ and $\langle\hat{a}_I^\dagger\hat{a}_J\rangle = N_I\delta_{IJ}$, and the expectation value of the stress energy-momentum tensor reduces to

$$\langle N_1,N_2,\ldots|\hat{T}_{\mu\nu}|N_1,N_2,\ldots\rangle = \sum_I N_I T_{\mu\nu}(f_I,f_I^*), \quad (22)$$

which is also finite and equal to the weighted sum over the stress energy-momentum tensors $T_{\mu\nu}(f_I,f_I^*)$ associated with each mode function $f_I(x)$. Note that there is no analog of Eq. (22) in the classical real scalar field theory; this is because the eigenstates (13) of the particle number operator satisfy $\langle\hat{\phi}(x)\rangle = 0$, and in this case, quantum fluctuations $[\langle\hat{\phi}^2(x)\rangle - \langle\hat{\phi}(x)\rangle^2]^{1/2}$ source the entire stress energy-momentum tensor (22). Note also the different purpose that the mode functions $f_I(x)$ serve in Eqs. (21) and (22); whereas in the former expression they are associated with the excitations of a classical field, in the latter, they represent the wave functions of the quantum particles, which can be also interpreted as N equal classical complex independent fields.

C. Statistical ensembles

Up to now, we have restricted our attention to pure states, corresponding to rays in Hilbert space. More generally, one may consider a statistical ensemble described by a density operator $\hat{\rho}$, that is, a self-adjoint non-negative operator $\hat{\rho} = \hat{\rho}^\dagger \geq 0$ of unit trace $\mathrm{Tr}(\hat{\rho}) = 1$. For the particular case in which this operator is diagonal with respect to the eigenvectors (13) of the particle number operator, $\hat{\rho}$ has the representation

$$\hat{\rho} = \sum_{N_1,N_2,\ldots}(p_{N_1N_2\ldots})|N_1,N_2,\ldots\rangle\langle N_1,N_2,\ldots|, \quad (23)$$

with the probabilities $0 \leq p_{N_1N_2\ldots} \leq 1$ satisfying $\sum_{N_1,N_2,\ldots}(p_{N_1N_2\ldots}) = 1$. Although this does not describe the most general situation, it is sufficient to describe, e.g., equilibrium systems at constant temperature T, in which case the probabilities $p_{N_1N_2\ldots}$ are subject to the Bose-Einstein thermal equilibrium distribution. A more detailed analysis of such thermal configurations lies beyond the scope of this article and will be studied in future work.

When dealing with mixed states, one needs to replace the expectation value that appears in the semiclassical Einstein equations (17) with the statistically averaged stress energy-momentum tensor $\langle\hat{T}_{\mu\nu}\rangle_{\rm stat} = \mathrm{Tr}(\hat{\rho}\hat{T}_{\mu\nu})$, with Tr denoting the trace. For a statistical ensemble of the form (23), it follows that

$$\mathrm{Tr}(\hat{\rho}\hat{T}_{\mu\nu}) = \sum_I\langle N_I\rangle_{\rm stat}T_{\mu\nu}(f_I,f_I^*). \quad (24)$$

This has the same form as the right-hand side of Eq. (22), with N_I replaced with its statistical average

$$\langle N_I\rangle_{\rm stat} \coloneqq \sum_{N_1,N_2,\ldots}(p_{N_1N_2\ldots})N_I, \quad (25)$$

and the previous result (22) is recovered by choosing all the $p_{N_1N_2\ldots}$'s equal to zero except for $p_{0,\ldots,N_I,0,\ldots} = 1$.

D. Semiclassical self-consistent configurations

In order to address a problem in semiclassical gravity it is convenient to introduce the notion of semiclassical self-consistent configurations [68] (see also Refs. [112,113]). A semiclassical self-consistent configuration $\{\mathcal{M}, ds^2; \hat{\phi}(x), \hat{\pi}(x), \mathscr{H}; |\psi\rangle \in \mathscr{H}\}$ consists of: (a) a spacetime manifold $\mathcal{M}$ equipped with a metric ds^2, (b) a quantum field theory $\hat{\phi}(x)$, $\hat{\pi}(x)$ with the Hilbert space $\mathscr{H}$ defined on this fixed classical background geometry, and (c) a state $|\psi\rangle$ in $\mathscr{H}$ such that the Klein-Gordon equation (2) and the semiclassical Einstein equations (17) are satisfied simultaneously at every point in the spacetime. This is a nontrivial task; in order to construct the Hilbert space $\mathscr{H}$, we need to determine the subspace X_+ of positive norm solutions $f_I(x)$ of the Klein-Gordon equation (2), and these solutions depend on the spacetime background which is obtained by solving the semiclassical Einstein equations (17), so that both the metric field and the quantum state need to be determined in a self-consistent way. In the case when the quantum theory consists of a real scalar field, we will say that a semiclassical self-consistent configuration constitutes a solution to the semiclassical real EKG theory (2) and (17).

Having said this, we have identified three scenarios for which the expectation value $\langle\hat{T}_{\mu\nu}\rangle$ of the stress energy-momentum tensor operator has a special structure: (i) coherent states (16), for which $\langle\hat{T}_{\mu\nu}\rangle$ is equal to the stress energy-momentum tensor of the corresponding classical solution $\langle\hat{\phi}(x)\rangle = \sum_I[\alpha_I f_I(x) + \alpha_I^* f_I^*(x)]$, saturating the quantum uncertainty principle; (ii) states with a definite number of particles (13), for which $\langle\hat{T}_{\mu\nu}\rangle$ is sourced by quantum fluctuations and represents a weighted sum over the classical stress energy-momentum tensors associated with the complex fields $f_I(x)$; and (iii) statistical ensembles described by a density operator $\hat{\rho}$ of the form (23), for which the statistical average $\mathrm{Tr}(\hat{\rho}\hat{T}_{\mu\nu})$ yields again a weighted sum of classical stress energy-momentum tensors. In the first case, the semiclassical system is identical to the classical EKG system for the *single, real*, free, minimally coupled scalar field $\sum_I [\alpha_I f_I(x) + \alpha_I^* f_I^*(x)]$. In the second and third cases, the semiclassical equations are equivalent to the classical EKG system for a *family of noninteracting, complex*, free, minimally coupled scalar fields $f_I(x)$ which need to form an orthonormal set of basis functions of the subspace X_+ of positive norm solutions of the Klein-Gordon equation. Note the different roles that the mode functions $f_I(x)$ play: in scenario i, they combine into a single real field, whereas in scenarios ii and iii, they all constitute independent complex fields. In this paper, we concentrate mainly on scenarios ii and iii; however, we also discuss scenario i for the case of a complex scalar field in Appendix B.

Up to this point in the presentation, we have intended to provide the reader with a general perspective of the problem of self-gravitating boson systems in the semiclassical theory. In the remainder of this article, we focus on static configurations, in which case there is a well-defined way of performing the decomposition (9).

