Cosmology in $f(Q)$ gravity: A unified dynamical systems analysis of the background and perturbations

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Motivated by the fact that cosmological models based on $f(Q)$ gravity are very efficient in fitting observational datasets at both background and perturbation levels, we perform a combined dynamical system analysis of both background and perturbation equations in order to examine the validity of this result through an independent method. We examine two studied $f(Q)$ models of the literature, namely, the power-law and the exponential ones. For both cases, we obtain a matter-dominated saddle point characterized by the correct growth rate of matter perturbations, followed by the transition to a stable dark-energy-dominated accelerated universe in which matter perturbations remain constant. Furthermore, analyzing the behavior of $f\sigma_8$, we find that the models fit observational data successfully, obtaining a behavior similar to that of the Lambda cold dark matter (ΛCDM) scenario, although the exponential model does not possess the latter as a particular limit. Hence, through the independent approach of dynamical systems, we do verify the results of observational confrontation, namely, that $f(Q)$ gravity can be considered as a very promising alternative to the ΛCDM concordance model.

I. INTRODUCTION

Einstein's theory of general relativity (GR) is the most successful theory to describe the gravitational interaction, and based on that, the Lambda cold dark matter (ΛCDM) scenario is the concordance cosmological model. However, this standard gravitational and cosmological paradigm faces theoretical and observational problems such as the nonrenormalizability of GR, cosmological-constant problem, coincidence problem, Hubble tension, σ_8 tension, etc. [1–7]. Hence, in the literature, one may find various modified theories of gravity [8–11] and several GR-based models beyond the ΛCDM one [12–14] that aim to alleviate a part or all of the above issues.

The usual way to construct gravitational modifications is to add extra terms in the Einstein-Hilbert Lagrangian, resulting in $f(R)$ gravity [15–17], Gauss-Bonnet and $f(G)$ gravity [18–20], cubic and $f(P)$ gravity [21–23], Horndeski/Galileon scalar-tensor theories [24,25], etc. Alternatively, one can add new terms to the equivalent formulation of gravity based on torsion, resulting in $f(T)$ gravity [11,26], $f(T, T_G)$ gravity [27–29], $f(T, B)$ gravity [30,31], scalar-torsion gravity [32], etc. Nevertheless, there is a third way to construct new classes of modified gravity, starting from the "symmetric teleparallel gravity," which is based on the nonmetricity scalar Q [33], and extending it to a function $f(Q)$ in the Lagrangian.

The modified theory of $f(Q)$ gravity leads to interesting cosmological phenomenology at the background level [34–58]. Additionally, it has been successfully confronted with various background and perturbation observational data, such as the cosmic microwave background, supernovae type Ia, baryonic acoustic oscillations, redshift space distortion (RSD), growth data, etc. [59–66]; this confrontation reveals that $f(Q)$ gravity may challenge the standard ΛCDM scenario. Finally, $f(Q)$ gravity comfortably passes the big bang nucleosynthesis (BBN) constraints too [67].

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Motivated by the exciting features of $f(Q)$ gravity, in this work, we employ the powerful mathematical tool of dynamical systems analysis in order to investigate for the first time the cosmological dynamics of $f(Q)$ cosmology at both background and perturbation levels. Such investigation can be used to further confirm the results obtained from the observational analysis. We mention that, usually, the dynamical systems approach is applied at the background level [68–85], however, relatively recently it was realized that the analysis can be applied at the perturbation level too [86–89]. Hence, with this combined analysis, we can determine both the background stable late-time solutions, as well as the growth of the structure formation, independent of the specific initial conditions. Moreover, we can examine how the matter perturbations backreact to the background solutions, too.

The work is organized as follows: In Sec. II, we present the field equations of $f(Q)$ gravity, from which one can arrive at the background and perturbed cosmological equations. Section III deals with the dynamical analysis of the combined system for the power-law and exponential models. Finally, the obtained results are summarized in Sec. IV.

