Collinear and triangular solutions to the coplanar and circular three-body problem in the parametrized post-Newtonian formalism

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This paper investigates the coplanar and circular three-body problem in the parametrized post-Newtonian (PPN) formalism, for which we focus on a class of fully conservative theories characterized by the Eddington-Robertson parameters β and γ. It is shown that there can still exist a collinear equilibrium configuration and a triangular one, each of which is a generalization of the post-Newtonian equilibrium configuration in general relativity. The collinear configuration can exist for arbitrary mass ratio, β, and γ. On the other hand, the PPN triangular configuration depends on the nonlinearity parameter β but not on γ. For any value of β, the equilateral configuration is possible, if and only if three finite masses are equal or two test masses orbit around one finite mass. For general mass cases, the PPN triangle is not equilateral as in the post-Newtonian case. It is shown also that the PPN displacements from the Lagrange points in the Newtonian gravity L_1, L_2, and L_3 depend on β and γ, whereas those to L_4 and L_5 rely only on β.

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I. INTRODUCTION

The three-body problem is among the classical ones in physics. It led to the notion of chaos [1]. On the other hand, particular solutions such as Euler's collinear solution and Lagrange's equilateral one [2,3] express regular orbits and they have still attracted interest, e.g., [4–8]. If one mass is zero and the other two masses are finite, the collinear solution and triangular one correspond to Lagrange points L_1, L_2, L_3, L_4, and L_5 as particular solutions for the coplanar restricted three-body problem.

In his pioneering work [9], Nordtvedt found that the position of the triangular points is very sensitive to the ratio of the gravitational mass to the inertial mass in gravitational experimental tests, where the post-Newtonian (PN) terms are not fully taken into account.

Krefetz [10] and Maindl [11] studied the restricted three-body problem in the PN approximation and found the PN triangular configuration for a general mass ratio between two masses. These investigations were extended to the PN three-body problem for general masses [12–17], and the PN counterparts for Euler's collinear [12,13] and Lagrange's equilateral solutions [14,15] were obtained. It should be noted that the PN triangular solutions are not necessarily equilateral for general mass ratios and they are equilateral only for either the equal mass case or two test masses. The stability of the PN solution and the radiation reaction at 2.5PN order were also examined [16,17].

In a scalar-tensor theory of gravity, a collinear configuration for three-body problem was discussed [18]. In addition to such fully classical treatments, a possible quantum gravity correction to the Lagrange points was proposed [19,20].

Moreover, the recent discovery of a relativistic hierarchical triple system including a neutron star [21] has generated renewed interest in the relativistic three-body problem and the related gravitational experiments [22–24].

The main purpose of the present paper is to reexamine the coplanar and circular three-body problem especially in the PPN formalism. One may ask if collinear and triangular configurations are still solutions for the coplanar three-body problem in the PPN gravity. If so, how large are the PPN effects of the three-body configuration? We focus on the Eddington-Robertson parameters β and γ, because the two parameters are the most important ones; β measures how much nonlinearity there is in the superposition law for gravity and γ measures how much space curvature is produced by unit rest mass [25,26]. Hence, preferred locations, preferred frames or a violation of conservation of total momentum will not be considered in this paper. We confine ourselves to a class of fully conservative theories. See, e.g., [27] for the celestial mechanics in this class of PPN theories.

This paper is organized as follows. In Sec. II, collinear configurations are discussed in the PPN formalism. Section III investigates PPN triangular configurations. In Sec. IV, the PPN corrections to the Lagrange points are examined. For brevity, the Lagrange points defined in Newtonian gravity are referred to as the Newtonian Lagrange points in this paper. Section V summarizes this

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paper. Throughout this paper, $G = c = 1$. A, B, and $C \in \{1, 2, 3\}$ label three masses.

II. COLLINEAR CONFIGURATION IN PPN GRAVITY

A. Euler's collinear solution in Newton gravity

Let us begin with briefly mentioning the Euler's collinear solution for the circular three-body problem in Newton gravity [2,3], for which each mass M_A ($A = 1, 2, 3$) at $\boldsymbol{x}_A$ is orbiting around the common center of mass (COM) at $\boldsymbol{x}_G$, and the orbital velocity and acceleration are denoted as $\boldsymbol{v}_A$ and $\boldsymbol{a}_A$, respectively. In this section, we suppose that three masses are always aligned, for which it is convenient to use the corotating frame with a constant angular velocity ω on the orbital plane chosen as the x-y plane.

Without loss of generality, we assume $x_1 > x_2 > x_3$ for $\boldsymbol{x}_A \equiv (x_A, 0)$. Let R_A denote the relative position of each mass M_A from the COM at $\boldsymbol{x}_G \equiv (x_G, 0)$. Namely, $R_A = x_A - x_G$. Note that $|R_A| \neq |\boldsymbol{x}_A|$ unless $x_G = 0$ is chosen. We define the relative vector between masses as $\boldsymbol{R}_{AB} \equiv \boldsymbol{x}_A - \boldsymbol{x}_B$, for which the relative length is $R_{AB} = |\boldsymbol{R}_{AB}|$. See Fig. 1 for a configuration of the Euler's collinear solution.

