Electric field-based quantization of the gauge invariant Proca theory

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We consider the gauge invariant version of the Proca theory, where besides the real vector field there is also the real scalar field. We quantize the theory such that the commutator of the scalar field operator and the electric field operator is given by a predefined three-dimensional vector field, say $\mathcal{E}$ up to a global prefactor. This happens when the field operators of the gauge invariant Proca theory satisfy the proper gauge constraint. In particular, we show that $\mathcal{E}$ given by the classical Coulomb field leads to the Coulomb gauge constraint making the vector field operator divergenceless. We also show that physically unreadable gauge constraints can have a strikingly simple $\mathcal{E}$-representation in our formalism. This leads to the discussion of Debye, Yukawa, etc. gauges. In general terms, we explore the mapping between classical vector fields and gauge constraints imposed on the operators of the studied theory.

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I. INTRODUCTION

The Proca theory delivers the simplest relativistic description of massive vector bosons [1,2]. As a result of that, it is of both phenomenological and theoretical interest.

In the phenomenological context, it captures some properties of ρ and ω mesons and the particles mediating weak interactions, W and Z bosons [1]. In addition to that, it is regarded as a promising extension of Maxwell's electrodynamics, the one taking into account the possibility that the photon may not be a massless particle after all. Thereby, various upper bounds on the photon mass are obtained by comparing the predictions of the Proca theory to actual experimental data (see e.g. Refs. [2,3] extensively discussing this physically rich topic). In the theoretical context, which is of main interest in this work, the Proca theory provides an elegant framework for the examination of various issues associated with the quantization of vector fields (see e.g. Refs. [1,4,5]).

We are interested in the Proca theory of the real vector field. Its classical Lagrangian density can be written as

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + \frac{m^2}{2}(V_\mu)^2, \qquad (1)$$

where V^μ is the vector field and m is the mass of spin-1 particles described by this theory after its quantization (see the Appendix for our conventions).

The important thing now is that theory (1) is manifestly noninvariant with respect to the gauge transformation. In fact, it is a gauge-fixed theory in the sense that field equations impose the Lorenz gauge constraint onto the vector field. This state of affairs can be easily changed by the replacement

$$V_\mu \to A_\mu + \frac{1}{e}\partial_\mu G, \qquad (2)$$

where the vector field A^μ and the real scalar field G are supposed to simultaneously change under the gauge transformation. Namely,

$$A_\mu \to A_\mu + \partial_\mu f, \qquad G \to G - ef, \qquad (3)$$

where f is a smooth real function of space-time coordinates and e is the unit of the electric charge.

Imposing (2) on (1), we see that the resulting Lagrangian density,

$$\mathcal{L}' = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{m^2}{2}\left(A_\mu + \frac{1}{e}\partial_\mu G\right)^2, \qquad (4)$$

is unaffected by the gauge transformation. For this reason, we will refer to the theory defined by (4) as the gauge invariant (GI) Proca theory. Such a theory was studied before, see e.g. Refs. [6,7], and it bears similarity to the Stueckelberg theory, which is reviewed in Ref. [8].

To proceed with the discussion of the GI Proca theory, one has to choose a gauge because the vector field is no longer Lorenz gauge fixed in (4). Besides the standard Coulomb gauge choice, which was e.g. enforced with the Lagrange multiplier technique in Ref. [6], the following intriguing gauge constraint was introduced in Ref. [7]

$$e\nabla \cdot \boldsymbol{A}_D = m^2 G_D. \tag{5}$$

It was labeled as the Coulomb gauge choice [7], but the rationale behind such a name was not provided. We believe that a proper name for such a gauge could be the Debye gauge, which will be carefully explained in this work. Anticipating this discussion, we have labeled the fields subjected to such a constraint with the appropriate subscript. Their quantization was studied in Ref. [7] by means of the Faddeev-Jackiw approach [9].

Our goal is to develop and discuss the quantization formalism, where gauge choices are labeled by the classical vector field $\boldsymbol{\mathcal{E}}$, which determines the commutator of the scalar and electric field operators. Thereby, we explore the mapping between such $\boldsymbol{\mathcal{E}}$ and the field operators of the GI Proca theory.