II. $f(Q)$ COSMOLOGY

In this section, we briefly review $f(Q)$ cosmology. The action of $f(Q)$ gravity is given by [33,34]

$$S = \int \left[-\frac{1}{16\pi G} f(Q) + \mathcal{L}_m \right] \sqrt{-g} d^4 x, \quad (1)$$

where g is the determinant of the metric $g_{\mu\nu}$ and $\mathcal{L}_m$ is the matter Lagrangian density. $f(Q)$ is an arbitrary function of the nonmetricity scalar [33]

$$Q = -\frac{1}{4} Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + \frac{1}{2} Q_{\alpha\beta\gamma} Q^{\gamma\beta\alpha} + \frac{1}{4} Q_\alpha Q^\alpha - \frac{1}{2} Q_\alpha \tilde{Q}^\alpha, \quad (2)$$

where $Q_\alpha \equiv Q_\alpha{}^\mu{}_\mu$ and $\tilde{Q}^\alpha \equiv Q_\mu{}^{\mu\alpha}$ are acquired from contractions of the nonmetricity tensor $Q_{\alpha\mu\nu} \equiv \nabla_\alpha g_{\mu\nu}$. Thus, the symmetric teleparallel equivalent of general relativity, and therefore general relativity, is recovered for $f(Q) = Q$.

Variation of action (1), and setting $8\pi G = 1$ for simplicity, leads to the field equations [34,35]

$$\frac{2}{\sqrt{-g}} \nabla_\alpha \left\{ \sqrt{-g} g_{\beta\nu} f_Q \left[-\frac{1}{2} L^{\alpha\mu\beta} + \frac{1}{4} g^{\mu\beta} (Q^\alpha - \tilde{Q}^\alpha) \right. \right.$$
$$\left. - \frac{1}{8} (g^{\alpha\mu} Q^\beta + g^{\alpha\beta} Q^\mu) \right] \right\} + f_Q \left[-\frac{1}{2} L^{\mu\alpha\beta} - \frac{1}{8} (g^{\mu\alpha} Q^\beta + g^{\mu\beta} Q^\alpha) \right.$$
$$\left. + \frac{1}{4} g^{\alpha\beta} (Q^\mu - \tilde{Q}^\mu) \right] Q_{\nu\alpha\beta} + \frac{1}{2} \delta_\nu^\mu f = T^\mu{}_\nu, \quad (3)$$

where $L^\alpha{}_{\mu\nu} = \frac{1}{2} Q^\alpha{}_{\mu\nu} - Q_{(\mu}{}^\alpha{}_{\nu)}$ is the disformation tensor, $T_{\mu\nu}$ is the matter energy-momentum tensor, and $f_Q \equiv \partial f / \partial Q$.

At the background level, we assume a homogeneous, isotropic, and spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, whose metric is of the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (4)$$

where t is the cosmic time, $a(t)$ is the scale factor, and x, y, z are the Cartesian coordinates. Note that in the FLRW metric, for the nonmetricity scalar, we obtain $Q = 6H^2$, where $H = \frac{\dot{a}}{a}$ is the Hubble function and the upper dot denotes derivative with respect to t. Imposing the splitting $f(Q) = Q + F(Q)$ and applying the FLRW metric, the corresponding Friedman equations are [33,34]

$$3H^2 = \rho + \frac{F}{2} - QF_Q, \quad (5)$$

$$(2QF_{QQ} + F_Q + 1)\dot{H} + \frac{1}{4}(Q + 2QF_Q - F) = -\frac{p}{2}, \quad (6)$$

with $F_Q \equiv \frac{dF}{dQ}$ and $F_{QQ} \equiv \frac{d^2F}{dQ^2}$. In the above equations, ρ and p are the energy density and pressure of the matter fluid, which for no interaction satisfy the conservation equation

$$\dot{\rho} + 3H(1 + w)\rho = 0, \quad (7)$$

with $w \equiv p/\rho$ as the matter equation-of-state parameter.