III. STATIC CASE

In addition to being globally hyperbolic, we now assume the spacetime $(\mathcal{M}, ds^2)$ to be static, which implies that there exists a preferred foliation $\mathcal{M} = \mathbb{R} \times \Sigma$ of the spacetime manifold such that the metric has the form

$$ds^2 = -\alpha^2(\vec{x})dt^2 + \gamma_{ij}(\vec{x})dx^i dx^j; \qquad (26)$$

i.e., the shift vector is zero, $\beta^i = 0$, and the lapse function $\alpha > 0$ and the induced three-metric γ_{ij} only depend on the spatial coordinates $\vec{x}$ on Σ. Introducing this ansatz into the Klein-Gordon equation (2), we obtain

$$\partial_t^2\phi - \alpha D^i(\alpha D_i\phi) + \alpha^2 {m_0}^2\phi = 0, \qquad (27)$$

where $\partial_t := \partial/\partial t$ is the partial derivative with respect to the time coordinate and D_i denotes the covariant derivative operator associated with the induced metric γ_{ij}. This equation contains no crossed terms of the form $\partial_t D_i$ and suggests the following ansatz for the basis functions,

$$f_I(t,\vec{x}) = \frac{1}{\sqrt{2\omega_I}} e^{-i\omega_I t} u_I(\vec{x}), \qquad (28)$$

with $\omega_I > 0$ and u_I a complex-valued function[6] of the spatial coordinates $\vec{x}$ only and where the factor $1/\sqrt{2\omega_I}$ has been introduced for future convenience. In this case, the Klein-Gordon equation (27) leads to the following eigenvalue problem for the square of the frequency ${\omega_I}^2$:

$$Hu_I := -\alpha D^j(\alpha D_j u_I) + \alpha^2 {m_0}^2 u_I = {\omega_I}^2 u_I. \qquad (29)$$

The linear operator H is formally self-adjoint on the Hilbert space Y of square-integrable functions $u: \Sigma \to \mathbb{C}$, with scalar product

$$\langle u_1, u_2\rangle := \int_\Sigma u_1^*(\vec{x}) u_2(\vec{x}) \frac{d\gamma}{\alpha(\vec{x})}, \quad u_1, u_2 \in Y. \qquad (30)$$

Indeed, one can check that, for a suitable definition of the domain $D(H)$ of the operator incorporating appropriate regularity and fall-off conditions, we have $\langle u_1, Hu_2\rangle = \langle Hu_1, u_2\rangle$ for all $u_1, u_2 \in D(H)$. Furthermore,

$$\langle u, Hu\rangle = \int_\Sigma (|Du(\vec{x})|^2 + {m_0}^2 |u(\vec{x})|^2)\alpha(\vec{x}) d\gamma, \qquad (31)$$

which is strictly positive for all $u \in D(H)$ different from zero. Hence, H is a symmetric positive operator, and since it commutes with complex conjugation, Neumann's theorem (see Theorem X.3 in Ref. [114]) implies that it possesses a positive self-adjoint extension. This offers the possibility of studying the eigenvalue problem (29) using the powerful tools of spectral theory for self-adjoint operators [115–117].

In the following, we shall assume H has a discrete spectrum with corresponding eigenvalues $0 < \omega_1^2 \le \omega_2^2 \le \ldots$, and associated eigenfunctions $u_1, u_2, \ldots$, which can be chosen such that

$$\langle u_I, u_J\rangle = \delta_{IJ}, \quad I, J = 1, 2, \ldots. \qquad (32)$$

Each of these eigenfunctions gives rise to a (complex-valued) solution of the Klein-Gordon equation of the form (28), which together with its complex conjugate solution $f_I^*(t,\vec{x})$ can easily be verified to satisfy the following properties,

$$(f_I, f_J) = -(f_I^*, f_J^*) = \delta_{IJ}, \quad (f_I, f_J^*) = 0, \qquad (33)$$

where $(\cdot,\cdot)$ denotes the inner product defined in Eq. (4).

If H has a pure discrete spectrum, then the functions u_I provide an orthonormal basis for Y, and the functions f_I and f_I^* defined by Eq. (28) provide a basis of complex-valued classical solutions of the Klein-Gordon equation, spanning the spaces of positive-frequency and "negative-frequency" solutions, respectively, which give rise to the spaces X_+ and X_+^* of the decomposition (9). Notice that this choice of the decomposition makes essential use of the staticity of the spacetime, i.e., the existence of the globally defined hypersurface-orthogonal timelike Killing vector field $\zeta = \partial_t$, where the functions f_I spanning the space X_+ have the

[6]Due to the fact that the operator H is real, we could in fact assume that the functions u_I are real valued. However, for later convenience (see Sec. IV), we shall only assume that the complex conjugate u_I^* of u_I is proportional to another member of the same basis, which we call $u_{I'}$. Note that $\omega_I = \omega_{I'}$.

property of being eigenstates $\pounds_\zeta f_I = -i\omega_I f_I$ of $\zeta = \partial_t$ with eigenvalues $-i\omega_I$, $\omega_I > 0$ (see the energy requirement of Ref. [103] for details). If H has a discrete and continuous spectrum (as will be the case for the boson star solutions discussed in the next section), the eigenfunctions f_I are incomplete; however, they may be completed by considering "generalized" eigenfunctions lying outside the Hilbert space Y, as it is usually done when dealing, for example, with free particles in Minkowski space. Alternatively, one can also consider "cutting off" the spatial domain Σ by replacing it with a compact subdomain $\Sigma_R \subset \Sigma$ with a smooth outer boundary $\partial\Sigma_R$ with large areal radius R and by solving the eigenvalue problem (29) on Σ_R with homogeneous Dirichlet conditions for u_I on $\partial\Sigma_R$. One then obtains a pure discrete spectrum at the cost of introducing the cutoff parameter R. However, as long as R is much larger than the size of the configuration, one would expect boundary effects to be negligible. Coming back to the case of free particles in Minkowski space, this is what one usually does when introducing a fictitious box of periodic boundary conditions and taking the limit of infinite volume at the end of the calculation. For the configurations that we construct in this paper, only the discrete spectrum will be excited.

For a static configuration as described by Eqs. (26) and (28), the projections of the semiclassical Einstein equations (17) normal and tangential to the hypersurface Σ reduce to the system [cf. Eqs. (2.4.10) and (2.5.4) in Ref. [118]]

$$R^{(3)} = 16\pi G\rho, \tag{34a}$$

$$R^{(3)}_{ij} - \frac{1}{\alpha} D_i D_j \alpha = 4\pi G[\gamma_{ij}(\rho - S) + 2S_{ij}], \tag{34b}$$

where $R^{(3)}_{ij}$ and $R^{(3)} := \gamma^{ij} R^{(3)}_{ij}$ refer to the three-dimensional Ricci tensor and Ricci scalar with respect to γ_{ij} and

$$\rho := n^\mu n^\nu \langle \hat{T}_{\mu\nu} \rangle, \tag{35a}$$

$$S_{ij} := (\delta_i{}^\mu + n_i n^\mu)(\delta_j{}^\nu + n_j n^\nu)\langle \hat{T}_{\mu\nu} \rangle \tag{35b}$$

are the expectation value of the energy density and the spatial stress tensor as measured by the so-called Eulerian observers (those moving along the normal direction to the spatial hypersurfaces), with $S := \gamma^{ij} S_{ij}$. For self-consistency with the staticity property, ρ and S_{ij} also need to be time independent, and the momentum flux given by

$$j_i := (\delta_i{}^\mu + n_i n^\mu) n^\nu \langle \hat{T}_{\mu\nu} \rangle \tag{35c}$$

must vanish. For an arbitrary state in $\mathscr{H}$, the energy density, the momentum flux, and the spatial stress tensor can be expressed in the form (A1) of Appendix A. As shown in this Appendix, ρ, j_i, and S_{ij} are time independent as long as $\langle \hat{a}_I \hat{a}_J \rangle = 0$ for all I, J, and $\langle \hat{a}_I^\dagger \hat{a}_J \rangle = 0$ whenever $\omega_I \neq \omega_J$. These conditions cannot be fulfilled for a nontrivial coherent state like in Eq. (16), where $\langle \hat{a}_I \hat{a}_J \rangle = \alpha_I \alpha_J$ is different from zero at least for some values of I and J. However, for a state with a definite number of particles as in Eq. (13), it follows that $\langle \hat{a}_I \hat{a}_J \rangle = 0$ and $\langle \hat{a}_I^\dagger \hat{a}_J \rangle = N_I \delta_{IJ}$, and we obtain

$$\rho = \sum_I \frac{N_I}{2\omega_I}\left[|Du_I|^2 + \left(\frac{\omega_I{}^2}{\alpha^2} + m_0{}^2\right)|u_I|^2\right], \tag{36a}$$

$$j_k = \sum_I \frac{N_I}{2}\frac{i}{\alpha}[(D_k u_I)u_I^* - u_I(D_k u_I^*)], \tag{36b}$$

$$S_{ij} = \sum_I \frac{N_I}{2\omega_I}\left\{(D_i u_I)(D_j u_I^*) + (D_j u_I)(D_i u_I^*) - \gamma_{ij}\left[|D_i u_I|^2 - \left(\frac{\omega_I{}^2}{\alpha^2} - m_0{}^2\right)|u_I|^2\right]\right\}, \tag{36c}$$

where we have abbreviated $|Du_I|^2 := \gamma^{ij}(D_i u_J)(D_j u_J^*)$ and where we recall that the functions u_I are subject to the orthogonality condition $\langle u_I, u_J \rangle = \delta_{IJ}$. If all the u_I's are chosen to be real valued, the momentum flux obviously vanishes. More generally, if u_I is complex valued, following the convention in footnote 6, $j_k = 0$ follows, provided that $N_I = N_{I'}$. See Appendix A for further information regarding these conditions and their necessity in the context of static and stationary states.