The coordinate origin $x = 0$ is chosen between M_1 and M_3, such that $R_1 > R_2 > R_3$, $R_1 > 0$ and $R_3 < 0$. By taking account of this sign convention, the equation of motion becomes

$$R_1\omega^2 = \frac{M_2}{R_{12}^2} + \frac{M_3}{R_{13}^2}, \tag{1}$$

$$R_2\omega^2 = -\frac{M_1}{R_{12}^2} + \frac{M_3}{R_{23}^2}, \tag{2}$$

$$R_3\omega^2 = -\frac{M_1}{R_{13}^2} - \frac{M_2}{R_{23}^2}. \tag{3}$$

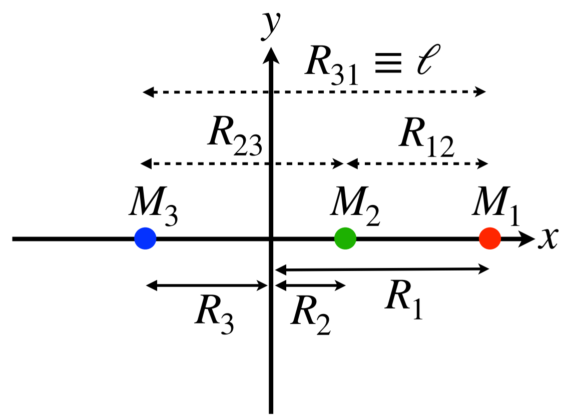

FIG. 1. Schematic figure for the collinear configuration of three masses.

We define the distance ratio as $z \equiv R_{23}/R_{12}$, which plays a key role in the following calculations. Note that $z > 0$ by definition. We subtract Eq. (2) from Eq. (1) and Eq. (3) from Eq. (2). By combining the results including the same angular velocity ω, we obtain a fifth-order equation for z as

$$(M_1 + M_2)z^5 + (3M_1 + 2M_2)z^4 + (3M_1 + M_2)z^3 - (M_2 + 3M_3)z^2 - (2M_2 + 3M_3)z - (M_2 + M_3) = 0, \tag{4}$$

for which there exists the only positive root [2,3]. In order to obtain Eq. (4), we do not have to specify the coordinate origin, e.g., $x_G = 0$. This is because Eq. (4) does not refer to any coordinate system. Once Eq. (4) is solved for z, we can obtain ω by substituting z into any of Eqs. (1)–(3).

B. PPN collinear configuration

In a class of fully conservative theories including only the Eddington-Robertson parameters β and γ, the equation of motion is [25,26]

$$\begin{aligned}\boldsymbol{a}_A = &-\sum_{B\neq A}\frac{M_B}{R_{AB}^2}\boldsymbol{n}_{AB} - \sum_{B\neq A}\frac{M_B}{R_{AB}^2}\left\{\gamma v_A^2 - 2(\gamma+1)(\boldsymbol{v}_A\cdot\boldsymbol{v}_B) + (\gamma+1)v_B^2 - \frac{3}{2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{v}_B)^2 - (2\gamma+2\beta+1)\frac{M_A}{R_{AB}} - 2(\gamma+\beta)\frac{M_B}{R_{AB}}\right\}\boldsymbol{n}_{AB} \\ &+ \sum_{B\neq A}\frac{M_B}{R_{AB}^2}\{\boldsymbol{n}_{AB}\cdot[2(\gamma+1)\boldsymbol{v}_A - (2\gamma+1)\boldsymbol{v}_B]\}(\boldsymbol{v}_A - \boldsymbol{v}_B) + \sum_{B\neq A}\sum_{C\neq A,B}\frac{M_B M_C}{R_{AB}^2}\left[\frac{2(\gamma+\beta)}{R_{AC}} + \frac{2\beta-1}{R_{BC}} - \frac{1}{2}\frac{R_{AB}}{R_{BC}^2}(\boldsymbol{n}_{AB}\cdot\boldsymbol{n}_{BC})\right]\boldsymbol{n}_{AB} \\ &- \frac{1}{2}(4\gamma+3)\sum_{B\neq A}\sum_{C\neq A,B}\frac{M_B M_C}{R_{AB}R_{BC}^2}\boldsymbol{n}_{BC} + O(c^{-4}),\end{aligned} \tag{5}$$

where

$$\boldsymbol{n}_{AB} \equiv \frac{\boldsymbol{R}_{AB}}{R_{AB}}. \tag{6}$$

For three aligned masses, Eq. (5) becomes the force-balance equation as

$$\ell\omega^2 = F_N + F_M + F_V\omega^2, \tag{7}$$

where we define $\ell \equiv R_{31}$, the mass ratio $\nu_A \equiv M_A/M$ for $M \equiv \sum_A M_A$, and