We can now introduce the effective, total energy density ρ_{eff} and pressure p_{eff}, respectively, as

$$\rho_{\text{eff}} \equiv \rho + \frac{F}{2} - QF_Q, \quad (8)$$

$$p_{\text{eff}} \equiv \frac{\rho(1+w)}{2QF_{QQ} + F_Q + 1} - \frac{Q}{2}, \quad (9)$$

and thus the corresponding total equation of state w_{eff} is given by

$$w_{\text{eff}} \equiv \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -1 + \frac{\Omega_m(1+w)}{2QF_{QQ} + F_Q + 1}. \quad (10)$$

We mention that for an accelerated universe one requires $w_{\text{eff}} < -\frac{1}{3}$. Finally, it proves convenient to introduce the energy density parameters as

$$\Omega_m = \frac{\rho}{3H^2}, \qquad \Omega_Q = \frac{\frac{F}{2} - QF_Q}{3H^2}, \quad (11)$$

and thus the first Friedman equation (5) becomes simply

$$\Omega_m + \Omega_Q = 1. \quad (12)$$

Let us proceed to the investigation of the linear perturbation level, focusing on the matter density contrast $\delta = \frac{\delta\rho}{\rho}$,

where $\delta\rho$ is the perturbation of the matter energy density. In particular, the evolution equation of the matter overdensity at the quasistatic limit is given by [34,63]

$$\ddot{\delta} + 2H\dot{\delta} = \frac{\rho\delta}{2(1 + F_Q)}, \qquad (13)$$

where the denominator of the last term accounts for the appearance of an effective Newton's constant. We mention that in the quasistatic limit the terms involving time derivatives in the perturbed equations are neglected, and only spatial derivative terms remain. It is worth mentioning that such an approximation is sufficient at small scales, well within the cosmic horizon [90].

III. DYNAMICAL SYSTEMS ANALYSIS

In this section, we construct the dynamical system of the background and perturbed equations, for a general function $F(Q)$. In this regard, we transform Eqs. (5)–(7) and (13) into a first-order autonomous system, by considering the following dynamical variables:

$$x = \frac{F}{6H^2}, \qquad y = -2F_Q, \qquad u = \frac{d(\ln \delta)}{d(\ln a)}. \qquad (14)$$

Hence, while the variables x, y are associated with the background evolution of the Universe, the variable u quantifies the growth of matter perturbations. Therefore, $u > 0$ signifies the growth and $u < 0$ indicates the decay of matter perturbations, whenever the matter density contrast δ is positive.

The background cosmological parameters Ω_m, Ω_Q and, $w_{\rm eff}$ can be expressed as

$$\Omega_m = 1 - x - y,$$
$$\Omega_Q = x + y,$$
$$w_{\rm eff} = -1 + \frac{(1 - x - y)(1 + w)}{2QF_{QQ} - \frac{y}{2} + 1}. \qquad (15)$$

Now, in terms of variables (14), the cosmological equations can be written as the following dynamical system:

$$x' = -\frac{\dot{H}}{H^2}(y + 2x), \qquad (16)$$

$$y' = 2y\frac{\dot{H}}{H^2}\frac{QF_{QQ}}{F_Q}, \qquad (17)$$

$$u' = -u(u + 2) + \frac{3(1 - x - y)}{(2 - y)} - \frac{\dot{H}}{H^2}u, \qquad (18)$$

where a prime stands for differentiation with respect to $\ln a$, and

$$\frac{\dot{H}}{H^2} = -\frac{3 - 3(x + y)}{4QF_{QQ} - y + 2}. \qquad (19)$$

The physical system is a product space of the background phase space $\mathbb{B}$, which includes the variables x, y, and the perturbed space $\mathbb{P}$, which consists of the variable u. Under the physical condition $0 \le \Omega_m \le 1$, the phase space of the combined system is

$$\Psi = \mathbb{B} \times \mathbb{P} = \{(x, y, u) \in \mathbb{R}^2 \times \mathbb{R} : 0 \le x + y \le 1\}. \qquad (20)$$

It is worth mentioning that the projection of orbits of the product space Ψ on space $\mathbb{B}$ reduces to the corresponding background orbits.