Taking into account Eqs. (36), and imposing $j_k = 0$, the Hamiltonian constraint (34a) and the trace and traceless parts of Eq. (34b) yield

$$R^{(3)} = 8\pi G \sum_I \frac{N_I}{\omega_I}\left[|Du_I|^2 + \left(\frac{\omega_I{}^2}{\alpha^2} + m_0{}^2\right)|u_I|^2\right], \tag{37a}$$

$$\frac{D^j D_j \alpha}{\alpha} = 8\pi G \sum_I \frac{N_I}{\omega_I}\left[\left(\frac{\omega_I{}^2}{\alpha^2} - \frac{m_0{}^2}{2}\right)|u_I|^2\right], \tag{37b}$$

$$\left[R^{(3)}_{ij} - \frac{1}{\alpha} D_i D_j \alpha\right]^{\mathrm{tf}} = 4\pi G \sum_I \frac{N_I}{\omega_I}[(D_i u_I)(D_j u_I^*) + (D_j u_I)(D_i u_I^*)]^{\mathrm{tf}}, \tag{37c}$$

where the superscript "tf" refers to the trace-free part with respect to γ_{ij}, i.e., $(A_{ij})^{\mathrm{tf}} := A_{ij} - \frac{1}{3}\gamma_{ij}(\gamma^{mn} A_{mn})$. These equations, together with the Klein-Gordon equation (29), constitute a nonlinear multieigenvalue problem for the frequencies ω_I describing a system of $N = \sum_I N_I$ identical quantum particles in self-gravitating equilibrium. Note that Eqs. (37) are also applicable to systems that are described in terms of a statistical ensemble of the form (23); in this case, N_I needs to be replaced with its statistical average $\langle N_I \rangle_{\mathrm{stat}}$ [see Eq. (25)].

In the next section, we further specialize these equations to the static, spherically symmetric case, and we show that for states of definite number of particles, as well as for statistical ensembles of the form (23), the resulting equations give rise to the ℓ-boson star configurations constructed in Ref. [75] (see also Refs. [76,77]) and even to more general solutions, a few examples of which are constructed numerically in Sec. V.

IV. STATIC SPHERICALLY SYMMETRIC CONFIGURATIONS

We now further specialize to a static, spherically symmetric spacetime, for which the three-metric can be expressed in the form

$$\gamma_{ij}dx^i dx^j = \gamma^2 dr^2 + r^2 d\Omega^2, \quad \gamma = \left(1 - \frac{2GM}{r}\right)^{-1/2}, \quad (38)$$

where M denotes the Misner-Sharp mass function and $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the standard line element on the unit two-sphere S^2. Furthermore, in these coordinates, the lapse α, the function γ, and the Misner-Sharp mass M only depend on the areal radius coordinate r. Because of the spherical symmetry, the mode solutions of the Klein-Gordon equation (29) can be assumed to be of the form

$$u_I(\vec{x}) = v_{n\ell}(r) Y^{\ell m}(\vartheta, \varphi), \qquad I = (n\ell m), \quad (39)$$

with $Y^{\ell m}$ denoting the standard spherical harmonics and where no sum in the total angular momentum number ℓ is considered. Note that $u_I(\vec{x})^* = (-1)^m u_{I'}(\vec{x})$ with $I' = (n, \ell, -m)$, such that the property assumed in footnote 6 is satisfied. Since the magnetic number m does not appear explicitly in the radial differential equation

$$-\frac{\alpha}{\gamma r^2}\left(\frac{\alpha r^2}{\gamma} v'_{n\ell}\right)' + \alpha^2\left[\frac{\ell(\ell+1)}{r^2} + m_0{}^2\right] v_{n\ell} = (\omega_{n\ell})^2 v_{n\ell} \quad (40)$$

that is obtained from Eq. (29) with the ansatz (39), the radial functions $v_{n\ell}(r)$ can be chosen to be independent of this number. Therefore, the eigenvalue problem (29) reduces to finding (for each $\ell = 0, 1, 2, \ldots$) a set of suitable radial basis functions $v_{n\ell}$ solving Eq. (40). Using the orthonormality property of the spherical harmonics, $\int_{S^2} Y^{\ell m} Y^{\ell' m' *} d\Omega = \delta_{\ell\ell'}\delta_{mm'}$, the orthogonality condition $\langle u_I, u_J\rangle = \delta_{IJ}$ reduces to

$$\int_0^\infty v_{n\ell}(r) v^*_{n'\ell}(r) \frac{\gamma(r)}{\alpha(r)} r^2 dr = \delta_{nn'}. \quad (41)$$

Assuming the functions α and γ are regular at the center $r = 0$, such that they have local expansions of the form $\alpha(r) = \alpha_0 + \alpha_2 r^2 + \cdots$ and $\gamma(r) = 1 + \gamma_2 r^2 + \cdots$, one can show that the local solution that is finite at $r = 0$ has the form $v_{n\ell}(r) \sim r^\ell$; see Ref. [75]. Likewise, as $r \to \infty$, we impose that the metric functions α and γ converge to 1 and that $v_{n\ell}(r)$ are bounded, which implies that they have the form $v_{n\ell}(r) \sim e^{-\sqrt{m_0{}^2 - (\omega_{n\ell})^2} r}$, with $0 < \omega_{n\ell} < m_0$. A further restriction arises from the identity $\langle u_I, H u_I\rangle = \omega_I{}^2$, which yields

$$\int_\Sigma \left\{ |Du_I|^2 + \left[m_0{}^2 - \frac{\omega_I{}^2}{\alpha^2}\right] |u_I|^2 \right\} \alpha d\gamma = 0 \quad (42)$$

and shows that $m_0{}^2 - \omega_I{}^2/\alpha^2$ cannot be positive everywhere, since otherwise it would follow from Eq. (42) that $u_I = 0$.

The functions α and γ must be determined by solving the static semiclassical Einstein field equations (37), where for consistency the right-hand side must be a spherically symmetric tensor.[7] This is clearly the case if only the ground state is populated, i.e., if $N_I = 0$ for all $I \neq (000)$, which gives rise to the standard boson star equations. More generally, one can demand that $N_{n\ell m} = 0$ for all $\ell > 0$, meaning that all the particles have zero angular momentum but may nevertheless be in excited energy states. This gives rise to the multistate boson star equations solved in Refs. [49–51]. Following the same arguments as in Appendix A of Ref. [75], in order for the expectation value of the stress energy-momentum tensor to be spherically symmetric, it is in fact sufficient to choose $N_{n\ell m}$ independent of the magnetic number m, such that

$$N_{n,\ell,-\ell} = N_{n,\ell,-(\ell-1)} = \ldots = N_{n,\ell,(\ell-1)} = N_{n,\ell,\ell}, \quad (43)$$

which implies that the total angular momentum vanishes, even if the individual particles posses angular momentum. Note that this choice also guarantees that the momentum flux is zero, as required for staticity. In other words, the excitation numbers $N_{n\ell m}$ are functions of the energy levels n and the total angular momentum ℓ, but not of the magnetic quantum number m. This is rather similar to the case of a kinetic gas, in which a one-particle distribution function depending only on the energy and the total angular momentum gives rise to a static, spherically symmetric configuration (see, for instance, Sec. 5.1 in Ref. [120]).

