Role of scalar mesons $a_0(980)$ and $a_0(1710)$ in the $D_s^+ \to \pi^0 K^+ K_S^0$ decay

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Recently, the BESIII Collaboration has measured the $D_s^+ \to \pi^0 K^+ K_S^0$ decay and reported the observation of scalar meson $a_0(1710)^+$ in the $K^+ K_S^0$ invariant mass spectrum. Based on the previous study about such a state in the $D_s^+ \to \pi^+ K_S^0 K_S^0$ decay, we perform a theoretical study of $a_0(1710)^+$ in the process $D_s^+ \to \pi^0 a_0(1710)^+ \to \pi^0 K^+ K_S^0$. In addition to the $a_0(1710)$ state, the contributions of K^* and $a_0(980)$ are also taken into account. Firstly, we consider the contributions from the tree diagrams of $K^{*+} \to K^+ \pi^0$ and $\bar{K}^{*0} \to \pi^0 \bar{K}^0$. Secondly, we describe the final state interaction of $K\bar{K}$ in the chiral unitary approach to study the contribution of $a_0(980)$, while the $a_0(1710)$ state is dynamically generated from the $K^* \bar{K}^*$ interaction, and then decays into $K^+ \bar{K}^0$. Since the final $K^+ K_S^0$ state is in pure isospin $I = 1$, the $D_s^+ \to \pi^0 K^+ K_S^0$ decay is an ideal process to study the $a_0(1710)^+$ and $a_0(980)^+$ resonances. It is found that the recent BESIII experimental measurements on the $K^+ K_S^0$, $\pi^0 K^+$, and $\pi^0 K_S^0$ invariant mass distributions can be well reproduced, which supports the molecular $K^* \bar{K}^*$ nature of the scalar $a_0(1710)$ resonance.

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I. INTRODUCTION

The $a_0(1710)$ resonance, an isospin partner of the scalar meson $f_0(1710)$, may have properties and structure similar to those of $f_0(1710)$ [1–7]. The $f_0(1710)$ resonance, with quantum numbers $J^{PC} = 0^{++}$, has been investigated in many previous theoretical works [8–18], where its structure has been studied from various perspectives. For example, in Ref. [19], it was shown that the $f_0(1710)$ resonance has a large component of $s\bar{s}$ in its wave function. In Refs. [20–22], it was considered as a scalar glueball. On the other hand, the $f_0(1710)$ can be viewed as a molecular state dynamically generated in the vector-vector coupled channel interactions [9–17]. Meanwhile, the $a_0(1710)$ with negative G parity is also dynamically generated in the isospin $I = 1$ sector [3–5], which couples mostly to the $K^* \bar{K}^*$ channel, as well as to the $\rho\omega$ and $\rho\phi$ channels. In Ref. [7], after extending the coupled channel vector meson-vector meson interactions to include the pseudoscalar meson-pseudoscalar meson interactions, a pole near the $K^* \bar{K}^*$ threshold identified as the $a_0(1710)$ state is also found.

In the molecular picture, the $f_0(1710)$ and $a_0(1710)$ states couple mostly to the $K^* \bar{K}^*$ channel, and their dominant decay channel is $K\bar{K}$ [3,7]. The BESIII experiment reported the first evidence for the interference between $f_0(1710)$ and $a_0(1710)$ in the amplitude analyses of the $D_s^+ \to \pi^+ K_S^0 K_S^0$ and $D_s^+ \to \pi^+ K^+ K^-$ decays [23,24]. It is found that there is a clear enhancement in the $K_S^0 K_S^0$ invariant mass distributions around 1.7 GeV, which is attributed to the contribution from the $a_0(1710)$ state. However, it is not seen in the $K^+ K^-$ invariant spectrum [24]. In fact, the $a_0(1710)^+$ state was previously reported by the *BABAR* experiment [25] in the process of $\eta_c \to a_0(1710)^+ \pi^-$, with the decay $a_0(1710)^+ \to \pi^+ \eta$. Its measured Breit-Wigner mass and width are $M_{a_0(1710)} = 1704 \pm 5 \pm 2$ MeV and $\Gamma_{a_0(1710)} = 110 \pm 15 \pm 11$ MeV, respectively [25].

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In our previous work [1], for the decay $D_s^+ \to \pi^+ K_S^0 K_S^0$, we found that the contribution from the K^* meson is crucial to the $a_0(1710)$ peak region. Furthermore, within the proposed mechanisms in Refs. [1,2], it is expected that the charged $a_0(1710)^+$ resonance will show up in the $K^+ K_S^0$ invariant mass distribution of the $D_s^+ \to \pi^0 K^+ K_S^0$ decay. Recently, the BESIII Collaboration has performed an amplitude analysis of the process $D_s^+ \to \pi^0 K^+ K_S^0$ [26], where the $a_0(1710)^+$ state was indeed observed in the $K^+ K_S^0$ invariant mass distribution. The fitted Breit-Wigner mass and width of $a_0(1710)^+$ state by the BESIII Collaboration are as following [26]:

$$M_{a_0(1710)} = 1817 \pm 8 \pm 20 \text{ MeV},$$
$$\Gamma_{a_0(1710)} = 97 \pm 22 \pm 15 \text{ MeV}. \quad (1)$$

Here, the fitted mass is close to the boundary region of the $K^+ \bar{K}^0$ invariant mass spectrum.[1] In the low energy region of the $K^+ \bar{K}^0$ line shape, there is a clear enhancement due to the scalar meson $a_0(980)$.

