Infrared renormalons in supersymmetric theories

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I argue that in large-N supersymmetric QCD infrared renormalons are absent in the conformal window, there is no need in conspiracy between the coefficient functions and the vacuum expectation values of at least some gluon operators—no factorials appear in the former and the latter vanish. Based on this conclusion, I conjecture that in supersymmetric gluodynamics (supersymmetric Yang-Mills theory without matter) at least the leading renormalon ambiguity disappears, which would be consistent with the fact that the gluon condensate vanishes in this theory, $\langle G^a_{\mu\nu} G^{\mu\nu a}\rangle = 0$.

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I. INTRODUCTION

Renormalons are described by a special class of graphs—the so-called bubble chains—which were identified in 1977 [1] (see also the review in Ref. [2]) as the source of a factorial divergence of perturbation theory at high orders.

The literature on this phenomenon is huge. The so-called Ünsal resurgence program [3] associated with the factorial divergence of perturbation theory proved to be very useful in many quantum-mechanics problems, in solutions of partial differential equations, and in some asymptotically free field theories in which the running of the coupling constant in the IR domain can be frozen in some way. In theories with a genuinely strong coupling IR regime, such as QCD, renormalons do appear only formally. In fact, the renormalon-associated factorial explosion at high orders is a spurious effect that emerges because formal expressions are used beyond their limit of applicability. What does that mean and what is to be done? In this introductory section I briefly explain the answers to these questions.

Let us consider the diagram shown in Fig. 1, representing a typical bubble chain in QCD for a leading renormalon. The IR contribution of this chain can be written as

$$D \propto Q^2 \int dk^2 \frac{k^2 \alpha(k^2)}{(k^2+Q^2)^3}, \tag{1}$$

where α is the running coupling constant and k^2 is the momentum running through the long gluon (dashed) line. It is obvious that at $k^2 \lesssim \Lambda^2$, Eq. (1) makes no sense because $\alpha(k^2)$ does not exist for such values of k^2; quarks and gluons do not exist either. It is straightforward to assess the corresponding ambiguity, which is of the order of $O(\Lambda^4/Q^4)$.

In the renormalon construction the factorial divergence in Eq. (1) is "demonstrated" in a completely formal way through the expansion of the running $\alpha(k^2)$,

$$\alpha(k^2) = \frac{\alpha(Q^2)}{1 - b_1 \frac{\alpha(Q^2)}{4\pi} \ln(Q^2/k^2)}, \tag{2}$$

with the subsequent expansion of the right-hand side in $\alpha(Q^2)$. Here b_1 is the first coefficient of the β function. Equation (2) describes the running constant $\alpha(k^2)$ expressed in terms of $\alpha(Q^2)$ with a fixed (large) Q^2. Expanding the denominator in Eq. (2) in powers $\alpha(Q^2)$, we arrive at the series

$$D(Q^2) \propto \frac{1}{Q^4} \alpha \sum_{n=0}^{\infty} \left(\frac{b_1 \alpha}{4\pi}\right)^n \int dk^2\, k^2 \left(\ln \frac{Q^2}{k^2}\right)^n, \qquad \alpha \equiv \alpha(Q^2), \tag{3}$$

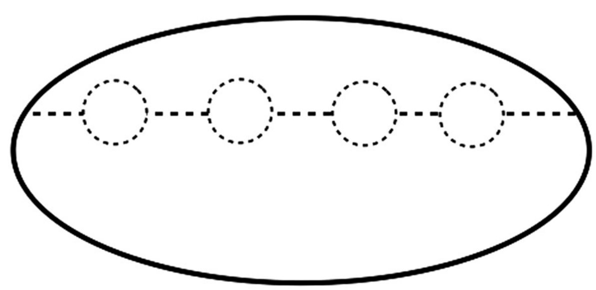

FIG. 1. Graph showing four loops renormalizing a gluon propagator (represented by the dotted line) attached to the quark loop. A renormalon is the sum over all such diagrams with n loops.

which can be rewritten as

$$D(Q^2) \propto \frac{\alpha(Q^2)}{2} \sum_{n=0}^{\infty} \left(\frac{b_1 \alpha(Q^2)}{8\pi}\right)^n \int dy\, y^n e^{-y},$$
$$y = 2 \ln \frac{Q^2}{k^2}. \tag{4}$$

The y integral in Eq. (4) taken from zero to infinity produces $n!$.