Assuming the validity of condition (43) and the spherically symmetric ansatz (38), Eqs. (37) reduce to

[7] A different approach to achieving this property was considered in Ref. [119], where the stress energy-momentum tensor is averaged over the spheres in order to get rid of the angular dependency. Our approach requires no such averaging.

TABLE II. Classification of the solutions to the static, spherically symmetric, semiclassical real EKG system. They represent self-gravitating equilibrium configurations of a definite number of identical quantum particles. These are all the cases obtained when combining two options for the radial quantum number n and three options for the total angular momentum number ℓ. The two options for n are: *i*) multiple values and *ii*) one value; while the three options for ℓ are: *i*) multiple values, *ii*) one value, and *iii*) fixed value $\ell = 0$. Note that some of these solutions are particular cases of others. In order to show this hierarchy more clearly, we enclose with a bracket solutions that are included in a more general solution, which is indicated with an arrow.

Name	n	ℓ	Relativistic	Newtonian
Multi-ℓ multistate boson star	$n_1, n_2, \ldots, n_p$	$\ell_1, \ell_2, \ldots, \ell_q$	Sec. V	⋯
Multistate ℓ-boson star	$n_1, n_2, \ldots, n_p$	ℓ_1	Sec. V	⋯
Multistate boson star	$n_1, n_2, \ldots, n_p$	0	[49–51]	[74]
Boson star	n_1	0	[1–6, 8]	[73, 121]
ℓ-Boson star	n_1	ℓ_1	[75–78]	[81–83]
Multi-ℓ boson star	n_1	$\ell_1, \ell_2, \ldots, \ell_q$	Sec. V	[83]

$$\frac{2GM'}{r^2} = \sum_{n\ell} \frac{\kappa_\ell N_{n\ell m}}{\omega_{n\ell}} \left[\frac{|v'_{n\ell}|^2}{\gamma^2} + \left(\frac{(\omega_{n\ell})^2}{\alpha^2} + {m_0}^2 + \frac{\ell(\ell+1)}{r^2} \right) |v_{n\ell}|^2 \right], \tag{44a}$$

$$\frac{1}{\alpha\gamma r^2} \left(\frac{r^2\alpha'}{\gamma} \right)' = \sum_{n\ell} \frac{\kappa_\ell N_{n\ell m}}{\omega_{n\ell}} \left[\left(2\frac{(\omega_{n\ell})^2}{\alpha^2} - {m_0}^2 \right) |v_{n\ell}|^2 \right], \tag{44b}$$

$$\frac{(\alpha\gamma)'}{r\alpha\gamma^3} = \sum_{n\ell} \frac{\kappa_\ell N_{n\ell m}}{\omega_{n\ell}} \left[\frac{|v'_{n\ell}|^2}{\gamma^2} + \frac{(\omega_{n\ell})^2}{\alpha^2} |v_{n\ell}|^2 \right], \tag{44c}$$

with $\kappa_\ell \coloneqq (2\ell+1)G$ and where the last identity was obtained by contracting the angular components of Eq. (37c) with the metric of the unit two-sphere, $\hat{g}_{AB}$, and making use of Eq. (44a). In addition, we have also used the identities $\sum_{m=-\ell}^{\ell} Y^{\ell m} Y^{\ell m *} = \frac{1}{4\pi}(2\ell+1)$ and $\sum_{m=-\ell}^{\ell} (\hat{\nabla}_A Y^{\ell m})(\hat{\nabla}^A Y^{\ell m *}) = \frac{1}{4\pi}\ell(\ell+1)(2\ell+1)$, where $\hat{\nabla}_A$ makes reference to the covariant derivative with respect to $\hat{g}_{AB}$ (see Appendix A in Ref. [75] for details). The full system of reduced static, spherically symmetric, semiclassical real EKG equations consists of Eqs. (40) and (44), where, due to the twice contracted Bianchi identities, Eq. (44b) can be omitted. Further, the eigenfunctions $v_{n\ell}$ should satisfy the normalization condition (41), although we can also do without this equation if we absorb the occupation numbers $N_{n\ell m}$ in the radial functions $v_{n\ell}$, as described in the next section. Once the functions $v_{n\ell}$ are known, the quantum field $\hat{\phi}(x)$ can be reconstructed using Eqs. (12), (28), and (39), which gives

$$\hat{\phi}(x) = \sum_{n\ell m} \frac{1}{\sqrt{2\omega_{n\ell}}} [\hat{a}_I e^{-i\omega_{n\ell} t} v_{n\ell}(r) Y^{\ell m}(\vartheta, \varphi) + \text{H.c.}]. \tag{45}$$

The ℓ-boson star configurations we have discussed in Refs. [75–77] are obtained by solving a particular case of this system, in which all the $N_{n\ell m}$'s vanish except the ones for $n = 0$ and some specific value of ℓ. In this case, after absorbing the factor $N_{0\ell m}/\omega_{0\ell}$ into the amplitude of $v_{0\ell}$, the system of Eqs. (44a), (44c), and (40) reduces precisely to the system (7a, 7b, 7c) of [75]. However, in contrast to the purely classical description in Refs. [75–77], which requires precisely $2\ell+1$ complex scalar fields, the semiclassical interpretation of the ℓ-boson stars becomes much more natural: they correspond to a particular excitation of a single real quantum spin zero field that describes a self-gravitating system of $(2\ell+1)N_{0\ell m}$ identical quantum particles of definite energy $E = \omega_{0\ell}$ and angular momentum $L = \sqrt{\ell(\ell+1)}$ (both evaluated in natural units). Furthermore, as is evident from the equations above, there are many other possible configurations that can be constructed in this way, involving excitations of different energy levels n and different total angular momentum numbers ℓ. We summarize these more general solutions and their subfamilies, as well as a few references to corresponding Newtonian configurations, in Table II. Numerical examples of some of these more general configurations are constructed in the next section.

V. NUMERICAL SOLUTIONS: A FEW EXAMPLES

In this section, we present numerical solutions to the static, spherically symmetric, semiclassical real EKG system described by Eqs. (40), (44a), and (44c). These solutions complement the mathematical analysis of the previous section. Specifically, we obtain three particular solutions as representative examples of the three types of solutions that have not been presented so far in the literature (see Table II), that is, a multi-ℓ boson star, a multistate ℓ-boson star, and a multi-ℓ multistate boson star.