As a next step we shall determine the dynamical evolution of the system by extracting its critical points and examining their stability. Physically, a stable point with $u > 0$ indicates an indefinite growth of matter perturbations and hence the system is not stable with respect to matter perturbations. However, a stable point with $u < 0$ indicates the decay of matter perturbations and therefore the system is asymptotically stable with respect to perturbations. Finally, when $u = 0$ at a stable point, this implies that matter perturbations remain constant. In summary, for a viable model, one desires to have an unstable or saddle point with $u > 0$, required for the description of the matter epoch of the Universe, in which the matter perturbations grow but which does not hold eternally, followed by a stable late-time attractor corresponding to acceleration but with $u = 0$.

In order to proceed to specific analysis, we need to specify the function F and hence determine the term $\frac{QF_{QQ}}{F_Q}$. In the following subsections, we will consider two specific models, which are known to lead to interesting cosmological phenomenology.

A. Model I: $F(Q) = \alpha(\frac{Q}{Q_0})^n$

We start our analysis by considering a power-law model with [34,60,62]

$$F(Q) = \alpha\left(\frac{Q}{Q_0}\right)^n, \qquad (21)$$

where α and n are two parameters and where $Q_0 = 6H_0^2$ is the present value of Q (note that applying the first Friedmann equation at present, α can be eliminated in terms of n and the present value Ω_{m0}). This model can describe the late-time Universe acceleration and it is also compatible with BBN constraints [67]. We mention that for $n = 0$ this model is equivalent to the concordance ΛCDM scenario, while for $n = 1$ the model reduces to the symmetric teleparallel equivalent of general relativity [34,60,62]. In this case, we have $QF_{QQ} = \frac{(1-n)y}{4}$ and hence the system (16)–(18) closes.

The corresponding dynamical system contains the following four critical points:

(i) Point A_1 (0,0,1): This point corresponds to a matter-dominated critical solution with the background parameters $\Omega_m = 1$ and $w_{\rm eff} = 0$. At the perturbation

level we have $u = 1$, which implies that the matter overdensity δ varies as the scale factor a and hence increases with the Universe expansion. The corresponding Jacobian matrix has the eigenvalues $-\frac{5}{2}, 3$, and $\frac{3}{2}(1 - n)$, therefore point A_1 for any value of n is always a saddle one. Hence, the trajectories pass through this point and leave it, as they are attracted by a late-time stable point. Thus, we conclude that this point could be the main candidate for describing the structure formation during the matter domination era at both the background and perturbation levels.

(ii) Point B_1 $(0, 0, -\frac{3}{2})$: At the background level, this point corresponds to matter domination, with $\Omega_m = 1$ and $w_{\mathrm{eff}} = 0$. However, this point could not describe the formation of structures at the perturbation level, since $u = -\frac{3}{2}$, and hence the matter overdensity δ varies as $a^{-\frac{3}{2}}$. The eigenvalues of the Jacobian matrix are $\frac{5}{2}, 3$, and $\frac{3}{2}(1 - n)$, and therefore this point is unstable for $n < 1$ and saddle for $n > 1$.

(iii) The curve of critical points C_1 $(1 - y, y, 0)$: Each point on this curve corresponds to a solution dominated by the effective dark-energy component, i.e., $\Omega_Q = 1$, in which the Universe is accelerated with a cosmological-constant-like behavior, namely, with $w_{\mathrm{eff}} = -1$. Furthermore, at the perturbation level we have $u = 0$, which implies that the matter perturbation remains constant. The corresponding eigenvalues are $-2, -3$, and 0, and since the curve is one dimensional with one vanishing eigenvalue, it is normally hyperbolic [69], and one can determine its stability by examining the signature of the remaining nonvanishing eigenvalues [69]. Therefore, we conclude that it is always stable. In summary, the curve C_1 describes the late-time dark-energy-dominated Universe, at both background and perturbation levels.

(iv) Curve of critical points D_1 $(1 - y, y, -2)$: Similar to C_1, this curve of critical points also corresponds to a cosmological-constant-like solution, i.e., with $w_{\mathrm{eff}} = -1$, dominated by the effective dark-energy component. Additionally, it is characterized by the decay of matter perturbations, since $u = -2$. However, it is a saddle point with eigenvalues $2, -3$, and 0. Hence, unlike C_1, curve D_1 cannot describe a late-time dark-energy-dominated universe at the perturbation level.