Observe, however, that a characteristic value of k^2 saturating the factorial is exponentially suppressed in n,

$$k^2 \sim Q^2 \exp\left(-\frac{n}{2}\right) \quad \text{or} \quad y \sim n. \tag{5}$$

The factorial explodes as we approach the domain $k^2 \lesssim \Lambda^2$, so that the denominator in Eq. (2) hits zero. This was explained in more detail in Refs. [4,5]. Thus, the factorial divergence at high orders in the case at hand is just a signature of the illegitimacy of using Eq. (1) at $k^2 \sim \Lambda^2$, that is, an artifact.

QCD and similar theories are self-consistent. Therefore, they must take care of their problems. The correct way to treat Fig. 1 is to use the Wilson operator product expansion (OPE). To this end, one introduces an auxiliary parameter μ. At strong coupling in confining theories it is assumed that $\mu \sim c\Lambda$, where a numerical factor c must be chosen, say, $c \sim 3$ or ~ 4, i.e., larger than Λ but not parametrically larger. The virtual momenta $k \geq \mu$ are included in the coefficient functions which become well defined. The contribution coming from $k \leq \mu$, i.e., from the soft domain, must be included in full in the vacuum expectation values (VEVs) of various operators. In the case of the leading renormalon (2) and (3), this operator is $G^a_{\mu\nu} G^{\mu\nu\, a}$ with the normal dimension four.

In the OPE, the renormalon issue becomes a nonproblem; it is replaced by the so-called *conspiracy*. The conspiracy implies that the coefficients $C_i(\mu)$ (which would contain renormalons under the formal procedure of letting $\mu \to 0$) must conspire with the gluon operators $O_i(\mu)$ so that the OPE sum $\sum_i C_i(\mu) \langle O_i(\mu) \rangle$ (where $\langle \ldots \rangle$ denotes the VEVs) is well defined and μ independent. Thus, rather than focusing on the "nonproblem" of renormalons, we must focus on the conspiracy mechanism. This works perfectly in nonsupersymmetric theories (see Refs. [4,5] and references therein).

Below, as a shorthand I will refer to the renormalon and the would-be factorial explosion in the coefficients $C_i(\mu)$ (emerging if one tries to send $\mu \to 0$ without implementing the proper conspiracy) as the *renormalon ambiguity*. The term "renormalon ambiguity" is awkward, but yet is widely used in the literature.

In supersymmetric Yang-Mills theories the mechanism of the conspiracy hits an obstacle [6] (see also Refs. [5,7]). Indeed, the VEV $\langle G^a_{\mu\nu} G^{\mu\nu\, a} \rangle = 0$ because $G^a_{\mu\nu} G^{\mu\nu\, a}$ is proportional to the trace of the energy-momentum tensor θ^μ_μ (up to an operator proportional to an equation of motion). This leaves the leading renormalon in Fig. 1 without a conspiracy partner. The issue was addressed in pure $\mathcal{N} = 1$ super-Yang-Mills theory (SYM) in Ref. [6] but not conclusively solved. One of the observations made in Ref. [6] was as follows.

In SYM theory, the isolation of the bubble chain graphs is quite nontrivial. The standard practice in QCD reduces to isolating the *matter* bubble chain, which is characterized by b_1^{matter} rather than b_1 [cf. Eq. (2)]. Isolating the matter bubble chain is easy and unambiguous, unlike the isolation of the gluon bubble chain which cannot be unambiguously defined in a gauge-invariant manner. Then, by default, one just replaces $b_1^{\text{matter}} \to b_1$.