The outline of this paper is the following. The concise summary of basic results concerning the Proca theory is provided in Sec. II. Next, our quantization procedure is introduced in Sec. III. Its features are then discussed in Sec. IV, where the electric field context of the proposed approach is laid out along with several illustrative examples. Finally, the summary of our work is presented in Sec. V, which is followed by the Appendix listing our conventions.

II. BASICS

We state below basic results concerning theories (1) and (4).

To begin, the independent variables of Proca theory (1) are fields V^i and their canonical conjugates

$$\pi_i = \partial_i V_0 - \partial_0 V_i. \tag{6}$$

Such a theory is canonically quantized by demanding that [1,4,5]

$$[V^i(t,\boldsymbol{x}), \pi^j(t,\boldsymbol{y})] = -\mathrm{i}\delta^{ij}\delta(\boldsymbol{y}-\boldsymbol{x}), \tag{7}$$

$$[V^i(t,\boldsymbol{x}), V^j(t,\boldsymbol{y})] = [\pi^i(t,\boldsymbol{x}), \pi^j(t,\boldsymbol{y})] = 0. \tag{8}$$

We note that

$$V^0 = -\frac{1}{m^2}\nabla \cdot \boldsymbol{\pi}, \tag{9}$$

which explains why V^0 is the dependent variable of theory (1). We also note that the canonical conjugate of V^0 vanishes.

Then, we remark that the variables of GI Proca theory (4), whose quantization will be discussed in Sec. III, are fields A^i and G as well as their canonical conjugates

$$\partial_i A_0 - \partial_0 A_i = \pi_i \tag{10}$$

and

$$\tilde{\pi} = \frac{m^2}{e}\left(A_0 + \frac{1}{e}\partial_0 G\right) = \frac{m^2}{e} V_0, \tag{11}$$

respectively. We note that the right-hand sides of (10) and (11) follow from mapping (2), which we assume in this work.

Finally, we have a few observations about $\boldsymbol{\pi}$ and $\tilde{\pi}$. First, Eqs. (9) and (11) imply that $\boldsymbol{\pi}$ and $\tilde{\pi}$ are linked via the *field constraint* [10]

$$\nabla \cdot \boldsymbol{\pi} = -e\tilde{\pi}. \tag{12}$$

Second, $\boldsymbol{\pi}$ and $\tilde{\pi}$ are gauge invariant. This means that, unlike A and G, they will not be equipped with a gauge-specific subscript below. Third, the physical content of $\boldsymbol{\pi}$, and so also of $\tilde{\pi}$ due to (12), is best seen from the fact that $\boldsymbol{\pi} = \boldsymbol{E}$, where $\boldsymbol{E} = -\partial_0 \boldsymbol{V} - \nabla V^0 = -\partial_0 \boldsymbol{A} - \nabla A^0$ is the electric field operator. Note that we use the same "electric field" terminology as in the theory of the massless electromagnetic field.

III. GAUGE ANSATZ AND COMMUTATION RELATIONS

We are interested in quantization of theory (4). In a nutshell, one may approach this problem in the following way.

To begin, one chooses the gauge constraint for the fields. For example, one may decide to work in the Coulomb gauge

$$\nabla \cdot \boldsymbol{A}_C = 0, \tag{13}$$

where the subscript indicates the gauge choice. Naturally, there are uncountably many other gauge choices, whose implications are not so obvious [see e.g. Eq. (5)].

Then, one figures out commutation relations between the fields and their canonical conjugates, which is a nontrivial task. Indeed, as they have to be consistent with the chosen gauge constraint, they are expected to differ from the canonical commutation relations.

We approach quantization of theory (4) somewhat differently. Namely, instead of imposing the specific gauge constraint in the form of the equation for the vector and scalar field operators, we require that

$$G_{\mathcal{E}}(t,\boldsymbol{x}) = e\int d^3z\, \boldsymbol{V}(t,\boldsymbol{z}) \cdot \boldsymbol{\mathcal{E}}(\boldsymbol{z}-\boldsymbol{x}), \tag{14a}$$

$$\boldsymbol{A}_{\mathcal{E}} = \boldsymbol{V} + \frac{1}{e}\boldsymbol{\nabla} G_{\mathcal{E}}, \tag{14b}$$

where $\boldsymbol{\mathcal{E}}$ is a time-independent $\mathbb{R}^3$-valued vector field and the appropriate subscript has been added to the fields to indicate their dependence on $\boldsymbol{\mathcal{E}}$. Equation (14a) can be seen as the ansatz, whereas Eq. (14b) expresses the fact that we rely on mapping (2), which also leads to

$$A^0_{\mathcal{E}} = \frac{e}{m^2}\tilde{\pi} - \frac{1}{e}\partial_0 G_{\mathcal{E}}. \tag{15}$$

All together, we will refer to (14) as the *gauge ansatz*.