Our analysis reveals that different critical points describe different modes of matter perturbations. Additionally, we mention that identical background critical points behave differently at the perturbation level. For instance, we showed that points A_1 and B_1 describe the decelerated matter-dominated era at the background level, but only point A_1 has the correct growth of matter structure. Interestingly, point A_1 is saddle and thus it provides the natural exit toward a late-time accelerated epoch. On the other hand, at late times, the curves of critical points C_1 and
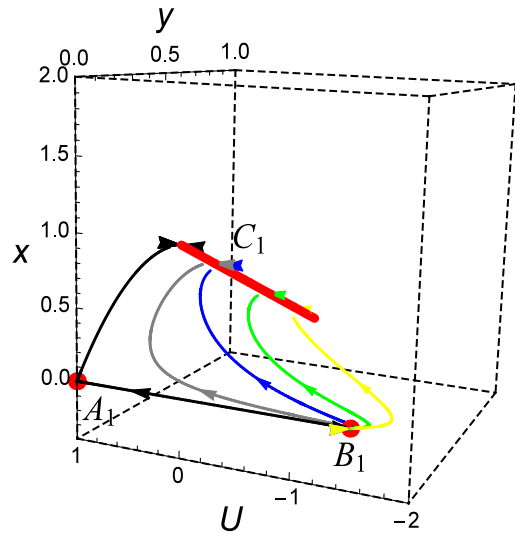

FIG. 1. The phase portrait of the system (16)–(18), for the power-law model I of (21) with $n = 0.5$. This particular example exhibits the evolution $B_1 \to A_1 \to C_1$.

D_1 are identical at the background level, describing the accelerated dark-energy-dominated epoch. However, only curve C_1 is physically and observationally interesting at the perturbation level, since it is stable and exhibits constant matter perturbations. Finally, examining for completeness whether there are critical points at infinity, we find that no such physical points exist.

In order to give the above information in a more transparent way, we display the phase portrait of the system (16)–(18) in Fig. 1. As we see, the system follows the orbit $B_1 \to A_1 \to C_1$. Furthermore, in Fig. 2 we present the evolution of the background parameters and the matter growth rate variable u, in terms of the redshift $z = \frac{a_0}{a} - 1$ with $a_0 = 1$ as the current scale factor. As we see, the model describes the transition from matter domination toward an accelerated dark-energy-dominated epoch.

In summary, the present power-law model can describe the desired thermal history of the Universe, both at the background and perturbation levels. Our analysis indicates that, in principle, the above hold for any value of n. Nevertheless, we should mention that a tuning of initial conditions is required in order to have a sufficiently long matter-dominated era.

Finally, in order to test the predictions on the matter growth with observational data, in Fig. 3 we provide the evolution of $f\sigma_8$. This quantity is defined as the product of the growth rate factor $f = u = \frac{d \ln \delta}{d \ln a}$ and the root-mean-square normalization of the matter power spectrum σ_8. The value of σ_8 usually depends on the model, however, here we have taken $\sigma_8 = 0.8$, which could alleviate the present σ_8 tension between the RSD and Planck data [2]. We mention here that we have checked that for $n > 0$ the evolution of $f\sigma_8$ coincides with that of the ΛCDM scenario, namely, with the case $n = 0$. From Fig. 3, we observe that models