In SYM theory this strategy does not work because of the absence of matter. In a subsequent paper [8] a study of the conspiracy in the two-dimensional supersymmetric $O(N)$ sigma model was carried out in which $\langle \theta^\mu_\mu \rangle$ vanished too. This model is exactly solvable for large N.[1]

Unfortunately, the study in Ref. [8] did not shed light on the issue of conspiracy in four-dimensional SYM for the following reason. Unlike four-dimensional SYM, two-dimensional $O(N)$ has a wider set of available dimension-two operators, and these extra operators (which do not reduce to θ^μ_μ) do indeed conspire to cancel the lowest-dimension renormalon ambiguity.[2]

In a bid to advance our understanding in four dimensions, I add matter to SYM theory, thus converting it to $\mathcal{N} = 1$ super-QCD (SQCD). The theory discussed below has the SU(N) gauge group and N_f massless matter fields (quark and squarks) in the fundamental representation. I will focus first on the conformal window in the Seiberg limit [11]. In this limit we let $N \to \infty$, keeping the 't Hooft coupling fixed [12]. The ratio N_f/N is a parameter that can be changed in a range that is specified below.

I argue that in this theory the renormalon factorials $n!$ and renormalon-associated ambiguities do not appear. No conspiracy with the gluon operator $G^a_{\mu\nu} G^{\mu\nu\, a}$ is needed, and $\langle G^a_{\mu\nu} G^{\mu\nu\, a} \rangle = 0$ is consistent. The VEVs of other gluon operators in the OPE are likely to be zeros too.

Based on these conclusions, I then return to SYM theory without matter and conjecture that renormalons are absent there too (at least, the leading one) and the conspiracy is not required.

The organization of the paper is as follows. In Sec. II I briefly describe SQCD and recall the main known facts.

[1] We had to consider the *next-to-leading order* in the $1/N$ expansion since the leading order is trivial due to factorization [9,10].

[2] Of course, in this case we can be certain in these cancellations even before isolating renormalons since the exact solution is well defined and has no ambiguities.

Section III contains my key assertions. There I present my arguments that renormalon ambiguities do not develop in the conformal window. In Sec. IV I briefly discuss SYM without matter. This theory is believed to be confining, and is definitely not conformal. However, even in this case it is natural to hypothesize that $C_i(Q^2)$ is well defined at least for the leading operator. Section V summarizes my conclusions. In the Appendix I carry out a direct comparison of the Adler functions in the Seiberg dual pairs. As is expected, they coincide in the IR limit when $Q^2 \to 0$, but differ when $Q^2 \neq 0$.

II. PRELIMINARIES

In this section I discuss some general aspects of SQCD and the 't Hooft (planar) limit [12], starting with the latter.

Preamble: The advantages of the $N \to \infty$ limit are as follows. Instantons and similar quasiclassical contributions are completely suppressed. This eliminates exponential terms $\sim e^{-S} \sim \exp(-C/\alpha)$. Moreover, the number of planar graphs does not grow *factorially* [13]. This leaves us with the IR and UV renormalons. The UV renormalons do not introduce ambiguities.

In SQCD, Seiberg proved [11,14] that two distinct supersymmetric theories—one with the $\mathrm{SU}(N)$ gauge group and the other with $\mathrm{SU}(N_f - N)$ plus an extra color-singlet "meson" superfield with a super-Yukawa coupling—are equivalent in the IR. This is the so-called "electric-magnetic" duality. Moreover, the gauge coupling β functions of both theories, electric and magnetic, are exactly determined by the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β functions [15] in terms of the anomalous dimensions of the matter fields. Although the anomalous dimensions $\gamma(\alpha)$ are not Bogomol'nyi-Prasad-Sommerfield protected and hence are not exactly calculable, their infrared limit γ_* is obtained from the requirement that the numerator in the NSVZ formula vanishes. In the electric theory,

$$\gamma_*(\alpha) = -\frac{3N - N_f}{N_f}. \tag{6}$$

The point $3N = N_f$ is the upper edge of the conformal window. The lower edge of the conformal window can be obtained from the dual "magnetic" theory (see Ref. [11]) in which[3]

$$N \to N_f - N, \quad \gamma \to \gamma_D \text{ and } (\gamma_D)_* = -\frac{2N_f - 3N}{N_f}, \tag{7}$$

implying that at the lower edge $\frac{3}{2}N = N_f$. Thus, the conformal window occupies the interval

$$\frac{3}{2}N \leq N_f \leq 3N. \tag{8}$$

The UV fixed point is at $\alpha = 0$, the UV and IR fixed points coincide at the edges of the conformal window. As was expected, the electric theory is weaker near the right edge of the conformal window, while the magnetic theory is weaker near the left edge.