The field $\boldsymbol{\mathcal{E}}$, whose meaning will be discussed in Sec. IV, defines the gauge in our formalism. In fact, it is easy to see that under $\boldsymbol{\mathcal{E}} \to \boldsymbol{\mathcal{E}}'$, $G_{\mathcal{E}}$ and $A_{\mathcal{E}}$ transform just as G and A in (3) with

$$f(t,\boldsymbol{x}) = \int d^3z\, \boldsymbol{V}(t,\boldsymbol{z}) \cdot [\boldsymbol{\mathcal{E}}(\boldsymbol{z}-\boldsymbol{x}) - \boldsymbol{\mathcal{E}}'(\boldsymbol{z}-\boldsymbol{x})]. \tag{16}$$

This time, however, f is operator valued. This is interesting because classical (c-number) gauge transformations are typically discussed in the context of gauge theories (see e.g. Sec. 2.5.2 of Ref. [11] for relevant remarks).

We are now ready to discuss equal-time commutators between the canonically related operators introduced in Sec. II. The nontrivial ones are

$$[G_{\mathcal{E}}(t,\boldsymbol{x}), \tilde{\pi}(t,\boldsymbol{y})] = \mathrm{i}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{y}-\boldsymbol{x}), \tag{17}$$

$$[A^i_{\mathcal{E}}(t,\boldsymbol{x}), \tilde{\pi}(t,\boldsymbol{y})] = \frac{\mathrm{i}}{e}\partial^y_i[\delta(\boldsymbol{y}-\boldsymbol{x}) - \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{y}-\boldsymbol{x})], \tag{18}$$

$$[G_{\mathcal{E}}(t,\boldsymbol{x}), \pi^j(t,\boldsymbol{y})] = -\mathrm{i}e\mathcal{E}^j(\boldsymbol{y}-\boldsymbol{x}), \tag{19}$$

$$[A^i_{\mathcal{E}}(t,\boldsymbol{x}), \pi^j(t,\boldsymbol{y})] = -\mathrm{i}\delta^{ij}\delta(\boldsymbol{y}-\boldsymbol{x}) + \mathrm{i}\partial^y_i\mathcal{E}^j(\boldsymbol{y}-\boldsymbol{x}), \tag{20}$$

where $\partial^y_i = \partial/\partial y^i$. These expressions trigger the following comments.

First, in order to verify these commutators, one can replace $G_{\mathcal{E}}$ and $A^i_{\mathcal{E}}$ in (17)–(20) with (14) and then use (7) to simplify the resulting expressions. Similarly, one may verify with the help of (8) that the remaining equal-time commutators between $G_{\mathcal{E}}$, $\boldsymbol{A}_{\mathcal{E}}$, $\boldsymbol{\pi}$, and $\tilde{\pi}$ identically vanish.

Second, we find these commutators remarkably compact and general. As expected, they do differ from canonical commutation relations: Eq. (17) is not equal to $\mathrm{i}\delta(\boldsymbol{y}-\boldsymbol{x})$, Eq. (20) is not equal to $-\mathrm{i}\delta^{ij}\delta(\boldsymbol{y}-\boldsymbol{x})$, and Eq. (18) as well as Eq. (19) do not vanish. The structure of (17)–(20) stems from the restrictions imposed by field constraint (12) and gauge ansatz (14); see Sec. IV C for additional relevant remarks. In particular, one may easily notice that (17) and (19) are interrelated via (12). The same remark applies to (18) and (20).

