Hawking-Page transition with reentrance and triple point in Gauss-Bonnet gravity

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In this paper, a new family of Hawking-Page transitions, i.e., Hawking-Page transitions with reentrance and a triple point, is introduced for the first time by investigating the Hawking-Page transition of hyperbolic AdS black holes in the extended thermodynamics in Gauss-Bonnet gravity. The reentrant Hawking-Page transition is composed of two Hawking-Page transitions with a high and a low Hawking-Page temperature, and the triple point corresponds to small black hole, massless black hole, and large black hole phases all coexisting. We discuss the temperatures of two branches of Hawking-Page transitions, which both depend on the pressure (i.e., the cosmological constant) and the Gauss-Bonnet constant. The pressure and the Gauss-Bonnet constant both increase the high Hawking-Page temperature and diminish the low Hawking-Page temperature. We also show the coexistence lines in the $P-T$ phase diagrams. The triple point and critical point in the phase diagrams of the Gauss-Bonnet AdS black hole systems are given, together with some interesting universal relations that only depend on the dimensions of spacetime. The reentrant Hawking-Page transition and triple point may correspond to the phase transition and triple point in the QCD phase diagram, following the spirit of the AdS/CFT correspondence. These results may improve the comprehension of the black hole thermodynamics in the quantum gravity framework and shed some light on the AdS/CFT correspondence beyond the classical gravity limit.

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I. INTRODUCTION

Black hole thermodynamics has attracted much attention in gravitational theory, as it is widely believed that black hole thermodynamics could provide further insight into the understanding of quantum gravity. In particular, the famous Hawking-Page transition [1] marked the beginning of an exciting period of exploring the holographic and quantum understanding of critical phenomena and phase transitions in the general AdS spacetime. Emparan *et al.* introduced a first-order phase transition of the RN-AdS (Reissner-Nordström-AdS) black hole [2,3] similar to the liquid-vapor phase transition of the van der Waals fluid. After treating the cosmological constant as thermodynamic pressure [4–10], a small black hole–large black hole phase transition of the RN-AdS black hole was established [11], which is precisely analogous to the liquid-vapor phase transition of the van der Waals fluid (see also [12,13] for reviews). Other interesting families of phase transitions have also been studied in black hole thermodynamics, including the reentrant phase transition [14,15] and the superfluid phase transition [16].

The Hawking-Page transition characterizes a first-order phase transition occurring between a large AdS black hole and a thermal AdS vacuum [1]. Since the AdS black hole phase dominates the partition function at a high temperature limit while the thermal AdS vacuum dominates at a low temperature limit, the thermal AdS gas will collapse to a stable, large black hole when the temperature increases. In particular, the Hawking-Page transition could be explained as the confinement-deconfinement phase transition of the gauge field [17], inspired by the AdS/CFT correspondence [18–20]. Very recently, there have been many studies about the Hawking-Page transition in different backgrounds [21–31], particularly regarding the Hawking-Page transition in the microcosmic [32–34] and holographic frameworks [35–37]. Thus, it is important to investigate the Hawking-Page transition since it should improve our understanding of the quantum and holographic properties of gravity in the spirit of the AdS/CFT correspondence.

Studies of the Hawking-Page transition also induce many applications about the mutual study on the particle physics, especially in the QCD phase diagram. Beyond the confinement-deconfinement phase transition, we also

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speculate the crossover that, in the holographic dictionary, corresponds to the Hawking-Page crossover of the Schwarzschild AdS black hole [38] in noncommutative spacetime, which is thought to be an effective description of quantum gravitational spacetime. This crossover is the critical endpoint of the deconfinement phase transition [39], which is a continuous phase transition in the QCD phase diagram. However, the QCD phase diagram has a complicated phase structure. For example, after considering the quarkyonic matter [40–46], a phase transition composed of the deconfinement phase transition and the quarkyonic transition, as well as a triple point for the hadronic matter, the quarkyonic matter, and the quark-gluon plasma, were introduced. There is also evidence from lattice QCD simulations [45]. Many recent studies presented the existence of this phase transition and triple point [40,46–60]. Then, it is natural to ask, how can this kind of phase transition and triple point be explained on the gravity side in the AdS/CFT correspondence?

This type of phase transition in the QCD phase diagram is the reentrant phase transition, which is composed of at least two phase transitions. The reentrant phase transition has previously been observed in a nicotine-water mixture [61], granular superconductors, liquid crystals, binary gases, ferroelectrics, and gels (see Ref. [62] and the references therein). In black hole thermodynamics, the reentrant phase transition is also presented in [14,15], appearing simultaneously with the triple point in the phase diagrams. In order to explain the reentrant phase transition and triple point of the QCD phase diagram on the gravity side, we focus on a new family of Hawking-Page transitions, i.e., the Hawking-Page transition with the reentrance and triple point, which can be found in Gauss-Bonnet gravity, as shown in this paper.

Gauss-Bonnet gravity is the quadratic order of higher-curvature gravities, and it has the following action:

$$S = \frac{1}{16\pi}\int \mathrm{d}^d x\sqrt{-g}(R - 2\Lambda + \alpha_{\mathrm{GB}}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2)), \tag{1}$$

where α_{GB} is the Gauss-Bonnet coupling constant and Λ is the cosmological constant. Gauss-Bonnet gravity has recently attracted considerable interest since the Gauss-Bonnet term appears naturally in the low-energy effective theories obtained from string theory, as the next-to-leading term; its presence can lead to qualitative changes in black hole physics. Gauss-Bonnet gravity preserves the property that the equations of motion involve only second derivatives of the metric; thus, it is possible to find explicit solutions. The black hole solutions of Gauss-Bonnet gravity have been found [63–66], and they exhibit notable effects on the spacetime by the Gauss-Bonnet correction in the action. The thermodynamics of these solutions are further studied in [67,68]. Especially in black hole chemistry, the reentrant phase transition is also found in Gauss-Bonnet gravity [69]. This is why we expect that the reentrant Hawking-Page transition exists in Gauss-Bonnet gravity. In this paper, the reentrant Hawking-Page transition is found for the first time by investigating the Hawking-Page transition of the hyperbolic Gauss-Bonnet AdS black hole in extended thermodynamics, which is composed of two Hawking-Page transitions. Many typical features of the reentrant Hawking-Page transition are given.

The paper is organized as follows: We revisit the extended thermodynamics of hyperbolic AdS black holes in Gauss-Bonnet gravity in the next section. In Secs. III and IV, we study the Hawking-Page transition with the reentrance and triple point in four and $d \geq 5$ dimensions, respectively. Finally, some concluding remarks are given.