At $N_f > 3N$ the electric theory is infrared free. At $N + 2 < N_f < \frac{3}{2}N$ its dual partner is infrared free. Thus, in these two domains there are no infrared renormalons, no conspiracy, and no VEVs of gluon operators. Of course, in the UV limit they are in the Landau regime and are not self-consistent unless embedded in a larger theory. I will not consider SQCD outside the conformal window. Inside the conformal window both the electric and magnetic theories flow in the IR to one and the same conformal theory.

To follow the strategy of nonsupersymmetric QCD, in what follows I will have to introduce an external source field. In QCD this role is usually played by a photon. Therefore, I will add a U(1) gauge superfield to the Seiberg model, thus combining SQCD and super-QED. The added U(1) field gauge, the "baryon" symmetry, will act as a source and will not be iterated in loops. The magnetic theory will support a dual U(1).

III. ARGUMENTS IN SQCD

Let us start when N_f is close to $3N$, namely,

$$N_f = 3N(1 - \epsilon), \qquad 0 < \epsilon \ll 1. \tag{9}$$

Then, in the electric theory the anomalous dimension at the IR fixed point γ_* is

$$\gamma_* \approx -\epsilon, \qquad |\gamma_*| \ll 1. \tag{10}$$

At large N the anomalous dimension takes the form

$$\gamma(\alpha) = -\frac{N\alpha}{2\pi} + \frac{1}{2}\left(\frac{N\alpha}{2\pi}\right)^2 + C\left[-3\left(\frac{N\alpha}{2\pi}\right)^2 + \frac{N_f}{N}\left(\frac{N\alpha}{2\pi}\right)^2\right] + \cdots, \tag{11}$$

where C is a numerical constant ~ 1 (see Ref. [16]). Note that at the right edge of the conformal window the term in the square brackets vanishes.

The corresponding IR fixed point for α,

$$\frac{N\alpha_*}{2\pi} \approx \epsilon \ll 1, \tag{12}$$

is achieved at $\mu \to 0$. I used only the leading term of Eq. (11) in Eq. (12). For the given values of N_f and N, the formula (12) gives the maximal value of α on the renormalization-group (RG) flow trajectory on the way from the

[3] In the dual theory γ_D depends not only on α but also on f; see Eq. (A3) and the discussion that follows it.

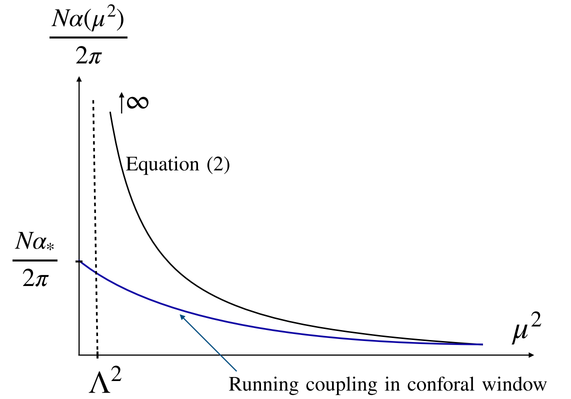

FIG. 2. Running coupling according to Eq. (2) leading to a formal factorial divergence in the bubble chain at high orders vs the conformal window coupling. Near the edges of the conformal window $\frac{N\alpha_*}{2\pi} \ll 1$, while in the middle $\frac{N\alpha_*}{2\pi} \lesssim 1$.

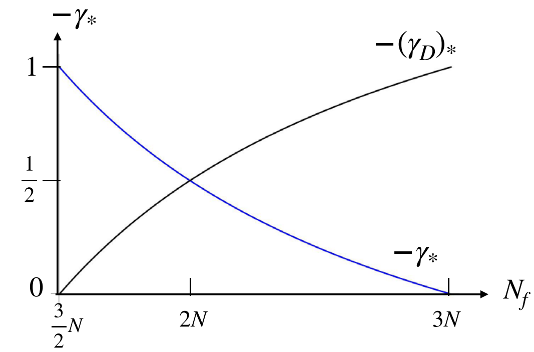

FIG. 3. Anomalous dimensions of matter fields as functions of N_f in the electric and magnetic theories. The magnetic γ is marked by the subscript D.