$$\begin{aligned}F_N = \frac{M}{\ell^2 z^2}[&1 - \nu_1 - \nu_3 + 2(1 - \nu_1 - \nu_3)z \\ &+ (2 - \nu_1 - \nu_3)z^2 + 2(1 - \nu_1 - \nu_3)z^3 \\ &+ (1 - \nu_1 - \nu_3)z^4],\end{aligned} \tag{8}$$

$$F_M = -\frac{M^2}{\ell^3 z^3}[\{2(\beta+\gamma)\nu_2 + (1+2\beta+2\gamma)\nu_3\}\nu_2 + \{(-1+4\beta+2\gamma)\nu_1 + 6(\beta+\gamma)\nu_2 + 3(1+2\beta+2\gamma)\nu_3\}\nu_2 z$$
$$+ \{(-5+12\beta+4\gamma)\nu_1 + 6(\beta+\gamma)\nu_2 - (1-10\beta-4\gamma)\nu_3\}\nu_2 z^2$$
$$+ \{2(\beta+\gamma)\nu_1^2 + 4(\beta+\gamma)\nu_2^2 - (7-14\beta-2\gamma)\nu_2\nu_3 + 2(\beta+\gamma)\nu_3^2 + ((-7+14\beta+2\gamma)\nu_2 + 2(1+2\beta+2\gamma)\nu_3)\nu_1\}z^3$$
$$+ \{(-1+10\beta+4\gamma)\nu_1 + 6(\beta+\gamma)\nu_2 + (12\beta+4\gamma-5)\nu_3\}\nu_2 z^4$$
$$+ \{3(1+2\beta+2\gamma)\nu_1 + 6(\beta+\gamma)\nu_2 + (-1+4\beta+2\gamma)\nu_3\}\nu_2 z^5 + \{(1+2\beta+2\gamma)\nu_1 + 2(\beta+\gamma)\nu_2\}\nu_2 z^6], \tag{9}$$

and

$$F_V = \frac{M}{(1+z)^2 z^2}[-\nu_1^2\nu_2 - 2\nu_1\nu_2(2\nu_1+\nu_2)z + \{\gamma\nu_1^3 + ((-2+4\gamma)\nu_2 + 3(1+\gamma)\nu_3)\nu_1^2$$
$$+ (2\nu_2+\nu_3)(\gamma\nu_2^2 + (1+2\gamma)\nu_2\nu_3 + \gamma\nu_3^2) + ((-1+5\gamma)\nu_2^2 + 8(1+\gamma)\nu_2\nu_3 + 3(1+\gamma)\nu_3^2)\nu_1\}z^2$$
$$+ 2(\nu_1+2\nu_2+\nu_3)\{\gamma\nu_1^2 + \gamma\nu_2^2 + (1+2\gamma)\nu_2\nu_3 + \gamma\nu_3^2 + ((1+2\gamma)\nu_2 + (3+2\gamma)\nu_3)\nu_1\}z^3$$
$$+ \{\gamma\nu_1^3 + 2\gamma\nu_2^3 - (1-5\gamma)\nu_2^2\nu_3 - 2(1-2\gamma)\nu_2\nu_3^2 + \gamma\nu_3^3 + ((1+4\gamma)\nu_2 + 3(1+\gamma)\nu_3)\nu_1^2$$
$$+ ((2+5\gamma)\nu_2^2 + 8(1+\gamma)\nu_2\nu_3 + 3(1+\gamma)\nu_3^2)\nu_1\}z^4 - 2\nu_2\nu_3(\nu_2+2\nu_3)z^5 - \nu_2\nu_3^2 z^6]. \tag{10}$$

By rearranging Eq. (5) for the collinear configuration by the same way as in Sec. II A, we find a seventh-order equation for z as

$$\sum_{k=0}^{7} A_k z^k = 0, \tag{11}$$

where the coefficients are

$$A_7 = \frac{M}{\ell}[-2(\beta+\gamma) - 2\nu_1 + 4(\beta+\gamma)\nu_3 + 2\nu_1^2 + 4\nu_1\nu_3 - 2(\beta+\gamma)\nu_3^2 - 2\nu_1^2\nu_3 - 2\nu_1\nu_3^2], \tag{12}$$

$$A_6 = 1 - \nu_3 + \frac{M}{\ell}[-(6\beta+7\gamma) - (6+2\beta+2\gamma)\nu_1 - (2-8\beta-11\gamma)\nu_3 + 4\nu_1^2 + (12+2\beta+2\gamma)\nu_1\nu_3 + (4-2\beta-4\gamma)\nu_3^2 + 2\nu_1^3 - 4\nu_1^2\nu_3 - 6\nu_1\nu_3^2 - 2\nu_3^3], \tag{13}$$