II. EXTENDED THERMODYNAMICS OF HYPERBOLIC AdS BLACK HOLE IN GAUSS-BONNET GRAVITY

In this section, we revisit the extended thermodynamics of hyperbolic AdS black holes in d-dimensional Gauss-Bonnet gravity. This black hole solution is well known to take the form [63–66]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{d-2,k}, \tag{2}$$

$$f(r) = \frac{r^2}{2\alpha}\left(1 - \sqrt{\frac{64\pi\alpha M}{(d-2)r^{d-1}} - \frac{64\pi\alpha P}{(d-2)(d-1)} + 1}\right) + k, \tag{3}$$

where $\mathrm{d}\Omega^2_{d-2,k}$ is the line element of a $(d-2)$-dimensional, maximally symmetric Einstein manifold with curvature $k = -1$ corresponding to the hyperbolic topology of the black hole horizon. Note that M is the mass of the black hole, and $\alpha = (d-3)(d-4)\alpha_{\mathrm{GB}}$ is a renormalized Gauss-Bonnet coupling constant. In this paper, we consider only the case $\alpha > 0$, i.e., $\alpha_{\mathrm{GB}} > 0$, since α_{GB} can be identified with the inverse string tension with a positive value if the theory is incorporated in string theory [63]. We take the spacetime dimension $d \geq 4$ since in $d = 4$ dimensions, though the Gauss-Bonnet term is a topological invariant and does not contribute to the spacetime, it has a notable effect on black hole thermodynamics resulting from a nontrivial black hole entropy. In the black hole chemistry framework, $P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi\ell^2}$ is the thermodynamic pressure associated with the cosmological constant Λ [4–10], with ℓ being the d-dimensional AdS radius. In higher derivative gravity, there are always two branches of black holes. In this paper, we only consider the case in Eq. (3), while another branch cannot reach the Schwarzschild limit and is unstable. Finally, one should note that in order to have a well-defined vacuum solution with $M = 0$ in $d \geq 5$ dimensions, the renormalized Gauss-Bonnet coupling constant α and pressure P have to satisfy the following constraint:

$$0 < \frac{64\pi\alpha P}{(d-2)(d-1)} \leq 1, \qquad d \geq 5. \tag{4}$$

We list the thermodynamical quantities, including the black hole mass, temperature, and entropy [67,68],

$$M = \frac{r_+^{d-5}(\alpha(d-1)(d-2) - (d-1)(d-2)r_+^2 + 16\pi P r_+^4)}{16\pi(d-1)}, \tag{5}$$

$$T = \frac{\alpha(d-2)(d-5) - (d-2)(d-3)r_+^2 + 16\pi P r_+^4}{4\pi(d-2)r_+(r_+^2 - 2\alpha)}, \tag{6}$$

$$S = \frac{1}{4} r_+^{d-2}\left(1 - \frac{2\alpha(d-2)}{(d-4)r_+^2}\right), \tag{7}$$

where r_+ is the event horizon radius of the black hole. The Gauss-Bonnet correction in the action leads to modification in the formula for the entropy of the black hole solution; the entropy is no longer proportional to the area of the black hole's event horizon but instead is given by a relationship depending on the Gauss-Bonnet term. The modified entropy from the Gauss-Bonnet term has a possible effect on the Hawking-Page transition, which is presented in the Appendix.

For the Gibbs free energy, by calculating the Euclidean action $\mathcal{I}$ of gravity and subtracting from it the Euclidean action of the thermal Gauss-Bonnet AdS vacuum [70–73], one can obtain

$$G = \beta\mathcal{I}. \tag{8}$$

This is consistent with the solution obtained similarly from the thermodynamic equation $G = H - TS$ [67,74,75]. In black hole chemistry, the Gibbs free energy of the Gauss-Bonnet AdS black hole is [67]

$$\begin{aligned} G &= H - TS = M - TS \\ &= \frac{-1}{16\pi(r_+^2 - 2\alpha)}\left(\frac{(d^2 - 5d - 96\pi\alpha P + 4)r_+^{d-1}}{(d-4)(d-1)}\right. \\ &\quad \left. - \frac{\alpha(d-8)r_+^{d-3}}{(d-4)} + \frac{16\pi P r_+^{d+1}}{(d-2)(d-1)} + \frac{2\alpha^2(d-2)r_+^{d-5}}{d-4}\right), \end{aligned} \tag{9}$$

which characterizes the canonical ensemble. Here the black hole mass M should be identified with the enthalpy H rather than the internal energy of the gravitational system [4]. The zero Gibbs free energy marks the Hawking-Page phase transition. Starting with the Gibbs free energy, the entropy and other thermodynamic quantities could also be calculated in turn by the Euclidean approach [70–73], which is consistent with the one from the other thermodynamic quantities using the first law of thermodynamics [67] and the Noether charge approach [68].

In what follows, we focus on the famous Hawking-Page transition, for which a thermodynamically stable state is given by the global minimum of G of the black hole and the background spacetime (with zero Gibbs free energy). To observe the phase transition, it is most useful to plot $G - T$ diagrams, fixing the other parameters. This means that the case with a negative G should be regarded as the Gauss-Bonnet black hole phase being thermodynamically favored over the background spacetime; the case with vanishing G just corresponds to the Hawking-Page transition point, which characterizes the phase transition between the Gauss-Bonnet black hole phase and the background spacetime phase. Especially in this case, the background spacetime is the massless AdS black hole (MBH) with a cosmological constant modified by the Gauss-Bonnet constant. When the Gauss-Bonnet constant α is vanishing, the spacetime reduces to the hyperbolic Schwarzschild AdS black hole, which does not contain the Hawking-Page transition.

III. REENTRANT HAWKING-PAGE TRANSITION AND TRIPLE POINT IN FOUR DIMENSIONS

In four dimensions, since the Gauss-Bonnet term is a topological invariant that does not contribute to the spacetime, the spacetime becomes the hyperbolic Schwarzschild AdS black hole, i.e.,

$$\begin{aligned} \mathrm{d}s^2 &= -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{2,-1}^2, \\ f(r) &= -\frac{2M}{r} + \frac{8}{3}\pi P r^2 - 1, \end{aligned} \tag{10}$$

which makes the discussion about the Hawking-Page transition more clear than the cases in higher dimensions. The thermodynamical quantities, mass and temperature of the system, reduce to

$$M = \frac{4\pi r_+^3}{3}P - \frac{1}{2}r_+, \tag{11}$$

$$T = 2Pr_+ - \frac{1}{4\pi r_+}, \tag{12}$$

which is exactly the same with the Schwarzschild AdS black hole, while the black hole entropy breaks the area law and takes the form [10,76]

$$S = \pi(r_+^2 - 4\alpha). \tag{13}$$

Thanks to this nontrivial black hole entropy, the Gauss-Bonnet term has a notable effect on black hole

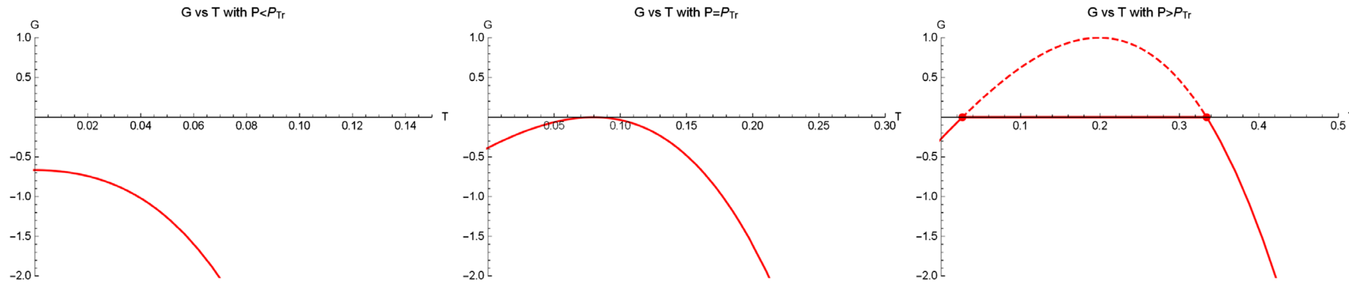


FIG. 1. Gibbs free energy vs temperature with different pressures (and $\alpha = 1$) in four dimensions. The globally stable states of the system are denoted by the thick lines. When $P > P_{\mathrm{Tr}}$, black holes undergo the reentrant Hawking-Page transition.

thermodynamics even in four dimensions. For example, the Gibbs free energy becomes

$$G = -\frac{2}{3}\pi P r_+^3 + r_+\left(8\pi\alpha P - \frac{1}{4}\right) - \frac{\alpha}{r_+}. \tag{14}$$