UV to the IR. Equation (2) is not valid and $N\alpha(k^2)/2\pi$ remains small at small values of k^2, as shown in Fig. 2. Moreover, the approach of α to α_* in the vicinity of the IR fixed point is power-like, not logarithmic. Under these circumstances, the renormalons do not develop, even formally, and the α series must be convergent. This was explained in great detail in Refs. [4,5,7]. Given that the gauge coupling α is always small in the regime at hand, I conclude that gluon operators have vanishing VEVs. The conspiracy is just not needed.

Now, let us move toward smaller values of N_f. Then, $|\gamma_*|$ increases and at $N_f = 2N$ reaches $\frac{1}{2}$. At this point one can pass to the dual theory—see Fig. 3 in which $|(\gamma_D)_*|$ decreases from $\frac{1}{2}$ down to zero as the value of N_f continues to decrease down to $\frac{3}{2}N$—but this is not necessary. Even if one continues with the electric theory, the maximal value of γ_* achieved at the left edge of the conformal window at $N_f = \frac{3}{2}N$ is $\gamma_* = 1$. Then,

$$\frac{N\alpha_*}{2\pi} \lesssim 1, \tag{13}$$

[cf. Eq. (11)] and α_* does not explode at small k^2 which would be needed for the factorial growth of coefficients to emerge. This can be proven by analysis of the dual theory. Indeed, $-(\gamma_* + (\gamma_D)_*) = 1$ for all N_f. Moreover, observable quantities in the IR in he electric and magnetic theories coincide identically; see the Appendix.[4]

An additional argument in favor of the statement of no renormalons/no conspiracy in the entire conformal window is that the values of γ_* and $(\gamma_D)_*$ [see Eq. (6)] are unambiguous and smooth functions of N_f. Everywhere inside the conformal window, we are in one and the same conformal phase, and there are no mass gaps and no irregularities due to opening mass thresholds.

Summarizing, SQCD in the conformal window exhibits no evidence for renormalon ambiguities which entails vanishing VEVs of at least some gluon operators.

Finally, I want to make a remark about nonsupersymmetric QCD with massless quarks. This theory also has a conformal window [17]. Close to its right edge,[5] i.e., at $N_f \lesssim 5.5N$, the infrared fixed point is at small values of α and, therefore, I expect no renormalon ambiguities and no VEVs. Unlike SYM, however, when we move to small enough N_f, say, $N_f \sim N$, there is a phase transition and, therefore, renormalon ambiguities, the conspiracy, and the OPE with the full set of VEVs reestablish themselves.

IV. CONJECTURE ON SUPERSYMMETRIC YANG-MILLS

Inspired by the conclusions of the previous section, in this section I ask what happens in pure SYM theory without matter. In the infrared this theory is certainly not conformal; rather, it is believed to be confining. What we know for sure is that the VEV of the leading dimension-four operator $G^a_{\mu\nu}G^{\mu\nu\, a}$ vanishes. This implies no leading renormalon ambiguity since $\mathcal{N} = 1$ SYM theory *per se* is unambiguous.

Let us have a look at the NSVZ β function in SYM. It can be written as

$$\frac{\partial(2\pi/N\alpha)}{\partial L} = \frac{3}{1 - \frac{N\alpha}{2\pi}}, \qquad L = \log\mu. \tag{14}$$

Equation (14) is *exact*; for more a detailed discussion see Ref. [18] and references therein. This implies that the value

[4]If one chooses to pass from the electric to the magnetic description at $N_f = 2N$, $\gamma_* = \frac{1}{2}$, one would see a cusp in Fig. 3. This is an artifact. In the "observable" quantities, such as the Adler functions, the predictions of the electric and magnetic theories coincide identically in the IR for each value of N_f and N, provided one replaces q_f by $(q_f)_D$.