Based on the BESIII experimental measurements [26], Ref. [27] has systematically studied these isovector scalar mesons and suggested that the $a_0(1710)$ [denoted as $a_0(1817)$], $a_0(1450)$, and $a_0(980)$ states are in the same isovector scalar meson family since they form a Regge trajectory. In addition, the $a_0(1710)$ state could also be regarded as a good isovector partner of the $X(1812)$ [27]. In a word, the nature of $a_0(1710)$ is still unclear, and further theoretical and experimental studies are necessary.

In this work, in addition to the contributions of the intermediate states K^{*+} and $\bar{K}^{*0}$ in the processes $D_s^+ \to \bar{K}^0 K^{*+} \to \pi^0 K^+ K_S^0$ and $D_s^+ \to K^+ \bar{K}^{*0} \to \pi^0 K^+ K_S^0$, we investigate the roles of the scalar mesons $a_0(980)^+$ and $a_0(1710)^+$ in the $D_s^+ \to \pi^0 K^+ K_S^0$ decay, where their contributions are included by taking into account the $K\bar{K}$ and $K^*\bar{K}^*$ final-state interactions within the chiral unitary approach [1,2,28–32].

This article is organized as follows. In Sec. II, we present the theoretical formalism for the $D_s^+ \to \pi^0 K^+ K_S^0$ decay, and in Sec. III, we show our numerical results and discussions, followed by a short summary in the last section.

II. FORMALISM

In this section, we present the theoretical formalism for the decay $D_s^+ \to \pi^0 K^+ K_S^0$, where we study the roles of the K^{*+} ($\bar{K}^{*0}$), $a_0(980)$, and $a_0(1710)$ states. The contribution of the $a_0(980)$ state is encoded in the S-wave $K\bar{K}$ final-state interactions [28–32], while the one of the $a_0(1710)^+$ state is in the S-wave $K^*\bar{K}^*$ final-state interactions [1,2].

The Cabibbo-favored process $D_s^+ \to \pi^0 K^+ \bar{K}^0$ can happen via the weak decay of the c quark into a W^+ boson and an s quark, followed by the W^+ boson decaying into a u quark and a $\bar{d}$ quark. In order to generate the states $\pi^0 K^{(*)+} \bar{K}^{(*)0}$, the $u\bar{s}$ ($s\bar{d}$) hadronize into $K^{(*)+}$ [$\bar{K}^{(*)0}$], and the $s\bar{d}$ ($u\bar{s}$), together with the $q\bar{q}$ ($=u\bar{u} + d\bar{d} + s\bar{s}$) pair created from the vacuum, hadronize into $\pi^0 \bar{K}^{(*)0}$ [$\pi^0 K^{(*)+}$]. Then the scalar meson $a_0(980)$ [$a_0(1710)$] could be dynamically generated from the S-wave $K^+\bar{K}^0$ [$K^{*+}\bar{K}^{*0}$] interaction.

A. The mechanism of $D_s^+ \to \bar{K}^0 K^{*+} (K^+ \bar{K}^{*0}) \to \pi^0 K^+ K_S^0$ reaction

Firstly, we study the decay of $D_s^+ \to \pi^0 K^+ K_S^0$ via the intermediate resonance K^{*+}, with $K^{*+} \to \pi^0 K^+$ in P wave. The vertex $D_s^+ \bar{K}^0 K^{*+}$ is also in P wave to keep the angular momentum conserved, and the effective interaction is taken as used in Refs. [1,29,33]. The hadron level diagram for the process $D_s^+ \to \bar{K}^0 K^{*+} \to \pi^0 K^+ K_S^0$ is depicted in Fig. 1, and the decay amplitude can be written as

$$\begin{aligned}\mathcal{M}_{K^{*+}} = {} & \frac{g_{D_s^+ \bar{K}^0 K^{*+}} g_{K^{*+} K^+ \pi^0}}{q^2_{K^{*+}} - m^2_{K^{*+}} + i m_{K^{*+}} \Gamma_{K^{*+}}} \\ & \times \Bigg[(m^2_{K^+} - m^2_{\pi^0}) \left(1 - \frac{q^2_{K^{*+}}}{m^2_{K^{*+}}}\right) \\ & + 2 p_1 \cdot p_3 \frac{m^2_{\pi^0} - m^2_{K^+} - m^2_{K^{*+}}}{m^2_{K^{*+}}} \\ & + 2 p_2 \cdot p_3 \frac{m^2_{\pi^0} - m^2_{K^+} + m^2_{K^{*+}}}{m^2_{K^{*+}}} \Bigg], \end{aligned} \quad (2)$$

where $q^2_{K^{*+}} = (p_1 + p_2)^2 = M^2_{\pi^0 K^+}$ is the four momenta square of the virtual K^{*+} meson, and $M_{\pi^0 K^+}$ is the invariant mass of the $\pi^0 K^+$ system.

As shown in Fig. 2, the decay of $D_s^+ \to \pi^0 K^+ K_S^0$ can also occur via the intermediate meson $\bar{K}^{*0}$. The corresponding decay amplitude for the process of $D_s^+ \to \pi^0 K^+ K_S^0$ of Fig. 2 can be easily obtained just by applying the substitution to $\mathcal{M}_{K^{*+}}$ with $p_2 \leftrightarrow p_3$, $K^{*+} \to \bar{K}^{*0}$, and $q_{K^{*+}} \to q_{\bar{K}^{*0}}$.