To consider the Hawking-Page transition, we explore the global stability of Gibbs free energy. This can be seen in the $G - T$ diagrams as shown in Fig. 1. One can see that the existence of the Hawking-Page transition depends on the maximum of the Gibbs free energy. There is a critical pressure P_{Tr} (we denote it as P_{Tr} since it just corresponds to the triple point in the phase diagram as shown later); for a black hole with $P = P_{\mathrm{Tr}}$, the maximum of the Gibbs free energy becomes zero, and hence $G \leq 0$, while for black holes with $P < P_{\mathrm{Tr}}$, the Gibbs free energy is always negative. This indicates that, when $P \leq P_{\mathrm{Tr}}$, the hyperbolic AdS black hole phase is globally preferred over the background spacetime phase, and the Hawking-Page transition will not occur here. For black holes with $P > P_{\mathrm{Tr}}$, see the right panel of Fig. 1. The maximum of the Gibbs free energy of black holes always divides the black holes into two branches. It is easy to check that the temperature is a monotonically increasing function of mass; thus, we can denote the two branches of black holes as follows: a small black hole with smaller mass and a large black hole with larger mass. It is obvious that the situation becomes interesting since there exist two zero free energy points. The globally stable states of the system are denoted by the thick lines. Here it is shown that the two zero free energy points both correspond to the Hawking-Page transitions, with a high Hawking-Page temperature and a low Hawking-Page temperature. If the temperature is lower than the low Hawking-Page temperature, the background spacetime, i.e., a massless black hole, should collapse into a small black hole, while a massless black hole should collapse into a large black hole when the temperature is larger than the large Hawking-Page temperature because, for these cases, the free energy of the black holes is lower than that of the background spacetime. When the temperature stays in the intermediate region between the high and low Hawking-Page temperature, the system favors a massless black hole phase. Therefore, black holes undergo a small black hole–massless black hole–large black hole Hawking-Page transition in this pressure region. This behavior is known as the reentrant Hawking-Page transition, which is composed of two first-order phase transitions (Hawking-Page transition) in our case.

Now we calculate the Hawking-Page temperature. After choosing a zero Gibbs free energy, we obtain the black hole radius of the Hawking-Page transition,

$$r_{\mathrm{HP}} = \frac{1}{4\sqrt{\pi P}}\sqrt{96\pi\alpha P \pm \sqrt{(3-96\pi\alpha P)^2 - 384\pi\alpha P} - 3}. \tag{15}$$

From Eq. (12), we can obtain the Hawking-Page temperature

$$T_{\mathrm{HP}} = T|_{r_+ = r_{\mathrm{HP}}}. \tag{16}$$

Since r_{HP} should be positive, we find that the (reentrant) Hawking-Page transition requires an additional condition

$$96\pi\alpha P \pm \sqrt{(3-96\pi\alpha P)^2 - 384\pi\alpha P} - 3 \geq 0, \tag{17}$$

which could be simplified as

$$\alpha P \geq \frac{3}{32\pi}. \tag{18}$$

The general behavior of the two branches of Hawking-Page temperature is illustrated in Fig. 2. It is clear that the (reentrant) Hawking-Page transition only arises when $\alpha \geq \frac{3}{32\pi P}$. Especially for the Gauss-Bonnet AdS black holes with $\alpha < \frac{3}{32\pi P}$, the Hawking-Page transition does not occur, which is consistent with the discussion about the hyperbolic AdS black hole in Einstein gravity (i.e., $\alpha = 0$). The Gauss-Bonnet constant α increases the high Hawking-Page temperature and diminishes the low Hawking-Page temperature.

Finally, we show that the coexistence line gives a whole picture of the reentrant Hawking-Page transition. This is plotted in the $P - T$ phase diagram as shown in Fig. 3. When the temperature of the system is fixed, there always exists a single Hawking-Page transition: a small black

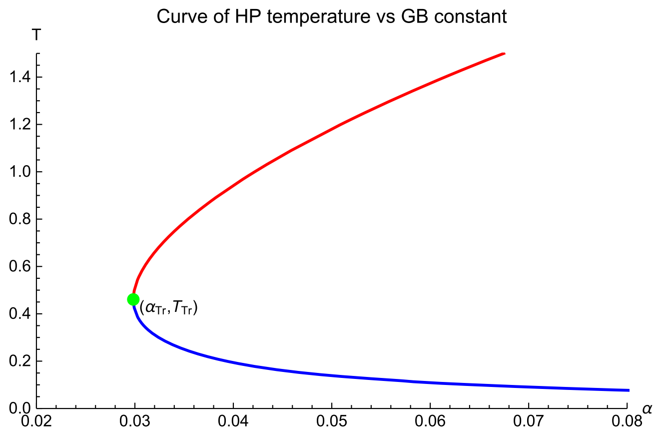


FIG. 2. Two branches of Hawking-Page temperature vs Gauss-Bonnet constant with $P = 1$ in four dimensions. The Gauss-Bonnet constant α increases the high Hawking-Page temperature and diminishes the low Hawking-Page temperature.

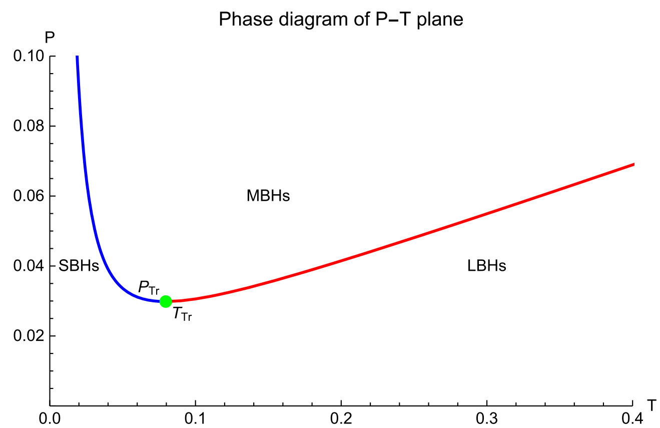


FIG. 3. Coexistence line: $P - T$ phase diagram (with $\alpha = 1$) in four dimensions. The triple point is highlighted. SBHs, LBHs, and MBHs correspond to the small black holes, large black holes, and massless black holes, respectively.

hole–massless black hole Hawking-Page transition at low temperature or a large black hole–massless black hole Hawking-Page transition at high temperature. When the pressure of the system is fixed, the reentrant Hawking-Page transition arises since the system undergoes two Hawking-Page transitions: a small black hole–massless black hole Hawking-Page transition at low temperature and a large black hole–massless black hole Hawking-Page transition at high temperature. The regimes observed in the $P - T$ phase diagram and described above can be understood as arising from a triple point where a small black hole, a massless black hole, and a large black hole all coexist. This triple point is located where the pressure of the Hawking-Page transition reaches its limiting value. Hence, it is easy to derive the triple point by making the two r_{HP} in Eq. (15) coincide, i.e.,

$$(3 - 96\pi\alpha P)^2 - 384\pi\alpha P = 0. \tag{19}$$

The pressure and temperature of the triple point are

$$P_{\mathrm{Tr}} = \frac{3}{32\pi\alpha}, \qquad T_{\mathrm{Tr}} = T_{\mathrm{HP}}|_{P=P_{\mathrm{Tr}}} = T|_{r_+=r_{\mathrm{Tr}},P=P_{\mathrm{Tr}}} = \frac{1}{4\pi\sqrt{\alpha}}, \tag{20}$$

with the black hole radius $r_{\mathrm{Tr}} = r_{\mathrm{HP}}|_{P=P_{\mathrm{Tr}}} = 2\sqrt{\alpha}$, while another triple point has a negative, thus unphysical, black hole radius. Note that only when $P > P_{\mathrm{Tr}}$ is there a reentrant Hawking-Page transition, while the Hawking-Page transition is vanishing when $P \leq P_{\mathrm{Tr}}$, which is consistent with the discussion about the $G - T$ diagrams. Namely, the triple point exactly corresponds to the black hole phase whose maximum of Gibbs free energy is zero. Moreover, it is interesting to introduce a universal relationship for the triple point,

$$\frac{P_{\mathrm{Tr}} r_{\mathrm{Tr}}}{T_{\mathrm{Tr}}} = \frac{3}{4}. \tag{21}$$