To proceed, from this point onward in addition to $\hbar = c = 1$, we also set $G = 1$, such that all quantities are dimensionless and measured in Planck units, although the solutions can be rescaled arbitrarily using a symmetry transformation, as we explain later. In practice, we solve the semiclassical real EKG system [Eqs. (40), (44a), and (44c)] expressed in the form

$$\psi''_{n\ell} = -\left[\gamma^2 + 1 - (2\ell+1)r^2\gamma^2\left(\frac{\ell(\ell+1)}{r^2} + {m_0}^2\right)(\psi_{n\ell})^2\right]\frac{\psi_{n\ell}{}'}{r} - \left(\frac{(\omega_{n\ell})^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - {m_0}^2\right)\gamma^2\psi_{n\ell}, \tag{46a}$$

$$\gamma' = \sum_{n\ell}\frac{2\ell+1}{2}r\gamma\left[\left(\frac{(\omega_{n\ell})^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2} + {m_0}^2\right)\gamma^2(\psi_{n\ell})^2 + (\psi'_{n\ell})^2\right] - \left(\frac{\gamma^2-1}{2r}\right)\gamma, \tag{46b}$$

$$\alpha' = \sum_{n\ell}\frac{2\ell+1}{2}r\alpha\left[\left(\frac{(\omega_{n\ell})^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - {m_0}^2\right)\gamma^2(\psi_{n\ell})^2 + (\psi'_{n\ell})^2\right] + \left(\frac{\gamma^2-1}{2r}\right)\alpha, \tag{46c}$$

where for convenience we have introduced the rescaled fields

$$\psi_{n\ell} = \sqrt{\frac{N_{n\ell m}}{\omega_{n\ell}}}v_{n\ell}. \tag{47}$$

Finally, one can choose an arbitrary value for the mass m_0, since solutions for a different value can then be obtained by a simple rescaling; see Eq. (50) below. In particular, we set $m_0 = 1$ for the numerical integrations but present the results in an m_0-independent form. Notice that from the normalization condition in Eq. (41) one can read off the number of particles in the different states using expression

$$N_{n\ell m} = \omega_{n\ell}\int_0^\infty (\psi_{n\ell})^2\frac{\gamma}{\alpha}r^2dr. \tag{48}$$

The choice of appropriate boundary conditions must guarantee that the boson star solutions are regular and asymptotically flat, and additionally that they possess finite total energy and finite energy density everywhere. Demanding regularity at the origin, i.e.,

$$\psi_{n\ell}(r) = \frac{\psi^0_{n\ell}}{2\ell+1}r^\ell, \tag{49a}$$

$$\psi'_{n\ell}(r) = \frac{\ell\psi^0_{n\ell}}{2\ell+1}r^{\ell-1}, \tag{49b}$$

$$\alpha(r) = 1, \tag{49c}$$

$$\gamma(r) = 1, \tag{49d}$$

when $r \to 0$, and a vanishing field at infinity, one obtains a nonlinear multiple-eigenvalue problem for the different mode frequencies $\omega_{n\ell}$. Here, $\psi^0_{n\ell}$ are some arbitrary positive constants related to the number of particles in the different energy levels, and with no loss of generality, we have fixed the value of the lapse function at the origin to 1. Notice that, since the system of equations is invariant under $(\alpha, \omega_{n\ell}) \mapsto \lambda(\alpha, \omega_{n\ell})$, with some positive arbitrary constant λ, one can always rescale the value of the lapse function in such a way that $\alpha(r \to \infty) = 1$, as we do later.

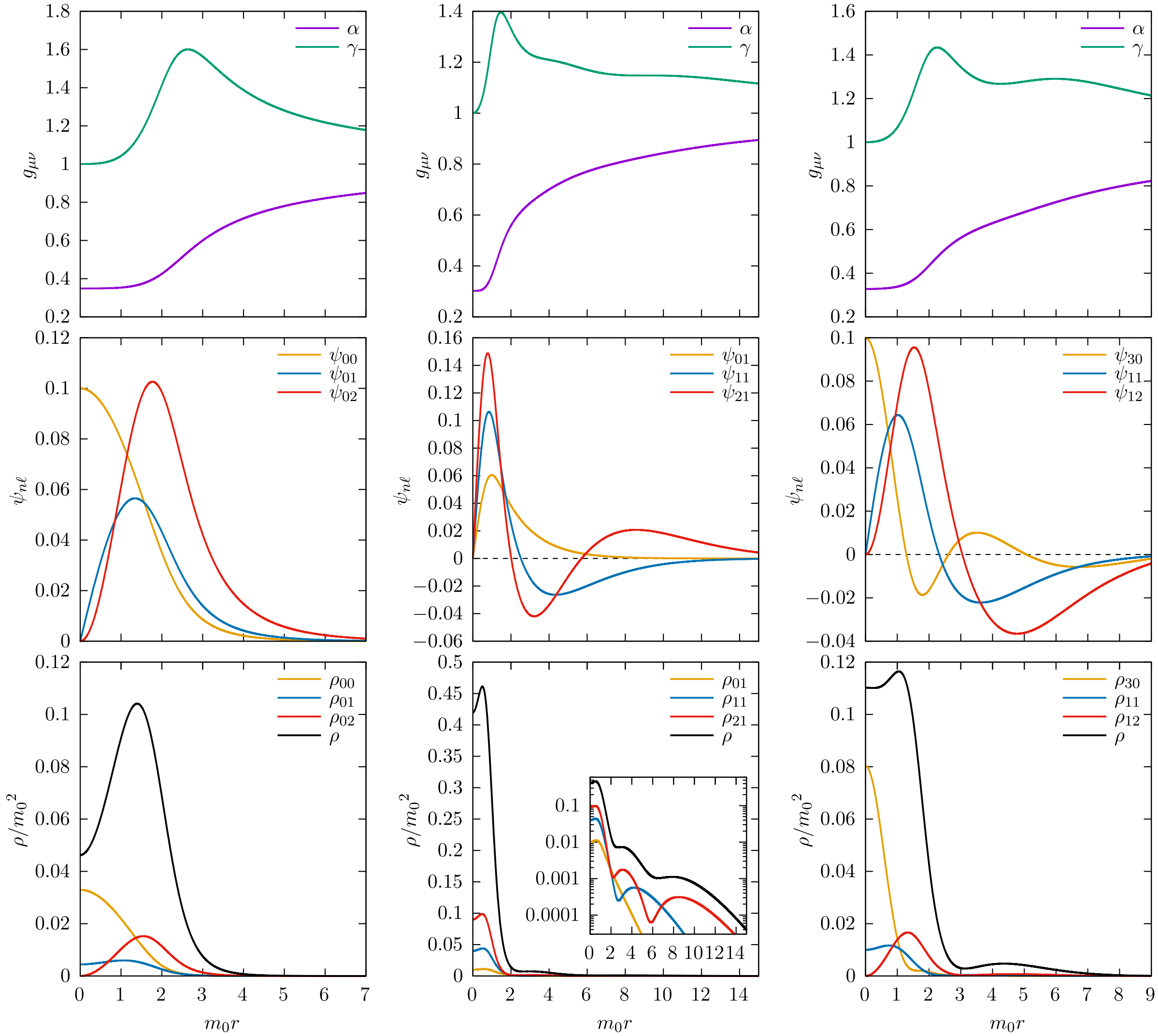
The integration of the system is performed numerically using a shooting algorithm to find the frequencies $\omega_{n\ell}$. To proceed, one integrates the system of Eqs. (46) outward, starting from the initial conditions in Eqs. (49) at a point very close to the origin (we used $r_0 = 5\times10^{-6}$), and search for the values of the frequencies $\omega_{n\ell}$ to match the asymptotic behavior of the mode functions until the shooting parameter converges to the desired accuracy. As we already mentioned, for simplicity, we have assumed $m_0 = 1$, although the solutions can be rescaled to an arbitrary value of the mass parameter using the invariance of the system under the transformation

$$m_0 \mapsto \lambda m_0, \qquad \omega_{n\ell} \mapsto \lambda\omega_{n\ell}, \qquad r \mapsto \lambda^{-1}r, \tag{50}$$

with the functions $\psi_{n\ell}$, α, and γ unchanged. Under this transformation, the occupation numbers (48) change according to $N_{n\ell m} \mapsto \lambda^{-2}N_{n\ell m}$. Note that, for instance, $\lambda \sim 10^{-50}$ in the case of an ultralight axion dark matter particle of mass $m_0 \sim 10^{-22}$ eV.

In Fig. 1, we present the results of our numerical solutions. In the left column—Fig. 1(a)—we present a multi-ℓ boson star solution. The configuration displayed is characterized by the quantum numbers $n = 0$ and $\ell = 0$, 1, 2.[8] Unlike the ℓ-boson stars introduced in Ref. [75], these new solutions do not require all the individual fields to have the same amplitude, and, still, the resulting spacetime is spherically symmetric. In the central column—Fig. 1(b)—we present a multistate ℓ-boson star solution. The configuration displayed is characterized by the quantum numbers $n = 0$, 1, 2 and $\ell = 1$. In the right column—Fig. 1(c)—we present a multi-ℓ multistate boson star solution. The

[8]Note that, as in our previous works, n refers to the number of nodes of the radial function in the interval $0 < r < \infty$. Hence, it should be compared to the "radial quantum number" in the theory of the hydrogen atom rather than the "principal quantum number."