Third, we have independently verified the above results in the two already introduced gauges, (5) and (13), where $\boldsymbol{\mathcal{E}}$ is given by (29) evaluated for $M = m$ and (27), respectively. We have done it via the Dirac bracket quantization technique adopted so as to enforce gauge constraints (5) and (13) (see Ref. [5] for the textbook introduction to such a quantization approach and Sec. IV A for the explanation of the above-listed choices of $\boldsymbol{\mathcal{E}}$).

IV. ELECTRIC FIELD PERSPECTIVE ON GAUGE ANSATZ

The quantum GI Proca theory is built of the vector field $A_{\mathcal{E}}$ and the scalar field $G_{\mathcal{E}}$. The role of $A_{\mathcal{E}}$ is clear: the electric and magnetic field operators are expressed in terms of $A_{\mathcal{E}}$, and so in such a sense this operator captures physics of the electromagnetic field. The question now is what is the role of $G_{\mathcal{E}}$. At first sight, it seems that the only role of $G_{\mathcal{E}}$ is to enforce the gauge invariance of the Lagrangian density. However, by looking at commutator (19), we realize that $G_{\mathcal{E}}$ also plays the role of the generator of the local shift of the electric field operator. To explain what we mean by saying so, we note that by combining (19) with the following well-known identity,

$$\exp(X)Y\exp(-X) = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \cdots, \tag{21}$$

it can be formally shown that

$$\exp[\mathrm{i}G_{\mathcal{E}}(t,\boldsymbol{x})]\boldsymbol{E}(t,\boldsymbol{y})\exp[-\mathrm{i}G_{\mathcal{E}}(t,\boldsymbol{x})] = \boldsymbol{E}(t,\boldsymbol{y}) + e\boldsymbol{\mathcal{E}}(\boldsymbol{y}-\boldsymbol{x}). \tag{22}$$

As both (19) and (22) particularly clearly expose the electric field context of $\boldsymbol{\mathcal{E}}$, we see the quantization procedure based on (14) as the electric field-based quantization scheme. Two remarks are in order now.

First, we use the term formal when we refer to (22) because we do not actually inquire if the operator $\exp[\pm\mathrm{i}G_{\mathcal{E}}(t,\boldsymbol{x})]$ is well defined. Second, we note that in the spirit of the Helmholtz theorem [12], one may consider the following decomposition of $\boldsymbol{\mathcal{E}}$,

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\Phi_{\mathcal{E}} + \boldsymbol{\nabla}\times\boldsymbol{F}_{\mathcal{E}}, \tag{23}$$

where $\Phi_{\mathcal{E}}$ and $\boldsymbol{F}_{\mathcal{E}}$ are classical time-independent scalar and vector fields, respectively. Formula (23) will guide our subsequent discussion.

A. Curl-free $\boldsymbol{\mathcal{E}}$

We study here gauges induced by

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\Phi_{\mathcal{E}}, \tag{24}$$

where $\Phi_{\mathcal{E}}$ is real valued.

To begin, we address the question of what is the relation between $G_{\mathcal{E}}$ and $\boldsymbol{A}_{\mathcal{E}}$ when (24) holds. After standard manipulations based on gauge ansatz (14), we find that

$$e\boldsymbol{\nabla}\cdot\boldsymbol{A}_{\mathcal{E}} = f_{\mathcal{E}}(-\mathrm{i}\boldsymbol{\nabla})G_{\mathcal{E}} + \Delta G_{\mathcal{E}}, \tag{25}$$

where $f_{\mathcal{E}}$ is defined via

$$\Phi_{\mathcal{E}}(\boldsymbol{r}) = \int \frac{d^3k}{(2\pi)^3}\frac{\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}{f_{\mathcal{E}}(\boldsymbol{k})} \tag{26}$$

and $f_{\mathcal{E}}(\boldsymbol{k}) = f^*_{\mathcal{E}}(-\boldsymbol{k})$ to ensure the real value of the above integral. We will refer to (25) as the *gauge constraint* to distinguish it from field constraint (12) and gauge ansatz (14). The *formal* character of (25) will be commented upon in Sec. V. We are now ready to discuss the previously mentioned Coulomb and Debye gauges.