IV. REENTRANT HAWKING-PAGE TRANSITION AND TRIPLE POINT IN $d \geq 5$ DIMENSIONS

The Hawking-Page transition in higher dimensions exhibits some differences from the case in four dimensions. One can only see the reentrant Hawking-Page transition behavior of Gauss-Bonnet AdS black holes for a range of pressure $P \in (P_{\mathrm{Tr}}, P_c)$. In this section, we first present the reentrant Hawking-Page transition in five and six dimensions; in other dimensions $d \geq 5$, the behavior is similar. Then, we derive the triple point and some universal relations from the Hawking-Page transition in n dimensions.

A. Reentrant Hawking-Page transition in five and six dimensions

In $d = 5$ dimensions, the Gibbs free energy and temperature of Gauss-Bonnet AdS black holes reduce to

$$G = -\frac{18\alpha^2 + 9\alpha r_+^2 + (3 - 72\pi\alpha P)r_+^4 + 4\pi P r_+^6}{48\pi(r_+^2 - 2\alpha)},$$
$$T = \frac{r_+(8\pi P r_+^2 - 3)}{6\pi(r_+^2 - 2\alpha)}. \tag{22}$$

In $d = 6$ dimensions, the corresponding thermodynamical quantities become

$$G = -\frac{r_+(20\alpha^2 + 5\alpha r_+^2 + (5 - 48\pi\alpha P)r_+^4 + 4\pi P r_+^6)}{80\pi(r_+^2 - 2\alpha)},$$
$$T = \frac{\alpha - 3r_+^2 + 4\pi P r_+^4}{4\pi r_+(r_+^2 - 2\alpha)}. \tag{23}$$

The general behavior of the Gibbs free energy is illustrated in Fig. 4, which has been plotted for different pressures in

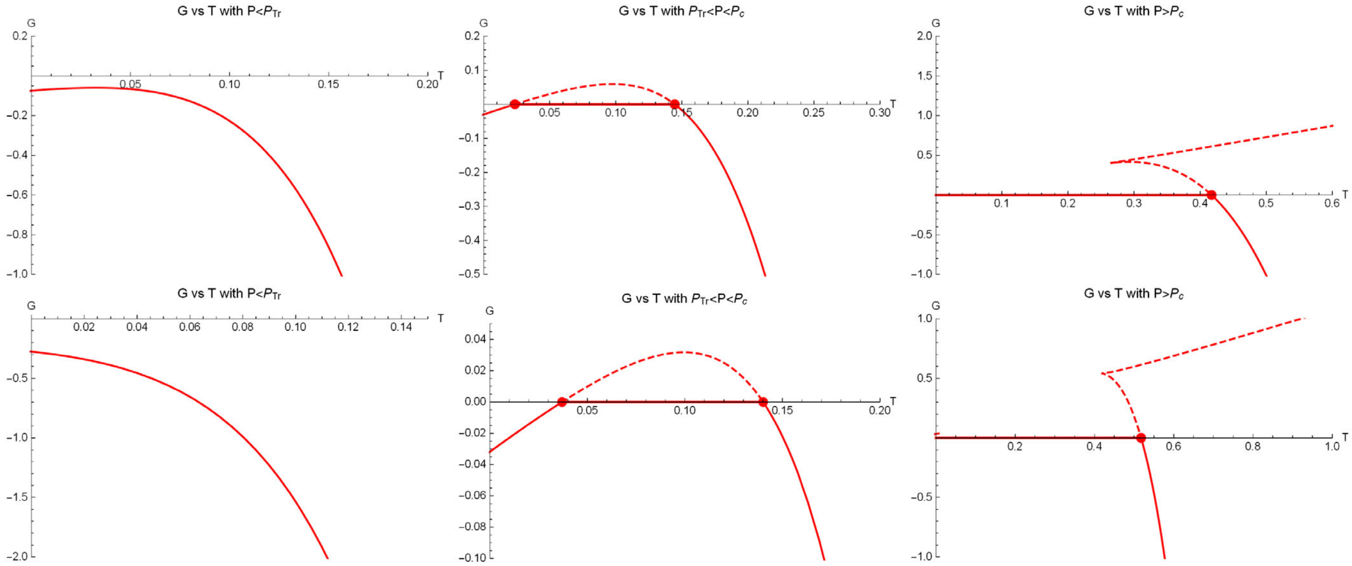


FIG. 4. Gibbs free energy vs temperature with different pressures (and $\alpha = 1$) in five and six dimensions. The top (bottom) three panels correspond to the cases in five (six) dimensions. The globally stable states of the systems are denoted by the thick lines. When $P_{\mathrm{Tr}} < P < P_c$, black holes undergo the reentrant Hawking-Page transition, while there is only a single Hawking-Page transition when $P > P_c$.

$G - T$ diagrams. The globally stable states of the systems are denoted by the thick lines. When the pressure is small ($P \leq P_{\mathrm{Tr}}$), the Gauss-Bonnet AdS black hole phase always has a smaller Gibbs free energy than the background spacetime phase, which indicates no Hawking-Page transition. When the pressure increases ($P_{\mathrm{Tr}} < P < P_c$), the reentrant Hawking-Page transition, composed of two Hawking-Page transitions, emerges, as shown in the middle diagram of Fig. 4. From the right diagram of Fig. 4, one can easily observe that there is a single Hawking-Page transition when the pressure increases ($P > P_c$) since the temperature of another Hawking-Page transition diverges or becomes negative. (Note that these ranges will be discussed later.)

Looking at the coexistence lines in the $P - T$ phase diagrams for the Hawking-Page transition in five and six dimensions in Fig. 5, the phase structure become more clear. There are two branches of Hawking-Page transitions. When a black hole crosses the solid line from left to right or bottom to top in the left branch, it undergoes a Hawking-Page transition from a small black hole to a massless black hole; when a black hole crosses the solid line from right to left or bottom to top in the right branch, it undergoes a Hawking-Page transition from a large black hole to a massless black hole. The reentrant Hawking-Page transitions are denoted by the thick lines, which can be bounded by the pressure of a triple point and an critical point. Namely, the reentrant Hawking-Page transition

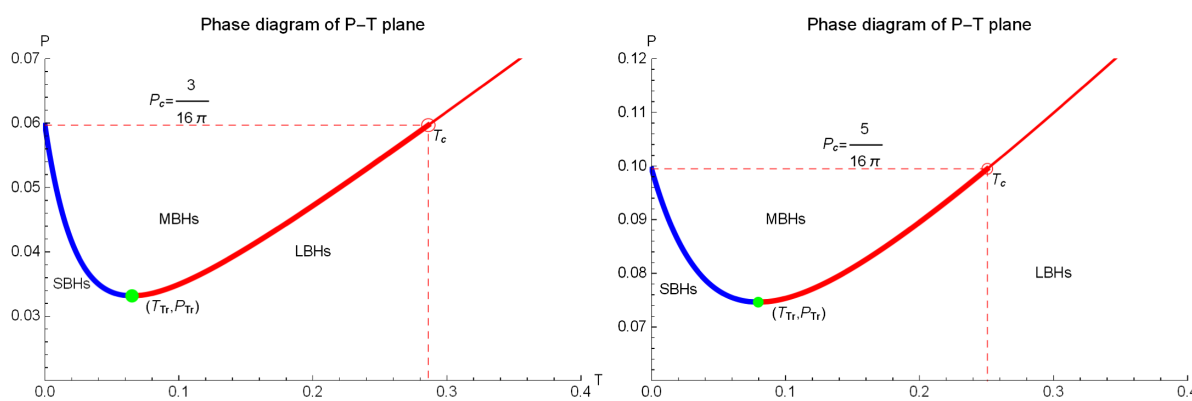


FIG. 5. Coexistence line: $P - T$ phase diagrams (with $\alpha = 1$) in five and six dimensions are plotted in the left and right panels, respectively. The reentrant Hawking-Page transitions are denoted by the thick lines. The triple points and critical points are highlighted. SBHs, LBHs, and MBHs correspond to the small black holes, large black holes, and massless black holes, respectively.

