Perturbative aspects of deformed Yang-Mills theory

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Center-stabilized $SU(N)$ Yang-Mills theories on $\mathbb{R}^3 \times S^1$ are QCD-like theories that can be engineered to remain weakly-coupled at all energy scales by taking the S^1 circle length L to be sufficiently small. In this regime, these theories admit effective long-distance descriptions as Abelian $U(1)^{N-1}$ gauge theories on $\mathbb{R}^3$, and semiclassics can be reliably employed to study nonperturbative phenomena such as color confinement and the generation of mass gaps in an analytical setting. At the perturbative tree level, the long-distance effective theory contains $(N-1)$ free photons with identical gauge couplings $g_3^2 \equiv g^2/L$. Vacuum-polarization effects, from integrating out heavy charged fields, lift this degeneracy to give $\lfloor \frac{N}{2} \rfloor$ distinct values, $g^2(\frac{2}{L}) \lesssim g_{3,\ell}^2 L \lesssim g^2(\frac{2\pi}{NL})$. In this work, we calculate these corrections to one-loop order in theories where the center-symmetric vacuum is stabilized by $2 \leq n_f \leq 5$ massive adjoint Weyl fermions with masses of order $m_\lambda \sim \frac{2\pi}{NL}$, (also known as "deformed Yang-Mills,") and show that our results agree with those found in previous studies in the $m_\lambda \to 0$ limit. Then, we show that our result has an intuitive interpretation as the running of the coupling in a "lattice momentum" in the context of the nonperturbative "emergent latticized fourth dimension" in the $N \to \infty$, fixed-NL limit.

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I. INTRODUCTION

Analytical methods to study the long-distance properties of four-dimensional asymptotically free non-Abelian gauge theories are few and far between; broadly speaking, it is a difficult problem to handle because the flow to strong coupling causes theoretical control over the system to be lost at low-energy scales. While there are known models that are well-behaved enough to be studied analytically, (e.g., Seiberg-Witten theory [1]) these typically require special structures such as supersymmetry, or otherwise make use of gauge-gravity duality arguments and string-inspired tools (such as in Ref. [2]).

Over the past years, studies performed on "center-stabilized" gauge theories on $\mathbb{R}^3 \times S^1$ have been remarkably fruitful for providing insight into the nonperturbative dynamics of four-dimensional gauge theories. These models are distinguished from the few known analytically-calculable models in four dimensions by the fact that they can be engineered to remain weakly coupled at all energy scales, so that a semiclassical expansion in terms of objects defined in the UV theory is reliable and self-consistent.

The basic idea behind these models is as follows: By compactifying $\mathbb{R}^4$ to $\mathbb{R}^3 \times S^1$, and "deforming" the pure Yang-Mills (YM) theory by adding a nonlocal and nonrenormalizable potential to the Lagrangian, the well-known deconfining phase transition (cf. thermal Yang-Mills [3]) at small circle lengths L can be circumvented, and the theory remains in the color-confining phase for all values of L. Adiabatic continuity to the full $\mathbb{R}^4$ theory of ultimate interest can therefore be argued on grounds that the theories share identical (nonspacetime) global symmetries for all $L \in [0, \infty]$. That is, they belong in the same "universality class" [4,5].

To be certain, the nonrenormalizable "deformed" theory that we are describing can be viewed as a lattice theory with a fixed finite lattice spacing [6]. On the other hand, it is also possible to define a UV-complete continuum theory with the same desired properties by introducing n_f S^1-periodic adjoint-representation fermion fields to the pure YM Lagrangian: The desired deformation potential is realized as the fermionic contribution to the dynamically-generated Gross-Pisarski-Yaffe (GPY) effective potential at energy scales below $\sim\frac{1}{L}$ [4,5,7,8]. If the fermions are massless,[1] this class of theories is referred to as QCD(adj) if $2 \leq n_f \leq 5$, and super Yang-Mills (SYM) if $n_f = 1$. It is

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[1]It should be noted that QCD(adj) with n_f massless fermions in its spectrum has a global chiral symmetry not shared by the $\mathbb{R}^4$ pure Yang-Mills and is therefore not covered by the aforementioned "universality class argument".

called "deformed Yang-Mills" (dYM), when the $2 \le n_f \le 5$ fermions are massive, or if the deformation potential is added "by hand," as in the lattice formulation.

From the theorist's perspective, one of the most alluring features of these admittedly artificial setups is that they admit a "weak-coupling regime" at $\Lambda NL \ll 2\pi$, (where Λ is the strong-coupling scale) in which the gauge coupling g^2 (and more pertinently, $g^2 N$) remains small at all energy scales. Thus, in this regime, the semiclassical expansion over high-energy monopole-instanton configurations is trustworthy, and can be reliably employed to study the effects of the nonperturbative physics on the low-energy theory. The result is a theoretical laboratory in which a wide variety of nonperturbative low-energy phenomena can be studied analytically. For example, color confinement, the generation of a nonperturbative mass gap [4,8], a deconfining phase transition [9,10], and certain aspects of chiral symmetry breaking [6]. For this reason, $\Lambda NL \ll 2\pi$ is sometimes also called the "calculable regime" in the context of center-stabilized $\mathbb{R}^3 \times S^1$ theories. For comprehensive reviews, see Refs. [11–13].

At leading perturbative order and finite N, the IR effective theory of $SU(N)$ dYM and QCD(adj) in the calculable regime is sometimes described as being "rather boring" [11], because the gauge sector describes $(N-1)$ free, massless photons in $\mathbb{R}^3$. When vacuum polarization effects are accounted for, the photons acquire $\lfloor \frac{N}{2} \rfloor$. These corrections have been calculated to one-loop order by various methods when the fermions are assumed to be massless; for $N = 2$, 3 QCD(adj) in Ref. [14], for SYM (i.e., $n_f = 1$) with arbitrary N in Ref. [10], and in QCD(adj) with arbitrary N in Ref. [15].

In this study, we derive a more general expression for these corrections that in particular covers the massive fermion case, for generic masses $m_1, \ldots m_{n_f}$ such that the theory remains in the center-symmetric and weak-coupling regime. The final result is contained in Eq. (2.13), and the bulk of our exposition explains how we arrive at this result. Our motivating aim is to confirm that the perturbative corrections to finite-N $SU(N)$ dYM theory yield no unpleasant surprises even when the stabilizer fermions are assumed to be heavy. This is a very reasonable assumption to make, since ultimately we are interested in obtaining insight on pure Yang-Mills on $\mathbb{R}^4$, and a continuum QCD-like theory that continues smoothly to pure YM should not contain light adjoint fermions in its IR spectrum.

Nevertheless, our results are not entirely devoid of novelty. Reference [16] showed that in the $N \to \infty$, $L \to 0$, fixed-NL limit of SYM, an emergent latticized fourth dimension appears, emerging out of the space of fields—even though we should expect that taking $L \to 0$ ought to result in a 3d theory. In particular, this emergent dimension exhibits $z = 2$ Lifschitz scaling invariance in SYM. In other words, the action is quartic, rather than quadratic, in the momentum, $\sim |\partial_y^2 \Phi|^2$, where ∂_y is the partial derivative in the emergent latticized dimension. Simply put, this is because in SYM, there is a discrete $\mathbb{Z}_N$ chiral symmetry (not to be confused with the $\mathbb{Z}_N$ center symmetry) that forbids monopole-instantons from contributing a bosonic potential of the form $\sim |\partial_y \Phi|^2$ in the semiclassical expansion. Such a potential is permitted, however, when the chiral symmetry is explicitly broken by a nonzero fermion mass, as is in the case we study here. We find a satisfying and intuitive interpretation of our massive correction in this emergent dimension as the flow towards *strong* coupling for *large* values of the "lattice momentum."

The rest of this paper is structured as follows: Section II contains an overview of the essentials of dYM theory in an effort to make this paper more self-contained. For the benefit of the impatient reader, we have placed our main result, Eq. (2.13) and its accompanying discussion, in Sec. II A 1. A discussion of this result in the context of the emergent latticized dimension of Ref. [16] is contained in Sec. II B 1.

Section III covers the derivation of Eq. (2.13) in detail, starting from the very beginning with the UV dYM Lagrangian. Since this calculation is fairly long and convoluted, we briefly summarize what we have done at the end of Secs. III A and III B to help the reader keep track of our progress. The main "meat" of the calculation, and therefore of this paper, is mostly contained in Sec. III C, especially Sec. III C 2.

In our calculation, we use the Mellin transform to rewrite certain infinite sums in a form that allows their asymptotic behavior to be more easily seen. The details of this manipulation, which is mostly just complex analysis in one variable, is given in the Appendix.

II. BACKGROUND, RESULTS AND DISCUSSION

A. Review of dYM: Perturbative aspects

Consider pure $SU(N)$ Yang-Mills theory on compactified $\mathbb{R}^3 \times S^1$, where the S^1 is a circle of circumference L,

$$S_{\mathrm{YM}}[\mathcal{A}, \mathcal{F}] = \int_{\mathbb{R}^3 \times S^1} \frac{1}{2g^2} \mathrm{tr}(\mathcal{F}^2). \tag{2.1}$$

This theory enjoys a global $\mathbb{Z}_N = Z(SU(N))$ center symmetry, as it only contains fields transforming in the adjoint representation of the gauge group. The action of this symmetry may be thought of as a "gauge"[2] transformation $g(x^\mu, x^4)\colon \mathbb{R}^3 \times S^1 \to SU(N)$ that is periodic over the S^1 modulo a $\mathbb{Z}_N$ factor,

[2]Though, of course, the $g(x^\mu, x^4)$ so defined is not a true gauge transformation by any means. That is, it is not a transition function between local trivializations of the principle bundle.

$$g(x^\mu, 0) = \omega g(x^\mu, L), \qquad \omega \equiv e^{i2\pi/N}. \tag{2.2}$$

This acts on the fundamental representation Polyakov loop Ω, the gauge holonomy along the S^1,

$$\Omega \equiv \mathcal{P} \exp i \int_0^L dx^4 \mathcal{A}_4, \tag{2.3}$$

as.[3]

$$\mathbb{Z}_N\colon \ \mathrm{tr}\Omega \to \omega \mathrm{tr}\Omega, \tag{2.4}$$

where $\mathcal{A}_4$ is the S^1 part of the gauge field $\mathcal{A}$.

At large L, the center symmetry is unbroken. That is to say, $\langle \mathrm{tr}\Omega^n \rangle = 0$ in the ground state for all $n \neq 0 \bmod N$. On the other hand, it is a well-known fact [3] that in the small-L limit, the theory undergoes a deconfining phase transition associated with the breaking of center symmetry. In this regime, where perturbative analyses can be trusted because of asymptotic freedom, Ref. [3] showed that the theory produces a (GPY) effective potential, $V_{\mathrm{pert.}}[\Omega]$,

$$V_{\mathrm{pert.}}[\Omega] = -\frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{|\mathrm{tr}\Omega^n|^2}{n^4} (1 + O(g^2)). \tag{2.5}$$

This result can be found by integrating out the Kaluza-Klein modes at one-loop order. This potential is minimized by Ω of the form $\Omega = \omega^k 1_N$ for any integer k, suggesting that the theory has N degenerate vacua related by the $\mathbb{Z}_N$ symmetry and describes a gluon plasma phase.

The basic idea behind $\mathbb{R}^3 \times S^1$ theories such as dYM is to reenforce the stability of the $\mathbb{Z}_N$ at small L by "flipping" the shape of the GPY potential, so to speak. This can be done in the most direct way by simply adding a "double-trace" term to the YM action,

$$\mathcal{L}_{\mathrm{dYM}} = \mathcal{L}_{\mathrm{YM}} + V_{\mathrm{deformed}}[\Omega], \tag{2.6a}$$

where

$$V_{\mathrm{deformed}}[\Omega] = \frac{1}{L^4} \sum_{n=1}^{\lfloor N/2 \rfloor} a_n |\mathrm{tr}\Omega^n|^2. \tag{2.6b}$$

But such a term is manifestly nonlocal, being defined in terms of a nonlocal operator. It is also nonrenormalizable, as it contains infinitely many irrelevant operators which blow up uncontrollably in the UV. As such, such a deformation of the theory may be considered problematic to those with a philosophical preference for continuum theories. We will return to address this objection later, and focus on the effects of the double-trace deformation potential on the IR theory for now.

The $\mathbb{Z}_N$ symmetry is said to be preserved if and only if the vacuum state of the theory satisfies $\langle \mathrm{tr}\Omega^n \rangle = 0$ for all $n \neq 0 \bmod N$, so the coefficients $a_n > 0$ in Eq. (2.6b) must each be chosen so as to dominate the dynamically generated $V_{\mathrm{pert.}}[\Omega]$. With the center symmetry stabilized, we can remove the gauge redundancy of Ω by choosing a diagonal representative from the class of physically equivalent minima,

$$\Omega = \omega^{(1-N)/2} \mathrm{diag}(1, \omega, \ldots, \omega^{N-1}). \tag{2.7}$$

This choice is in fact unique, up to permutations of the coefficients corresponding to Weyl reflections that can also be gauged-fixed away by working in the (affine) Weyl chamber. This allows us to write $\langle \Omega \rangle$ as a physically meaningful expectation value despite its uncontracted matrix indices.[4]

This vacuum expectation value (VEV) precipitates a simulacrum of the Higgs mechanism in which the $\mathcal{A}_4$ field plays the role of an adjoint Higgs field. The gauge group generators left unbroken by $\langle \Omega \rangle$ form a Cartan subalgebra $\mathfrak{t} \subset su(N)$, generating the maximal torus $U(1)^{N-1} \subset SU(N)$. The Higgs mechanism endows fields in $\mathfrak{t}^\perp$ [i.e., fields that carry charge under the $U(1)^{N-1}$] with an effective mass $\geq \frac{2\pi}{NL} \equiv m_W$, the so-called Abelianization scale.