(a) Multi-ℓ boson star (b) Multi-state ℓ-boson star (c) Multi-ℓ multi-state boson star

FIG. 1. For examples, we describe here three particular solutions, each one of a different type, as indicated in the subfigures. For all cases, we show the metric components $\alpha(r)$ and $\gamma(r)$ (top panels); the state components $\psi_{n\ell}(r)$ (middle panels); and the energy density $\rho(r)$ as well as the individual components $\rho_{n\ell}(r)$ (bottom panels). Note that ρ can be obtained simply by summing its components (including all degenerate states in m). Thus, we have $\rho = \rho_{00} + 3\rho_{01} + 5\rho_{02}$ in (a), $\rho = 3\rho_{01} + 3\rho_{11} + 3\rho_{21}$ in (b), and $\rho = \rho_{30} + 3\rho_{11} + 5\rho_{12}$ in (c). We show more properties of these solutions in Table III. Finally, we note that the actual numerical integration domain in r is much larger than what is shown (for clarity) in the figures.

configuration displayed is characterized by the quantum numbers $n = 1$, 3 and $\ell = 0$, 1, 2. In all three cases, the lapse function α and the metric component γ of each configuration are shown in the top panels, the radial profiles for the different wave functions $\psi_{n\ell}$ that are excited in the configuration are shown in the middle panels, and the total energy density and the density of the individual energy levels are shown in the bottom panel. In Table III, we report the main numbers associated with these configurations.

In this section, we presented just three particular solutions as illustrative examples of the three new types of boson stars described in this work. However, going into further analysis of such solutions is beyond the scope of this article. Instead, we expect to carry out a more detailed study in a separate work.

VI. DISCUSSION AND CONCLUSIONS

We have shown that static, spherically symmetric boson star configurations [1–14] and many of their generalizations (including ℓ-boson stars [75–79] and multistate boson stars [49–51]), arise naturally within the semiclassical gravity approximation in quantum field theory. Furthermore, we have found new possible generalizations, namely, the multi-ℓ multistate boson stars, that represent the most general solutions to the N-particle, static,

TABLE III. Quantum numbers, amplitudes, eigenfrequencies, particle numbers in each state, and total particle number for the configurations presented in Fig. 1. Note that for the case of an ultralight axion $m_0 \sim 10^{-50}$ in natural units; hence, in this example, the particle numbers are large for the configurations presented.

Name	n	ℓ	$\psi^0_{n\ell}/m_0{}^\ell$	$\omega_{n\ell}/m_0$	$m_0{}^2N_{nl}$	$m_0{}^2N$	Fig.
	0	0	0.1	0.5278	0.0195		
Multi-ℓ boson star	0	1	0.2	0.6453	0.0243	0.8134	1(a)
	0	2	0.4	0.7736	0.1442		
	0	1	0.3	0.7438	0.0289		
Multistate ℓ-boson star	1	1	0.6	0.8235	0.1150	1.3884	1(b)
	2	1	0.9	0.8792	0.3189		
	3	0	0.1	0.8679	0.0133		
Multi-ℓ multistate boson star	1	1	0.3	0.7497	0.0439	1.2745	1(c)
	1	2	0.5	0.8247	0.2259		

spherically symmetric, semiclassical real EKG system, in which the total number of particles is definite.

Our approach is based on the expansion of a single, real, free quantum scalar field in terms of a linear combination of creation and annihilation operators. We then construct the Hilbert space by successive applications of creation operators on the vacuum state, that we assume is well defined (which is guaranteed for the static configurations considered in this article). Taking particular Fock states with a definite number of particles, the expectation value of the renormalized stress energy-momentum tensor operator takes the same form as its counterpart in a classical theory with N fields. Each of these N fields accounts for one excitation mode of the quantum field and corresponds to the quantum particles of the system. The number of particles contained in each mode is then fixed by the number of classical fields with the same quantum numbers and can be chosen as a free parameter. In this way, standard boson stars correspond to the population of only one mode (the ground state) of the quantum field, for which $n = \ell = m = 0$. Other self-gravitating scalar configurations such as ℓ-boson stars and multistate boson stars are naturally related to them and are just different manifestations of the quantum fluctuations of the same scalar field with different modes populated.

Regarding ℓ-boson stars in particular, we would like to highlight an important difference between the classical and semiclassical interpretations: the construction of the classical solutions described in our previous works might be considered somehow “artificial,” in the sense that they must consist of a particular combination of $2\ell + 1$ independent, complex scalar fields having the exact same radial amplitude in order to constitute as a whole a spherically symmetric matter distribution. On the other hand, within the semiclassical gravity approximation, the same ℓ-boson star configurations are obtained more naturally, starting from a single real quantum scalar field in a state that describes a distribution of particles populating a given energy level and containing angular momentum. Apart from providing a more natural explanation for their existence, the semiclassical description of ℓ-boson stars should have other interesting applications. For example, a relevant problem is analyzing the dynamical stability of these configurations in semiclassical gravity. This is clearly a much more difficult problem that requires generalizing our framework for spacetimes being perturbed off a static one.

Further, in the realm of an ultralight scalar field/fuzzy dark matter component, the diversity of self-gravitating structures that might model the dark matter halos is greatly enhanced in the semiclassical theory. Standard $\ell = 0$ boson stars have been considered as a possible explanation for the dark matter distribution in small galaxies, such as dwarf spheroidals [32]. However, it is clear that these configurations alone are not sufficient to describe the properties observed in galaxies spanning different size scales. On the one hand, the radius of stable $\ell = 0$ boson stars decreases when their total mass increases, in stark contrast to what is being observed in real galaxies. On the other hand, their mass density decreases exponentially with the radius at large distances, and they cannot accommodate the flatness of the rotation curves observed in spiral galaxies (although they can still describe the core of galaxies like the Milky Way [122]). The multi-ℓ multistate configurations offer the possibility to account for the desired behavior of the dark matter distribution in large galaxies beyond the core region. Likewise, several examples of anomalous halo systems recently reported in Ref. [123] could potentially be explained by means of scalar field configurations with nontrivial values of n, ℓ, and m. The connection between the multi-ℓ multistate boson star solutions and the configurations obtained in numerical simulations [27–31] is left as an open question.

With regard to dark compact objects, the state of maximal compactness of ℓ-boson stars has been identified to grow with ℓ, as discussed in Ref. [78]. This could lead to potential observational signatures in the gravitational wave spectra resulting from the merger of these objects [24].

In this article, we have only presented some representative cases by solving numerically the spherically symmetric, static semiclassical system of equations for certain quantum numbers; however, a more exhaustive analysis is required in order to compare our solutions with the astrophysical data. In conclusion, the interpretation of boson stars within the semiclassical gravity approximation considerably increases the possibility for boson fields to describe more realistic astrophysical systems.