FIG. 3. The evolution of $f\sigma_8$ as a function of the redshift, for the power-law model I of (21), for various values of n. The data are taken from [91,92].
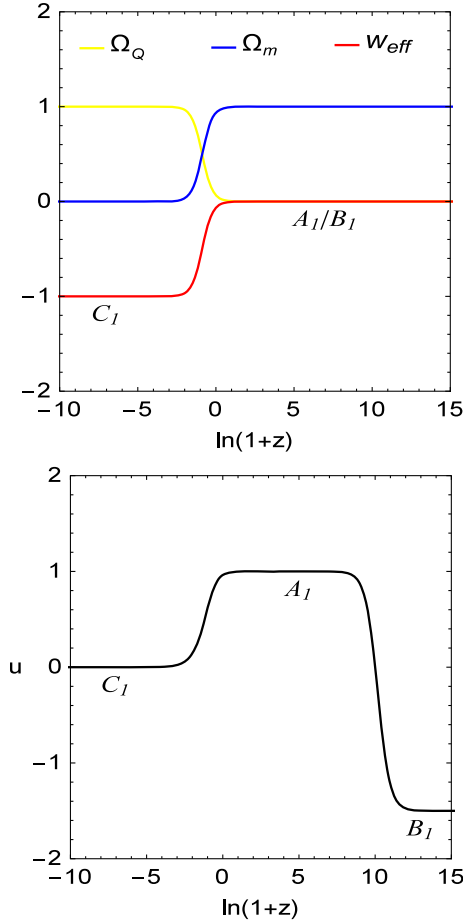
FIG. 2. Upper: evolution of the density parameters of matter Ω_m and of nonmetricity (dark energy) Ω_Q, as well as of the total, effective equation-of-state parameter w_{eff}, as functions of the redshift, for the power-law model I of (21) with $n = 0.5$. Lower: evolution of the perturbation variable $u = \frac{d(\ln \delta)}{d(\ln a)}$.

with $n < 0$ have a smaller value of $f\sigma_8$, however, the data prefer comparatively larger values. Hence, models with $n < 0$ are not favored by the data. Additionally, it is worth noting that observational data still favor $n < 1$ [60–62,65]. Thus, from our analysis we can conclude that the condition $0 < n < 1$ is required in order to acquire consistency with observations.

B. Model II: $F(Q) = Qe^{\beta\frac{Q_0}{Q}} - Q$

In this subsection, we consider the exponential model [63]

$$F(Q) = Qe^{\beta\frac{Q_0}{Q}} - Q, \tag{22}$$

with β as the only dimensionless parameter. For $\beta = 0$, the model is equivalent to GR without a cosmological constant, however, the interesting feature is that for $\beta \neq 0$ this model can fit observations in a very satisfactory way, although it does not include a cosmological constant [63]. Note that
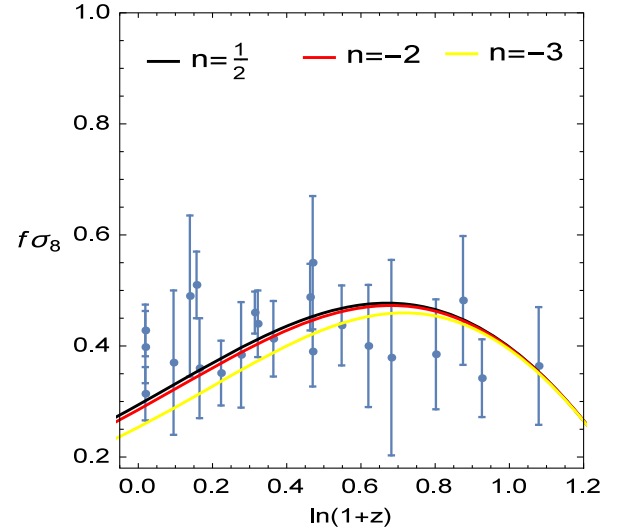
applying the first Friedmann equation at present, β can be eliminated in terms of Ω_{m0} [63]. Additionally, since at early times $Q \gg Q_0$, the model tends to GR limit and hence it trivially passes the BBN constraints [63,67].

In this case, we have $QF_{QQ} = \frac{(x+1)^2 + x(y-2) + \frac{y^2}{4} - 1}{x+1}$, and therefore the dynamical system (16)–(18) has four curves of critical points. In what follows, we shall describe the properties of each curve.

(i) Curve of critical points A_2 $(-\frac{y}{2}, y, 1)$: This curve corresponds to a matter scaling solution with $\Omega_m = 1 - \frac{y}{2}$ and $w_{\text{eff}} = 0$. The corresponding Jacobian matrix has the eigenvalues $-\frac{5}{2}$, 3, and 0, and thus the corresponding points are always saddle. Furthermore, since $u = 1$, it is implied that the matter perturbations grow, and hence the solution is of interest from the structure formation point of view.