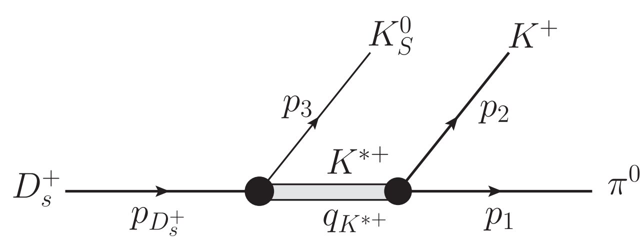

FIG. 1. The decay of $D_s^+ \to \pi^0 K^+ K_S^0$ via the intermediate vector K^{*+}.

[1] If we take $M_{D_s^+} = 1968.34$ MeV and $m_{\pi^0} = 134.98$ MeV, we get the maximum value of the $K^+\bar{K}^0$ invariant mass $(M_{K^+\bar{K}^0})_{\max} = 1833.36$ MeV.

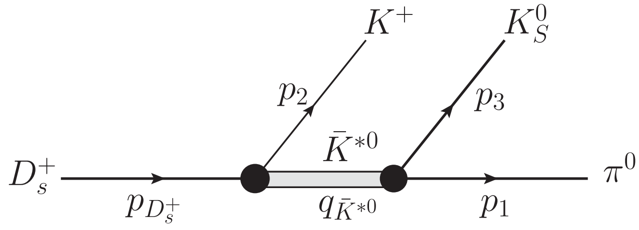

FIG. 2. The decay of $D_s^+ \to \pi^0 K^+ K_S^0$ via the intermediate vector $\bar{K}^{*0}$.

The coupling constants $g_{D_s^+ \bar{K}^0 K^{*+}}$, $g_{K^{*+}K^+\pi^0}$, $g_{D_s^+ \bar{K}^{*0} K^+}$, and $g_{\bar{K}^{*0}\bar{K}^0\pi^0}$ are determined from the experimental partial decay widths of $D_s^+ \to \bar{K}^0 K^{*+}$, $K^{*+} \to K^+\pi^0$, $D_s^+ \to K^+\bar{K}^{*0}$, and $\bar{K}^{*0} \to \bar{K}^0\pi^0$, respectively. These coupling constants are related to the corresponding partial decay widths as

$$\Gamma_{D_s^+ \to \bar{K}^0 K^{*+}} = \frac{g^2_{D_s^+ \bar{K}^0 K^{*+}}}{2\pi} \frac{|\mathbf{P}|^3}{m^2_{K^{*+}}}, \tag{3}$$

$$\Gamma_{K^{*+} \to K^+\pi^0} = \frac{g^2_{K^{*+}K^+\pi^0}}{6\pi} \frac{|\mathbf{P}|^3}{m^2_{K^{*+}}}, \tag{4}$$

$$\Gamma_{D_s^+ \to K^+ \bar{K}^{*0}} = \frac{g^2_{D_s^+ K^+ \bar{K}^{*0}}}{2\pi} \frac{|\mathbf{P}|^3}{m^2_{\bar{K}^{*0}}}, \tag{5}$$

$$\Gamma_{\bar{K}^{*0} \to \bar{K}^0\pi^0} = \frac{g^2_{\bar{K}^{*0}\bar{K}^0\pi^0}}{6\pi} \frac{|\mathbf{P}|^3}{m^2_{\bar{K}^{*0}}}, \tag{6}$$

where $\mathbf{P}$ is the three-momentum of one of the two final particles in the rest frame of the decay particle, which is given by

$$|\mathbf{P}| = \frac{\lambda^{1/2}(M_0^2, m_1^2, m_2^2)}{2M_0}, \tag{7}$$

with the Källen function $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$, and M_0 is the mass of the initial decay particle, while m_1 and m_2 are the masses of the two final particles, respectively.

Note that from the partial decay width, one can only get the module squared of the coupling constant. Here, these coupling constants are taken as real and positive as done in Refs. [1,29]. The partial decay width of $K^* \to K\pi$ is taken from the particle data group (PDG) [8]. While for the $D_s^+ \to K^{*+}\bar{K}^0$ decay, we take $\mathrm{Br}(D_s^+ \to \bar{K}^0 K^{*+}) = (5.4 \pm 1.2)\%$ from Ref. [8], then one can easily obtain $\mathrm{Br}(D_s^+ \to K^+\bar{K}^{*0}) = (12.7 \pm 2.8)\%$ with the recent BESIII measurement of $\frac{\mathrm{Br}(D_s^+ \to \bar{K}^{*0}K^+)}{\mathrm{Br}(D_s^+ \to \bar{K}^0 K^{*+})} = 2.35^{+0.42}_{-0.23_{\mathrm{stat}}} \pm 0.10_{\mathrm{syst}}$ [26]. The obtained results for these coupling constants used in this work are listed in Table I.