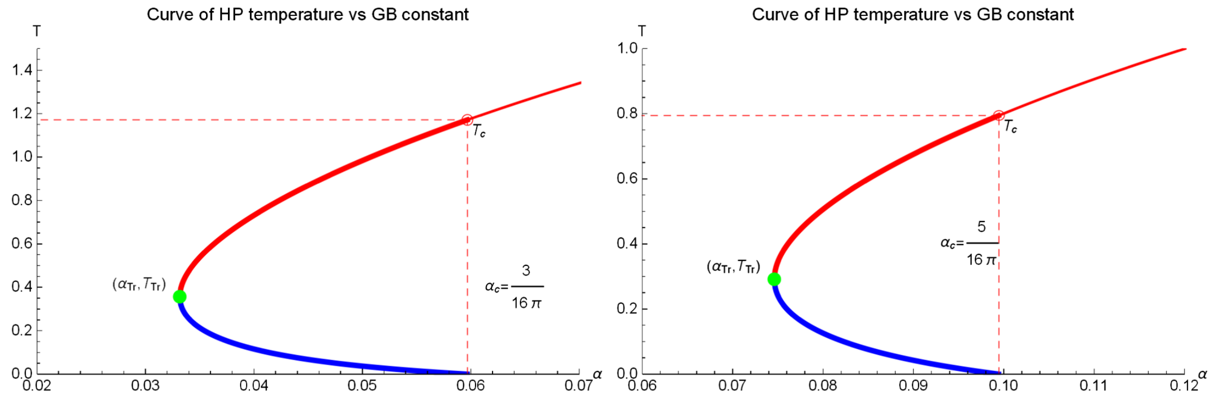


FIG. 6. Two branches of the Hawking-Page temperature vs the Gauss-Bonnet constant with $P = 1$ in five and six dimensions, shown in the left and right figures, respectively. The Gauss-Bonnet constant α increases the high Hawking-Page temperature and diminishes the low Hawking-Page temperature.

behavior of Gauss-Bonnet AdS black holes only exists for a range of pressure $P \in (P_{\rm Tr}, P_c)$. In particular, $(P_{\rm Tr}, T_{\rm Tr})$ denote the pressure and temperature of the triple point, where a small black hole, a massless black hole, and a large black hole all coexist. Therefore, we can derive the triple point using the property that it exactly corresponds to the black hole phase whose maximum of the Gibbs free energy is zero, which will be shown in the next subsection. Moreover, from Fig. 5, one can also see that for Gauss-Bonnet AdS black holes with fixed Gauss-Bonnet constant α, the pressure P increases the large Hawking-Page temperature and diminishes the small Hawking-Page temperature.

We denote the pressure and temperature of the critical point as (P_c, T_c). If the pressure approaches the critical value P_c, the left branch of the Hawking-Page temperature will diverge and not become zero, which is shown in the right diagram of Fig. 4. When the pressure is beyond the pressure of the critical point, the small black hole has a negative temperature, which is physically unacceptable. As a result, the left branch of the Hawking-Page transition is vanishing, and only the right branch is left. Then, one cannot observe the reentrant Hawking-Page transition. We derive the critical point in the next subsection following the property that the critical point corresponds to the two-phase coexistence state having zero Gibbs free energy, and its pressure P_c coincides with the pressure of another two-phase coexistence state with a diverging Hawking-Page temperature.

The two branches of Hawking-Page temperature can be obtained from the zero Gibbs free energy as well, which leads to

$$R^3 + \frac{(d-2)(d^2-5d-96\pi\alpha P+4)}{16\pi(d-4)P}R^2 - \frac{\alpha(d-8)(d-2)(d-1)}{16\pi(d-4)P}R + \frac{\alpha^2(d-2)^2(d-1)}{8\pi(d-4)P} = 0, \tag{24}$$

where $r_+ = \sqrt{R}$ is inserted. This is a classical cubic equation whose roots can be analytically obtained. Since the roots have a complicated form, we do not present them or the corresponding Hawking-Page temperature here. We plot the two branches of the Hawking-Page temperature vs the Gauss-Bonnet constant in five and six dimensions in Fig. 6. The Hawking-Page temperature follows a similar property, as the Gauss-Bonnet constant α increases the large Hawking-Page temperature and diminishes the low Hawking-Page temperature. One should note that the Hawking-Page transition with a high temperature only happens in Gauss-Bonnet AdS spacetime with $\alpha > \alpha_{\rm Tr}$, while the branch of the Hawking-Page transition with a low temperature only happens in Gauss-Bonnet AdS spacetime with $\alpha_{\rm Tr} < \alpha < \alpha_c$. We calculate their values in the next subsection.

B. Triple points in $d \geq 5$ dimensions

The triple point [with thermodynamic quantities $(P_{\rm Tr}, T_{\rm Tr}, r_{\rm Tr})$] corresponds to the black hole phase whose maximum of the Gibbs free energy is zero, which can be calculated by

$$G|_{P=P_{\rm Tr}, T=T_{\rm Tr}, r=r_{\rm Tr}} = 0, \qquad \left.\frac{\partial G}{\partial r_+}\right|_{P=P_{\rm Tr}, T=T_{\rm Tr}, r_+=r_{\rm Tr}} = 0. \tag{25}$$

The former leads to Eq. (24), while the latter can be simplified as

$$16\pi P R^3 + R^2((d-2)(d-3) - 96\pi\alpha P) - \alpha(d-2)(d-9)R + 2\alpha^2(d-2)(d-5). \tag{26}$$

Combining these two equations and Eq. (5), we obtain the triple point

$$P_{\mathrm{Tr}} = \frac{d(d-1)(d-4)}{64(d-2)\pi\alpha}, \qquad T_{\mathrm{Tr}} = \frac{1}{2\sqrt{2}\pi\sqrt{\frac{\alpha(d-2)}{d-4}}},$$
$$r_{\mathrm{Tr}} = \sqrt{\frac{2\alpha(d-2)}{d-4}}. \tag{27}$$

In order to derive the critical point with thermodynamic quantities (P_c, T_c, r_c), we first need to find the two-phase coexistence state [with $(P_c,\ T_{\mathrm{div}},\ r_{\mathrm{div}})$] with a diverging Hawking-Page temperature, which requires the conditions

$$G|_{P=P_c, T=T_{\mathrm{div}}, r_+=r_{\mathrm{div}}} = 0, \qquad T|_{P=P_c, r_+=r_{\mathrm{div}}} = \infty. \tag{28}$$