We can now perform the path integral around the center-symmetric vacuum. Working perturbatively, (the treatment of the nonperturbative physics is left to Sec. II B) weak-coupling ensures that the $\mathcal{A}_4$ fluctuations around $\langle \Omega \rangle$, of mass $\gtrsim \sqrt{g^2 N} m_W$, can only effect small corrections to the effective action. Weak coupling, in turn, is guaranteed by the weak-coupling assumption $m_W \gg \Lambda$—meaning that all dynamic charged fields can be safely integrated out before the onset of strong coupling as we carry the theory towards the infrared. When the dust settles, we are left with a weakly-coupled $U(1)^{N-1}$ gauge theory containing no light charged fields in its spectrum. In fewer words, everything works out fine.

In settings where it is desirable to have a theory that respects both locality and UV-completeness, and yet preserve center symmetry at all scales, we can opt to have $V_{\mathrm{deformed}}[\Omega]$ generated dynamically as well, by adding

[3]According to the modern viewpoint, this $\mathbb{Z}_N$ belongs to a class of "generalized" global symmetries, which act on operators with nontrivial spatial extent. In this context, Eq. (2.4) defines the symmetry [17].

[4]To be certain, the action of the center $\mathbb{Z}_N$ in this gauge is, with Ω_i denoting the ith diagonal of Ω [13],

$$\mathbb{Z}_N\colon \ \Omega_i \to \begin{cases} \omega \Omega_{i-1} & i \neq 1, \\ \omega \Omega_N & i = 1. \end{cases} \tag{2.8}$$

The cyclic permutation is necessitated by gauge-fixing to the Weyl chamber.

sufficiently light, or massless, S^1-periodic adjoint fermions λ_I to the theory [8], rather than inserting the deformation potential "by hand." In such a setting, the periodicity requirement $\lambda_I(x^\mu, x^4) = +\lambda_I(x^\mu, x^4 + L)$ prohibits a thermal interpretation for the S^1, which must therefore be taken to be a spatial circle.

For the theory with $1 \leq n_f \leq 5$ Weyl fields[5] of masses m_I indexed by $1 \leq I \leq n_f$, the dynamically generated (GPY) effective potential is [5,10,18]

$$V[\Omega] = -\frac{1}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} \left[2 - \sum_{I=1}^{n_f} (nLm_I)^2 K_2(nLm_I)\right] |\mathrm{tr}\Omega^n|^2, \tag{2.9}$$

where K_2 is the modified Bessel function of order 2. It is not hard to find constraints on the m_I for each value n_f that stabilize the center-symmetric Ω in Eq. (2.7); see e.g., Ref. [18]. We also note in passing that the $n_f = 1$ potential vanishes (in fact, to all perturbative orders) in the massless case, and is center-unstable otherwise. This particular case is known as super Yang-Mills (SYM), in which the UV theory enjoys an exact $\mathcal{N} = 1$ supersymmetry, which allows many aspects of its rich nonperturbative physics to be calculated exactly. But SYM is outside of the scope of this study, along with the massless QCD(adj), and we henceforth only consider $2 \leq n_f \leq 5$ and $m_I > 0$.

Assuming the fermion masses to be roughly equal, it turns out that center stability requires $m_I \lesssim m_W$. In particular, this means that we can assume that the m_I are $O(m_W)$ so that the fermions disappear from the low-energy theory, and the effective action can be written on $\mathbb{R}^3$ as

$$S_{3d} = \int_{\mathbb{R}^3} \sum_{a,b=1}^{N} \kappa_{ab} F^a_{\mu\nu} F^b_{\mu\nu} + (A_4 \text{ and higher-order terms}) \tag{2.10a}$$

for Abelian field strengths $F^a_{\mu\nu}$ and $\mathbb{R}^3$ indices $\mu, \nu \in \{1, 2, 3\}$ and Lie algebra indices $a, b \in \{1 \ldots N\}$. There is also a a neutral scalar field A^a_4 in the IR theory, which descends from the Abelian part of $\mathcal{A}_4$ and corresponds to the oscillations of the eigenvalues of Ω around the center-symmetric VEV. But this field receives a $(\text{mass})^2 \sim g^2 N m_W^2$ correction from the GPY potential and can be integrated out by moving the theory to still lower energies.

The quantity κ_{ab} in Eq. (2.10a) is the quantum-corrected photon coupling matrix,

$$\kappa_{ab} = \frac{m_W^{-1}}{16\pi} \left(\frac{8\pi^2}{Ng^2} \delta_{ab} + O(1)\right), \qquad g^2 \equiv g^2(4\pi/L), \tag{2.10b}$$

[5]The theory loses asymptotic freedom for $n_f > 5$.

where in Eq. (2.10b), the gauge coupling is normalized with respect to the $L \to \infty$, $m_I \to 0$ limit

$$\Lambda^{b_0} = \mu^{b_0} \exp\left(-\frac{8\pi^2}{g^2(\mu) N}\right), \qquad b_0 \equiv \frac{11 - 2n_f}{3}, \tag{2.11}$$

and b_0 is the one-loop coefficient of the beta function of $(g^2 N)^{-1}$ in that limit. As stated before, these corrections have been calculated in previous studies for arbitrary N and $1 \leq n_f \leq 5$ in the limit $m_I = 0$. Our calculation generalizes to the massive case, and is a new result. We also believe it to be a nontrivial problem in terms of significance (as we will argue in this section), as well as difficulty (which we will demonstrate in the next section).

1. The one-loop corrections to κ_{ab}

Compared to the writing out the matrix entries of κ_{ab} explicitly in the Cartan-Weyl basis, [given in Eq. (3.46)] it is more enlightening to present its eigenvalues, κ_ℓ,

$$\sum_{a,b=1}^{N} \kappa_{ab} F^a_{\mu\nu} F^{b\mu\nu} = \sum_{\ell=1}^{N} \kappa_\ell \tilde{F}^\ell_{\mu\nu} \tilde{F}^{\ell\mu\nu}, \tag{2.12a}$$

where

$$\tilde{F}^\ell_{\mu\nu} \equiv \frac{1}{\sqrt{N}} \sum_{a=1}^{N} \omega^{-\ell a} F^a_{\mu\nu}, \quad \kappa_\ell \equiv \frac{1}{N} \sum_{a,b=1}^{N} \omega^{\ell(a-b)} \kappa_{ab}, \tag{2.12b}$$

where again $\omega = e^{i\frac{2\pi}{N}}$, so Eq. (2.12) are really just discrete Fourier transforms in the indices a, b. Then, assuming center-stabilizing fermion masses m_I,[6]

$$\begin{aligned}\kappa_\ell &\equiv \frac{1}{4} g_{3,\ell}^{-2} \\ &= \frac{m_W^{-1}}{16\pi} \left[\frac{8\pi^2}{Ng^2(m_W e^{-\gamma})} + \left(\frac{11 - 2n_f}{3}\right) \log \frac{1}{\sin \pi \frac{\ell}{N}} \right. \\ &\quad \left. + \frac{2}{3} \sum_{I=1}^{n_f} W_\ell\left(\frac{m_I}{m_W}\right)\right] \qquad \text{for } 1 \leq \ell \leq N - 1. \end{aligned} \tag{2.13}$$

$W_\ell = W_{N-\ell}$ is an $O(1)$ pure function of the masses m_I in units of m_W, which enjoys the following properties:

$$W_\ell(0) = 0 \qquad \text{for all integers } \ell, \tag{2.14a}$$

[6]Please note that the $\ell = N$ mode must be excluded from the spectrum as it corresponds to the trace of $F^a_{\mu\nu}$, (as can be seen from the definition,) which is unphysical in our theory; it would have been a physical mode if we had instead chosen our gauge group to be $U(N)$.

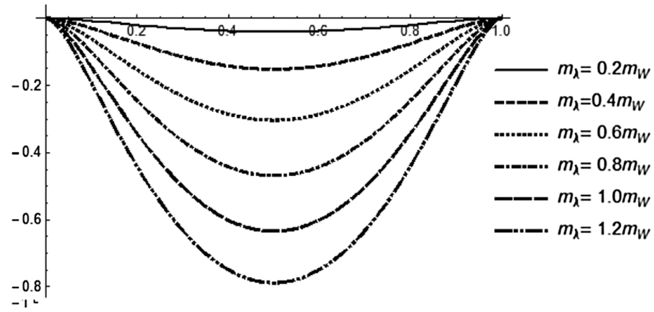


FIG. 1. A numerical plot of W_ℓ as a function of $\frac{\ell}{N}$ for some select values of m, in units of m_W. The graphs were plotted with Eq. (2.15b) for $m = 1.0m_W$ and $m = 1.2m_W$, and with (2.15a) otherwise.

$$W_\ell\left(\frac{m}{m_W}\right) \to \log\left(\frac{m_W e^{-\gamma}}{m \sin\pi\frac{\ell}{N}}\right) \quad \text{monotonically, as } m/m_W \to \infty, \tag{2.14b}$$

$$W_\ell\left(\frac{m}{m_W}\right) \to 0 \qquad \text{as } \frac{\ell}{N} \to 0, 1, \tag{2.14c}$$

$$W_\ell(\tau') < W_\ell(\tau) < 0 \quad \text{for } 0 < \tau < \tau', \tag{2.14d}$$

$$W_{\lfloor N/2\rfloor}(\tau) < W_{\ell'}(\tau) < W_\ell(\tau) \le 0 \quad \text{for } \ell < \ell' < \lfloor N/2\rfloor, \qquad \tau > 0. \tag{2.14e}$$

A plot of W_ℓ as a function of $\frac{\ell}{N}$ for a few select values of m is given in Fig. 1.

Equation (2.13) is written so that all the information about the one-loop corrections due to the fermion masses is encoded in the pure function W_ℓ. In particular, Eq. (2.14d) implies that the $\ell = 1$ mode receives a vanishingly-small mass correction in the $N \to \infty$ limit; conversely, Eq. (2.14e) implies the $\lfloor N/2\rfloor$ mode receives the largest mass correction. Equations (2.14a) and (2.14b) together imply that $\kappa_1 \ge \kappa_\ell \ge \kappa_{\lfloor N/2\rfloor}$ for all m_I and ℓ. Note that in order for our results to make sense, we must require $\kappa_{\lfloor N/2\rfloor} > 0$; we will discuss the conditions that fulfill this requirement later.

We present two analytic expressions for W_ℓ with different convergence properties; one expression holds for $m < m_W$, and the other, for $m \gtrsim m_W$. The former of these is

$$W_\ell\left(\frac{m}{m_W}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}\left(\frac{m}{m_W}\right)^{2n} \times \left[\zeta(2n+1) - \mathrm{Re}(\mathrm{Li}_{2n+1}(e^{2\pi i\frac{\ell}{N}}))\right] \quad \text{for } (m/m_W) < 1, \tag{2.15a}$$

where ζ is the Riemann zeta function, and Li_s is the polylogarithm function of order s. This expression fails to converge when $m > m_W$[7]; it is in this regime where our second expression is more useful,

$$W_\ell\left(\frac{m}{m_W}\right) = \log\left(\frac{m_W e^{-\gamma}}{m \sin\pi\frac{\ell}{N}}\right) + \sum_{p=1}^{\infty}\left\{2K_0\left(2\pi p\frac{m}{m_W}\right) - K_0\left(2\pi\left(p-1+\frac{\ell}{N}\right)\frac{m}{m_W}\right) - K_0\left(2\pi\left(p-\frac{\ell}{N}\right)\frac{m}{m_W}\right)\right\}. \tag{2.15c}$$

K_0 is the modified Bessel function of order 0. Equation (2.15b) is one-loop exact for all m, but it is more useful at large $m \gtrsim m_W$, where it may be very well-approximated by the first term of the series, as $K_0(t) \sim e^{-t}$ at large t. Conversely, Eq. (2.15b) is less useful at small m/m_W as $K_0(t) \sim \log t$ at small t.

Since $W_\ell \le 0$, the massive correction competes against the massless-limit corrections encoded in the log-sine term. Indeed, by taking $m_I \gg m_W$, the Ith fermion decouples from the theory,[8] $n_f \to (n_f - 1)$, up to an overall renormalization of g^2N, or, equivalently, a redefinition of the strong-coupling scale Λ.

We can also use Eq. (2.11) to define a "lattice-renormalized" 't Hooft coupling λ_ℓ,

$$\frac{1}{\lambda_\ell} \equiv b_0 \log\left(\frac{m_W e^{-\gamma}}{\Lambda \sin\pi\frac{\ell}{N}}\right) \quad \text{for } 1 \le \ell \le N-1. \tag{2.16a}$$

Then assuming for convenience equal fermion masses $m_I = m_\lambda$ for all $I = 1\ldots n_f$, and abbreviating $W_\ell(\frac{m_\lambda}{m_W}) = W_\ell$, Eq. (2.13) can be written in a neater form,

$$\kappa_\ell = \frac{m_W^{-1}}{16\pi}\left(\frac{1}{\lambda_\ell} + \frac{2n_f}{3}W_\ell\right) \quad \text{for } 1 \le \ell \le N-1. \tag{2.16b}$$

The dependence of κ_ℓ on $\frac{\ell}{N}$ is illustrated in Fig. 2 for a few sample values of m_λ, for fixed $n_f = 4$ and $m_W = e^3\Lambda$. Given a fixed value of $m_W/\Lambda = \frac{2\pi}{\Lambda NL} \gg 1$, it is a straightforward

[7]Observing that the terms in the square brackets of (2.15a) are absolutely bounded for all $n \ge 1$, and that

$$\frac{(2n)!}{(n!)^2 2^{2n}} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}, \tag{2.15b}$$

the root test gives a radius of convergence of $(m/m_W) < 1$.