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APPENDIX A: STRESS ENERGY-MOMENTUM TENSOR

In this Appendix, we summarize the results for the energy density (35a), momentum flux (35c), and spatial stress tensor (35b) of an arbitrary quantum state on a static spacetime that are used in Sec. III. These expressions are

$$\rho = \sum_{I,J} \frac{1}{4\sqrt{\omega_I\omega_J}} \left\{ \langle \hat{a}_I \hat{a}_J \rangle \left[\left(-\frac{\omega_I\omega_J}{\alpha^2} + m_0{}^2 \right) u_I u_J + (D_k u_I)(D^k u_J) \right] e^{-i(\omega_I+\omega_J)t} + \langle \hat{a}_I^\dagger \hat{a}_J \rangle \left[\left(+\frac{\omega_I\omega_J}{\alpha^2} + m_0{}^2 \right) u_I^* u_J + (D_k u_I^*)(D^k u_J) \right] e^{+i(\omega_I-\omega_J)t} + \text{c.c.} \right\}, \tag{A1a}$$

$$j_k = \sum_{I,J} \frac{1}{4\sqrt{\omega_I\omega_J}} \frac{i}{\alpha} \{ \langle \hat{a}_I \hat{a}_J \rangle [-\omega_J (D_k u_I) u_J - \omega_I u_I (D_k u_J)] e^{-i(\omega_I+\omega_J)t} + \langle \hat{a}_I^\dagger \hat{a}_J \rangle [-\omega_J (D_k u_I^*) u_J + \omega_I u_I^* (D_k u_J)] e^{+i(\omega_I-\omega_J)t} - \text{c.c.} \}, \tag{A1b}$$

and

$$S_{ij} = \sum_{I,J} \frac{1}{4\sqrt{\omega_I\omega_J}} \left\{ \langle \hat{a}_I \hat{a}_J \rangle \left[(D_i u_I)(D_j u_J) + (D_j u_I)(D_i u_J) - \gamma_{ij} \left(\left(+\frac{\omega_I\omega_J}{\alpha^2} + m_0{}^2 \right) u_I u_J + (D_k u_I)(D^k u_J) \right) \right] e^{-i(\omega_I+\omega_J)t} + \langle \hat{a}_I^\dagger \hat{a}_J \rangle \left[(D_i u_I^*)(D_j u_J) + (D_j u_I^*)(D_i u_J) - \gamma_{ij} \left(\left(-\frac{\omega_I\omega_J}{\alpha^2} + m_0{}^2 \right) u_I^* u_J + (D_k u_I^*)(D^k u_J) \right) \right] e^{+i(\omega_I-\omega_J)t} + \text{c.c.} \right\}, \tag{A1c}$$

where c.c. stands for complex conjugation.

In this article, we are interested in *static configurations*, that is in states satisfying the property that ρ and S_{ij} are time-independent and $j_k = 0$, such that the expectation value of the stress energy-momentum tensor is compatible, through the semiclassical Einstein equations, with the static spacetime metric (26). It is clear from Eqs. (A1) that ρ, j_k, and S_{ij} are time independent if $\langle \hat{a}_I \hat{a}_J \rangle = 0$ for all I, J and $\langle \hat{a}_I^\dagger \hat{a}_J \rangle = 0$ whenever $\omega_I \neq \omega_J$. Furthermore, assuming the convention in footnote 6, the additional condition $\langle \hat{N}_I \rangle = \langle \hat{N}_{I'} \rangle$ implies that $j_k = 0$.

The question of whether these conditions are also necessary for a static configuration is more subtle. To proceed, we introduce the notion of *stationary states*, defined as states in the Schrödinger picture which are invariant under time translations. In quantum mechanics, time translations are represented by unitary operators of the form $\hat{\mathcal{U}}[t_0, t] = \exp[-i\hat{H}(t - t_0)]$, where the Hamiltonian $\hat{H}$ is the generator of the transformations. If a state $|\psi\rangle$ is invariant under time evolution, then $\hat{\mathcal{U}}[t_0, t]|\psi\rangle = |\psi\rangle$, apart from a possible phase that does not contribute to physical observables. Hence, stationary states are eigenvectors of the Hamiltonian operator. If, in addition, the condition $\langle \hat{N}_I \rangle = \langle \hat{N}_{I'} \rangle$ is satisfied, the state is called *static*.

As we show now, static states give rise to static configurations. To prove this, we first observe that the Hamiltonian can be defined as

$$\hat{H} := \int_{\Sigma_t} \hat{T}_{\mu\nu} k^\mu n^\nu d\gamma, \tag{A2}$$

with n_ν and $d\gamma$ given as in Eq. (4), and $k = \partial_t$ the timelike Killing vector field associated with the static symmetry. Taking Σ_t to be a $t = \text{const}$ hypersurface, such that $k^\nu = \alpha n^\nu$, and using Eqs. (19), (26), (28), (29), (30), and (32) and integration by parts, we obtain

$$\hat{H} = \sum_I \hat{N}_I \omega_I. \tag{A3}$$

In particular, it follows that each of the states $|N_1, N_2, \ldots\rangle$ is an eigenfunction of $\hat{H}$ with energy $E = \sum_I N_I \omega_I$. Any stationary state can be written as a superposition of eigenfunctions of the form $|N_1, N_2, \ldots\rangle$ which have the same definite value of the energy E (note that if E is degenerate this superposition might contain several such eigenfunctions.) Since $\omega_I > 0$ for each I, it follows that $\hat{a}_I \hat{a}_J |N_1, N_2, \ldots\rangle$ is either zero or is an eigenfunction of $\hat{H}$ with energy smaller than E. This implies that $\langle \hat{a}_I \hat{a}_J \rangle = 0$ for all I, J. Similarly, $\hat{a}_I |N_1, N_2, \ldots\rangle$ is either zero or is an eigenfunction of $\hat{H}$ with energy $E - \omega_I$, which implies that $\langle \hat{a}_I^\dagger \hat{a}_J \rangle = 0$ if $\omega_I \neq \omega_J$. These conditions imply that ρ, j_k, and S_{ij} are time independent. Therefore, if in addition the state is static, we conclude that the configuration is static.

An interesting question is whether or not static configurations can only be sourced by static states. For a real scalar field, we have not been able to obtain a particular realization of a nonstatic state that is associated with a static configuration. However, in the next Appendix, we provide such a counterexample in the complex scalar field theory.

APPENDIX B: COMPLEX SCALAR FIELDS

In this Appendix, we generalize the static, spherically symmetric, semiclassical real EKG system to the case of a complex scalar field. The main interest of this extension is that, as we are going to demonstrate below (and in stark contrast to the real case), when the field is complex, there are coherent states which are compatible with a static spacetime background.

The construction of the semiclassical, complex EKG system is very similar to that presented in the main text, so for the sake of simplicity, we will only stress the main differences with respect to the real case. When expressed in terms of the creation ($\hat{a}_I^\dagger$, $\hat{b}_I^\dagger$) and annihilation ($\hat{a}_I$, $\hat{b}_I$) operators, a complex scalar field takes the form

$$\hat{\phi}(x) = \sum_I [\hat{a}_I f_I(x) + \hat{b}_I^\dagger f_I^*(x)], \tag{B1}$$

where $f_1, f_2, \ldots$ is an orthonormal set of basis functions of the subspace X_+, and the creation and annihilation operators satisfy the standard commutation relations $[\hat{a}_I, \hat{a}_J^\dagger] = [\hat{b}_I, \hat{b}_J^\dagger] = \delta_{IJ}$, with all other possible commutators vanishing. The Hilbert space $\mathscr{H}$ can be constructed by successive application of creation operators on the vacuum state. The elements of this construction $|N_1^a, \ldots, N_K^a, \ldots, N_1^b, \ldots, N_K^b, \ldots\rangle$ are the eigenstates of the particle and antiparticle number operators $\hat{N}_I^a = \hat{a}_I^\dagger \hat{a}_I$ and $\hat{N}_I^b = \hat{b}_I^\dagger \hat{b}_I$, respectively. As in the real case, the coherent states are defined as the eigenstates of the annihilation operators $\hat{a}_I$ and $\hat{b}_I$. Note that the main difference with respect to the real field is that the complex theory contains particles and antiparticles, although they are indistinguishable unless we introduce interactions with other fields that break the degeneracy, which is not the case here.