From the latter, one can directly find $r_{\mathrm{div}} = \sqrt{2\alpha}$. Inserting this into the former, i.e., Eq. (24), we obtain the pressure of the critical point,

$$P_c = \frac{(d-1)(d-2)}{64\pi\alpha}. \tag{29}$$

Inserting P_c again into Eqs. (24) and (5), we obtain

$$T_c = \frac{-d^2 + (\sqrt{d^2-6d+17}+12)d - \sqrt{d^2-6d+17} - 23}{8\sqrt{2}\pi\sqrt{\alpha(d-4)(\sqrt{d^2-6d+17}+3)}}, \tag{30}$$

$$r_c = \sqrt{\frac{2\alpha(\sqrt{d^2-6d+17}+3)}{d-4}}. \tag{31}$$

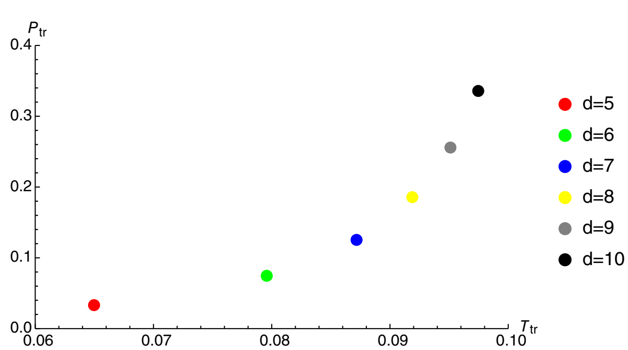
Considering the constraint of parameters in Eq. (4), or equivalently $P \le \frac{(d-1)(d-2)}{64\pi\alpha}$, one finds that the pressure of the critical point for the reentrant Hawking-Page transition is just the upper bound of the physical pressure of the Gauss-Bonnet AdS black hole. Therefore, in $d \ge 5$ dimensions, the single Hawking-Page transition is forbidden, as the corresponding pressures of Gauss-Bonnet AdS black holes are $P > P_c = \frac{(d-1)(d-2)}{64\pi\alpha}$). In conclusion, the hyperbolic Gauss-Bonnet AdS black hole always exhibits a reentrant Hawking-Page transition.

We show the triple points and critical points of diverse dimensions in Figs. 7 and 8. When the dimension d increases, the temperature of the triple point increases, while that of the critical point always decreases. We also plot the $P - T$ phase diagrams of diverse dimensions in Fig. 9; one can see the reentrant Hawking-Page transition behavior of Gauss-Bonnet AdS black holes for a range of pressures $P \in (P_{\mathrm{Tr}}, P_c)$ in arbitrary dimensions. On the other hand, from the range $\frac{d(d-4)(d-1)}{(d-2)64\pi\alpha} < P < \frac{(d-1)(d-2)}{64\pi\alpha}$ for the reentrant Hawking-Page transition, i.e., $\frac{d(d-4)(d-1)}{(d-2)64\pi} < \alpha P < \frac{(d-1)(d-2)}{64\pi}$, one can easily conclude that if the pressure P is fixed, the reentrant Hawking-Page transition will only happen in Gauss-Bonnet AdS spacetime with $\alpha_{\mathrm{Tr}} = \frac{d(d-4)(d-1)}{(d-2)64\pi P} < \alpha < \frac{(d-1)(d-2)}{64\pi P} = \alpha_c$.

C. Some universal relations

In this subsection, we demonstrate some universal relations and constants associated with the Hawking-Page phase transition since we expect that they will provide a foundation for understanding black hole thermodynamics and other special properties of (other) black holes in AdS spacetime in the quantum and holographic frameworks. Universal relations and constants have important applications in understanding a physical theory.

In particular, from the critical phenomenon and phase transition of thermodynamical systems, some universal relations and constants emerge. For classical thermodynamical systems, we take the van der Waals fluid with the equation of state $(P + \frac{a}{v^2})(v - b) = kT$ as an example. From the investigation of the critical point and liquid-vapor phase transition, it is easy to introduce the famous universal relation



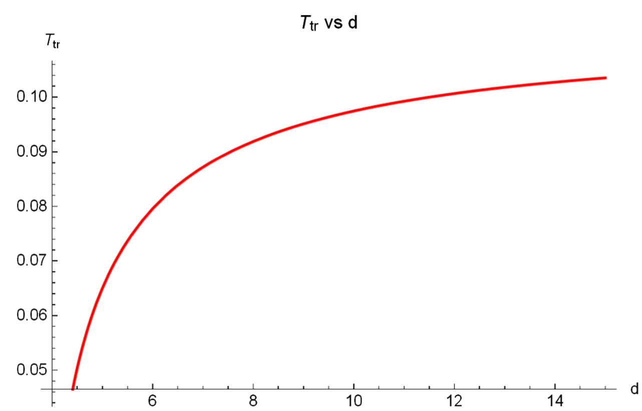


FIG. 7. Triple points (with $\alpha = 1$) in diverse dimensions.

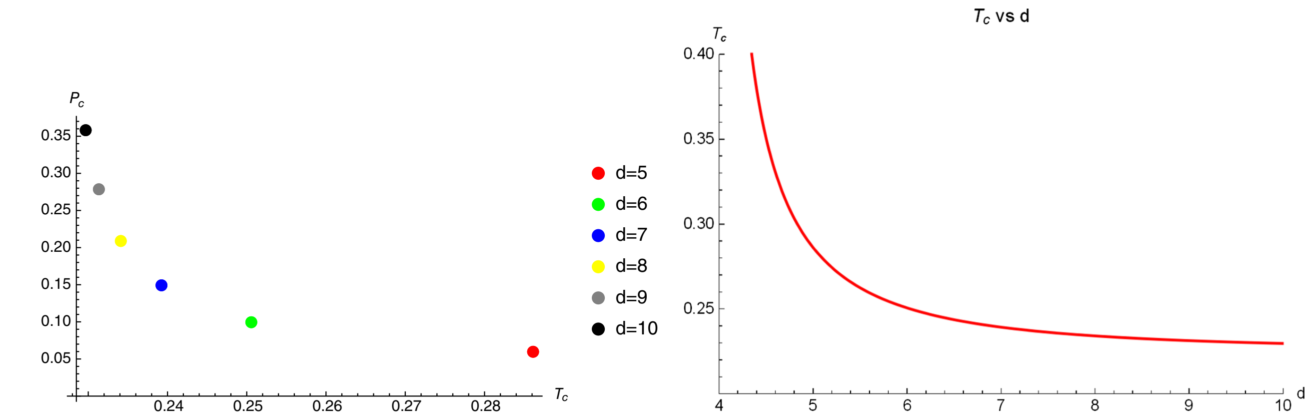


FIG. 8. Critical points (with $\alpha = 1$) in diverse dimensions.

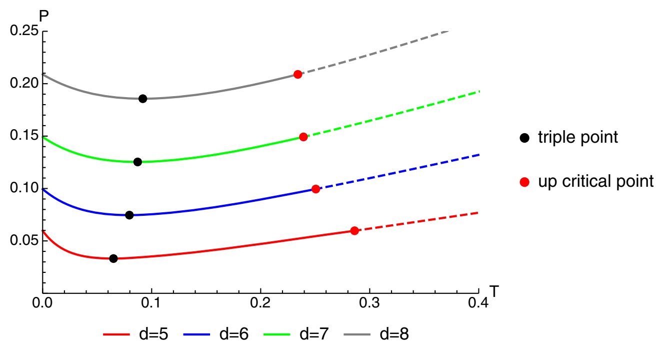


FIG. 9. The $P-T$ phase diagrams (with $\alpha = 1$) in diverse dimensions. The reentrant Hawking-Page transitions are denoted by the solid lines, while the forbidden single Hawking-Page transitions are denoted by the dashed lines.

$$\rho_V = \frac{P_c v_c}{kT_c} = \frac{3}{8}. \tag{32}$$

As for quantum thermodynamical systems, for instance, black holes, a similar universal relation,

$$\rho_B = \frac{P_c v_c}{T_c} = \frac{2d-5}{4d-8}, \tag{33}$$

was also found from the investigation of the critical point of a small black hole–large black hole phase transition of charged AdS black holes in the extended thermodynamics [11,14]. These ratios are all universal numbers independent of the parameters of the systems, e.g., a, b for the van der Waals fluid and q for charged AdS black holes.