[8]Assuming, of course, that the theory still remains in the center-symmetric regime.

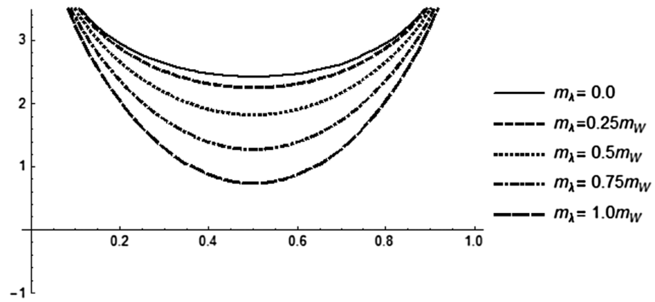


FIG. 2. A numerical plot of κ_ℓ in units of $\frac{m_W^{-1}}{16\pi}$, as a function of $\frac{\ell}{N}$ for some select values of m_λ, for $n_f = 4$ and $m_W = e^3\Lambda$. The plot diverges at $\ell/N \to 0, 1$ (not depicted) due to the log-sine running in $\frac{1}{\lambda_\ell}$.

exercise in numerical analysis to find constraints on m_λ so that $\kappa_{\lfloor N/2 \rfloor} > 0$ in order for our result to make sense. Conversely, $\mathbb{Z}_N$-stability requires that e.g., $m_\lambda \lessapprox 1.08 m_W$ for $n_f = 4$, and this bound can in particular be saturated by taking $m_W/\Lambda \gtrapprox e^3$. On the other hand, $\mathbb{Z}_N$ stability requires $m_\lambda \lessapprox 1.2 m_W$ for $n_f = 5$, and saturation of that bound would require $m_W/\Lambda \gtrapprox e^9$, which is substantially larger.

Equations (2.13), (2.15a), and (2.15b) comprise the main results of this paper; they are derived in detail in Sec. III, with reference to some results from Appendix. The rest of this section discusses how to interpret these results in the context of the nonperturbative physics of dYM theory, particularly with regards to the "emergent latticized dimension" of Ref. [16].

B. Review of dYM: Nonperturbative aspects

Let us now very quickly summarize the derivation of the low-energy effective Lagrangian in dYM theory at leading order in the semiclassical expansion. The basic idea is essentially the same as Polyakov's version of confinement in the Georgi-Glashow (GG) model in $(2+1)$ dimensions [19], although there are crucial differences due to the intrinsically four-dimensional nature of dYM theory. The reader interested in a more detailed exposition is referred to Refs. [4,20,21].

The contribution of the nonperturbative physics to the path integral in a weakly-coupled Euclidean QFT can be approximated to first exponential order by summing over classical field configurations that are inundated by a "gas" of weakly-interacting minimal-action instantons. This is the so-called dilute-instanton gas approximation, and it is applicable in dYM because weak coupling can be reliably assumed to hold at all scales provided that $NL\Lambda \ll 2\pi$.

In addition to a topological charge $Q \sim \int \mathrm{tr} \mathcal{F} \wedge \mathcal{F} = \frac{1}{N}$, the instantons of dYM theory carry a magnetic charge ($\sim \int F$) under the $U(1)^{N-1}$—they are essentially 't Hooft-Polyakov monopoles, with $\mathcal{A}_4$ again standing in for the adjoint Higgs field. In particular, we call them monopole-instantons.

Among these, there are $(N-1)$ "BPS"[9] monopoles, each carrying a magnetic charge corresponding to a simple root α_i of the gauge group. In distinction to the three-dimensional Polyakov model, in $\mathbb{R}^3 \times S^1$ theories there is also an Nth "twisted," or Kaluza-Klein, (KK) monopole, which carries charge $\sum_{i=1}^{N-1}(-\alpha_i) \equiv \alpha_N$, the affine root. In addition to these, there are also the antiparticles carrying charge $-\alpha_i$. In a sense, the $(N-1)$ BPS + KK monopole-instantons can be thought of as the "dissociation" of the BPST instanton in four-dimensional $SU(N)$ Yang-Mills into N subconstituents [22,23].

We unfortunately do not have exact expressions for the monopole-instantons outside of the supersymmetric $n_f = 1, m = 0$ case, but as it turns out, they will not be required as far as our presentation is concerned. We need only know that these charged objects interact with a long-range Coulombic interaction, and have a nonlinear "A_4/Higgs condensate" core of size $\sim m_W^{-1}$. In addition, there is also a "medium-range" ($\sim 1/g\sqrt{N} m_W$) Yukawa interaction arising from A_4/Higgs exchange.

Every insertion of a monopole-instanton in the path integral comes with three translation zero modes and a Boltzmann suppression factor $e^{-S_0} \sim (NL\Lambda)^{b_0} \ll 1$, where $S_0 \approx \frac{8\pi^2}{Ng^2(m_W)}$ is the one-monopole action. This means the typical monopole-instanton separation $d \sim e^{S_0/3}$ is much greater than the monopole diameter $\sim m_W^{-1}$, allowing us to ignore the contribution from paths with overlapping monopole-instanton cores. It also means that we can ignore the effects of A_4 exchange. The proliferation of magnetic charges in the vacuum gives rise to a potential for the photon. This potential which is most conveniently described in terms of the dual photon σ^a, defined as

$$\frac{1}{2}\varepsilon_{\mu\nu\rho}\kappa_{ab}F^a_{\mu\nu} = \frac{1}{16\pi}\partial_\rho \sigma^b. \tag{2.17}$$

Written in terms of σ^a, the IR behavior of dYM theory is described to first order in the semiclassical expansion, by the 3d Lagrangian $\mathcal{L}_{3d,\mathrm{dual}}$,

$$\mathcal{L}_{3d,\mathrm{dual}} = \frac{1}{2(8\pi)^2}\kappa^{-1}_{ab}\partial_\mu\sigma^a\partial_\mu\sigma^b + \zeta\sum_{k=1}^{N}[1-\cos(\sigma^{k+1}-\sigma^k)], \tag{2.18}$$

where $\sigma^{N+1} \equiv \sigma^1$, and κ^{-1}_{ab} is the inverse[10] of κ_{ab}, and ζ is the monopole fugacity,

[9] This is a common abuse of terminology; outside of the supersymmetric case, the BPS bound cannot be saturated because the "Higgs" potential $V[\Omega]$ cannot be set to zero. So, strictly speaking we are expanding around "almost-BPS" configurations.

[10] Actually, it should be the pseudoinverse since the eigenvalue corresponding to the $\ell = N$ mode diverges, but the difference is immaterial since the $\ell = N$ mode is unphysical.

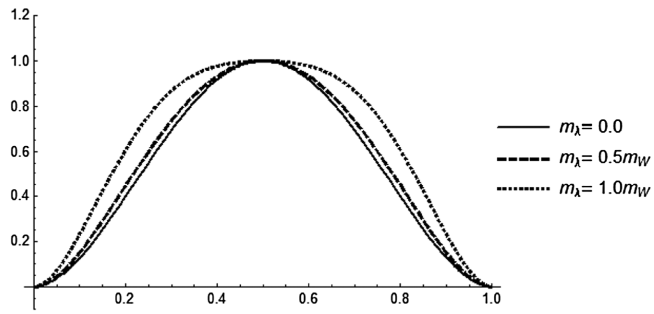


FIG. 3. A numerical plot of $m^2_{\sigma,\ell}/m^2_{\sigma,N/2}$ as a function of $\frac{\ell}{N}$, for $n_f = 4$ and $m_W = e^4\Lambda$.

$$\zeta \equiv A m_W^3 (g^2 N)^{-2} e^{-8\pi^2/Ng^2(m_W)}. \tag{2.19}$$

$A(\{m_I\}, n_f)$ is an $O(1)$ preexponential factor.[11] In the Fourier basis, the 3d dual photon Lagrangian is, to quadratic accuracy in the fields,

$$\mathcal{L}_{3d,\mathrm{dual}} = \sum_{\ell=1}^{N-1} \left[\frac{\kappa_\ell^{-1}}{(8\pi)^2} |\partial_\mu \tilde{\sigma}^\ell|^2 + \zeta \sin^2\left(\pi \frac{\ell}{N}\right) |\tilde{\sigma}^\ell|^2 \right] + O(\tilde{\sigma}^4), \tag{2.20}$$

where $\tilde{\sigma}^\ell$ is the discrete Fourier transform of σ^a,

$$\tilde{\sigma}^\ell \equiv \frac{1}{\sqrt{N}} \sum_{a=1}^{N} \omega^{-\ell a} \sigma^a. \tag{2.21}$$

From this expression we can read off the dual-photon masses squared,

$$m^2_{\sigma,\ell} \sim \zeta \sin^2\left(\pi \frac{\ell}{N}\right) \kappa_\ell. \tag{2.22}$$

Let us take ΛNL to be sufficiently small so that $\frac{2}{3} n_f \lambda_\ell W_\ell \ll 1$ can be treated as a small correction for all ℓ. In that case, we can write a mass-corrected expression for the scaling behavior of the k-wall thicknesses. Recalling (2.16a),

$$\frac{m_{\sigma,k}}{m_{\sigma,1}} \approx \frac{\sin \pi \frac{k}{N}}{\sin \frac{\pi}{N}} \left(\frac{\lambda_1}{\lambda_k}\right)^{1/2} \left[1 + \frac{n_f}{3} (\lambda_k W_k - \lambda_1 W_1)\right]. \tag{2.23}$$

The multiplicative sine factor is the expected tree-level scaling behavior; the factor of $(\lambda_1/\lambda_k)^{1/2}$ is due to the one-loop corrections in the massless limit, and the factor in the square brackets gives the massive correction. The dependence of $m^2_{\sigma,\ell}$ on ℓ in units of $m^2_{\sigma,N/2}$ is graphically depicted in Fig. 3.

1. Emergent dimension at large N: A 4d interpretation of the mass correction

Let us now consider the large-N limit. To do this, we simultaneously take $N \to \infty$ and $L \to 0$ whilst keeping NL constant so as to stay inside the weak-coupling regime.[12] This is known as the "Abelian large-N limit" [11]. In this setup, we can treat $\frac{\ell}{N} \in [0, 1]$ as though it were on a continuum, and the potential in Eq. (2.20) has an interpretation as the kinetic energy on a latticized and compact fourth dimension, with a quadratic (as opposed to quartic, as is the case in SYM) dependence on a lattice momentum p_y. But what is the scale of this momentum? Since the mass gap for the dual photon $m^2_{\sigma,1}$ vanishes in the large-N limit, the only remaining mass scale to characterize the low-energy theory is $m^2_{\sigma,N/2} \equiv m_{N/2}$, the (Debye) mass of the heaviest dual photon,

$$m_{N/2} \sim m_W \lambda^{-3/2} e^{-1/2\lambda}, \tag{2.24a}$$

where

$$\lambda \equiv \frac{Ng^2(m_W)}{8\pi^2} \approx \lambda_{N/2}, \tag{2.24b}$$

up to small corrections. This allows us to define $p_{y,\ell}$ as an honest-to-goodness lattice momentum,

$$p_{y,\ell} \equiv m_{N/2} \sin\left(\pi \frac{\ell}{N}\right). \tag{2.25}$$

We can also read off the two-point function directly from (2.20). Defining $x^M \equiv (\vec{x}, y)$, $p_M \equiv (\vec{p}, p_y)$, and momentarily disregarding the massive correction,

$$\int d^4x e^{ip_M x^M} \langle \sigma(x_M) \sigma(0) \rangle \sim (\lambda_\ell \vec{p}^2 + \lambda p_y^2)^{-1}. \tag{2.26}$$

We observe that there is a restored Lorentz symmetry which is broken by an anomalous scaling dimension $\Delta = b_0 \lambda$ as $\lambda_\ell \sim p_y^{b_0\lambda}$.

Put another way, the dual-photon coupling λ_ℓ exhibits logarithmic running in the lattice momentum $p^y \sim \sin(\pi \frac{\ell}{N})$,

$$p_y \frac{d}{dp_y} \left(\frac{1}{\lambda_\ell}\right) = p_y \frac{d}{dp_y} \left(b_0 \log \frac{1}{p_y} + (\mathrm{const})\right) = -b_0. \tag{2.27}$$

[11] As an aside, let us note that calculating the prefactor $A(\{m_I\}, n_f)$ is a highly nontrivial open calculation, and has only been performed in the SYM case, first in [24], and later corrected in [9,10]. This is because it involves matrix determinants in a monopole-instanton background, for which we do not even have an exact analytic expression, as mentioned. We make no attempt to calculate A here.

[12] The 't Hooft coupling g^2N is under control in this limit, as can be seen from Eq. (2.11).

In particular, the scaling behavior is opposite to that of the $\mathbb{R}^4$ theory [cf. Eq. (2.11)],

$$\mu \frac{d}{d\mu}\left(\frac{1}{\lambda(\mu)}\right) = +b_0. \tag{2.28}$$

We can also show how this analogy can be extended to encompass the mass-correction terms $\sim W_\ell$. The one-loop correction to the coupling due to a single adjoint fermion with mass m in an $SU(N)$ theory on $\mathbb{R}^4$, renormalized at some scale μ in the $\overline{\text{MS}}$ scheme is [cf. Eq. (3.38)]

$$\begin{aligned}\mathcal{M}^{\text{fermion}}_{\mathbb{R}^4}(P) &= -2\int_0^1 dx\, x(1-x)\log\left(\frac{P^2 x(1-x)+m^2}{\mu^2}\right)\\ &\sim \begin{cases} -\frac{2}{3}\log(\frac{P}{\mu}) & P^2 \gg m^2,\\ -\frac{2}{3}\log(\frac{m}{\mu}) & P^2 \ll m^2,\end{cases}\end{aligned} \tag{2.29}$$

where x is a Feynman parameter. We can compare this with our result of the contribution in the $\mathbb{R}^3\times S^1$ theory, which can be read off from (2.13),

$$\begin{aligned}\mathcal{M}^{\text{fermion}}_{\mathbb{R}^3\times S^1,\ell} &= \frac{2}{3}\left[\log\left(\sin\pi\frac{\ell}{N}\right) + W_\ell\left(\frac{m}{m_W}\right)\right]\\ &\sim \begin{cases} +\frac{2}{3}\log\sin(\pi\frac{\ell}{N}) & m_W \gg m,\\ -\frac{2}{3}\log(\frac{m}{m_W}) & m_W \ll m.\end{cases}\end{aligned} \tag{2.30}$$

This result is consistent with our interpretation of p_y as a momentum, with the mass correction behaving as we should expect in the $\mathbb{R}^4$ theory, albeit with opposite momentum-scaling behavior in p_y.