We say that $\boldsymbol{\mathcal{E}}$ induces the Coulomb gauge when

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\Phi_C, \qquad \Phi_C = \frac{1}{4\pi r}, \tag{27}$$

where $\boldsymbol{\nabla} = (\partial/\partial r^i)$ and $r = |\boldsymbol{r}|$. Such a terminology is supported by two observations. First, it is natural in our formalism because such $\boldsymbol{\mathcal{E}}$ is given by the negative gradient of the Coulomb potential originating from the unit charge. Second, a simple calculation shows that $f_{\mathcal{E}}(-\mathrm{i}\boldsymbol{\nabla}) = -\Delta$ here, which leads to the divergenceless vector field via (25). Properly labeling the fields, we have

$$(G_C, \boldsymbol{A}_C) = (G_{\mathcal{E}}, \boldsymbol{A}_{\mathcal{E}}) \quad \text{for} \quad \boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\Phi_C, \tag{28}$$

where the vector field satisfies gauge constraint (13) in the traditional nomenclature.

In full analogy to the above reasoning, the gauge induced by

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\Phi_D, \qquad \Phi_D = \frac{\exp(-Mr)}{4\pi r} \tag{29}$$

will be called the Debye gauge ($M > 0$). We have proposed this name because such $\boldsymbol{\mathcal{E}}$ is given by the negative gradient of the Debye potential describing the screening of the unit charge in plasmas and electrolytes.

As far as the relation between $G_{\mathcal{E}}$ and $\boldsymbol{A}_{\mathcal{E}}$ is concerned, we find $f_{\mathcal{E}}(-\mathrm{i}\boldsymbol{\nabla}) = -\Delta + M^2$ in the Debye gauge. Then, it follows from (25) that the fields in such a gauge satisfy

$$e\boldsymbol{\nabla}\cdot\boldsymbol{A}_D = M^2 G_D. \tag{30}$$

Note that previously stated gauge constraint (5) is the $M = m$ version of (30).

Next, we observe that the gauge constraint satisfied by the fields nontrivially depends on the magnitude and the direction of $\boldsymbol{\mathcal{E}}$ (the magnitude and the sign of $\Phi_{\mathcal{E}}$). This can be illustrated by the introduction of the following two gauges.

We define the primed Coulomb gauge by saying that it is induced in our formalism by

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\Phi_{C'}, \qquad \Phi_{C'} = \beta\Phi_C = \frac{\beta}{4\pi r}, \tag{31}$$

where $\beta > 0$. The sensitivity of the gauge constraint to the change of the magnitude of $\boldsymbol{\mathcal{E}}$ is now seen by comparing (13) to

$$e\boldsymbol{\nabla}\cdot\boldsymbol{A}_{C'} = \frac{\beta - 1}{\beta}\Delta G_{C'}, \tag{32}$$

which is satisfied by the fields in the primed Coulomb gauge.

Furthermore, we consider the gauge induced by

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\Phi_Y, \qquad \Phi_Y = -\Phi_D = -\frac{\exp(-Mr)}{4\pi r}, \tag{33}$$

where the subscript refers to the fact that such $\boldsymbol{\mathcal{E}}$ is given by the negative gradient of the Yukawa potential obtained for the unit strength of the internucleon interactions. The fields in so defined Yukawa gauge satisfy

$$e\boldsymbol{\nabla}\cdot\boldsymbol{A}_Y = 2\Delta G_Y - M^2 G_Y. \tag{34}$$

The difference between (30) and (34) illustrates the sensitivity of the gauge constraint to the global change of the direction of $\boldsymbol{\mathcal{E}}$.

Moving on, we note that new gauges can be obtained by superposing fields $\boldsymbol{\mathcal{E}}$. For fields $\boldsymbol{\mathcal{E}}$ given by (24), this typically leads to the complicated relation between $G_{\mathcal{E}}$ and $\boldsymbol{A}_{\mathcal{E}}$ due to the reciprocal additivity law for $f_{\mathcal{E}}$. Namely, if

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\Phi_{\mathcal{E}'} - \boldsymbol{\nabla}\Phi_{\mathcal{E}''} - \cdots, \tag{35}$$

then

$$\frac{1}{f_{\mathcal{E}}} = \frac{1}{f_{\mathcal{E}'}} + \frac{1}{f_{\mathcal{E}''}} + \cdots. \tag{36}$$