$$A_5 = 2 + \nu_1 - 2\nu_3 + \frac{M}{\ell}[-3(2\beta+3\gamma) - 3(2+2\beta+2\gamma)\nu_1 - (6-11\gamma)\nu_3 + (12+6\beta+2\gamma)\nu_1\nu_3 + (12+6\beta-2\gamma)\nu_3^2 + 6\nu_1^3 - 6\nu_1\nu_3^2 - 6\nu_3^3], \tag{14}$$

$$A_4 = 1 + 2\nu_1 - \nu_3 + \frac{M}{\ell}[-2\beta - 4\gamma - (2\beta+8\gamma)\nu_1 - (6+6\beta-8\gamma)\nu_3 - (6+4\beta-2\gamma)\nu_1^2 + (4+2\beta-2\gamma)\nu_1\nu_3 + (12+8\beta-4\gamma)\nu_3^2 + 6\nu_1^3 + 2\nu_1^2\nu_3 - 4\nu_1\nu_3^2 - 6\nu_3^3], \tag{15}$$

$$A_3 = -1 + \nu_1 - 2\nu_3 + \frac{M}{\ell}[2\beta + 4\gamma + (6+6\beta-8\gamma)\nu_1 + (2\beta+8\gamma)\nu_3 - (12+8\beta-4\gamma)\nu_1^2 - (4+2\beta-2\gamma)\nu_1\nu_3 + (6+4\beta-2\gamma)\nu_3^2 + 6\nu_1^3 + 4\nu_1^2\nu_3 - 2\nu_1\nu_3^2 - 6\nu_3^3], \tag{16}$$

$$A_2 = -2 + 2\nu_1 - \nu_3 + \frac{M}{\ell}[6\beta + 9\gamma + (6-11\gamma)\nu_1 + (6+6\beta+6\gamma)\nu_3 - (12+6\beta-2\gamma)\nu_1^2 - (12+6\beta+2\gamma)\nu_1\nu_3 + 6\nu_1^3 + 6\nu_1^2\nu_3 - 6\nu_3^3], \tag{17}$$

$$A_1 = -1 + \nu_1 + \frac{M}{\ell}[6\beta + 7\gamma + (2-8\beta-11\gamma)\nu_1 + (6+2\beta+2\gamma)\nu_3 - (4-2\beta-4\gamma)\nu_1^2 - (12+2\beta+2\gamma)\nu_1\nu_3 - 4\nu_3^2 + 2\nu_1^3 + 4\nu_1\nu_3^2 + 6\nu_1^2\nu_3 - 2\nu_3^3], \tag{18}$$

$$A_0 = \frac{M}{\ell}[2\beta + 2\gamma - 4(\beta+\gamma)\nu_1 + 2\nu_3 + 2(\beta+\gamma)\nu_1^2 - 4\nu_1\nu_3 - 2\nu_3^2 + 2\nu_1^2\nu_3 + 2\nu_1\nu_3^2]. \tag{19}$$

It follows that Eq. (11) recovers the PN collinear configuration by Eq. (13) of Ref. [13] if and only if $\beta = \gamma = 1$. The uniqueness is because the number of the parameters β, γ is two for eight coefficients $A_0, \ldots, A_7$.

From Eq. (7) for z obtained above, the angular velocity ω_{PPN} of the PPN collinear configuration is obtained as

$$\omega_{PPN} = \omega_N\left(1 + \frac{F_M}{2F_N} + \frac{F_V}{2\ell}\right), \tag{20}$$

where $\omega_N = (F_N/\ell)^{1/2}$ is the Newtonian angular velocity. The subscript N denotes the Newtonian case.

III. TRIANGULAR CONFIGURATION IN PPN GRAVITY

A. Lagrange's equilateral solution in Newtonian gravity

In this subsection, we suppose that the three masses are in coplanar and circular motion with keeping the same separation between the masses, namely $R_{AB} = a$ for a constant a.

It is convenient to choose the coordinate origin as the COM,

$$\sum_A M_A \boldsymbol{x}_A = 0, \tag{21}$$

for which the equation of motion for each mass in the equilateral triangle configuration takes a compact form as [2]

$$\frac{d^2\boldsymbol{x}_A}{dt^2} = -\frac{M}{a^3}\boldsymbol{x}_A. \tag{22}$$

See e.g., Eq. (8.6.5) in Ref. [2] for the derivation of Eq. (22). A triangular configuration is a solution, if the Newtonian angular velocity ω_N satisfies

$$(\omega_N)^2 = \frac{M}{a^3}. \tag{23}$$

The orbital radius ℓ_A of each mass around the COM is [2]