III. PERTURBATIVE ANALYSIS: THEORY AND PRACTICE

The remainder of this paper mainly focuses on deriving and calculating loop integrals and Matsubara sums. Our approach is extremely straightforward—essentially identical to the analysis of a thermal gauge theory at temperatures $T = 1/L$, but for the fact that our S^1 is spacelike rather than timelike. This means, in particular for the fermions, that the S^1 momenta ω_n assume integer values $\omega_n = \frac{2\pi n}{L}$, rather than half-integer $\omega_n = \frac{2\pi}{L}(n+\frac{1}{2})$. As the calculation is rather involved, our presentation will try to go into as much detail as we can without being overly cumbersome. For the convenience of the reader, we will summarise the contents of Secs. III A and III B at the end of their respective sections.

Let us start by defining our notation. We will use $M, N \in \{1,2,3,4\}$ for Euclidean indices on $\mathbb{R}^3\times S^1$, (with x^4 the coordinate on S^1) and $\mu,\nu\in\{1,2,3\}$ for indices on the $\mathbb{R}^3$. We will use $a,b,c,i,j,k\in\{1,\ldots N\}$ to denote Lie algebra indices.

We also define the (over-complete) Cartan-Weyl basis on $su(N)$,

$$(H_i)_{ab} = \delta_{ia}\delta_{ib} = \text{diag}(0,\ldots,\overbrace{1}^{i\text{'th}},\ldots,0) \quad 1\le i\le N, \tag{3.1a}$$

which span the Cartan subalgebra $\mathfrak{t}$. These are accompanied by the raising and lowering operators spanning $\mathfrak{t}^\perp$, the orthogonal complement of $\mathfrak{t}$,

$$(E_{\beta_{ij}})_{ab} = \delta_{ai}\delta_{bj}, \tag{3.1b}$$

for N-component vectors β_{ij} in the root lattice of $su(N)$, which in our basis are written

$$\begin{aligned}\beta^a_{ij} &\equiv \delta^a_i - \delta^a_j\\ &= (0,\ldots\overbrace{1}^{i\text{'th}},\ldots,\overbrace{-1}^{j\text{'th}},\ldots,0), \quad 1\le i\le j\le N.\end{aligned} \tag{3.1c}$$

Perhaps a bit idiosyncratically, we say that the subscripts on β_{ij} are a set of antisymmetric indices labeling the roots of $su(N)$: $\beta_{ij} = -\beta_{ji}$, and the superscript a denotes its ath vector component.

$E_{\beta_{ij}}, E_{-\beta_{ij}}$ are respectively raising and lowering operators for the $su(2)$ subalgebra associated with the root β_{ij},

$$[H_i,H_j]=0,\quad [H_k,E_{\beta_{ij}}]=\beta^k_{ij}E_{\beta_{ij}},\quad [E_{\beta_{ij}},E_{-\beta_{ij}}]=\sum_k \beta^k_{ij}H_k, \tag{3.2a}$$

(no sums over i, j). We also have

$$E^\dagger_{\beta_{ij}} = E_{-\beta_{ij}} = E_{\beta_{ji}}, \qquad H^\dagger_k = H_k. \tag{3.2b}$$

These generators are normalized as

$$\text{tr}[H_iH_j] = \delta_{ij}, \qquad \text{tr}[E_\beta E_{-\beta'}] = \delta_{\beta\beta'}. \tag{3.2c}$$

In the interest of brevity, we will frequently abuse notation and treat β as though it were the index on the root space and omit the subscripts ij, as we have just done above. To avoid confusion, there will be no implicit sum over $su(N)$ indices unless otherwise specified.

As a matter of convenience, we normalize the components of $su(N)$-valued fields ψ as

$$\psi(x^\mu,x^4) = \frac{1}{2}\sum_k \psi^k(x^\mu,x^4)H_k + \frac{1}{\sqrt{2}}\sum_\beta \psi^\beta(x^\mu,x^4)E_\beta, \tag{3.3a}$$

obeying Hermiticity conditions,

$$(\psi^k)^* = \psi^k, \qquad (\psi^\beta)^* = \psi^{-\beta}, \tag{3.3b}$$

and constrained by a trace-free condition,

$$\sum_k \psi^k(x^\mu, x^4) = 0, \tag{3.3c}$$

so that the expansion (3.3a) is unique although it is written in terms of an overcomplete basis.

A. Formal setup: Beginnings

Let us start with a four-dimensional Euclidean $SU(N)$ gauge theory with non-Abelian field strength $\mathcal{F}_{MN}$ and n_f two-component massive adjoint fermions λ_I. As we are performing a perturbative calculation, the vacuum angle is "invisible" to us, so we might as well set the fermion masses to be real and the topological angle $\theta = 0$,

$$\mathcal{L}_{4d} = \mathrm{tr}\left[\frac{1}{2g^2}(\mathcal{F}_{MN})^2 + 2i\sum_{I=1}^{n_f}\left(\bar{\lambda}_{I\dot{\alpha}}\bar{\sigma}^{M\dot{\alpha}\alpha}(\nabla_M\lambda_I)_\alpha + \frac{m_I}{2}\lambda_{I\alpha}\lambda_{I\beta}\varepsilon^{\alpha\beta} + \frac{m_I}{2}\bar{\lambda}_I{}^{\dot{\alpha}}\bar{\lambda}_I{}^{\dot{\beta}}\varepsilon_{\dot{\alpha}\dot{\beta}}\right)\right]. \tag{3.4}$$

∇_M is the covariant derivative on adjoint-representation fields

$$\nabla_M \equiv \partial_M + i[\mathcal{A}_M, \cdot], \tag{3.5}$$

and $\bar{\sigma}^M = (i\vec{\sigma}, 1_2)$ are the Euclidean sigma matrices.

Formally integrating out the high-energy ($\gtrsim m_W$) degrees of freedom around the center-symmetric Ω gives us the effective 3d Lagrangian, (2.10a).

We want to explicitly integrate out the high-energy ($\gtrsim m_W$) degrees of freedom to obtain the effective 3d Lagrangian, (2.10a) to find the one-loop corrections to κ_{ab}, the photon-coupling matrix. The methods we use can also be applied almost verbatim to find ρ_{ab}, the corrected scalar couplings, as well as M_{ab} the scalar masses. Since these are not as interesting to us, we simply quote their Fourier-transformed results in Eqs. (3.55) and (3.56).

Following Abbott's approach, (e.g., Ref. [25],) we use an adapted background field gauge method to calculate vacuum polarization. This is fairly standard textbook material, but to review, first, we treat the $su(N)$-valued gauge field A_M as the sum of a "classical" background field and a "quantum" high-frequency field,

$$\mathcal{A}_M = \underbrace{A_M}_{\text{classical}} + \underbrace{g a_M}_{\text{quantum}}. \tag{3.6}$$

The normalization is for convenience. We say that these fields have two complementary expressions of gauge symmetry for $U, \tilde{U}; \mathbb{R}^3 \times S^1 \to SU(N)$,

$$A_M \to U(A_M + i\partial_M)U^{-1}, \qquad a_M \to U a_M U^{-1} \quad \text{(gauge transformation under } U\text{)}, \tag{3.7a}$$

$$a_M \to \tilde{U}(a_M + i\partial_M)\tilde{U}^{-1}, \qquad A_M \to \tilde{U} A_M \tilde{U}^{-1} \quad \text{(gauge transformation under } \tilde{U}\text{)}. \tag{3.7b}$$

Anticipating a 3d and Abelian theory, we take A_M to be Abelian and trivial over x^4, and call its field strength F_{MN},

$$\partial_4 A_M = 0, \qquad [A_M, A_N] = 0, \tag{3.8a}$$

$$F_{MN} \equiv \partial_M A_N - \partial_N A_M. \tag{3.8b}$$

We want to fix the gauge under $\tilde{U}$ in order to integrate out a_M, which, as we will see, are basically W-bosons. To do this, we would impose the condition

$$D^M a_M + ig[a^M, a_M] = 0, \tag{3.9a}$$

where D_M is the "covariant derivative with connection A_M",

$$D_M \equiv \partial_M + i[A_M, \cdot], \tag{3.9b}$$

which can be done by adding a Gaussian term to the Lagrangian,

$$\Delta\mathcal{L}_a = \mathrm{tr}(\nabla^M a_M)^2, \tag{3.10}$$

and a Lagrangian $\mathcal{L}_c$ for scalar-yet-Grassmannian $su(N)$-valued ghost fields c, $\bar{c}$.

Since A_4 has a nonzero VEV, we must write

$$A_4 \equiv \frac{\phi}{L} + A_4^0, \tag{3.11a}$$

where ϕ is the (constant in x^μ) VEV,

$$\phi \equiv -iL\log\Omega, \tag{3.11b}$$

and A_4^0 represents the fluctuations around the VEV, but we can mostly ignore A_4^0 as it is $U(1)^{N-1}$-neutral and therefore not involved in the corrections to κ_{ab} at one-loop. Gauge-fixing the center-symmetric Ω as in Eq. (2.7), ϕ has vector components

$$\phi^k = \frac{\pi}{N}\sum_{\beta>0}\beta^k = \frac{2\pi}{N}\left(\frac{N+1}{2} - k\right), \tag{3.12}$$

where the sum in Eq. (3.12) is over the positive roots. In particular, this means

$$\phi \cdot \beta_{ij} = \frac{2\pi}{N}(j-i). \tag{3.13}$$

All together, the gauge-fixed Lagrangian has the form

$$\mathcal{L} = \underbrace{\mathcal{L}_{cl}}_{\text{classical fields}} + \underbrace{\mathcal{L}_a + \Delta\mathcal{L}_a}_{\text{W-bosons}} + \underbrace{\mathcal{L}_c}_{\text{ghosts}} + \underbrace{\mathcal{L}_\lambda}_{\text{fermions}} + O(\hbar^3), \tag{3.14}$$

where $\mathcal{L}_a$ contains the $\sim Aaa$, $AAaa$ terms upon expanding $\mathcal{L}_{4d}$ in terms of A_M and a_M, and similarly for $\mathcal{L}_c$ and $\mathcal{L}_\lambda$. We observe that by choosing A_M to be Abelian, the Abelian parts of each of the quantum fields $a, \lambda, c, \bar{c}$ cannot contribute to κ_{ab} at one-loop order, so we may forget about them altogether for the rest of this analysis.

Let us take any $su(N)$-valued field ψ and simultaneously expand in the KK modes and the Cartan-Weyl basis, recalling our convention as in Eq. (3.3a),

$$\psi(x^\mu, x^4) = \frac{1}{2}\sum_{k,z} e^{i\frac{2\pi z}{L}x^4}\psi^{k,z}(x^\mu)H_k + \frac{1}{\sqrt{2}}\sum_{\beta,z} e^{i\frac{2\pi z}{L}x^4}\psi^{\beta,k}(x^\mu)E_\beta,$$

so that

$$iD_4\psi = \frac{1}{\sqrt{2}}\sum_{z,\beta} e^{i\frac{2\pi z}{L}x^4}\left(\frac{2\pi z + \phi\cdot\beta}{L}\right)\psi^\beta E_\beta + (\text{Abelian and } O(\hbar^2) \text{ parts}), \tag{3.15}$$

so asymptotically, the derivative operator iD_4 diagonalizes with eigenvalues

$$iD_4 \to \frac{2\pi z + \phi\cdot\beta_{ij}}{L}, \qquad z \text{ integer}, \tag{3.16}$$

so that fields in $\mathfrak{t}^\perp$ with charge β and circle momentum $2\pi z/L$ of a field with mass m obtains an effective 3d $(\text{mass})^2$,

$$m^2 + m_W^2[Nz + (i-j)]^2 \geq m_W^2. \tag{3.17}$$

Thus only $U(1)^{N-1}$-neutral and x^4-trivial fields survive in the IR theory at scales $\ll m_W$, consistent with our hypotheses on A_M.

1. Summary of Sec. III A

We outlined the background field approach to perturbation theory. With an eye toward the infrared theory, we set the background A_M to be x^4-trivial and Abelian, and showed that only "quantum fields" proportional to the broken gauge generators may contribute to the corrections of κ_{ab} at one-loop order. We further showed that this assumption is self-consistent, because all fields with nonvanishing x^4 momentum or carrying charge under the $U(1)^{N-1}$ acquire an effective mass $\geq m_W$ through the Higgs mechanism.