B. The mechanism of $D_s^+ \to \pi^0 a_0(980)^+ \to \pi^0 K^+ K_S^0$ reaction

For the production of the $a_0(980)$ resonance, we need to produce the $\pi^0 K^+ \bar{K}^0$ in the first step, and then the $K^+\bar{K}^0$ final-state interaction generates the $a_0(980)$ state. Following Ref. [31], for the hadronization of $s\bar{d}$ and $u\bar{s}$ into a pair of pseudoscalar mesons, we can write

$$\sum_{i=u,d,s} s\bar{q}_i q_i \bar{d} = M_{3i}M_{i2} = (M^2)_{32}, \tag{8}$$

$$\sum_{i=u,d,s} u\bar{q}_i q_i \bar{s} = M_{1i}M_{i3} = (M^2)_{13}, \tag{9}$$

where M is the $SU(3)$ quark pair $q_i\bar{q}_j$ matrix, and it is defined as [11,34]

$$M = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}. \tag{10}$$

Accordingly, at the hadron level, if we consider the η and η' mixing as in Ref. [35], we can write the matrix M in terms of the pseudoscalar (P) mesons,

$$P = \begin{pmatrix} \frac{\eta}{\sqrt{3}} + \frac{\pi^0}{\sqrt{2}} + \frac{\eta'}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta}{\sqrt{3}} - \frac{\pi^0}{\sqrt{2}} + \frac{\eta'}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{\eta}{\sqrt{3}} + \frac{\sqrt{6}\eta'}{3} \end{pmatrix}. \tag{11}$$

Then, the hadronization processes at the quark level in Eqs. (8) and (9) can be expressed at the hadron level as

$$(M^2)_{32} \to (P \cdot P)_{32} = \pi^+ K^- - \frac{1}{\sqrt{2}}\pi^0 \bar{K}^0, \tag{12}$$

TABLE I. Two-body decays and the corresponding coupling constants.

Decay process	Partial decay width (MeV)	Coupling constant	Value
$D_s^+ \to \bar{K}^0 K^{*+}$	$(0.71 \pm 0.16) \times 10^{-10}$ (Ref. [8])	$g_{D_s^+ \bar{K}^0 K^{*+}}$	$(1.05 \pm 0.12) \times 10^{-6}$
$K^{*+} \to K^+\pi^0$	16.9 ± 0.3 (Ref. [8])	$g_{K^{*+}K^+\pi^0}$	3.23 ± 0.03
$D_s^+ \to K^+\bar{K}^{*0}$	$(1.66 \pm 0.37) \times 10^{-10}$ (Refs. [8,26])	$g_{D_s^+ \bar{K}^{*0} K^+}$	$(1.62 \pm 0.18) \times 10^{-6}$
$\bar{K}^{*0} \to \bar{K}^0\pi^0$	15.8 ± 0.2 (Ref. [8])	$g_{\bar{K}^{*0}\bar{K}^0\pi^0}$	3.13 ± 0.02

$$(M^2)_{13} \to (P \cdot P)_{13} = \pi^+ K^0 + \frac{1}{\sqrt{2}} \pi^0 K^+. \quad (13)$$

Together with the other final state K^+ (hadronizatized directly from $u\bar{s}$) or $\bar{K}^0$ (hadronizatized directly from $s\bar{d}$), we obtain the $D_s^+ \to \pi^0 K^+ \bar{K}^0$ reaction at tree level [shown in Fig. 3(a)] with the above mechanism as

$$\mathcal{M}^{\rm tree}_{\pi^0 K^+ \bar{K}^0} = \frac{V_{\pi^0 K^+ \bar{K}^0}}{\sqrt{2}}, \quad (14)$$

where we take $V_{\pi^0 K^+ \bar{K}^0}$ as a constant, as done in most previous theoretical works [36–41].

Next, the final-state interaction of $K^+ \bar{K}^0$ will produce the $a_0(980)$ state. The corresponding diagram is shown in Fig. 3(b), and the corresponding amplitude is defined as $M^{\rm FSI}_{K^+ \bar{K}^0}$. Then, the total decay amplitude for the $D_s^+ \to \pi^0 K^+ K^0_S$ reaction shown in Fig. 3, including the contributions from tree and final-state-interaction diagrams, can be easily obtained as follows[2]:

$$\begin{aligned} \mathcal{M}_{a_0(980)} &= \mathcal{M}^{\rm tree}_{\pi^0 K^+ \bar{K}^0} + \mathcal{M}^{\rm FSI}_{K^+ \bar{K}^0} \\ &= \frac{V_{\pi^0 K^+ \bar{K}^0}}{\sqrt{2}} [1 + G_{K\bar{K}}(M_{K^+ \bar{K}^0}) \\ &\quad \times t_{K^+ \bar{K}^0 \to K^+ \bar{K}^0}(M_{K^+ \bar{K}^0})], \end{aligned} \quad (15)$$

with $M_{K^+ \bar{K}^0}$ the invariant mass of the $K^+ \bar{K}^0$ system, and $V_{\pi^0 K^+ \bar{K}^0}$ is a free parameter, which will be determined by the experimental data.

In Eq. (15), $G_{K\bar{K}}$ is the $K\bar{K}$ loop function, which is given by

$$G_i = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(P-q)^2 - m_1^2 + i\epsilon} \frac{1}{q^2 - m_2^2 + i\epsilon}, \quad (16)$$

where m_1 and m_2 are the masses of the two mesons in the loop of the ith channel, and P and q are the four-momenta of the two-meson system and the second meson, respectively. The loop function of Eq. (16) is logarithmically divergent. There are two methods to solve this singular integral, either using the three-momentum cutoff method, or the dimensional regularization method. In our work, we adopt the cutoff method, and perform the integral for q in Eq. (16) with a cutoff $|\vec{q}_{\rm max}| = 903$ MeV to regularize the loops (see Ref. [42] for more details).