Now we explore the universal relations from the Hawking-Page transition. Since in the extended thermodynamics of AdS black holes the specific volume v is always identified with the horizon radius r_+, rather than the thermodynamic volume V, we study $\rho_{\rm HP} = \frac{P_{\rm HP} r_{\rm HP}}{T_{\rm HP}}$ at the Hawking-Page transition point with the thermodynamical quantities $(P_{\rm HP}, r_{\rm HP}, T_{\rm HP})$. We first consider the d-dimensional Schwarzschild-AdS black hole. There exists a Hawking-Page transition with the thermodynamical quantities $r_{\rm HP} = \sqrt{\frac{(d-1)(d-2)}{16\pi P}}$ and $T_{\rm HP} = \sqrt{\frac{4(d-2)P}{(d-1)\pi}}$. Then we obtain a universal relation

$$\rho_{\rm HP} = \frac{P_{\rm HP} r_{\rm HP}}{T_{\rm HP}} = \frac{(d-1)}{8}. \tag{34}$$

For the case in our paper, the situation becomes subtle. We can simplify this as $\rho_{\rm HP} = \frac{P_{\rm HP} r_{\rm HP}}{T_{\rm HP}} = \frac{1}{2-\frac{1}{4\pi P r_{\rm HP}^2}}$. Although $r_{\rm HP}$ has a complicated form, we can insert the four-dimensional case in Eq. (15) as an example and find a nonuniversal ratio $\rho_{\rm HP}$ dependent on the Gauss-Bonnet constant α. When we generalize the discussion to the reentrant Hawking-Page transition, we find the universal ratios for the triple point and critical point,

$$\rho_{\rm Tr} = \frac{P_{\rm Tr} r_{\rm Tr}}{T_{\rm Tr}} = \frac{1}{16}(d-1)d, \tag{35}$$

$$\rho_c = \frac{P_c r_c}{T_c} = -\frac{(d-2)(d-1)(\sqrt{d^2-6d+17}+3)}{4(d^2-(\sqrt{d^2-6d+17}+12)d+\sqrt{d^2-6d+17}+23)}. \tag{36}$$

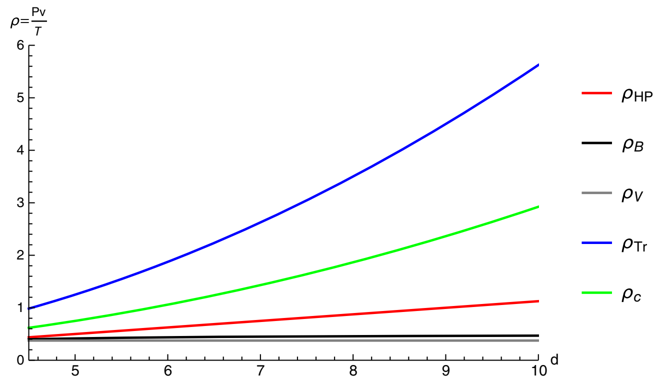


FIG. 10. Some universal ratios vs dimensions.

All the above ratios are illustrated in Fig. 10, which are universal and independent of the parameters of the systems.

On the other hand, since the temperature and pressure of the triple points and critical points share similar dependence on the Gauss-Bonnet constant α, $T \sim \frac{1}{\sqrt{\alpha}}$, $P \sim \frac{1}{\alpha}$, with different d-dependent coefficients, we introduce some other universal relations,

$$\frac{T_{\mathrm{Tr}}}{T_c} = -\frac{4(d-4)\sqrt{\frac{\sqrt{d^2-6d+17}+3}{d-2}}}{d^2-\sqrt{d^2-6d+17}(d-1)-12d+23},$$
$$\frac{P_{\mathrm{Tr}}}{P_c} = \frac{d(d-4)}{(d-2)^2}, \tag{37}$$

which both only depend on the dimensions d.

V. CONCLUSION

In this paper, we investigate the Hawking-Page transition of hyperbolic Gauss-Bonnet AdS black holes in extended thermodynamics. When $d \geq 4$, a new family of Hawking-Page transitions, i.e., the reentrant Hawking-Page transition, is found for the first time, which is composed of two Hawking-Page transitions with a high and a low Hawking-Page temperature. We also find the triple point where a small black hole, massless black hole, and large black hole all coexist. We calculate the temperature of two branches of Hawking-Page transitions, which both depend on the pressure (i.e., the cosmological constant and the Gauss-Bonnet constant). It is shown that pressure P and the Gauss-Bonnet constant α both increase the high Hawking-Page temperature and diminish the low Hawking-Page temperature. We present the $P-T$ phase diagrams of the Gauss-Bonnet AdS black hole. We find that the reentrant Hawking-Page transition always exists in four dimensions, while in $d > 4$ dimensions, it can only be seen for a range of pressure $P \in (P_{\mathrm{Tr}}, P_c)$, which is just the pressure of the triple point and an critical point. Above the pressure (temperature) of the critical points, the $d > 4$-dimensional Gauss-Bonnet AdS black hole systems undergo a single Hawking-Page transition. The triple points and the critical points for arbitrary dimensional Gauss-Bonnet AdS black hole systems are given, together with some interesting universal relations that only depend on the dimensions d and are independent of the parameters of the systems.

It is well known that the Hawking-Page transition could be explained as the confinement-deconfinement phase transition of the gauge field [17], inspired by the AdS/CFT correspondence [18–20]. The deconfinement phase transition reduces to a crossover [39], which implies a critical endpoint at a certain chemical potential μ. It is speculated that, in the holographic dictionary, this crossover corresponds to the Hawking-Page crossover of the Schwarzschild-AdS black hole [38] in noncommutative spacetime, which is thought to be an effective description of quantum gravitational spacetime. Following a similar spirit, we conjecture that the reentrant Hawking-Page transition and the triple point may be explained as the reentrant phase transition and the triple point in the QCD phase diagram. However, something more rigorously quantitative is very difficult here since string loops cannot be calculated yet in any of the backgrounds thought to be dual to gauge theory. The duality between the first-order phase transitions is present on spherically compactified spaces, while the reentrant Hawking-Page transition is given for the Gauss-Bonnet AdS black hole with the horizon typology being hyperbolic. Thus, it is important to explore the reentrant Hawking-Page transition for the spherical AdS black holes, which is left as a future task. It is also interesting to study the reentrant Hawking-Page transitions in different backgrounds and to generalize the study to the microcosmic and holographic frameworks. For example, one can consider the effect of the general higher-derivative terms on the Hawking-Page transition of the AdS black holes [77–88] and non-Schwarzschild AdS black holes [89–94], as well as the charged and rotating AdS black holes.

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APPENDIX: NEGATIVE ENTROPY AND HAWKING-PAGE TRANSITION

From the entropy of the hyperbolic Gauss-Bonnet AdS black hole, i.e., Eq. (7), one can find negative entropy, which seems to restrict some values of the parameters and thus provides an additional effect on the Hawking-Page transition. However, the constraint from negative black hole entropy does not make sense. Actually, as discussed in [66,68], there is an additive ambiguity in the definition of the entropy which can be appropriately chosen to avoid the problem of negative black hole entropy, in both the Euclidean approach and the Noether charge approach. They argue instead that the occurrence of negative entropy reflects a deficiency in the methods used to calculate the entropy; one can add an arbitrary constant to the entropy of all the black hole solutions in some family of solutions without affecting the first law. By changing the choice of this constant, one can clearly arrange for all black hole solutions to have positive entropy. Then, the ambiguous place of negative entropy becomes the point at which one has to decide which classical black hole solution we assign zero entropy to; if one make this choice appropriately, all black hole solutions will have positive entropy. Thus, in black hole thermodynamics, the negative entropy can be removed by an appropriate choice of zero entropy.