B. The one-loop Wilsonian action

We can write an expression for the Wilsonian effective action $\Gamma[A]$ by formally integrating out the quantum fields under the path integral sign. Setting the vacuum energy to zero,

$$\begin{aligned}\Gamma[A] &= -\log\left[\int DaD\bar{c}Dc\prod_I^{n_f}(D\bar{\lambda}_I D\lambda_I)e^{-\int\mathcal{L}[A,c,\lambda_I,a]}\right] \\ &= \int_{\mathbb{R}^3}\left[\frac{L}{4g^2}(F^k_{\mu\nu})^2 + \frac{L}{2g^2}(\partial_\mu A_4^k)^2\right] \\ &\quad + \sum_{s=0,\frac{1}{2},1}\sum_{f_s}\chi(s)\operatorname{Tr}\log(-D^2_{(s)} + m^2_{f_s,s}) \\ &\quad + (\text{higher loop contributions}).\end{aligned} \tag{3.18}$$

There is a lot of notation to define in Eq. (3.18), but it will make life easier by formatting the problem so that the entire nontrivial part of the calculation is contained in the single expression "$\operatorname{Tr}\log(-D^2_{(s)} + m^2_{f_s,s})$", which we will only have to evaluate once to cover all the relevant cases, rather than having to work with massive or massless, spinor, scalar, and vector integrals separately.

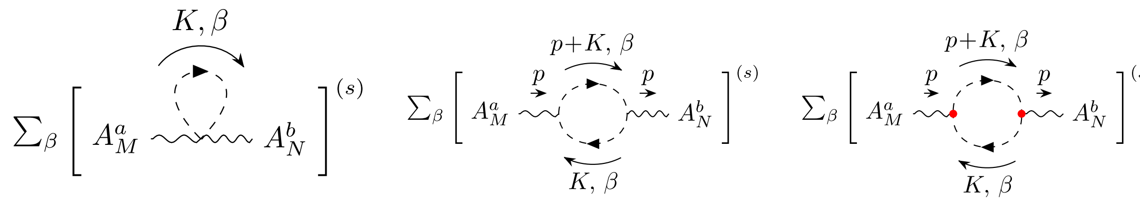
"Tr" refers to the trace over the respective Hilbert spaces, and $-D^2_{(s)}$ is a differential operator defined in Eq. (3.21). The terms on the third row of Eq. (3.18) are due to the W-bosons a, ($s=1$,) the gauge ghosts $c, \bar{c}$, ($s=0$,) and the fermions λ_I ($s=1/2$). The $s=1/2$ term is obtained by doubling then halving the trace-log of the massive Weyl operator,

$$\sum_{I=1}^{n_f}\frac{1}{2}\operatorname{Tr}\log(i\bar{\sigma}\cdot D + im_I) \equiv \sum_{I=1}^{n_f}\frac{1}{4}\operatorname{Tr}\log(-D^2_{(1/2)} + m_I^2). \tag{3.19}$$

$\sum_{f_s}$ is a sum over flavor indices I when $s=1/2$ and $m^2_{f_s,s} = 0$ for $s \neq 1/2$. $\chi(s)$ is a prefactor determined by the statistics of each field,

$$\chi(s) \equiv \begin{cases} -1 & (s=0), \\ -1/4 & (s=1/2), \\ +1/2 & (s=1). \end{cases} \tag{3.20}$$

To define $-D^2_{(s)}$, let A, B denote indices in the spin-s irrep of the (Euclidean) Lorentz group. Then



(a) $(\Pi_2^{(s)})^{ab}_{MN}$ (b) $(\Pi_1^{(s)})^{ab}_{MN}$ (c) $(\Pi_F^{(s)})^{ab}_{MN}$

FIG. 4. Representations of the loop integrals in Eq. (3.27) in terms of Feynman diagrams (a–c). The Σ_F vertex is distinguished from the Σ_1 vertex with a (red) dot.

$$
\begin{aligned}
(-D_s^2)_{AB} &= -D_M D^M \delta_{AB} + F^k_{MN}(H_k^{\mathrm{adj}})(\sigma^{(s)}_{MN})_{AB} \\
&= \left[(i\partial_\mu)^2 + \left(i\partial_3 + \frac{\phi^k}{L}(H_k^{\mathrm{adj}})\right)^2\right](\delta^{(s)})_{AB} \\
&\quad + (\Sigma_1^{(s)})_{AB} + (\Sigma_2^{(s)})_{AB} + (\Sigma_F^{(s)})_{AB}, \qquad (3.21)
\end{aligned}
$$

(with an implicit sum over k,) where $(H_k^{\mathrm{adj}}) \equiv [H_k, \cdot]$, and

$$(\Sigma_F^{(s)})_{AB} \equiv F^k_{MN}(H_k^{\mathrm{adj}})(\sigma^{(s)}_{MN})_{AB}, \qquad (3.22a)$$

$$(\Sigma_1^{(s)})_{AB} \equiv -iA^k_M \overset{\leftrightarrow}{\partial}{}^M (H_k^{\mathrm{adj}})(\delta^{(s)})_{AB}, \qquad (3.22b)$$

$$(\Sigma_2^{(s)})_{AB} \equiv [A^k_M(H_k^{\mathrm{adj}})]^2(\delta^{(s)})_{AB}, \qquad (3.22c)$$

(again, with an implicit sum over k). $\Sigma_1^{(s)}$ and $\Sigma_2^{(s)}$ are respectively the 3- and 4-point interactions of a charged adjoint field, and $\Sigma_F^{(s)}$ is the spin-field coupling term responsible for asymptotic freedom in non-Abelian theories. $\delta^{(s)}$ and $\sigma^{(s)}_{MN}$ are respectively the identity matrix and the generators of rotations in the spin-s representation. Explicitly, (and abusing notation slightly by mixing indices)

$$
(\sigma^{(s)}_{MN})_{AB} = \begin{cases} 0 & (s=0), \\ \frac{i}{4}(\bar{\sigma}_{[M}\sigma_{N]})_{AB} & (s=1/2), \\ -i(\delta_{AM}\delta_{NB} - \delta_{AN}\delta_{MB}) & (s=1). \end{cases} \qquad (3.23)
$$

so the one-loop correction to the Wilsonian can be written, to quadratic order in A_M, as

$$
\begin{aligned}
\mathrm{Tr}\log\left(\frac{-D_s^2 + m_s^2}{-\partial^2 + m_s^2}\right) &= \mathrm{Tr}\left(\frac{\Sigma_2^{(s)}}{-\partial^2 + m_s^2}\right) \\
&\quad - \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\Sigma_1^{(s)}}{-\partial^2 + m_s^2}\right)^2 + \left(\frac{\Sigma_F^{(s)}}{-\partial^2 + m_s^2}\right)^2\right] \\
&\quad + O(A^3). \qquad (3.24)
\end{aligned}
$$

There is no $\sim\Sigma_F\Sigma_1$ cross-term because the trace of $\sigma^{(s)}_{MN}$ vanishes. Expanding in a Fourier basis to quadratic order in the fields, Eq. (3.18) becomes

$$
\begin{aligned}
\Gamma[A^k_M;\mu] = 2\sum_{a,b}\int \frac{d^3p}{(2\pi)^3}\Big[&A^{\mu a}\kappa_{ab}(p^2\delta_{\mu\nu} - p_\mu p_\nu)A^{\nu b} \\
&+ A^a_4\left(p^2\rho_{ab} + \frac{L}{2g^2}M^2_{ab}\right)A^b_4\Big] + O(A^3). \qquad (3.25)
\end{aligned}
$$

We note in passing that the GPY potential $V[\Omega]$ still appears in Eq. (3.25) through M^2_{ab}, its second derivative.

Now we are ready to draw some Feynman diagrams. Let $p^M = (p^\mu, 0)$ denote the external momentum of A_M, and for convenience, define an effective loop-momentum $K^M_{(\beta,z)}$,

$$K^M \equiv K^M_{(\beta,z)} \equiv \left(k^\mu, \frac{2\pi z + \phi\cdot\beta}{L}\right). \qquad (3.26)$$

Using Equations (3.22a), (3.22b), (3.22c), and (3.24) and reading off from (3.25), we can write down the corrections to the $\sim A^a_M A^b_N$ term in the action,

$$
\begin{aligned}
(\Pi_2^{(s)})^{ab}_{MN} &\equiv \frac{1}{2}\frac{\delta^2}{\delta A^a_M \delta A^b_N}\mathrm{Tr}\left(\frac{\Sigma_2^{(s)}}{-\partial^2 + m_s^2}\right) \\
&= \frac{1}{L}\sum_{z,\beta}\int\frac{d^3k}{(2\pi)^3} d(s)\beta^a\beta^b\delta_{MN} \\
&\quad \times \left[\frac{1}{(K^2 + \Delta_s^2)} + \frac{(1-2x)^2p^2}{2(K^2 + \Delta_s^2)^2}\right], \qquad (3.27a)
\end{aligned}
$$

$$
\begin{aligned}
(\Pi_1^{(s)})^{ab}_{MN} &\equiv \frac{1}{2}\frac{\delta^2}{\delta A^a_M \delta A^b_N}\left[-\frac{1}{2}\mathrm{Tr}\left(\frac{\Sigma_1^{(s)}}{-\partial^2 + m_s^2}\right)^2\right] \\
&= -\frac{1}{L}\sum_{z,\beta}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2}d(s)\beta^a\beta^b \\
&\quad \times \int_0^1 dx \frac{4K_MK_N + (1-2x)^2p_Mp_N}{(K^2 + \Delta_s)^2}, \qquad (3.27b)
\end{aligned}
$$

$$
\begin{aligned}
(\Pi_F^{(s)})^{ab}_{MN} &\equiv \frac{1}{2}\frac{\delta^2}{\delta A^a_M \delta A^b_N}\left[-\frac{1}{2}\mathrm{Tr}\left(\frac{\Sigma_F^{(s)}}{-\partial^2 + m_s^2}\right)^2\right] \\
&= -\frac{1}{L}\sum_{z,\beta}\int\frac{d^3k}{(2\pi)^3}c(s)\beta^a\beta^b \\
&\quad \times \int_0^1 dx \frac{2(p^2\delta_{MN} - p_Mp_N)}{(K^2 + \Delta_s)^2}. \qquad (3.27c)
\end{aligned}
$$

These integrals are pictorially represented by the Feynman diagrams in Fig. 4. In each of the integrals above we have employed a Feynman parameter x, and shifted our loop momentum $K^M \to K^M - xp^M$. We have also defined an effective $(\text{mass})^2$, Δ_s [not to be confused with the 3d effective $(\text{mass})^2$ in Eq. (3.17)]

$$\Delta_s \equiv m_s^2 + p^2 x(1-x), \tag{3.28}$$

We have further defined $d(s)$, the number of spin states in the spin-s representation, and $c(s)$, the spin-field coupling coefficient,[13]

$$d(s) \equiv \mathrm{tr}(\delta^{(s)}) = \begin{cases} 1 & (s=0), \\ 4 & (s=1/2), \\ 4 & (s=1), \end{cases}$$
$$c(s) \equiv \mathrm{tr}(\sigma^{(s)}_{MN}\sigma^{(s)MN}) = \begin{cases} 0 & (s=0), \\ 1 & (s=1/2), \\ 2 & (s=1), \end{cases} \tag{3.29}$$

where the traces are over the spin indices, which we have omitted. The rest of our report will be largely concerned with evaluating these three integrals.

1. Summary of Sec. III B

We introduced some formal notation to write down the one-loop effective action in a more compact form, Eq. (3.18). This allowed us to write the integrals of each of a, λ, c in terms of the loop integrals $\Pi_2^{(s)}$, [Eq. (3.27a)] $\Pi_1^{(s)}$, [Eq. (3.27b)], and $\Pi_F^{(s)}$ [Eq. (3.27c)]. As we will see, the evaluation of these integrals are by no means a trivial task, but we will make them much more tractable with a handful of clever manipulations.

C. Outline of the calculation

We have written the integrals in (3.27) to superficially respect the Euclidean Lorentz group $SO(4)$, but to evaluate them we must rewrite (3.27) to reflect the broken rotational symmetry $SO(4) \to SO(3)$. Symmetry considerations tell us that averaging $K_M K_N$ must give

$$(\overline{K_M K_N})_{(\beta,z)} = \frac{k^2}{3}\delta_{\mu\nu}\mathcal{P}^{\mu\nu}_{MN} + \left(\frac{2\pi z + \phi\cdot\beta}{L}\right)^2 \mathcal{P}^{44}_{MN}, \tag{3.30a}$$

where $\mathcal{P}^{\mu\nu}_{MN}$ and $\mathcal{P}^{44}_{MN}$ are projection operators to $\mathbb{R}^3$ and S^1, respectively

$$\mathcal{P}^{\mu\nu}_{MN} \equiv \delta^\mu_M \delta^\nu_N, \tag{3.30b}$$

$$\mathcal{P}^{44}_{MN} \equiv \delta^4_M \delta^4_N. \tag{3.30c}$$

Integrating over the angular coordinates and summing the three graphs in Eq. (3.27), we get

$$\begin{aligned}(\Pi^{(s)})^{ab}_{\mu\nu} &\equiv \sum_{\mathcal{I}=2,1,F} (\Pi^{(s)}_{\mathcal{I}})^{ab}_{\mu\nu} \\ &= \sum_{z,\beta}\beta^a\beta^b \int_0^1 dx \int_0^\infty \frac{d(kL)}{2\pi^2}(kL)^2 \left\{ \left[\frac{(1-2x)^2}{2}d(s) - 2c(s)\right](p^2\delta_{\mu\nu} - p_\mu p_\nu) S_1(b,\omega_s L) \right. \\ &\quad \left. + \frac{d(s)}{L^2}\delta_{\mu\nu}\left[S_0(b,\omega_s L) - \frac{2}{3}(kL)^2 S_1(b,\omega_s L)\right]\right\}. \end{aligned} \tag{3.31a}$$