The transition amplitude $t_{K^+ \bar{K}^0 \to K^+ \bar{K}^0}$, which is dependent on the invariant mass $M_{K^+ \bar{K}^0}$, can be obtained in the chiral unitary approach by solving the Bethe-Salepter equation [43–45],

[2]It is worth mentioning that we have neglected the πK final-state interactions, where the scalar meson $\kappa(700)$ appears. This is because its contribution is very small compared with the dominant contribution from the K^* meson.

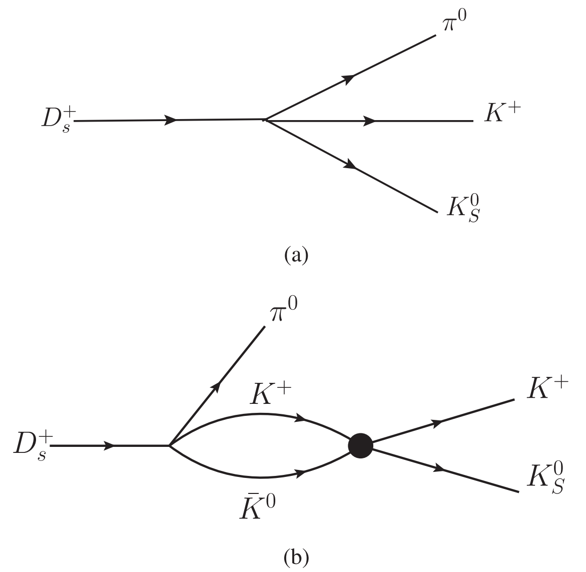

FIG. 3. The mechanisms of the $D_s^+ \to \pi^0 K^+ K^0_S$ decay. (a) tree diagram and (b) the final-state interaction of $K^+ \bar{K}^0$ to produce the $a_0(980)^+$ state.

$$T = [1 - VG]^{-1} V, \quad (17)$$

where V is a 2×2 matrix with the transition potentials between the isospin channels $K\bar{K}$ and $\pi\eta$. The transition amplitudes $t_{K^+ \bar{K}^0 \to K^+ \bar{K}^0}$ in particle basis can be related to the one in isospin basis,

$$t_{K^+ \bar{K}^0 \to K^+ \bar{K}^0} = t_{K\bar{K} \to K\bar{K}}. \quad (18)$$

With the isospin multiplets $K = (K^+, K^0), \bar{K} = (\bar{K}^0, -K^-)$, and $\pi = (-\pi^+, \pi^0, \pi^-)$, the 2×2 matrix V can be easily obtained as follows [28–32,42]:

$$\begin{aligned} V_{K\bar{K} \to K\bar{K}} &= -\frac{1}{4f^2} s, \\ V_{K\bar{K} \to \pi\eta} &= \frac{\sqrt{6}}{12 f^2} \left(3s - \frac{8}{3} m_K^2 - \frac{1}{3} m_\pi^2 - m_\eta^2 \right), \\ V_{\pi\eta \to K\bar{K}} &= V_{K\bar{K} \to \pi\eta}, \\ V_{\pi\eta \to \pi\eta} &= -\frac{1}{3f^2} m_\pi^2, \end{aligned} \quad (19)$$

where $f = 93$ MeV is the pion decay constant, and s is the invariant mass squared of the pseduoscalar-psedudoscalar system.

In Fig. 4, we show the real and imaginary parts of the transition amplitude $t_{K^+ \bar{K}^0 \to K^+ \bar{K}^0}$ as a function of the invariant mass of the $K^+ \bar{K}^0$ system. One can see that there is a clear structure around 980 MeV, which is associated to the $a_0(980)$ resonance.

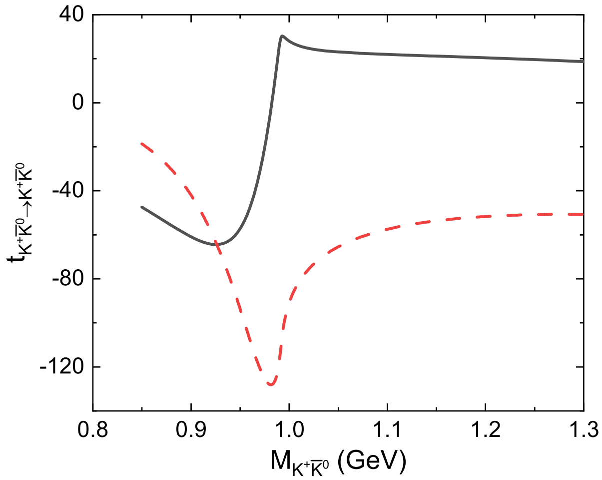

FIG. 4. Real (solid curve) and imaginary (red-dashed curve) parts of the transition amplitude $t_{K^+\bar{K}^0\to K^+\bar{K}^0}$ as a function of the invariant mass of the $K^+\bar{K}^0$ system.