This can be illustrated by the consideration of the Coulomb-Yukawa gauge, which we define as the gauge induced by

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\Phi_C - \boldsymbol{\nabla}\Phi_Y = -\boldsymbol{\nabla}\left(\frac{1}{4\pi r} - \frac{\exp(-Mr)}{4\pi r}\right). \tag{37}$$

A quick calculation shows that in this case $f_{\mathcal{E}}(-\mathrm{i}\boldsymbol{\nabla}) = (\Delta/M)^2 - \Delta$, which results in

$$e\boldsymbol{\nabla}\cdot\boldsymbol{A}_{CY} = \frac{1}{M^2}\Delta(\Delta G_{CY}). \tag{38}$$

Note that such a gauge constraint resembles neither (13) nor (34) despite the fact that it is induced by the superposition of the fields $\mathcal{E}$ leading to the Coulomb and Yukawa gauges. This is the consequence of the fact that $f_{\mathcal{E}} \neq f_{\mathcal{E}'} + f_{\mathcal{E}''} + \cdots$ when (35) holds. We mention in passing that the $M = m$ version of the operator G_{CY} was used in Ref. [13] to construct the finite-energy charged state in Proca theory (1).

Finally, we note the trivial possibility of choosing $\mathcal{E} = \mathbf{0}$. This sets $G_{\mathcal{E}} = 0$, removing the scalar field from the theory. Such a gauge choice is known in the literature as the unitary gauge (see e.g. Ref. [14]). In our formalism, the term null gauge seems to be more appropriate.

B. Divergence-free $\mathcal{E}$

We briefly comment here upon gauges induced by

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\nabla} \times \boldsymbol{F}_{\mathcal{E}}, \tag{39}$$

where $\boldsymbol{F}_{\mathcal{E}}$ is $\mathbb{R}^3$ valued.

For a general function $\boldsymbol{F}_{\mathcal{E}}$, we are unsure how to derive the closed-form expression for the gauge constraint akin to (25). Thus, we focus on the specific results inspired by the discussion from Sec. IVA. Namely, we consider

$$\boldsymbol{F}_{\mathcal{E}}(\boldsymbol{r}) = \boldsymbol{d} \int \frac{d^3k}{(2\pi)^3} \frac{\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}{g_{\mathcal{E}}(\boldsymbol{k})}, \tag{40}$$

where $\boldsymbol{d} \in \mathbb{R}^3$ is the constant vector and $g_{\mathcal{E}}(\boldsymbol{k}) = g^*_{\mathcal{E}}(-\boldsymbol{k})$. It can be then found via (14) that the fields of the GI Proca theory satisfy the following formal gauge constraint:

$$e\boldsymbol{d}\cdot(\boldsymbol{\nabla} \times \boldsymbol{A}_{\mathcal{E}}) = g_{\mathcal{E}}(-\mathrm{i}\boldsymbol{\nabla})G_{\mathcal{E}}. \tag{41}$$

To see how all this works in practice, one may choose $\boldsymbol{F}_{\mathcal{E}}$ to be given by

$$\boldsymbol{d}\frac{\beta}{4\pi r}, \qquad \pm\boldsymbol{d}\frac{\exp(-Mr)}{4\pi r}, \qquad \boldsymbol{d}\left(\frac{1}{4\pi r} - \frac{\exp(-Mr)}{4\pi r}\right). \tag{42}$$

From the results presented in Sec. IVA, it is clear that these choices lead to $g_{\mathcal{E}}(-\mathrm{i}\boldsymbol{\nabla})$ equal to

$$-\frac{1}{\beta}\Delta, \qquad \pm(-\Delta + M^2), \qquad (\Delta/M)^2 - \Delta, \tag{43}$$

respectively. The corresponding gauge constraints are obtained by combining (41) with (43), the $\mathcal{E}$ fields associated with them are given by the curl of the vector fields listed in (42).