We also write out the (44) part, which are needed to renormalize

$$\begin{aligned}(\Pi^{(s)})^{ab}_{44} &\equiv \sum_{\mathcal{I}} (\Pi^{(s)}_{\mathcal{I}})^{ab}_{44} \\ &= \sum_{z,\beta}\beta^a\beta^b \int_0^1 dx \int_0^\infty \frac{d(kL)}{2\pi^2}(kL)^2 \left\{ \left[\frac{(1-2x)^2}{2}d(s) - 2c(s)\right] p^2 S_1(b,\omega_s L) \right. \\ &\quad \left. + \frac{d(s)}{L^2}[S_0(b,\omega_s L) - 2S_2(b,\omega_s L)]\right\}, \end{aligned} \tag{3.31b}$$

where $(\Pi^{(s)})^{ab}_{\mu\nu}$ and $(\Pi^{(s)})^{ab}_{44}$ are defined in the obvious way

$$(\Pi^{(s)})^{ab}_{MN} \equiv (\Pi^{(s)})^{ab}_{\mu\nu}\mathcal{P}^{\mu\nu}_{MN} + (\Pi^{(s)})^{ab}_{44}\mathcal{P}^{44}_{MN}, \tag{3.31c}$$

[13]Note that $d(1/2) = 4$ for us, because we doubled the number of polarizations in Eq. (3.19); this is already compensated for by an additional factor of $1/2$ in front of the fermion determinant in Eq. (3.18).

and we have also defined

$$\omega_s \equiv \sqrt{k^2 + \Delta_s}, \qquad b \equiv \phi \cdot \beta, \tag{3.31d}$$

and dimensionless sums over the KK modes, $S_{0,1,2}$,

$$S_1(b, \omega L) \equiv \sum_{n\in\mathbb{Z}} \frac{1}{[(2\pi n + b)^2 + (\omega L)^2]^2},$$
$$S_2(b, \omega L) \equiv \sum_{n\in\mathbb{Z}} \frac{(2\pi n + b)^2}{[(2\pi n + b)^2 + (\omega L)^2]^2}. \tag{3.31e}$$

The third sum, S_0, is a standard result. It can be evaluated exactly by e.g., Matsubara summation,

$$\begin{aligned} S_0(b, \omega L) &\equiv \sum_{n\in\mathbb{Z}} \frac{1}{(2\pi n + b)^2 + (\omega L)^2} \\ &= \frac{1}{2\omega L} + \frac{1}{2\omega L}\mathrm{Re}\left(\frac{1}{e^{L\omega + ib} - 1}\right) \\ &\equiv I_0^{\mathrm{vac}}(\omega L) + \delta I_0(b, \omega L). \end{aligned} \tag{3.31f}$$

where we have defined a function $I_0^{\mathrm{vac}} \equiv \frac{1}{2\omega L}$ that falls off as a negative power in ωL, and another, $\delta I_0 \equiv \frac{1}{2\omega L}\mathrm{Re}(\frac{1}{e^{L\omega+ib}-1})$, that falls off exponentially.

Since the summand of S_0 is monotone decreasing in $|n|$, differentiation commutes with summation, so $S_{1,2}$ can be trivially evaluated by taking derivatives of both sides of Eq. (3.31f),

$$\begin{aligned} S_1(b, \omega L) &= -\frac{\partial}{\partial(\omega L)^2} S_0(b, \omega L) \\ &\equiv I_1^{\mathrm{vac}}(\omega L) + \delta I_1(b, \omega L), \end{aligned} \tag{3.32a}$$

$$\begin{aligned} S_2(b, \omega L) &= \frac{\partial}{\partial(\omega L)^2}[(\omega L)^2 S_0(b, \omega L)] \\ &\equiv I_2^{\mathrm{vac}}(\omega L) + \delta I_2(b, \omega L), \end{aligned} \tag{3.32b}$$

where $I_{1,2}^{\mathrm{vac}}$. and $\delta I_{1,2}$ are defined in terms of derivatives of I_0^{vac}. and δI_0, respectively, in the obvious ways as suggested by the notation. The point is that we can split the integrals in Eq. (3.31a),

$$(\Pi^{(s)})^{ab}_{\mu\nu} = (\Pi^{(s),\mathrm{vac}})^{ab}_{\mu\nu} + (\delta\Pi^{(s)})^{ab}_{\mu\nu}, \tag{3.33}$$

by collecting the $I_{0,1,2}^{\mathrm{vac}}$. terms into $(\Pi^{(s),\mathrm{vac}})^{ab}_{\mu\nu}$, and the $\delta I_{0,1,2}$ terms into $(\delta\Pi^{(s)})^{ab}_{\mu\nu}$, and similarly for $(\Pi^{(s)})^{ab}_{44}$. We will call these the *vacuum integral* and *pseudothermal integral* contributions respectively, and we consider them separately in the following.

The basic idea is this. We can see by the asymptotics that the $(\Pi^{(s),\mathrm{vac}})^{ab}_{MN}$ integrals remain unchanged in the $L \to \infty$ limit. This means we can evaluate those integrals in terms of the familiar loop integrals in $\mathbb{R}^4$, in a way we show explicitly. Obviously these integrals are UV divergent, but they can be renormalized in the $\overline{\mathrm{MS}}$ scheme in the usual way. On the other hand, the $SO(4)$-breaking, L-dependent parts of $(\Pi^{(s)})^{ab}_{MN}$ are contained entirely within $(\delta\Pi^{(s)})^{ab}_{MN}$; the exponential decay of the $\delta I_{0,1,2}$ means that the integrands of $(\delta\Pi^{(s)})^{ab}_{\mu\nu}$ are uniformly convergent in kL. Then we may use the identities (3.32a) and (3.32b) to integrate by parts in kL and obtain a much more tractable expression.

1. The vacuum integrals

We can evaluate the loop integrals in $\Pi^{(s),\mathrm{vac}}$. by "undoing" an integral over an auxiliary continuous variable k_4. For example, (defining $\omega \equiv \sqrt{k^2 + \Delta}$ for positive Δ)

$$\begin{aligned} \int \frac{d^3k}{(2\pi)^3} L I_0^{\mathrm{vac}} &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega} \\ &= \int \frac{d^3k}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dk_4}{2\pi}\frac{1}{(k_4)^2 + k^2 + \Delta}, \end{aligned}$$

thus mapping the integral over $k \in \mathbb{R}^3$ to one over $\tilde{k} \in \mathbb{R}^4$. Then we regulate the expression by taking the analytic continuation to $d \equiv 4 - \varepsilon$ dimensions. In summary,

$$\int \frac{dk}{2\pi^2} k^2 L I_1^{\mathrm{vac}} \to \mu^{-\varepsilon}\int \frac{d^d\tilde{k}}{(2\pi)^d}\frac{1}{(\tilde{k}^2 + \Delta)^2}, \tag{3.34a}$$

$$\int \frac{dk}{2\pi^2} k^2 L^3 I_2^{\mathrm{vac}} \to \mu^{-\varepsilon}\int \frac{d^d\tilde{k}}{(2\pi)^d}\frac{(k_4)^2}{(\tilde{k}^2 + \Delta)^2}, \tag{3.34b}$$

$$\int \frac{dk}{2\pi^2} k^2 L I_0^{\mathrm{vac}} \to \mu^{-\varepsilon}\int \frac{d^d\tilde{k}}{(2\pi)^d}\frac{1}{(\tilde{k}^2 + \Delta)}. \tag{3.34c}$$

The expressions on the lhs are the relevant $\mathbb{R}^3$ integrals, and μ is the $\overline{\mathrm{MS}}$ scale of the theory. "→" means "analytically continues to". On the other hand, we also have the following series of relations under the integral sign,

$$\begin{aligned} \frac{\tilde{k}^2}{d}\delta^{MN}|_{\mathbb{R}^d} &\equiv \overline{(\tilde{k}^M\tilde{k}^N)}|_{\mathbb{R}^d} \leftarrow [\overline{(k^\mu k^\nu)}\mathcal{P}^{\mu\nu}_{MN} + (k_4)^2\mathcal{P}^{44}_{MN}]|_{\mathbb{R}^3\times S^1} \\ &= \left[\frac{k^2}{3}\mathcal{P}^{\mu\nu}_{MN}\delta_{\mu\nu} + (k_4)^2\mathcal{P}^{44}_{MN}\right]|_{\mathbb{R}^3\times S^1}, \end{aligned} \tag{3.35}$$

where $\mathcal{P}^{44}_{MN}$ and $\mathcal{P}^{\mu\nu}_{MN}$ are the projectors defined in (3.30b) and (3.30c). Combining these expressions, the vacuum integrals can be rewritten as integrals in $d = (4 - \varepsilon)$ dimensions by restoring the $SO(4)$ symmetry,

$$
\begin{aligned}
(\Pi^{(s),\mathrm{vac}})^{ab}_{MN} &\equiv \sum_{\mathcal{I}} (\Pi^{(s),\mathrm{vac.}}_{\mathcal{I}})^{ab}_{MN} \\
&= \sum_\beta \beta^a\beta^b \int_0^1 dx \int \frac{d^d\tilde{k}}{(2\pi)^d} \frac{\mu^{4-d}}{(\tilde{k}^2+\Delta_s)^2} \left\{ \left[\frac{d-2}{2}\tilde{k}^2 + \Delta_s\right] d(s)\delta_{MN} \right. \\
&\quad \left. + \left[\frac{(1-2x)^2}{2} d(s) - 2c(s)\right] [(p^2\delta_{\mu\nu} - p_\mu p_\nu)\mathcal{P}^{\mu\nu}_{MN} + p^2\mathcal{P}^{44}_{MN}] \right\}. \qquad (3.36)
\end{aligned}
$$

Expanding in powers of $1 \gg \varepsilon > 0$, it is easy to regulate the $\tilde{k}$ integral to get a convergent result. The (Abelian part of the) UV counterterm, $\delta Z_s \mathrm{tr} F_{MN}F^{MN}$, contributes diagrammatically,

$$
\left[A^a_M \sim\!\otimes\!\sim A^b_N \right]^{\mathrm{CT},(s)} = \delta Z_s \sum_\beta \beta^a\beta^b \left[(p^2\delta_{\mu\nu} - p_\mu p_\nu)\mathcal{P}^{\mu\nu}_{MN} + p^2\,\mathcal{P}^{44}_{MN}\right]. \qquad (3.37a)
$$

So, for each s we choose

$$
-\delta Z_s \equiv \frac{1}{32\pi^2}\left(\frac{d(s)}{3} - 4c(s)\right)\left(\frac{2}{\varepsilon} - \gamma + \log 4\pi\right), \qquad (3.37b)
$$

and the sum of the three *regulated* vacuum integrals is therefore

$$
\begin{aligned}
(\tilde{\Pi}^{(s),\mathrm{vac}})^{ab}_{MN} &\equiv (\Pi^{(s),\mathrm{vac}})^{ab}_{MN} + (\text{counterterms}) \\
&= \sum_\beta \frac{\beta^a\beta^b}{32\pi^2} \int_0^1 dx [d(s)(1-2x)^2 - 4c(s)] \\
&\quad \times [(p^2\delta_{\mu\nu} - p_\mu p_\nu)\mathcal{P}^{\mu\nu}_{MN} + p^2\mathcal{P}^{44}_{MN}] \log\left(\frac{\mu^2}{\Delta_s}\right). \qquad (3.38)
\end{aligned}
$$

2. The pseudothermal integrals

Now we consider the pseudothermal integrals. Using Eqs. (3.32a) and (3.32b), we can simplify the loop integrals immensely by integrating by parts by changing variables $\frac{\partial}{\partial(\omega L)^2} = \frac{1}{2kL}\frac{\partial}{\partial(kL)}$. We find that all boundary terms vanish, and the results are, in summary,

$$
\int_0^\infty d(kL)(kL)^2 \delta I_1(b, \omega L) = \frac{1}{2}\int_0^\infty d(kL)\delta I_0(b,\omega L), \qquad (3.39a)
$$

$$
\int_0^\infty d(kL)(kL)^4 \delta I_1(b,\omega L) = \frac{3}{2}\int_0^\infty d(kL)(kL)^2\delta I_0(b,\omega L), \qquad (3.39b)
$$

$$
\int_0^\infty d(kL)(kL)^2\delta I_2(b,\omega L) = -\frac{1}{2}\int_0^\infty d(kL)(\omega L)^2 \delta I_0(b,\omega L). \qquad (3.39c)
$$

Plugging into Eqs. (3.31a) and (3.31b), the pseudothermal integrals may be written

$$
\begin{aligned}
(\delta\Pi^{(s)})^{ab}_{\mu\nu} &= \sum_\beta \frac{\beta^a\beta^b}{8\pi^2}\int_0^1 dx[d(s)(1-2x)^2 - 4c(s)] \\
&\quad \times (p^2\delta_{\mu\nu} - p_\mu p_\nu) R^b_0(\sqrt{\Delta_s}L), \qquad (3.40)
\end{aligned}
$$

where we have defined[14]

$$
\begin{aligned}
R^b_0(\sqrt{\Delta}L) &\equiv \int_0^\infty d(kL)\cdot \delta I_0\left(b, \sqrt{(kL)^2 + \Delta L^2}\right) \\
&= \int_0^\infty \frac{d(kL)}{2\sqrt{(kL)^2+\Delta L^2}} \mathrm{Re}\left(\frac{1}{e^{\sqrt{(kL)^2+\Delta L^2}+ib} - 1}\right) \\
&= \sum_{n=1}^\infty K_0(n\sqrt{\Delta}L)\cos(n\phi\cdot\beta_{ij}), \qquad (3.41)
\end{aligned}
$$

where K_0 is the modified Bessel function of order 0. This represents the only remaining nontrivial sum, as far as the corrections to κ_{ab} are concerned. We have not given an expression for $(\delta\Pi^{(s)})^{ab}_{44}$ as it is not needed to find κ_{ab}.