In a classical thermodynamic system, the zero entropy is chosen by the Nernst heat theorem, which states that the entropy at absolute zero should be a constant, i.e., zero without loss of generality, since there is no thermal motion at absolute zero. In a quantum thermodynamic system, the question is subtle. Considering the Fermi gas at absolute zero, the nonzero number of microscopic states leads to a nonzero entropy of the system, which is actually a relative zero entropy since it does not contribute to thermodynamic quantities or laws in the thermodynamical limit. When generalizing to quantum information theory, the discussion is more open; the entropies can be negative when considering quantum entangled systems, which reflects a fact related to quantum nonseparability [95,96]. The negative entropy of Gauss-Bonnet black holes has no physical significance in the thermodynamical limit and will not affect the Hawking-Page transition.

On the other hand, on the classical gravity side, the black hole characterizes some features of a classical thermodynamic system, i.e., the liquid-vapor-like phase transition [11]. For a physical system in classical thermodynamics, it should contain degrees of freedom in statistical physics, which means that it should take positive entropy. In this sense, one can try to consider the influence of negative entropy on the Hawking-Page transition, which could be a reference study for the property of gravity. From Eq. (7), the positive entropy provides a constraint for the size of the black hole, i.e.,

$$r_+ \geq r_s = \sqrt{\frac{2\alpha(d-2)}{(d-4)}}. \tag{A1}$$

In $d = 4$ dimensions, the constraint reduces to $r_+ \geq 2\sqrt{\alpha_{\rm GB}}$ (or equivalently, $T \geq T_s = 4P\sqrt{\alpha_{\rm GB}} - \frac{1}{8\pi\sqrt{\alpha_{\rm GB}}}$), which contributes to the phase structure of the Hawking-Page transition. From the $G - T$ diagram in the left panel of Fig. 11, we find that the small black hole branch and the small massless black hole become physically unacceptable. After highlighting the globally stable states of the system by the thick lines, we find that the Hawking-Page transition occurring at the low temperature disappears, which is physically unacceptable. The reentrant Hawking-Page transition is vanishing, and there is a single Hawking-Page transition occurring at the high temperature between a massless black hole and a large black hole. This can also be seen from the right panel of Fig. 11. The left blue line denotes the lower bound of temperature T_s. In the low temperature region $T < T_s$, there is no black hole phase.

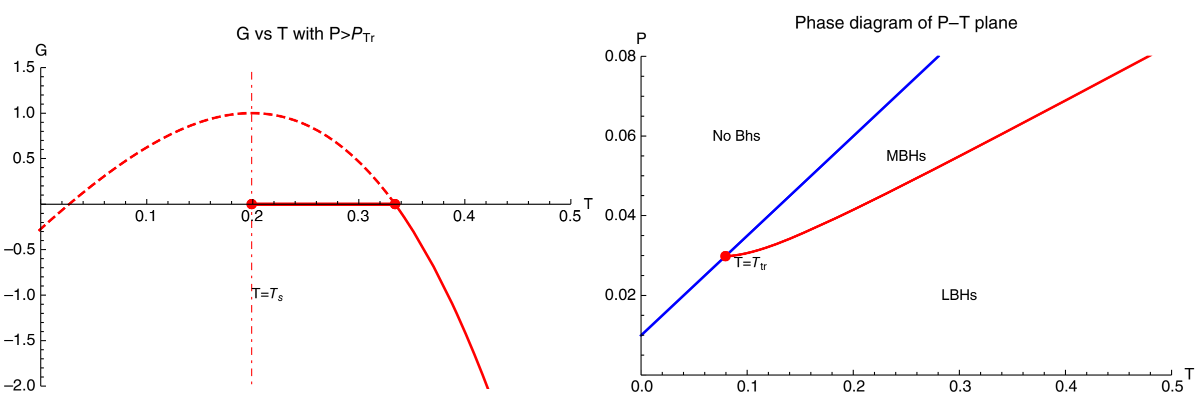

FIG. 11. The $G - T$ diagram and the $P - T$ phase diagram (with $\alpha = 1$) in four dimensions after considering the negative entropy. The reentrant Hawking-Page transition is vanishing under the positive entropy constraint, and there is a single Hawking-Page transition between a massless black hole and a large black hole. BHs, LBHs, and MBHs correspond to the black holes, large black holes, and massless black holes, respectively.

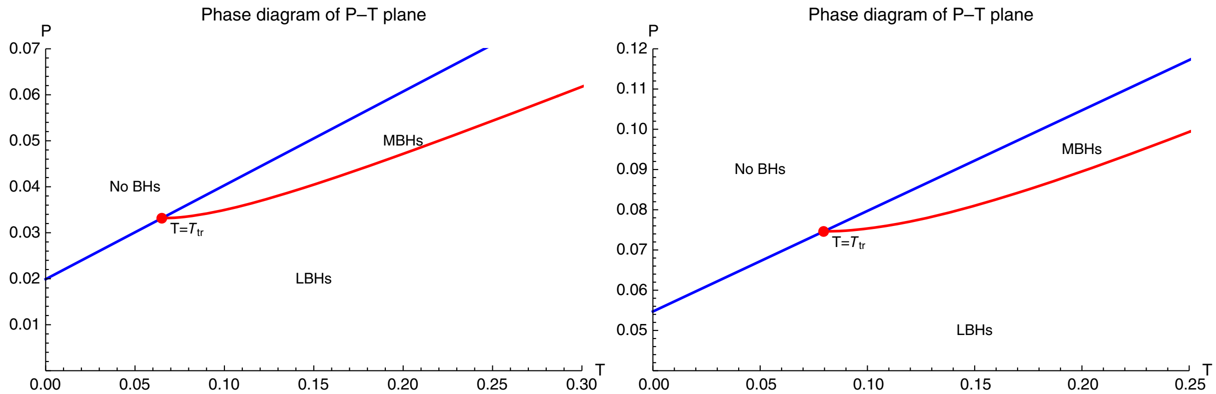

FIG. 12. The $P - T$ phase diagrams under the positive entropy constraint (with $\alpha = 1$) in five (left) and six (right) dimensions. There is a single Hawking-Page transition between a massless black hole and a large black hole as well. BHs, LBHs, and MBHs correspond to the black holes, large black holes, and massless black holes, respectively.

Obviously, there is only a single Hawking-Page transition in the $P - T$ phase diagram.

In higher dimensions, the entropy constraint in Eq. (A1) leads to a lower bound of the black hole temperature,

$$T \geq T_s = \frac{4(d-2)\alpha P}{(d-4)r_s} - \frac{(d^2-d-8)}{16\pi r_s}. \qquad \text{(A2)}$$

It is easy to check that this curve in the $P - T$ diagram goes through the triple points $(T_{\rm Tr}, P_{\rm Tr})$. Then, the phase structure of higher dimensional black holes is similar to the four-dimensional case. We plot the $P - T$ phase diagrams under the positive entropy constraint (i.e., the lower bound of temperature T_s shown as blue lines) in five (left) and six (right) dimensions in Fig. 12. There is always a single Hawking-Page transition between a massless black hole and a large black hole, and the reentrant Hawking-Page transition disappears because of the positive entropy constraint.

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