C. The mechanism of $D_s^+ \to \pi^0(K^*\bar{K}^*)^+ \to K_S^0K^+\pi^0$ reaction

Following Ref. [1], the same mechanism that produced $(K^*\bar{K}^*)^0$ can also produce $(K^*\bar{K}^*)^+$, and after the π^0 and $K^{*+}\bar{K}^{*0}$ pair produced, the final-state interactions of K^{*+} and $\bar{K}^{*0}$ will lead to the dynamical generation of the $a_0(1710)^+$ state, which then decays into K^+ and K_S^0 in the final state. The rescattering diagram for the $D_s^+ \to \pi^0(K^*\bar{K}^*)^+ \to \pi^0 a_0(1710)^+ \to \pi^0K^+K_S^0$ decay is shown in Fig. 5. The decay amplitude can be written as

$$\mathcal{M}_{a_0(1710)} = \frac{V_{\pi^0K^{*+}\bar{K}^{*0}}}{2}\tilde{G}_{K^*\bar{K}^*}(M_{K^+K_S^0}) \times \frac{g_{K^*\bar{K}^*}g_{K\bar{K}}}{M^2_{K^+K_S^0} - M^2_{a_0(1710)^+} + iM_{a_0(1710)^+}\Gamma_{a_0(1710)^+}}. \tag{20}$$

It is worth mentioning that, in the present work, all the model parameters for the intermediate $a_0(1710)$ are determined in Ref. [1], and we take $M_{a_0(1710)^+} = 1777$ MeV and $\Gamma_{a_0(1710)^+} = 148$ MeV.

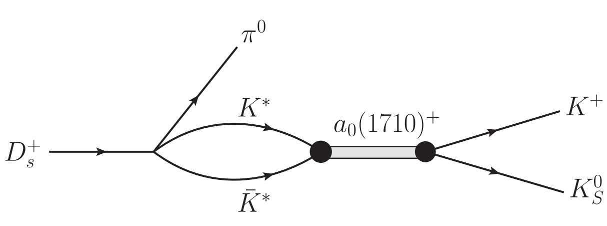

FIG. 5. Feynman diagram for the $D_s^+ \to \pi^0(K^*\bar{K}^*)^+ \to \pi^0 a_0(1710)^+ \to \pi^0K^+K_S^0$ decay, where $a_0(1710)$ is dynamically generated by the $K^*\bar{K}^*$ final-state interaction.

D. Invariant mass distributions of the $D_s^+ \to \pi^0K^+K_S^0$ decay

With the above theoretical formalism, we can write the total decay amplitude of $D_s^+ \to \pi^0K^+K_S^0$ as follows:

$$\mathcal{M} = \mathcal{M}_{K^{*+}} + \mathcal{M}_{\bar{K}^{*0}} + e^{i\theta_1}\mathcal{M}_{a_0(980)} + e^{i\theta_2}\mathcal{M}_{a_0(1710)}, \tag{21}$$

where θ_1 is the relative phase between contribution from K^* (sum of K^{*+} and $\bar{K}^{*0}$) and $a_0(980)$, while θ_2 stands for the relative phase between contribution K^* and $a_0(1710)$. The amplitude of Eq. (21) depends on two invariant masses $M_{K^+K_S^0}$ and $M_{\pi^0K^+}$.

Then, the double differential decay width for $D_s^+ \to \pi^0K^+K_S^0$ is given by[3]

$$\frac{d^2\Gamma}{dM_{K^+K_S^0}dM_{\pi^0K^+}} = \frac{M_{K^+K_S^0}M_{\pi^0K^+}}{128\pi^3m^3_{D_s^+}}|\mathcal{M}|^2. \tag{22}$$

Finally, one can easily obtain the invariant mass distributions $d\Gamma/dM_{K^+K_S^0}$ and $d\Gamma/dM_{\pi^0K^+}$, by integrating Eq. (22) over each of the invariant mass variables. For example, for a given value of $M_{K^+K_S^0}$, the upper and lower limits for $M_{\pi^0K^+}$ are fixed as [8]

$$\begin{aligned}(M^2_{\pi^0K^+})_{\max} &= (E^*_{\pi^0} + E^*_{K^+})^2 \\ &\quad - \left(\sqrt{E^{*2}_{\pi^0} - m^2_{\pi^0}} - \sqrt{E^{*2}_{K^+} - m^2_{K^+}}\right)^2, \\ (M^2_{\pi^0K^+})_{\min} &= (E^*_{\pi^0} + E^*_{K^+})^2 \\ &\quad - \left(\sqrt{E^{*2}_{\pi^0} - m^2_{\pi^0}} + \sqrt{E^{*2}_{K^+} - m^2_{K^+}}\right)^2,\end{aligned}$$

here $E^*_{\pi^0}$ and $E^*_{K^+}$ are the energies of π^0 and K^+ in the $K^+K_S^0$ rest frame, respectively,

$$\begin{aligned}E^*_{\pi^0} &= \frac{m^2_{D_s^+} - M^2_{K^+K_S^0} - m^2_{\pi^0}}{2M_{K^+K_S^0}}, \\ E^*_{K^+} &= \frac{M^2_{K^+K_S^0} - m^2_{K_S^0} + m^2_{K^+}}{2M_{K^+K_S^0}}.\end{aligned} \tag{23}$$

III. RESULTS AND DISCUSSION

With these above formalism, we calculate the invariant mass distributions of the $D_s^+ \to \pi^0K^+K_S^0$ decay. In the numerical analysis, we have five free parameters: (1) the factor $V_{\pi^0K^{*+}\bar{K}^{*0}}$ for the weak and hadronization strength related to the production of intermediate $\pi^0(K^*\bar{K}^*)^+$ at tree level; (2) $V_{\pi^0K^+\bar{K}^0}$ for the weight of the contribution from the intermediate $a_0(980)^+$ state; (3) relative phase θ_1;

[3]We take $|K^0\rangle = \frac{1}{\sqrt{2}}(|K_S^0\rangle + |K_L^0\rangle)$ and $|\bar{K}^0\rangle = \frac{1}{\sqrt{2}}(|K_S^0\rangle - |K_L^0\rangle)$, where we have neglected the effect of CP violation.