Summing the result with the vacuum contribution,

$$
\begin{aligned}
&(\Pi^{(s),\mathrm{vac}})^{ab}_{\mu\nu} + (\delta\Pi^{(s)})^{ab}_{\mu\nu} + (\text{counterterms}) \\
&\quad = \frac{(p^2\delta_{\mu\nu} - p_\mu p_\nu)}{8\pi^2}\int_0^1 dx[d(s)(1-2x)^2 - 4c(s)] \\
&\quad \times \left(\mathcal{R}^{ab}_0(\sqrt{\Delta_s}L) + \delta^{ab} N \log\frac{\mu}{\sqrt{\Delta_s}}\right), \qquad (3.42)
\end{aligned}
$$

where we have defined

[14]The integral in the second line can be carried out by expanding in series in $|e^{-\sqrt{(kL)^2+\Delta L^2} - ib}| < 1$.

$$\mathcal{R}_0^{ab}(\sqrt{\Delta}L) \equiv \sum_{i,j} \beta_{ij}^a \beta_{ij}^b R_0^{\phi\cdot\beta_{ij}}(\sqrt{\Delta}L) = \sum_{i,j} \beta_{ij}^a \beta_{ij}^b \sum_{n=1}^{\infty} K_0(n\sqrt{\Delta}L)\cos(n\phi\cdot\beta_{ij}). \tag{3.43}$$

In Appendix A 1 we explicitly show that

$$R_0^b(\sqrt{\Delta}L) = \frac{1}{2}\log\frac{\sqrt{\Delta}L}{4\pi} + \tilde{R}_0^b(\sqrt{\Delta}L), \tag{3.44}$$

where $\tilde{R}_0^b(t)$ is a pure function that has a power series expansion around $t = 0$ for fixed $b \in (0, 2\pi)$. We know that Eq. (3.44) must be true because the running of the coupling g^2 must freeze out at scales below m_W. Equation (3.44) allows us to disregard the p^2 dependence in $R_0^b(\sqrt{\Delta}L)$ as higher-derivative corrections, and integrate over the Feynman parameter x trivially. Recalling Eqs. (3.29) and (3.20),

$$\sum_{s=0,1}\chi(s)\left(4c(s) - \frac{d(s)}{3}\right) = \frac{11}{3}, \tag{3.45a}$$

and

$$\chi(1/2)\left(4c(1/2) - \frac{d(1/2)}{3}\right) = -\frac{2}{3}, \tag{3.45b}$$

we have

$$\kappa_{ab} = \frac{m_W^{-1}}{16\pi}\left[\frac{8\pi^2}{Ng^2(\frac{4\pi}{L})}\delta_{ab} + \frac{1}{N}\sum_{i,j}\beta_{ij}^a\beta_{ij}^b\left(\frac{11}{3}\tilde{R}_0^{\phi\cdot\beta_{ij}}(0) - \frac{2}{3}\sum_{I=1}^{n_f}\tilde{R}_0^{\phi\cdot\beta_{ij}}(m_I L)\right)\right]. \tag{3.46}$$

An expression for $\tilde{R}_0^{\phi\cdot\beta_{ij}}(t)$ is derived in Eq. (A17). All that remains now is to diagonalize Eq. (3.46).

D. The sums over β: Linear algebra on the root lattice

Let us consider the sums over the root vectors β. It is not hard to show by standard Fourier analysis that, for any integer n,

$$C_n^{ab} := \sum_{i,j}\beta_{ij}^a\beta_{ij}^b\cos\left(\frac{2\pi}{N}n(i-j)\right) = 2(N\delta_{N\equiv n}\delta^{ab} - 1)\cos\left(\frac{2\pi}{N}n(a-b)\right), \tag{3.47}$$

where

$$\delta_{n\equiv k} = \begin{cases} 1 & n\equiv k, \\ 0 & n\not\equiv k. \end{cases} \tag{3.48}$$

The relation "≡" is to be understood here as equality in the mod N sense (we instead use "≔" to denote "is defined to be" for this subsection).

The matrix in Eq. (3.47) is diagonalized by the (trace-free) eigenvectors u_ℓ with vector components

$$(u_\ell)^b := e^{i\frac{2\pi}{N}\ell b}, \qquad 1 \le \ell \le N-1, \tag{3.49}$$

and have eigenvalues indexed by ℓ,

$$\sum_{b=1}^{N} C_n^{ab}(u_\ell)^b = N(2\delta_{n\equiv N} - \delta_{n\equiv\ell} - \delta_{n\equiv N-\ell})(u_\ell)^a. \tag{3.50}$$

Plugging Eq. (3.50) into Eq. (3.43), and recalling $m_W = \frac{2\pi}{NL}$, we can read off the eigenvalues $\mathcal{R}_{0\ell}$ of $\mathcal{R}_0^{ab}$,

$$\mathcal{R}_{0\ell}(mL) = N\sum_{p=1}^{\infty}\left\{2K_0\left(2\pi p\frac{m}{m_W}\right) - K_0\left[2\pi\left(p-\frac{\ell}{N}\right)\frac{m}{m_W}\right] - K_0\left[2\pi\left(p-1+\frac{\ell}{N}\right)\frac{m}{m_W}\right]\right\}. \tag{3.51}$$

When $m \gtrsim m_W$, this series is very well-approximated by the $p = 1$ term. However, some extra work is needed to extract information about the $m \ll m_W$ case. In Appendix A 2, we perform the sum over p by taking the Mellin transform and find [cf. Eq. (A24)]

$$\mathcal{R}_{0\ell}(mL) = N\left[\gamma + \log\frac{m}{m_W}\sin\pi\frac{\ell}{N} + W_\ell\left(\frac{m}{m_W}\right)\right], \tag{3.52a}$$

where, as mentioned before, W_ℓ is an $O(1)$ function such that $W_\ell(0) = 0$, and has a power series expansion for $\tau \equiv (m/m_W) < 1$,

$$W_\ell(\tau) = \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}\left(\frac{i\tau}{2}\right)^{2n}[\zeta(2n+1) - \mathrm{Re}(\mathrm{Li}_{2n+1}e^{2\pi i\frac{\ell}{N}})], \quad \tau < 1. \tag{3.52b}$$

This is exactly Eq. (2.15a), and Eq. (2.15b) follows directly from Eqs. (3.51) and (3.52a). Putting everything together, we finally obtain Eq. (2.13),

$$\kappa_\ell = \frac{m_W^{-1}}{16\pi}\left[\frac{8\pi^2}{Ng^2(m_W e^{-\gamma})} + b_0\log\frac{1}{\sin\pi\frac{\ell}{N}} + \frac{2}{3}\sum_I^{n_f} W_\ell\left(\frac{m_I}{m_W}\right)\right], \quad 1 \le \ell \le N-1. \tag{3.53}$$

Note that although heretofore the fermion masses only appeared in the combination mL, Eqs. (3.52a) and (3.53) suggest that they are in fact more naturally measured in units of m_W, as we should expect.

On the other hand, the 44 parts of the integrals also give us M^2_{ab}, the scalar (mass)^2 matrix. Omitting the intermediate steps,

$$M^2_{ab} = g^2 \sum_\beta \sum_{n=1}^{\infty} \frac{\beta^a \beta^b}{4\pi^2 L^2} \left[\sum_I^{n_f} (m_I L)^2 K_2(n m_I L) - \frac{2}{n^2} \right] \times \cos(n\beta \cdot \phi), \tag{3.54}$$

where K_2 is the modified Bessel function of order 2. This matches the result from taking the second derivative of the GPY potential, (2.9), which serves as a "sanity check" on our calculations. For completeness, we present M^2_ℓ, the physical scalar (masses)^2,

$$M^2_\ell = g^2 N m_W^2 \left[\sum_I^{n_f} F_\ell\left(\frac{m_I}{m_W}\right) - F_\ell(0) \right], \tag{3.55a}$$

where

$$F_\ell(\tau) \equiv \frac{\tau^2}{4\pi^2} \sum_{p=1}^{\infty} \left\{ K_2\left[2\pi\left(p - 1 + \frac{\ell}{N}\right)\tau\right] + K_2\left[2\pi\left(p - \frac{\ell}{N}\right)\tau\right] - 2K_2(2\pi p \tau) \right\}. \tag{3.55b}$$

We also present ρ_ℓ, the eigenvalues of ρ_{ab},

$$\rho_\ell = \kappa_\ell + \frac{m_W^{-1}}{96\pi}\left[1 - \sum_I^{n_f} X_\ell\left(\frac{m_I}{m_W}\right)\right], \tag{3.56a}$$

where X_ℓ is an $O(1)$ function defined in terms of W_ℓ,

$$X_\ell(\tau) = 1 + 4\tau^2 \frac{d}{d\tau^2} W_\ell(\tau). \tag{3.56b}$$

Equations (3.55) and (3.56) are only presented for completeness, although they may be found without too much difficulty using the methods described in this paper.

IV. FUTURE DIRECTIONS

In this study we have derived an explicit one-loop expression for the eigenvalues of κ_{ab}, the polarization operator of the $SU(N)$ dYM theory with massive fermions, and provisionally surveyed some properties of the emergent fourth dimension. It would be interesting to numerically examine the effect of these one-loop corrections on the k-string tensions, (as was done for SYM in Ref. [26],) but to do so would require us to compute the matrix determinants in the monopole measure, ζ—a daunting task (see the discussion in Footnote 11).

Additionally, the topological angle θ dependence in Yang-Mills theory has been the subject of much attention [27–31]; we should also like to examine the dependence of the k-string tensions on the topological angle θ as well as on the circle length L, at the tree-order level, to compare against results on the lattice.

Finally, we would also like to further study the confining properties of dYM outside of the calculable regime, $NL\Lambda \gg 1$ and its conjectured continuity with the small $NL\Lambda$ regime, on the lattice.

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APPENDIX: THE MELLIN TRANSFORM, AND SOME RESULTS

In this appendix we explicitly evaluate the sums over p in (3.51) to obtain an expression for κ_ℓ in terms of analytic functions. To this end, we introduce the Mellin transform, an integral transform on real-valued functions.

Definition A.1. The Mellin transform $\mathcal{M}$ is an integral transform defined on the space of real integrable functions $f\colon \mathbb{R}^+ \to \mathbb{R}$ as

$$\varphi(s) \equiv \mathcal{M}_s[f(t)] \equiv \int_0^\infty dt t^{s-1} f(t). \tag{A1a}$$

In particular, for each $\lambda > 0$,

$$\mathcal{M}_s[f(\lambda t)] = \lambda^{-s} \mathcal{M}_s[f(t)]. \tag{A1b}$$

The inverse transform $\mathcal{M}^{-1}$ is, formally,

$$f(t) = \mathcal{M}_t^{-1}[\varphi(s)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds t^{-s} \varphi(s), \tag{A1c}$$

where c is some real number chosen so that the integral in (A1c) converges (see Sec. 2.5 of [32]). Usually what this means is to take the sum over the residues of the poles of $\varphi(s)$ on the real half-line, $s \in (-\infty, c]$. To illustrate with a simple example, let us compute the Mellin transform of $f(t) = e^{-t}$, and its inverse.

Example A.1. *Directly from the definition,*

$$\mathcal{M}_s[e^{-t}] \equiv \int_0^\infty dt t^{s-1} e^{-t} = \Gamma(s). \tag{A2}$$

Now consider the inverse transform. Since $\Gamma(s)$ has poles at $s = 0, -1, -2\ldots$, we evaluate the integral by limiting the integration contour $c \to 0^+$ and closing the contour over the $\mathrm{Re}(s) < 0$ half-plane. The integral over the arc goes to zero at large radius, so

$$\lim_{c\to 0^+}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dst^{-s}\Gamma(s) = \sum_{n=0}^{\infty}\operatorname*{res}_s(t^{-s}\Gamma(s),-n) \overset{\text{A.4}}{=} \frac{(-t)^n}{n!} = e^{-t}, \tag{A3}$$

as expected, because near the poles of $\Gamma(s)$,

$$\Gamma(s) = \frac{(-1)^n}{n!}[(s+n)^{-1} + \psi^{(0)}(1+n)] + O(s+n), \quad n = 0,1,2\ldots \tag{A4}$$

where $\psi^{(0)}(z) \equiv \frac{d}{dz}\log(\Gamma(z))$ *is the polygamma function (of order 0).*

1. Proof of equation (3.44)

We are now prepared to prove Eq. (3.44) and derive a series expression for $\tilde{R}_0^b$. The idea is to perform the sums over n in "Mellin space," then transform back to "mass space" to obtain a series expansion in t. Like in Example A.1, the inverse transform involves evaluating the residue of a chain of poles on the real axis.

To begin, we observe the Mellin transform of the modified Bessel function of order ν, K_ν, is known to be [33,35]

$$\mathcal{M}_s[K_\nu(t)] = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right). \tag{A5}$$

Plugging this into Eq. (3.41),

$$\mathcal{M}_s[R_0^b(t)] = \sum_{n=1}^{\infty}\mathcal{M}_s[K_0(nt)]\cos(nb) \overset{\text{A.1b}}{=} 2^{s-3}\Gamma\left(\frac{s}{2}\right)^2\sum_{n=1}^{\infty}n^{-s}(e^{inb}+e^{-inb}) \overset{\text{A.7}}{=} 2^{s-3}\Gamma\left(\frac{s}{2}\right)^2(\mathrm{Li}_s e^{ib} + \mathrm{Li}_s e^{-ib}), \tag{A6}$$

where Li_s is the polylogarithm function of order s,

$$\sum_{k=1}^{\infty}\frac{e^{ikb}}{k^s} = \mathrm{Li}_s e^{ib}, \qquad b \text{ real}, \qquad s > 0. \tag{A7}$$

Changing back to the original variable t,

$$R_0^b(t) = \mathcal{M}_t^{-1}\mathcal{M}_s[R_0^b(t')] = \frac{1}{2\pi i}\int_{-i\infty+c}^{+i\infty+c} dst^{-s}2^{s-3}\Gamma\left(\frac{s}{2}\right)^2(\mathrm{Li}_s e^{ib} + \mathrm{Li}_s e^{-ib}). \tag{A8}$$

Note that the integral in Eq. (A8) is over the order s of the polylogarithm, rather than its argument. As in Example A.1, we can evaluate this integral by letting the integration contour approach the imaginary axis from the right, $c \to 0^+$, and close the contour over the half-plane $\mathrm{Re}(s) \le 0$. The polylogarithm terms are regular for all s for real $0 < b < 2\pi$, so we are left with the residues from the chain of poles at $s = 0, -2, -4\ldots$, where the gamma function diverges.