$$\ell_1 = a\sqrt{\nu_2^2 + \nu_2\nu_3 + \nu_3^2}, \tag{24}$$

$$\ell_2 = a\sqrt{\nu_1^2 + \nu_1\nu_3 + \nu_3^2}, \tag{25}$$

$$\ell_3 = a\sqrt{\nu_1^2 + \nu_1\nu_2 + \nu_2^2}. \tag{26}$$

B. PPN orbital radius

We suppose again that three masses in circular motion are in a triangular configuration with a constant angular velocity ω. By noting that a vector in the orbital plane can be expressed as a linear combination of $\boldsymbol{x}_1$ and $\boldsymbol{v}_1$, Eq. (5) becomes

$$-\omega^2\boldsymbol{x}_1 = -(\omega_N)^2\boldsymbol{x}_1 + g_1(\omega_N)^2\boldsymbol{x}_1 + \frac{\sqrt{3}M}{16a}\frac{\nu_2\nu_3(\nu_2-\nu_3)(16\beta-1-9\nu_1)}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\omega_N\boldsymbol{v}_1, \tag{27}$$

where Eq. (23) is used and

$$g_1 = \frac{M}{a}\left[\left(2\beta + \gamma + (\nu_2+\nu_3)(\nu_2+\nu_3-1) - \frac{7}{16}\nu_2\nu_3\right) + \frac{3}{16}\frac{\nu_2\nu_3\{9\nu_2\nu_3 + 2(\nu_2+\nu_3)(8\beta-5)\}}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\right]. \tag{28}$$

By a cyclic permutation, we obtain the similar equations for M_2 and M_3.

The second and third terms in the right-hand side of Eq. (27) are the PPN forces. The second term is parallel to $\boldsymbol{x}_1$, whereas the third term is parallel to $\boldsymbol{v}_1$ Note that $\boldsymbol{v}_1$ is not parallel to $\boldsymbol{x}_1$ in circular motion.

The location of the COM in the fully conservative theories of PPN [28,29] remains the same as that in the PN approximation of general relativity [30,31]

$$\boldsymbol{G}_{PN} = \frac{1}{E}\sum_A M_A\boldsymbol{x}_A\left[1 + \frac{1}{2}\left(v_A^2 - \sum_{B\neq A}\frac{M_B}{R_{AB}}\right)\right], \tag{29}$$

where E is defined as

$$E \equiv \sum_A M_A\left[1 + \frac{1}{2}\left(v_A^2 - \sum_{B\neq A}\frac{M_B}{R_{AB}}\right)\right]. \tag{30}$$

This coincidence allows us to obtain the PPN orbital radius ℓ_A^{PPN} around the COM by straightforward calculations. The orbital radius of M_1 is formally obtained as

$$(\ell_1^{PPN})^2 = (\ell_1)^2 + \frac{aM}{2}\left(1 - \frac{a^3\omega_N^2}{M}\right) \times (-2\nu_1^2\nu_2^2 - 2\nu_2^2\nu_3^2 - 2\nu_3^2\nu_1^2 + 2\nu_1\nu_2^3 + \nu_2\nu_3^3 + \nu_2^3\nu_3 + 2\nu_3^3\nu_1 - 2\nu_1^2\nu_2\nu_3 + \nu_1\nu_2^2\nu_3 + \nu_1\nu_2\nu_3^2), \tag{31}$$

and the similar expressions of ℓ_2^{PPN} and ℓ_3^{PPN} for the orbital radius of M_2 and M_3 are obtained.

Unless the second term of the right-hand side in Eq. (31) vanishes, the difference between ℓ_1^{PPN} and ℓ_1 would make our computations rather complicated. However, it vanishes because ω_N satisfies Eq. (23). As a result, the PPN orbital radius remains the same as the Newtonian one. Namely, $\ell_A^{PPN} = \ell_A$.

C. Equilateral condition

First, we discuss a condition for an equilateral configuration.

For Eq. (27) to hold, the coefficient of the velocity vector $\boldsymbol{v}_1$ must vanish, because there are no other terms including $\boldsymbol{v}_1$. The coefficient is proportional to $\nu_2\nu_3(\nu_2-\nu_3)$. The same thing is true also of M_2 and M_3. For any value of β, therefore, the equilateral configuration in the PPN gravity can be present if and only if three finite masses are equal or two test masses orbit around one finite mass.

Note that one can find a very particular value of β satisfying

$$16\beta - 1 - 9\nu_1 = 0, \tag{32}$$

which leads to the vanishing coefficient of the velocity vector $\boldsymbol{v}_1$. However, this choice is very unlikely, because the particular value of β is dependent on the mass ratio ν_1 and it is not universal. Hence, this case will be ignored.

D. PPN triangular configuration for general masses

Next, let us consider a PPN triangle configuration for general masses. For this purpose, we introduce a nondimensional parameter ε_{AB} at the PPN order, such that each side length of the PPN triangle can be expressed as

$$R_{AB} = a(1+\varepsilon_{AB}). \tag{33}$$

The equilateral case is achieved by assuming $\varepsilon_{AB}=0$ for every masses. See Fig. 2 for the PPN triangular configuration.