(4) relative phase θ_2; and (5) a global normalization factor C, which is needed to normalize the theoretical invariant mass distributions to the events obtained by the BESIII Collaboration.

We perform χ^2 fit to the BESIII measurements of the $\pi^0 K^+$, $\pi^0 K^0_S$, and $K^+ K^0_S$ invariant mass distributions. There are 129 data points in total. The obtained χ^2 is $\chi^2/(129-5) = 0.33$, and the fitted parameters are $C = (2.3 \pm 0.1) \times 10^{14}$, $V_{\pi^0 K^{*+} \bar{K}^{*0}} = (2.2 \pm 0.4) \times 10^{-4}$, $V_{\pi^0 K^+ \bar{K}^0} = (1.3 \pm 0.1) \times 10^{-4}$, $\theta_1 = (184.7 \pm 13.2)^\circ$, and $\theta_2 = (293.9 \pm 18.1)^\circ$. It should be stressed that the fitted parameter $V_{\pi^0 K^{*+} \bar{K}^{*0}}$ is close to the value 1.69×10^{-4} that is determined from the contribution of $a_0(1710)^0$ to the $D^+_s \to \pi^+ K^0_S K^0_S$ decay [1]. In fact, if we fix $V_{\pi^0 K^{*+} \bar{K}^{*0}}$ as 1.69×10^{-4}, we can also obtain a very good fit with $\chi^2/(129-4) = 0.34$, and the fitted parameters are similar to these values obtained above within uncertainties.

With the fitted central values of the model parameters, we firstly show the theoretical results in Fig. 6 for the $\pi^0 K^+$ invariant mass distributions, where the red-solid curve stands for the total contribution. While the blue-dashed, green-dotted, pink-dot dashed, and purple-dot dashed curves correspond to the contributions from the intermediate states $a_0(1710)^+$, K^{*+}, $\bar{K}^{*0}$, and $a_0(980)^+$, respectively. It is shown that our theoretical calculations cannot describe well these structures around $M_{\pi^0 K^+} \sim 0.7$ and 1.4 GeV, which implies that there should be other contributions around those energies. While the higher tail of the $\pi^0 K^+$ line shape is mostly from the reflections of the $a_0(980)$ and the $\bar{K}^{*0}$, and the contribution from $\bar{K}^{*0}$ is dominant above $M_{\pi^0 K^+} \sim 1.4$ GeV.

Next, in Fig. 7, the theoretical results for the $\pi^0 K^0_S$ invariant mass distributions are shown. It is easily seen that the experimental data can be well reproduced within our reaction mechanisms, particularly the shoulder structure from the threshold to $M_{\pi^0 K^0_S} \sim 0.8$ GeV. Again, the higher tail of the $\pi^0 K^0_S$ line shape is from the $a_0(980)$ resonance

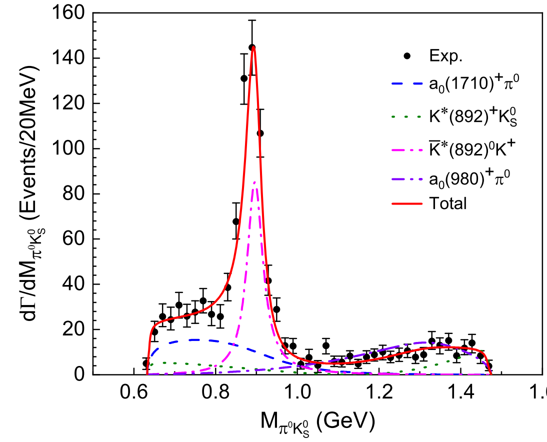

FIG. 7. Invariant mass distributions of $\pi^0 K^0_S$ for the $D^+_s \to \pi^0 K^+ K^0_S$ decay, compared with the experimental data taken from Fig. 2(b) of Ref. [26].

and K^{*+}, which may indicate that the contribution from the vector meson $K^{*+}(1410)$ is small and negligible. Yet, its contribution was included in the analysis by the BESIII Collaboration [26].

Finally, in Fig. 8, we show the theoretical results on the $K^+ K^0_S$ invariant mass distributions. From Fig. 8, one can see that our numerical results can reproduce the $a_0(980)^+$ and $a_0(1710)^+$ peaks. The bump structure in the energy region from 1.2 to 1.4 GeV is mainly from the reflection of the K^{*+} meson. Furthermore, as for the $D^+_s \to \pi^+ K^0_S K^0_S$ reaction [1], the K^{*+} and $\bar{K}^{*0}$ mesons contribute significantly to the peak region of the $a_0(1710)^+$ resonance, though the contribution of $a_0(1710)^+$ is the largest one, compared to the K^* contributions. To better investigate the role of $a_0(1710)$ resonance, future measurements could do some cuts and remove these contributions from the $K^*(892)$ mesons.