The stress energy-momentum tensor operator associated with a complex scalar field takes the form

$$\hat{T}_{\mu\nu} = (\nabla_\mu \hat{\phi})(\nabla_\nu \hat{\phi})^\dagger + (\nabla_\nu \hat{\phi})(\nabla_\mu \hat{\phi})^\dagger - g_{\mu\nu}[(\nabla_\alpha \hat{\phi})(\nabla^\alpha \hat{\phi})^\dagger + {m_0}^2 \hat{\phi}\hat{\phi}^\dagger]. \tag{B2}$$

Introducing the decomposition (B1) into (B2), we obtain

$$\hat{T}_{\mu\nu} = \sum_{I,J} [\hat{a}_I \hat{a}_J^\dagger T_{\mu\nu}(f_I, f_J^*) + \hat{a}_I \hat{b}_J T_{\mu\nu}(f_I, f_J) + \hat{b}_I^\dagger \hat{a}_J^\dagger T_{\mu\nu}(f_I^*, f_J^*) + \hat{b}_I^\dagger \hat{b}_J T_{\mu\nu}(f_I^*, f_J)], \tag{B3}$$

with $T_{\mu\nu}(f_I, f_J)$ as in Eq. (20).

If the spacetime is static, the energy density (35a), momentum flux (35c), and spatial stress tensor (35b) take the form

$$\begin{aligned}
\rho = \sum_{I,J} \frac{1}{2\sqrt{\omega_I \omega_J}} \Bigg\{ &\langle \hat{a}_I^\dagger \hat{a}_J \rangle \left[\left(+ \frac{\omega_I \omega_J}{\alpha^2} + {m_0}^2 \right) u_I^* u_J + (D_k u_I)(D^k u_J^*) \right] e^{+i(\omega_I - \omega_J)t} \\
&+ \langle \hat{a}_I \hat{b}_J \rangle \left[\left(- \frac{\omega_I \omega_J}{\alpha^2} + {m_0}^2 \right) u_I u_J + (D_k u_I)(D^k u_J) \right] e^{-i(\omega_I + \omega_J)t} \\
&+ \langle \hat{b}_I^\dagger \hat{a}_J^\dagger \rangle \left[\left(- \frac{\omega_I \omega_J}{\alpha^2} + {m_0}^2 \right) u_I^* u_J^* + (D_k u_I^*)(D^k u_J^*) \right] e^{+i(\omega_I + \omega_J)t} \\
&+ \langle \hat{b}_I^\dagger \hat{b}_J \rangle \left[\left(+ \frac{\omega_I \omega_J}{\alpha^2} + {m_0}^2 \right) u_I^* u_J + (D_k u_I^*)(D^k u_J) \right] e^{+i(\omega_I - \omega_J)t} \Bigg\},
\end{aligned} \tag{B4a}$$

$$
\begin{aligned}
j_k = \sum_{I,J} \frac{1}{2\sqrt{\omega_I \omega_J}} \frac{i}{\alpha} \{ & \langle \hat{a}_I^\dagger \hat{a}_J \rangle [-\omega_J (D_k u_I^*) u_J + \omega_I u_I^* (D_k u_J)] e^{-i(\omega_I - \omega_J)t} \\
& + \langle \hat{a}_I \hat{b}_J \rangle [-\omega_J (D_k u_I) u_J - \omega_I u_I (D_k u_J)] e^{-i(\omega_I + \omega_J)t} + \langle \hat{b}_I^\dagger \hat{a}_J^\dagger \rangle [+\omega_J (D_k u_I^*) u_J^* + \omega_I u_I^* (D_k u_J^*)] e^{+i(\omega_I + \omega_J)t} \\
& + \langle \hat{b}_I^\dagger \hat{b}_J \rangle [-\omega_J (D_k u_I^*) u_J + \omega_I u_I^* (D_k u_J)] e^{+i(\omega_I - \omega_J)t} \},
\end{aligned} \tag{B4b}
$$

and

$$
\begin{aligned}
S_{ij} = \sum_{I,J} \frac{1}{2\sqrt{\omega_I \omega_J}} \Bigg\{ & \langle \hat{a}_I^\dagger \hat{a}_J \rangle \left[(D_i u_I^*)(D_j u_J) + (D_j u_I^*)(D_i u_J) - \gamma_{ij} \left(\left(-\frac{\omega_I \omega_J}{\alpha^2} + m_0{}^2 \right) u_I^* u_J + (D_k u_I^*)(D^k u_J) \right) \right] e^{+i(\omega_I - \omega_J)t} \\
& + \langle \hat{a}_I \hat{b}_J \rangle \left[(D_i u_I)(D_j u_J) + (D_j u_I)(D_i u_J) - \gamma_{ij} \left(\left(+\frac{\omega_I \omega_J}{\alpha^2} + m_0{}^2 \right) u_I u_J + (D_k u_I)(D^k u_J) \right) \right] e^{-i(\omega_I + \omega_J)t} \\
& + \langle \hat{b}_I^\dagger \hat{a}_J^\dagger \rangle \left[(D_i u_I^*)(D_j u_J^*) + (D_j u_I^*)(D_i u_J^*) - \gamma_{ij} \left(\left(+\frac{\omega_I \omega_J}{\alpha^2} + m_0{}^2 \right) u_I^* u_J^* + (D_k u_I^*)(D^k u_J^*) \right) \right] e^{+i(\omega_I + \omega_J)t} \\
& + \langle \hat{b}_I^\dagger \hat{b}_J \rangle \left[(D_i u_I^*)(D_j u_J) + (D_j u_I^*)(D_i u_J) - \gamma_{ij} \left(\left(+\frac{\omega_I \omega_J}{\alpha^2} + m_0{}^2 \right) u_I^* u_J + (D_k u_I^*)(D^k u_J) \right) \right] e^{-i(\omega_I - \omega_J)t} \Bigg\},
\end{aligned}
$$

where we have used Wick's ordering to obtain these expressions. Comparing them with those of Eq. (A1), we find that, apart from an overall factor of 2, they differ in some of the expectation values which are quadratic in creation and annihilation operators. Of course, if we choose a state with a definite number of particles, for which $\langle \hat{a}_I^\dagger \hat{a}_J \rangle = N_I \delta_{IJ}$ and $\langle \hat{a}_I \hat{b}_J \rangle = \langle \hat{a}_I^\dagger \hat{b}_J^\dagger \rangle = \langle \hat{b}_I^\dagger \hat{b}_J \rangle = 0$, Eqs. (A1) and (B4) coincide, and we recover the relations in Eqs. (36). The same holds true for a state with a definite number of antiparticles. Furthermore, it is even possible to have a state with a definite number of particles *and* antiparticles simultaneously; in that case, we only have to replace N_I by $N_I^a + N_I^b$ in Eqs. (36). This is because the stress energy-momentum tensor sources gravity, and this tensor does not differentiate between particles and antiparticles. However, if the field is complex, it is still possible to guarantee a static source even if the system is not in an eigenstate of the Hamiltonian operator. This is what happens for coherent states that involve a single mode I of the particle or antiparticle sector. For these states, $\langle \hat{a}_I^\dagger \hat{a}_J \rangle = |\alpha_I|^2 \delta_{IJ}$, $\langle \hat{a}_I \hat{b}_J \rangle = \langle \hat{b}_I^\dagger \hat{a}_J^\dagger \rangle = \langle \hat{b}_I^\dagger \hat{b}_J \rangle = 0$ (where in this case we have assumed that we only have particles), and it is easy to convince oneself that we recover Eqs. (36) with N_I replaced by $|\alpha_I|^2$ and no sum over I. Note that this is not possible in the real case, where the coefficients $\langle \hat{a}_I \hat{a}_J \rangle = (\alpha_i)^2 \delta_{IJ}$ and $\langle \hat{a}_I^\dagger \hat{a}_J^\dagger \rangle = (\alpha_I^*)^2 \delta_{IJ}$ of a coherent state give rise to a time dependency of the source terms that is not compatible with a static metric. The main difference of these solutions with respect to those with a definite number of particles is that the expectation value of the scalar field evolves nontrivially in time.

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