In order to fix the degree of freedom corresponding to a scale transformation, we follow Ref. [15] to suppose that the arithmetic mean of the three side lengths is unchanged as

$$\frac{R_{12}+R_{23}+R_{31}}{3} = a\left[1+\frac{1}{3}(\varepsilon_{12}+\varepsilon_{23}+\varepsilon_{31})\right]. \tag{34}$$

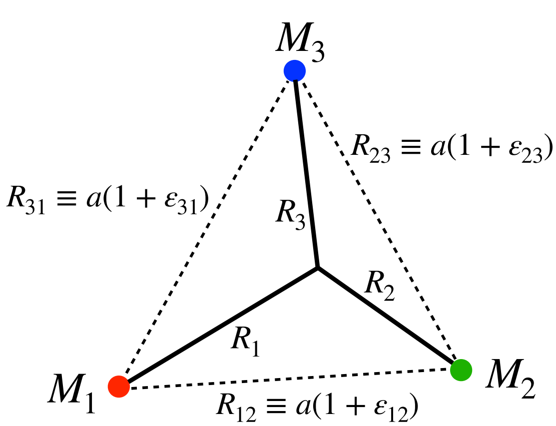

FIG. 2. Schematic figure for the PPN triangular configuration of three masses. An inequilateral triangle is described by the parameter ε_{AB}. R_A coincides with ℓ_A in the Newtonian limit, for which ε_{AB} vanishes.

The left-hand side of Eq. (34) is a in the Newtonian case, which leads to

$$\varepsilon_{12}+\varepsilon_{23}+\varepsilon_{31} = 0. \tag{35}$$

This is a gauge fixing in ε_{AB}.

In terms of ε_{AB}, Eq. (27) is rearranged as

$$\begin{aligned}-\omega^2\boldsymbol{x}_1 = {}& -(\omega_N)^2\boldsymbol{x}_1 - \frac{3}{2}\frac{(\omega_N)^2}{\nu_2^2+\nu_2\nu_3+\nu_3^2}\\ &\times\Big[\{\nu_2(\nu_1-\nu_2-1)\varepsilon_{12}+\nu_3(\nu_1-\nu_3-1)\varepsilon_{31}\}\boldsymbol{x}_1\\ &+\sqrt{3}\nu_2\nu_3(\varepsilon_{12}-\varepsilon_{31})\frac{\boldsymbol{v}_1}{\omega_N}\Big]+\boldsymbol{\delta}_1,\end{aligned} \tag{36}$$

where

$$\boldsymbol{\delta}_1 = g_1(\omega_N)^2\boldsymbol{x}_1 + \frac{\sqrt{3}M\nu_2\nu_3(\nu_2-\nu_3)(16\beta-1-9\nu_1)}{16a(\nu_2^2+\nu_2\nu_3+\nu_3^2)}\omega_N\boldsymbol{v}_1. \tag{37}$$

By a cyclic permutation, the equations for M_2 and M_3 can be obtained.

A triangular equilibrium configuration can exist if and only if the two conditions (A) and (B) are simultaneously satisfied; (A) Each mass satisfies Eq. (36), and (B) the configuration is unchanged in time.

Equation (36) is the equation of motion for M_1. To be more accurate, therefore, ω in Eq. (36) should be denoted as ω_1. Similarly, we introduce ω_2 and ω_3 in the equations of motion for M_2 and M_3, respectively. Then, condition (B) means $\omega_1=\omega_2=\omega_3$.

Condition (A) is equivalent to condition (A2); The coefficient of $\boldsymbol{v}_A$ in the equation of motion vanishes as

$$\varepsilon_{12}-\varepsilon_{31}-\frac{M}{24a}(\nu_2-\nu_3)(16\beta-1-9\nu_1)=0, \tag{38}$$

$$\varepsilon_{23}-\varepsilon_{21}-\frac{M}{24a}(\nu_3-\nu_1)(16\beta-1-9\nu_2)=0, \tag{39}$$

$$\varepsilon_{31}-\varepsilon_{23}-\frac{M}{24a}(\nu_1-\nu_2)(16\beta-1-9\nu_3)=0. \tag{40}$$