Unfortunately, the poles of $\Gamma(s/2)^2$ are of order 2, so evaluating the residues with the integrand of (A8), as is, would involve the expression $\frac{d}{ds}\mathrm{Li}_s e^{ib}$, which produces a result that is even more opaque than our original expression.

However, a known identity [see Eq. (25.13.3) in [32]] relates the polylogarithms to the Hurwitz zeta function, ζ,

$$i^{-s}\mathrm{Li}_s(e^{ib}) + i^s\mathrm{Li}_s(e^{-ib}) = \frac{(2\pi)^s}{\Gamma(s)}\zeta\left(1-s,\frac{b}{2\pi}\right), \quad 0 < b < 2\pi. \tag{A9}$$

Where $\zeta(z,x)$ satisfies,

$$\zeta(z,x) = \sum_{n=1}^{\infty}(n+x)^{-z}, \qquad \mathrm{Re}(s) > 1 \quad \text{and} \quad x \neq 0,1,2\ldots. \tag{A10}$$

We can sum the expression in Eq. (A9) with $b \to 2\pi - b$ and divide by $(i^s + i^{-s})$ to rewrite the integrand of (A8) as

$$t^{-s}\mathcal{M}_s[R_0^b(t')] \overset{\text{A.8}}{=} 2^{-3}\Gamma\left(\frac{s}{2}\right)^2(\mathrm{Li}_s(e^{ib}) + \mathrm{Li}_s(e^{-ib}))\left(\frac{t}{2}\right)^{-s} \overset{\text{A.9}}{=} \frac{2^{-4}}{(1+i^{-2s})}\frac{\Gamma(\frac{s}{2})^2}{\Gamma(s)}\left(\frac{it}{4\pi}\right)^{-s} \times\left[\zeta\left(1-s,\frac{b}{2\pi}\right) + \zeta\left(1-s,1-\frac{b}{2\pi}\right)\right]. \tag{A11}$$

This is helpful because the factor of $\Gamma(s)^{-1}$ in Eq. (A9) reduces the order of the poles by one, and the zetas in the parentheses in Eq. (A11) are regular except at $s = 0$, so the poles at $s = -2, -4, -6\ldots$ are simple.

a. The residue at $s = 0$

Near $s = 0$, the zeta terms diverge like $1/s$,

$$\zeta\left(1-s,\frac{b}{2\pi}\right) + \zeta\left(1-s,1-\frac{b}{2\pi}\right) = -\frac{2}{s} - \psi^{(0)}\left(\frac{b}{2\pi}\right) - \psi^{(0)}\left(1-\frac{b}{2\pi}\right) + O(s). \tag{A12}$$

So we must also look at the series expansion of the regular terms in (A11) near $s = 0$,

$$(1+i^{-2s})^{-1}\left(\frac{it}{4\pi}\right)^{-s}=\frac{1}{2}-\frac{1}{2}\log\frac{t}{4\pi}s+O(s^2). \quad \text{(A13)}$$

Combining Eqs. (A12), (A13), and (A4), we find our famous logarithmic term

$$\begin{aligned}&\operatorname*{res}_s(t^{-s}\mathcal{M}_s[R_0^b],0)\\&\quad=\frac{1}{2}\log\frac{t}{4\pi}-\frac{1}{4}\left[\psi^{(0)}\left(\frac{b}{2\pi}\right)+\psi^{(0)}\left(1-\frac{b}{2\pi}\right)\right]. \quad \text{(A14)}\end{aligned}$$

b. The residues at $s=-2,-4,-6\ldots$

Since the poles at $s=-2,-4,-6\ldots$ can only contribute terms $\sim t^{2n}$ for $n=1,2,3\ldots$, we have proven our claim in (3.44), so we are actually *done*, but since we have already done most of the work,

$$\begin{aligned}&\operatorname*{res}_s(t^{-s}\mathcal{M}_s[R_0^b],-2n)\\&\quad=\frac{1}{4}\frac{(2n)!}{(n!)^2}\left(\frac{it}{4\pi}\right)^{2n}\left[\zeta\left(1+2n,\frac{b}{2\pi}\right)+\zeta\left(1+2n,1-\frac{b}{2\pi}\right)\right]\\&\quad\overset{A.16}{=}\frac{-1}{4(n!)^2}\left(\frac{it}{4\pi}\right)^{2n}\left[\psi^{(2n)}\left(\frac{b}{2\pi}\right)+\psi^{(2n)}\left(1-\frac{b}{2\pi}\right)\right],\\&\qquad n=1,2,3\ldots \quad \text{(A15)}\end{aligned}$$

where $\psi^{(2n)}$, the polygamma function of order $2n$, is related to ζ by [32,36]

$$\psi^{(2n)}(z)=-(2n)!\zeta(2n+1,z),\quad n=1,2,3\ldots \quad \text{(A16)}$$

Putting our results together,

$$\begin{aligned}\tilde{R}_0^b(t)=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{t}{4\pi}\right)^{2n}\\\times\left[\psi^{(2n)}\left(\frac{b}{2\pi}\right)+\psi^{(2n)}\left(1-\frac{b}{2\pi}\right)\right]. \quad \text{(A17)}\end{aligned}$$

Plugging this result into Eq. (3.46) and taking the massless limit, $t=0$, the correction to the photon coupling matches that of the SYM result derived in Ref. [10].

2. Derivation of Eq. (3.52)

Now, consider our expression for $\mathcal{R}_{0\ell}$, the eigenvalues of $\mathcal{R}_0^{ab}$, Eq. (3.51). Let us define

$$\nu_\ell(\tau)\equiv\sum_{p=1}^{\infty}K_0\left[\left(\frac{\ell}{N}+p-1\right)\tau\right]+K_0\left[\left(-\frac{\ell}{N}+p\right)\tau\right], \quad \text{(A18a)}$$

$$\xi(\tau)\equiv\sum_{p=1}^{\infty}2K_0(p\tau), \quad \text{(A18b)}$$

$$\tau\equiv 2\pi\frac{m}{m_W}=NLm. \quad \text{(A18c)}$$

Starting with ν_ℓ: the intermediate steps are largely the same as in the preceding subsection,

$$\begin{aligned}\nu_\ell(\tau)&\equiv\mathcal{M}_t^{-1}\mathcal{M}_s\left\{\sum_{p=1}^{\infty}K_0\left[\tau\left(p-1+\frac{\ell}{N}\right)\right]+K_0\left[\tau\left(p-\frac{\ell}{N}\right)\right]\right\}\\&\overset{A.1b}{=}\mathcal{M}_t^{-1}\left[\tau^{-s}2^{s-2}\Gamma\left(\frac{s}{2}\right)^2\sum_{p=1}^{\infty}\left[\left(p-1+\frac{\ell}{N}\right)^{-s}+\left(p-\frac{\ell}{N}\right)^{-s}\right]\right.\\&\overset{A.10}{=}\sum_{\text{poles}}\operatorname*{res}_s\left[\tau^{-s}2^{s-2}\Gamma\left(\frac{s}{2}\right)^2\left[\zeta\left(s,1-\frac{\ell}{N}\right)+\zeta\left(s,\frac{\ell}{N}\right)\right],\{\text{pole}\}\right]\\&=\frac{\pi}{\tau}+\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{\tau}{2}\right)^{2n}\left[\zeta'\left(-2n,1-\frac{\ell}{N}\right)+\zeta'\left(-2n,\frac{\ell}{N}\right)\right.\\&\qquad\left.+\cancelto{0}{\left[\zeta\left(-2n,1-\frac{\ell}{N}\right)+\zeta\left(-2n,\frac{\ell}{N}\right)\right]}\left(\psi(n+1)-\log\frac{\tau}{2}\right)\right]\\&\overset{A.19}{=}\frac{\pi}{\tau}+\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{\tau}{2}\right)^{2n}\left[\zeta'\left(-2n,1-\frac{\ell}{N}\right)+\zeta'\left(-2n,\frac{\ell}{N}\right)\right]\\&\overset{A.20}{=}\frac{\pi}{\tau}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2}\left(\frac{i\tau}{4\pi}\right)^{2n}\left(\mathrm{Li}_{2n+1}e^{2\pi i\frac{\ell}{N}}+\mathrm{Li}_{2n+1}e^{-2\pi i\frac{\ell}{N}}\right),\end{aligned}$$

The expression in the third line has order-2 poles at $s = 0, -2, -4\ldots$ (from the gammas) and a simple pole at $s = 1$, (from the zetas) and $\zeta'(s,x) \equiv \frac{d}{ds}\zeta(s,x)$.

The striked out term in the fourth line vanishes because [32,37]

$$\zeta(-2n,x) = -\frac{B_{2n+1}(x)}{2n+1} = \frac{B_{2n+1}(-x)}{2n+1}, \quad x \text{ real}, \quad n = 0,1,2\ldots \tag{A19}$$

where $B_k(x)$ is the Bernoulli polynomial of order k, which has parity $(-1)^k$ under $x \to 1-x$.

Lastly, the final line follows from [see Eq. (13) in [34]]

$$\zeta'(-2n,1-x) + \zeta'(-2n,x) = \frac{(2n)!}{(2\pi i)^{2n}}(\mathrm{Li}_{2n+1}e^{2\pi i x} + \mathrm{Li}_{2n+1}e^{-2\pi i x}), \quad x \text{ real}, \quad n = 0,1,2\ldots \tag{A20}$$

and since $\mathrm{Li}_1(z) = -\log(1-z)$,

$$\nu_\ell(\tau) = \frac{\pi}{\tau} + \log\frac{1}{2\sin\pi\frac{\ell}{N}} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}\left(\frac{i\tau}{4\pi}\right)^{2n}(\mathrm{Li}_{2n+1}e^{2\pi i\frac{\ell}{N}} + \mathrm{Li}_{2n+1}e^{-2\pi i\frac{\ell}{N}}). \tag{A21}$$

To solve for ξ, we observe that, for the plain (Riemann) zetas $\zeta(z,0) \equiv \zeta(z)$ [33,38],

$$\zeta'(-2n) = \frac{(2n)!}{(2\pi i)^n}\zeta(2n+1), \tag{A22}$$

and the sum over p goes through almost verbatim. The result is

$$\xi(\tau) = \sum_{\text{poles}}\operatorname*{res}_s\left[\tau^{-s}2^{s-2}\Gamma\left(\frac{s}{2}\right)^2\zeta(s), \{\text{pole}\}\right] = \frac{\pi}{\tau} + \gamma + \log\frac{\tau}{4\pi} + \sum_{n=1}^{\infty}\left(\frac{i\tau}{4\pi}\right)^{2n}\frac{(2n)!}{(n!)^2}\zeta(2n+1). \tag{A23}$$

As before, the poles are located at $s = 0, -2, -4\ldots.$ and $s = 1$. So together,

$$N^{-1}\mathcal{R}_{0\ell} = \xi - \nu_\ell = \gamma + \log\left(\sin\pi\frac{\ell}{N}\tau\right) + W_\ell(\tau), \tag{A24}$$

where W_ℓ has a power series expansion in τ,

$$W_\ell(\tau) \equiv \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}\left(\frac{i\tau}{2}\right)^{2n}\Big[\zeta(2n+1) - \frac{1}{2}(\mathrm{Li}_{2n+1}e^{2\pi i\frac{\ell}{N}} + \mathrm{Li}_{2n+1}e^{-2\pi i\frac{\ell}{N}})\Big]. \tag{A25a}$$

As mentioned in Footnote 7, the root test shows that this infinite series diverges for $m \geq m_W$. In that case, W_ℓ can be approximated by

$$W_\ell(\tau) = \gamma + \log\sin\pi\frac{\ell}{N}\tau + 2K_0(2\pi\tau) - K_0\left[2\pi\left(1-\frac{\ell}{N}\right)\tau\right] - K_0\left(2\pi\frac{\ell}{N}\tau\right) + O(e^{-2\pi\tau}). \tag{A25b}$$

Manipulating the series expansions for $\zeta(2n+1)$ and Li_{2n+1}, Eqs. (A7) and (A10), we can also write $W_\ell(\tau)$ purely in terms of elementary functions,

$$W_\ell(\tau) = 2\sum_{k=1}^{\infty}[(k^2+\tau^2)^{-1/2} - k^{-1}]\sin^2\left(\pi\frac{\ell}{N}k\right). \tag{A25c}$$

We note that Eq. (A25c) converges much more slowly than the previous ones, but on the other hand, it clearly shows that W_ℓ is strictly negative, and goes to zero as $\frac{\ell}{N} \to 0$. Observing that

$$\frac{d^2}{dx^2}\mathrm{Li}_{2n+1}(e^{ix}) = -\mathrm{Li}_{2n-1}(e^{ix}), \tag{A26}$$

a similar argument shows that W_ℓ is strictly concave up in ℓ for all $1 \leq \ell \leq \lfloor N/2 \rfloor$.

Finally, Eq. (3.51) shows that $\mathcal{R}_{0\ell} \to 0$ rapidly as $\tau \to \infty$, so (A24) demands

$$W_\ell(\tau) \to \log\frac{e^{-\gamma}}{\tau\sin\pi\frac{\ell}{N}}, \qquad \tau \to \infty, \tag{A27}$$

which concludes the proofs for the statements we made about W_ℓ in Eq. (2.14).

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