(ii) Curve of critical points B_2 $(-\frac{y}{2}, y, -\frac{3}{2})$: Similar to A_2, this curve corresponds to a matter scaling solution. However, since $u < 0$, we deduce that the matter perturbations decay and therefore it cannot describe the growth of structures during the matter-dominated epoch. It corresponds to an unstable node with eigenvalues $\frac{5}{2}$, 3, and 0.

(iii) Curves of critical points C_2 $(1 - y, y, 0)$ and D_2 $(1 - y, y, -2)$: Both these curves correspond to accelerated solutions, dominated by the nonmetricity component. The stability and cosmological properties of curves C_2 and D_2 are exactly the same as the curves C_1 and D_1. Finally, we find that only C_2 is interesting for the late-time Universe at the perturbation level.

Similar to model I of the previous subsection, we see that the inclusion of perturbations distinguishes critical points that are equivalent at the background level. Hence,
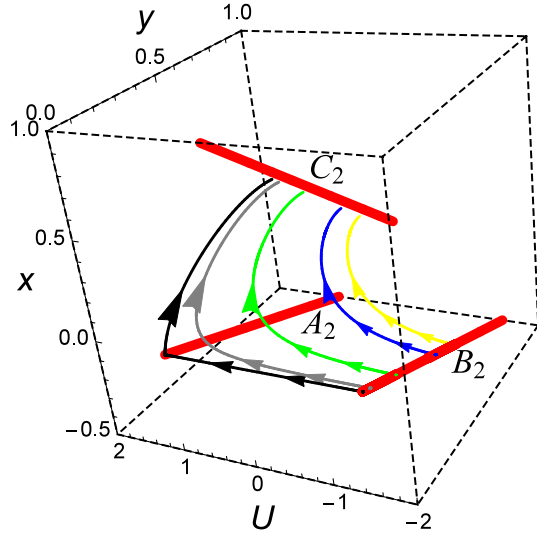
FIG. 4. The phase portrait of the system (16)–(18), for the exponential model II of (22). This particular example exhibits the evolution $B_2 \rightarrow A_2 \rightarrow C_2$.

FIG. 6. The evolution of $f\sigma_8$ as a function of the redshift, for the exponential model II of (22). The data are taken from [91,92].

from the combined background and perturbation analysis, we find that only curve A_2 is physically interesting to describe the matter-dominated epoch, where matter perturbations are generated. On the other hand, curve C_2 corresponds to late-time dark-energy domination, with a fixed evolution of matter perturbations, as observations require. Finally, for this model, we also find no critical points at infinity.

In Fig. 4, we present the phase-space evolution, describing the transition $B_2 \rightarrow A_2 \rightarrow C_2$. Furthermore, in Fig. 5 we depict the evolution of the background cosmological parameters and the growth rate u, where we observe the transition from matter domination toward a late-time dark-energy-dominated epoch. As we mentioned above, it is interesting that, even though the present model does not possess a ΛCDM limit for any parameter choice, the corresponding dynamics is qualitatively similar with that of ΛCDM (see Fig. 5). Hence, since the model is free from the cosmological-constant problem, it may be considered as slightly preferred over the ΛCDM scenario, constituting an interesting alternative.