From Eqs. (38)–(40) and the gauge fixing as $\varepsilon_{12}+\varepsilon_{23}+\varepsilon_{31}=0$, we obtain

$$\begin{aligned}\varepsilon_{12} = {}&\frac{M}{72a}[(\nu_2-\nu_3)(16\beta-1-9\nu_1)\\ &-(\nu_3-\nu_1)(16\beta-1-9\nu_2)],\end{aligned} \tag{41}$$

$$\begin{aligned}\varepsilon_{23} = {}&\frac{M}{72a}[(\nu_3-\nu_1)(16\beta-1-9\nu_2)\\ &-(\nu_1-\nu_2)(16\beta-1-9\nu_3)],\end{aligned} \tag{42}$$

and

$$\varepsilon_{31} = \frac{M}{72a}[(\nu_1 - \nu_2)(16\beta - 1 - 9\nu_3) - (\nu_2 - \nu_3)(16\beta - 1 - 9\nu_1)]. \quad (43)$$

Therefore, the PPN triangle is inequilateral depending on β via ε_{AB} but not on γ. This suggests that also the PPN Lagrange points corresponding to L_4 and L_5 are sensitive to β but are free from γ, as shown in Sec. IV.

It follows that Eqs. (41)–(43) recover the PN counterpart of Eqs. (26)–(28) of Ref. [15] if and only if $\beta = 1$. The uniqueness is because the PPN parameter is only β for three equations as Eqs. (41)–(43).

Condition (B) is satisfied, if $\omega_1 = \omega_2 = \omega_3 \equiv \omega_{PPN}$, where ω_{PPN} means the angular velocity of the PPN configuration. By substituting Eqs. (41) and (43) into Eq. (36), ω_{PPN} is obtained as

$$\omega_{PPN} = \omega_N(1 + \delta_\omega), \quad (44)$$

where, by using Eq. (28), the PPN correction δ_ω is

$$\begin{aligned}\delta_\omega &= \frac{3}{4}\frac{\nu_2(\nu_1-\nu_2-1)\varepsilon_{12}+\nu_3(\nu_1-\nu_3-1)\varepsilon_{31}}{\nu_2^2+\nu_2\nu_3+\nu_3^2} - \frac{1}{2}g_1 \\ &= -\frac{M}{48a}\{64\beta+24\gamma-1-42(\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1)\}. \quad (45)\end{aligned}$$

There is a symmetry among M_1, M_2, M_3 in the second line of Eq. (45), which means that δ_ω is the same for all bodies. Condition (B) is thus satisfied.

IV. PPN CORRECTIONS TO THE LAGRANGE POINTS

A. PPN Lagrange points L_1, L_2, and L_3

In this section, we discuss PPN modifications of the Lagrange points that are originally defined in the restricted three-body problem in Newton gravity. We choose $\nu_A = 1 - \nu$, $\nu_B = \nu$, and $\nu_C = 0$, where ν is the mass ratio of the secondary object (a planet).

First, we seek PPN corrections to L_1, L_2, and L_3. There are three choices of how to correspond M_1, M_2, and M_3 to the Sun, a planet and a test mass in the collinear configuration. Indeed the three choices lead to the Lagrange points L_1, L_2, and L_3.

We consider the collinear solution by Eq. (11). We denote the physical root for Eq. (11) as $z = z_N(1 + \varepsilon)$ for the Newtonian root z_N with using a small parameter ε ($|\varepsilon| \ll 1$) at the PPN order. We substitute z into Eq. (11) and rearrange it to obtain ε as

$$\varepsilon = -\frac{\sum_{k=0}^{7} A_k^{PPN}(z_N)^k}{\sum_{k=1}^{6} kA_k^N(z_N)^k}, \quad (46)$$

where $O(\varepsilon^2)$ is discarded because of being at the 2PN order, and A_k^N and A_k^{PPN} denote the Newtonian and PPN parts of A_k, respectively, as $A_k = A_k^N + \varepsilon A_k^{PPN}$ ($A_0^N = 0$ and $A_7^N = 0$ because there are no counterparts in the Newtonian case).

Equation (46) is used for calculating the PPN corrections to L_1, L_2, and L_3. The PPN displacement from the Newtonian Lagrange point L_1 is thus obtained as

$$\delta_{PPN}R_{23} \equiv R_{23} - (R_{23})_N = \frac{\varepsilon z_N}{(1 + z_N)^2}\ell + O(\ell\varepsilon^2), \quad (47)$$

where M_1, M_2, and M_3 are chosen as a planet, a test mass, and the Sun, respectively.

Similarly, the PPN displacement from the Newtonian Lagrange point L_2 becomes

$$\delta_{PPN}R_{31} \equiv R_{31} - (R_{31})_N = \frac{\varepsilon z_N}{(1 + z_N)}\ell + O(\ell\varepsilon^2), \quad (48)$$

where M_1, M_2, and M_3 are chosen as the Sun, a planet, and a test mass, respectively. The PPN displacement from the Newtonian Lagrange point L_3 is

$$\delta_{PPN}R_{23} \equiv R_{23} - (R_{23})_N = \frac{\varepsilon z_N}{(1 + z_N)}\ell + O(\ell\varepsilon^2), \quad (49)$$

where M_1, M_2, and M_3 are chosen as a planet, the Sun. and a test mass, respectively. Here, a value of z_N depends on L_1, L_2, or L_3, which is given by Eq. (4).