[5]The exact position of the left edge is still unknown.

$$\frac{N\alpha_*}{2\pi} = 1 \tag{15}$$

is the maximal value of the coupling constant that can be achieved in the α running according to the asymptotic freedom formula. The value (15) is approached from below as follows [18]:

$$\alpha_* - \alpha(\mu) \sim \text{const}(\mu - \Lambda)^{1/2}, \qquad \mu \gtrsim \Lambda \tag{16}$$

(see Fig. 1 in Ref. [18]). Since the regimes (2), (15), and (16) are drastically different, it is plausible that SYM is free from the bubble-chain ambiguities.

A related question immediately comes to mind: is it possible in principle that *confining* super-Yang-Mills theories, as opposed to superconformal theories, are compatible with the Euclidean OPEs for two-point functions that have VEVs of certain operators in OPE vanishing? After all, confining SYM theories must have a mass gap and, at $N \to \infty$, the spectral densities in the two-point functions must look like a comb built from delta functions at the positions of the meson states.

The answer to this question in its most extreme formulation was found long ago, and it was in positive. In 1978, Migdal [19] asked: what is the best possible accuracy to which $\log Q^2$ can be approximated by an infinite sum of infinitely narrow discrete mesons in the spectral function. The answer is as follows.

If the mesons are placed at the zeros of a Bessel function, with well-defined residues, then no $(\Lambda^2/Q^2)^k$ corrections to $\log Q^2$ will appear in the Euclidean OPE; all corrections will be *exponentially* suppressed at large Q^2. Much later, it was realized [20] that just this situation takes place in the holographic QCD model suggested in Ref. [21]. The authors found that, on the one hand, the bare-quark-loop logarithm is represented in their anti–de Sitter/QCD model as an infinite sum over excited mesons, and on the other hand, the Euclidean OPE is of the form

$$\log Q^2 + \sum_i a_i \exp(-b_i Q^2/\Lambda^2), \tag{17}$$

that is, there are no power corrections at all.

V. CONCLUSIONS

In this paper I revisited a long-standing problem of renormalons in supersymmetric Yang-Mills theories caused by the impossibility of the required conspiracy in the OPE. I argued that in SQCD this problem is absent in the Seiberg conformal window: there are no reasons for the factorial growth. In SYM without matter the situation may be similar. This latter statement is a hypothesis.

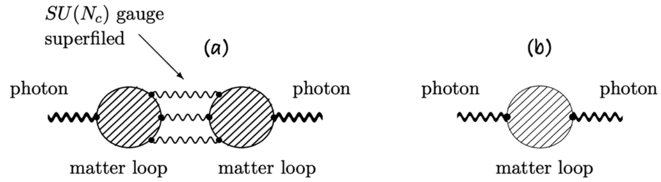

FIG. 4. Graph determining the Adler D functions in Euclidean space. Upon analytic continuation to Minkowski space, their imaginary part reduces to the total cross section of $V_\mu \to$ matter. All quasidisconnected graphs of the type depicted in (a) vanish due to supersymmetry.

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APPENDIX: ADLER FUNCTION IN THE IR LIMIT

In Sec. III I mention that all quantities observable in the IR limit in electric and magnetic theories coincide identically. Now I want to demonstrate this using the example of the exact Adler function D relatively recently calculated in Ref. [22]. The Adler function is determined by the current-current two-point correlator. In this sense it can be viewed as a supersymmetric analog of the study of e^+e^- annihilation to hadrons. This result in the IR limit is not new: it was indirectly obtained in Ref. [23] [see the discussion after Eq. (2.27)], where the infrared limit of D was shown to be related by supersymmetry (SUSY) to the triangular 't Hooft FFR anomaly which, in turn, had been matched in dual theories long ago [11]. R in FFR is the anomaly-free R charge.[6]

The demonstration presented below is *direct*. It is based on the superfield calculation of graphs presented in Fig. 4.

Since our focus is on strong interactions, we can truncate e^+e^- and consider the two-point function of (virtual) photons, as in Fig. 4. Its imaginary part gives the cross section for matter production. Thus, I combine SUSY QCD with SUSY QED. For simplicity, I will assume that all electric charges of the matter fields are one,

$$q_f = 1. \tag{A1}$$

The exact formula for the Adler function $D(Q^2)$ takes the form [22]

[6]The same relation could be obtained from Refs. [24–26] in which the so-called a maximization was required. It is worth emphasizing that the result thus obtained is applicable only at the IR fixed points (but generally not along the RG flow away from the fixed points; see my remark at the end of this appendix).

$$D(Q^2) = \frac{3}{2} N \sum_f q_f^2 [1 - \gamma(\alpha(Q^2))], \tag{A2}$$

where the sum runs over all flavors and γ is the anomalous dimension of the matter fields (all matter fields belong to the fundamental representation of color). In our formulation of the problem γ is the same for all flavors. Equation (A2) is valid for $Q^2 \in [0, \infty]$.