Finally, in Fig. 6, we investigate the evolution of $f\sigma_8$, using $\sigma_8 = 0.7$. As we observe, the behavior is comparable with that of the ΛCDM paradigm. In summary, the unified dynamical systems analysis confirms the observational investigation at background and perturbation levels performed in [63,65].
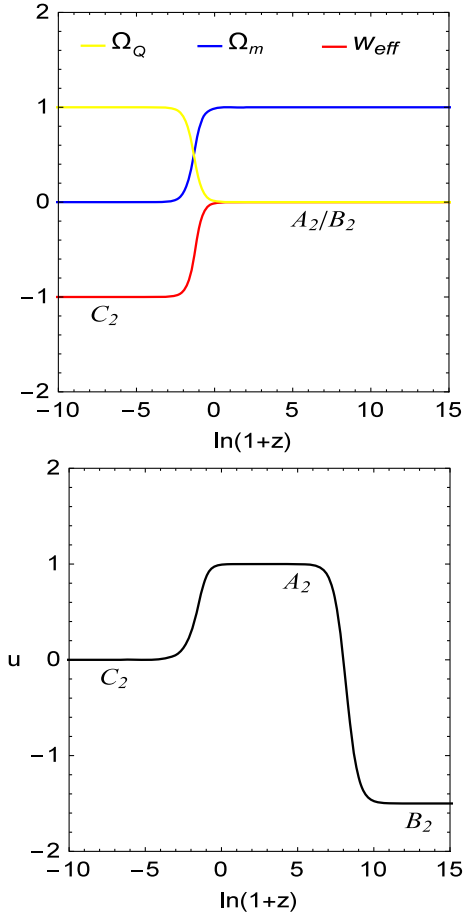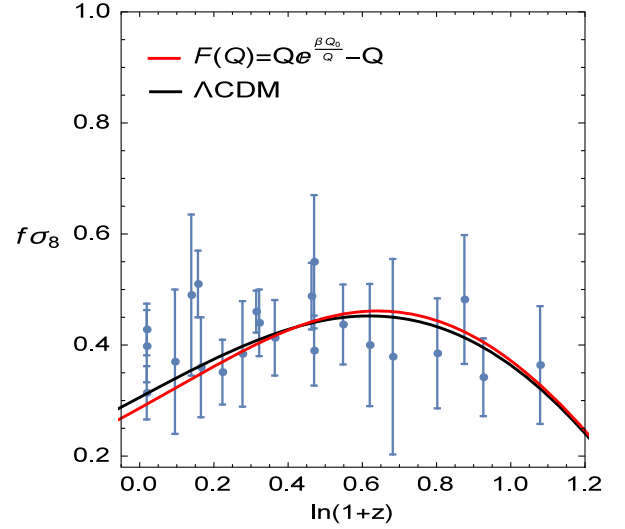

FIG. 5. Upper: evolution of the density parameters of matter Ω_m and of nonmetricity (dark energy) Ω_Q, as well as of the total, effective equation-of-state parameter w_{eff}, as functions of the redshift, for the exponential model II of (22). Lower: evolution of the perturbation variable $u = \frac{d(\ln \delta)}{d(\ln a)}$.

IV. CONCLUSIONS

Motivated by the fact that cosmological models based on $f(Q)$ gravity are very efficient in fitting observational datasets at both background and perturbation levels [63,65], in the present work we performed a combined dynamical systems analysis of both background and

perturbation equations in order to examine the validity of this result through an independent method.

After transforming the background and perturbation equations into an autonomous system, we focused on two studied $f(Q)$ models of the literature, namely, the power-law and the exponential ones. Because of the extra variable related to matter perturbations, each background critical point split into two points, characterized by different behavior of matter perturbations and stability.

Concerning the power-law model, we obtained a matter-dominated saddle point characterized by the correct growth rate of matter perturbations, followed by the transition to a stable dark-energy-dominated accelerated universe in which matter perturbations remain constant. Furthermore, we studied the growth of matter perturbations by analyzing the behavior of $f\sigma_8$, and we found that the model fits observational data successfully if the exponent lies within $0 < n < 1$, in which case we obtained a behavior similar to that of the ΛCDM scenario.

Concerning the exponential model, we also found curves of points corresponding to matter domination and matter perturbation growth, and the fact that they are saddle points provides a successful transition to the stable late-time dark-energy-dominated solution with constant matter perturbations. The interesting feature of this model is that this desired behavior is obtained although the model does not possess the ΛCDM scenario as a particular limit, namely, it arises solely from the nonmetricity structure. Additionally, we found that, while the power-law model resembles ΛCDM cosmology for $n < 1$, the exponential model resembles the latter for any choice of the model parameter.

In summary, the combined dynamical analysis at the background and perturbation levels do verify the results of observational confrontation, showing through an independent way that $f(Q)$ gravity, and specifically, the exponential model, can be considered as a very promising alternative to the ΛCDM concordance model.

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