B. PPN Lagrange points L_4 and L_5

Next, we discuss PPN corrections to the Lagrange points L_4 and L_5, for which we consider the PPN triangular solution. Let a denote the orbital separation between the primary object and the secondary one, which equals to $R_{12} = \ell(1 + \varepsilon_{12})$. Therefore, $\ell = a(1 - \varepsilon_{12}) + O(a\varepsilon^2)$, where ε^2 denotes the second order in ε_{AB}. By using this for R_{23} and R_{31}, we obtain $R_{23} = a(1 + \varepsilon_{23} - \varepsilon_{12}) + O(a\varepsilon^2)$, and $R_{31} = a(1 + \varepsilon_{31} - \varepsilon_{12}) + O(a\varepsilon^2)$.

The PPN displacement from the Newtonian Lagrange point L_4 (and L_5) with respect to the Sun is obtained as

$$\begin{aligned}\delta_{PPN}R_{31} &\equiv R_{31} - a \\ &= a(\varepsilon_{31} - \varepsilon_{12}) + O(a\varepsilon^2) \\ &= -\frac{\nu(16\beta - 10 + 9\nu)}{24}M + O\left(\frac{M^2}{a}\right), \quad (50)\end{aligned}$$

where $\nu_1 = 1 - \nu$, $\nu_2 = \nu$, and $\nu_3 = 0$ are used in the last line.

In the similar manner, the PPN displacement from the Newtonian Lagrange point L_4 (and L_5) with respect to the planet

TABLE I. The PPN displacement from the Newtonian Lagrange points of the Sun-Jupiter system. The PPN corrections to L_1, L_2, L_3, and L_4 are listed in this table, where the sign convention for L_1, L_2, L_3 is chosen along the direction from the Sun to the Jupiter, and the correction to L_5 is identical to that to L_4. The PPN displacement for L_4 is two-dimensional and hence they are indicated by the deviations from the Sun and from the Jupiter.

Lagrange points	PPN displacement [m]
L_1	$-0.000051 + 40.00\beta - 9.905\gamma$
L_2	$0.000040 - 50.27\beta + 12.40\gamma$
L_3	$0.000122 + 1.424\beta + 0.01882\gamma$
$L_4(L_5)$-Sun	$-0.05875 \times (-9.991 + 16\beta)$
$L_4(L_5)$-Jupiter	$-61.53 \times (-1 + 16\beta)$

$$\begin{aligned}\delta_{PPN}R_{23} &\equiv R_{23} - a \\ &= a(\varepsilon_{23} - \varepsilon_{12}) + O(a\varepsilon^2) \\ &= -\frac{(1-\nu)(16\beta - 1 - 9\nu)}{24}M + O\left(\frac{M^2}{a}\right). \end{aligned} \quad (51)$$

Equation (51) can be obtained more easily from Eq. (50) if the correspondence as $1 - \nu \leftrightarrow \nu$ is used.

C. Example: The Sun-Jupiter case

The PPN corrections to the L_1, L_2, and L_3 can be expressed as a linear function in β and γ. The PPN corrections to L_4 and L_5 are in a linear function only of β. The results for the Sun-Jupiter system are summarized in Table I, where the sign convention is chosen along the direction from the Sun to a planet.

Before closing this section, we mention gravitational experiments. The lunar laser ranging experiment put a constraint on $\eta \equiv 4\beta - \gamma - 3$ as $|\eta| < O(10^{-4})$ [32,33]. If one wishes to constrain $1 - \beta$ at the level of $O(10^{-4})$ by using the location of the Lagrange points, the Lagrange point accuracy of about a few millimeters (e.g., for L_4) is needed in the solar system, though this is very unlikely in the near future.

On the other hand, possible PPN corrections in a three-body system may be relevant with relativistic astrophysics in, e.g., a relativistic hierarchical triple system and a supermassive black hole with a compact binary [34–38]. This subject is beyond the scope of the present paper.

V. CONCLUSION

The coplanar and circular three-body problem was investigated for a class of fully conservative theories in the PPN formalism, characterized by the Eddington-Robertson parameters β and γ.

The collinear configuration can exist for arbitrary mass ratio, β and γ. On the other hand, the PPN triangular configuration depends on the nonlinearity parameter β but not on γ. This is far from trivial, because the parameter β is not separable from γ apparently at the level of Eq. (5). For any value of β, the equilateral configuration in the PPN gravity is possible, if and only if three finite masses are equal or two test masses orbit around one finite mass. For general mass cases, the PPN triangle is not equilateral.

We showed also that the PPN displacements from the Newtonian Lagrange points L_1, L_2, and L_3 depend on both β and γ, while those to L_4 and L_5 rely only upon β. It is left for future to study the stability of the PPN configurations.

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