C. Gauge constraints vs commutation relations

Let us consider a gauge constraint written in the form $\Upsilon = 0$. We will say that it is consistent with equal-time commutation relations, written for the fields belonging to some set $\mathcal{X}$, when $[\Upsilon(t,\boldsymbol{x}), X(t,\boldsymbol{y})] = 0$ for all $X \in \mathcal{X}$. For example, the consistency of gauge constraint (30) with commutation relations (17)–(20) requires $[e\boldsymbol{\nabla}\cdot\boldsymbol{A}_D(t,\boldsymbol{x}) - M^2 G_D(t,\boldsymbol{x}), X(t,\boldsymbol{y})] = 0$ for $X = G_D, \boldsymbol{A}_D, \boldsymbol{\pi}, \tilde{\pi}$.

We note that it can be easily verified that gauge constraints (13), (30), (32), (34), and (38) are consistent with commutation relations (17)–(20). It goes without saying that this happens when the right-hand sides of (17)–(20) are evaluated with the corresponding fields $\mathcal{E}$: (27), (29), (31), (33), and (37), respectively. For a general curl-free $\mathcal{E}$ given by (24), it can be formally shown that gauge constraint (25) is consistent with (17)–(20).

We also note that similar self-consistency checks can be performed for the divergence-free $\mathcal{E}$ discussed in Sec. IV B. Namely, it can be shown that (41) is formally consistent with (17)–(20) when (40) holds, which can be also individually verified for the specific cases listed in (42).

V. SUMMARY

We have discussed how GI Proca theory (4) can be quantized with the help of gauge ansatz (14). Such an ansatz is parametrized by the classical vector field $\mathcal{E}$, which determines the commutator of the scalar field operator and the electric field operator (19).

In several special cases, we have found an explicit mapping between the field $\mathcal{E}$ and the gauge constraint satisfied by the fields of the GI Proca theory. In particular, we have discussed the mapping

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\left(\frac{1}{4\pi r}\right) \mapsto \boldsymbol{\nabla}\cdot\boldsymbol{A}_C = 0, \tag{44}$$

which gives a new meaning to the term Coulomb gauge, a very suggestive one in our opinion. While discussing other cases, we have found that unreadable gauge constraints can have a strikingly simple $\mathcal{E}$-representation in our formalism, which we find remarkable. One of the simplest illustrations supporting such an observation is the following:

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\nabla}\left(\frac{\exp(-Mr)}{4\pi r}\right) \mapsto e\boldsymbol{\nabla}\cdot\boldsymbol{A}_D = M^2 G_D, \tag{45}$$

which defines the Debye gauge in our nomenclature. Further support for the above observation is provided by comparing gauge constraints (32), (34), and (38) to the fields $\mathcal{E}$ associated with them (31), (33), and (37), respectively. We note that, to the best of our knowledge, none of these three gauge constraints has been previously mentioned in the literature. We also note that another batch of unusual gauge constraints, having simple $\mathcal{E}$-representation, can be obtained by combining (41) with (43).

In a more general context, we have proposed the relation between curl-free $\mathcal{E}$ given by (24) and the gauge constraint satisfied by the fields of the GI Proca theory (25). Such a result has a formal character because it involves the pseudodifferential operator $f_{\mathcal{E}}(-\mathrm{i}\boldsymbol{\nabla})$, where the function $f_{\mathcal{E}}$ can be in general nonanalytic or singular for well-defined $\mathcal{E}$. If such complications are present, then there is the question of what (25) really means. These somewhat intriguing ambiguities do not affect our gauge ansatz-based considerations (14), which do not rely on the form of the gauge constraint satisfied by the fields. Similar remarks apply to formal result (41), which has been obtained for the particular class of divergence-free $\mathcal{E}$.

Finally, we would like to emphasize the efficiency of the discussed formalism. Indeed, our quantization procedure is carried out all at once for different gauges labeled by $\mathcal{E}$. This is illustrated by the general character of commutation relations (17)–(20).

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APPENDIX: CONVENTIONS

We adopt the Heaviside-Lorentz system of units in its $\hbar = c = 1$ version. Greek and Latin indices of tensors take values 0,1,2,3 and 1,2,3, respectively. The metric signature is $(+---)$. 3-vectors are written in bold, e.g. $x = (x^\mu) = (x^0, \boldsymbol{x})$. We use the Einstein summation convention, $(X_{\mu\cdots})^2 = X_{\mu\cdots}X^{\mu\cdots}$, and $\Delta = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$. The complex conjugation is denoted as $*$.

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