Supersymmetry implies that all "semiconnected" graphs of the type presented in Fig. 4(a) vanish [22]. Only the graph in Fig. 4(b) contributes. Conceptually, the derivation of Eq. (A2) is somewhat similar to that of the NSVZ formula [15]. The anomalous dimensions of the matter fields from Ref. [15] were used to determine the IR values γ_*. They were also instrumental in the studies of the superconformal R symmetries in four dimensions and their relation to the a theorem [24–26].

A few words are in order here about the magnetic component of Seiberg's dual pair. It contains an additional "meson" color-neutral superfield $\mathcal{M}^i_j$ with a certain superpotential. Hence, in addition to the gauge coupling, a (super-)Yukawa coupling is present too,

$$\mathcal{W}_D = f \mathcal{M}^i_j \mathcal{Q}_i \bar{\mathcal{Q}}^j. \tag{A3}$$

The anomalous dimension γ_D depends not only on g_D^2, but also on the superpotential coupling constant f, as does the beta function for f. While the superpotential is not renormalized, the coupling f still runs due to the emergence of the Z factors in the matter kinetic terms. If the number of dual colors (i.e., $N_f - N$) and the number of flavors N_f are large, then

$$\beta_f = -(N_f - N)\frac{\alpha}{2\pi} + c_f \frac{|f|^2}{4\pi^2}, \tag{A4}$$

where c_f is a positive number depending on the structure of the matrix superfield $\mathcal{M}^i_j$, for instance, $c_f \sim N_f$. The RG flow of $|f|^2$ toward the IR slightly depends on the initial conditions. If we switch off α, then $|f|^2$ is IR free. However, the gluon contribution has the opposite effect. Even if at an intermediate scale μ the constant $\alpha < |f|^2$, under the RG flow toward the IR α will go up while $|f|^2$ will go down so that eventually they undergo a crossover; at this point, α will become larger and will force $|f|^2$ to run according to the asymptotic freedom law, and then the cycle reverses. In the IR they both hit an IR fixed point which is determined by Eq. (7). On the other hand, starting at point A in Fig. 3 in Ref. [27], one will never see the Landau growth of $|f|^2$ in the IR; the IR limit is the point C in this figure. As explained in Ref. [14], the value of $|f|^2$ at the IR fixed point can be rescaled by any field rescaling preserving Eq. (5.6) in Ref. [14], where $1/\mu = f$.

Equation (6) implies that in the infrared limit

$$1 - \gamma_* = 1 + \frac{3N - N_f}{N_f} = \frac{3N}{N_f}, \tag{A5}$$

$$D_* = \frac{9}{2} N^2. \tag{A6}$$

What changes in passing to the magnetic theory? First of all, the U(1) charge becomes

$$q_D = \frac{N}{N_f - N}, \tag{A7}$$

instead of (A1) (see Ref. [11]). Second, in the dual magnetic theory, the number of colors is $N_f - N$. Taking these changes into account in the IR limit [see, e.g., Eq. (7)],

$$1 - (\gamma_D)_* = 1 + \frac{3(N_f - N) - N_f}{N_f} = \frac{3(N_f - N)}{N_f}, \tag{A8}$$

$$D_* = \frac{9}{2} N^2, \tag{A9}$$

where I used Eq. (A7) for the dual U(1) charge. We see that the "observable" total cross sections in the Seiberg dual pair coincide in the IR for all values of N_f/N.

At the same time, it is quite obvious that at $Q^2 \nrightarrow 0$ the electric and magnetic Adler functions differ. Their ratio at the UV fixed point (i.e., $Q^2 \to \infty$) varies from $\frac{3}{2}$ to 3 depending on the position in the conformal window.

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