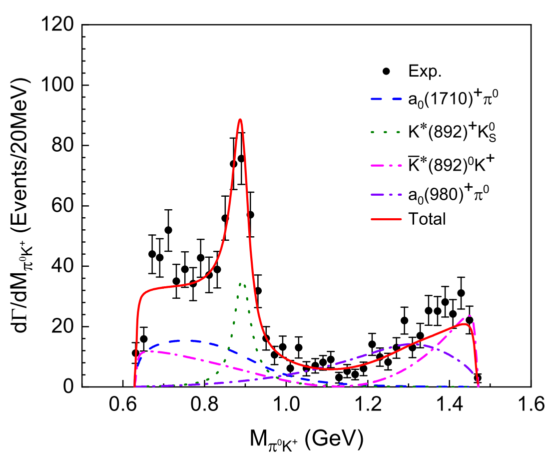

FIG. 6. Invariant mass distributions of $\pi^0 K^+$ for the $D^+_s \to \pi^0 K^+ K^0_S$ decay, compared with the experimental data taken from Fig. 2(c) of Ref. [26].

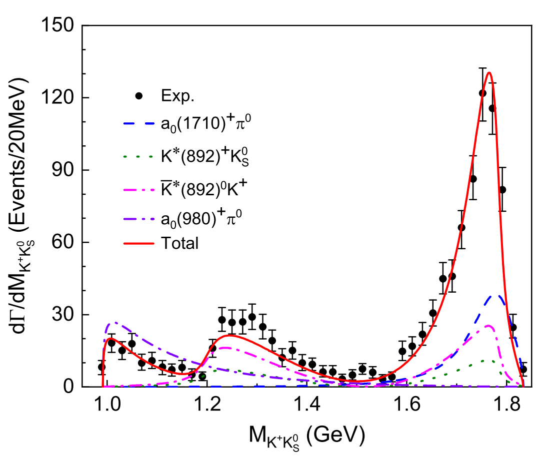

FIG. 8. Invariant mass distributions of $K^+ K^0_S$ for the $D^+_s \to \pi^0 K^+ K^0_S$ decay, compared with the experimental data taken from Fig. 2(a) of Ref. [26].

It is worth mentioning that in Ref. [29], the $a_0(980)$ state was studied in the $D_s^+ \to \pi^0 K^+ \bar{K}^0$ decay by including the contributions from the triangle diagrams involving the intermediate K^* and ρ resonances, and then the $K\bar{K}$ and $\pi\eta$ pairs fuse to generate the $a_0(980)$ state. It is found that the signal of the $a_0(980)$ state is also clearly seen [29]. In this work, for the production of $a_0(980)$, we rely on the production of the $\pi^0 K^+ \bar{K}^0$, where all particles are produced in S wave. Yet, it is easy to see that the mechanism proposed in Ref. [29] should be suppressed due to the highly off shell effect of the K^* propagator when the $K^+\bar{K}^0$ invariant mass is close to the $a_0(980)^+$ mass. Furthermore, to include such contributions more free parameters are needed. In addition, we have checked the S-wave final-state interactions of π^0 and K^+ $(\bar{K}^0)$, and found that these two contributions are much smaller compared with the contributions of the K^{*+} and $\bar{K}^{*0}$. Nevertheless, the model proposed in the present work can give a reasonable description of the experimental data for the $D_s^+ \to \pi^0 K^+ K_S^0$ decay and constitutes a further theoretical effort in studying the roles of the $a_0(980)$ and $a_0(1710)$ states in the relevant reaction.

IV. SUMMARY

In the present work, we investigated the Cabibbo-favored process $D_s^+ \to \pi^0 K^+ K_S^0$. In addition to the contributions from the tree diagrams of $K^{*+} \to \pi^0 K^+$ and $\bar{K}^{*0} \to \pi^0 K_S^0$, the S-wave $(K^*\bar{K}^*)^+$ final-state interaction within the chiral unitary approach, where the $a_0(1710)^+$ state is dynamically generated was also considered. While for the $K\bar{K}$ final-state interaction, we took into account the $a_0(980)^+$ state, which is a dynamically generated resonance from the coupled-channel pseudoscalar-pseudoscalar interaction. Considering all these contributions, we calculated the three invariant mass distributions of $K^+K_S^0$, $\pi^0 K^+$, and $\pi^0 K_S^0$, to which $a_0(1710)^+$, $a_0(980)^+$, K^{*+}, and $\bar{K}^{*0}$ contribute, respectively.

We showed that the new experimental data on the invariant mass distributions of $d\Gamma/dM_{\pi^0 K^+}$, $d\Gamma/dM_{K^+K_S^0}$, and $d\Gamma/dM_{\pi^0 K_S^0}$ measured by the BESIII Collaboration [26] can be reproduced. We found that the vector meson $K^*(892)$ plays a crucial role in the $a_0(1710)^+$ peak region, and the contribution from $K^*(1410)$ was not needed. The higher tail of the πK line shapes are mainly due to the reflection effects of the $a_0(980)$ resonance and $K^*(892)$ mesons.

We would like to stress that the decay of $D_s^+ \to \pi^0 K^+ K_S^0$ is a good platform to study the isospin one scalar mesons, since the final $K^+K_S^0$ system is in pure isospin $I = 1$. The experimental measurements on the $D_s^+ \to \pi^0 K^+ K_S^0$ reaction by the BESIII Collaboration support the $K^*\bar{K}^*$ nature of the $a_0(1710)$ state, and the $a_0(980)$ state is a dynamically generated state from the coupled-channel interactions of pseudoscalar meson and pseudoscalar meson. Furthermore, thanks to the important role played by the scalar mesons $a_0(980)$ and $a_0(1710)$ in the $D_s^+ \to \pi^0 K^+ K_S^0$ reaction, the accurate data on this reaction can be used to improve our knowledge about these scalar mesons.

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[1] X. Zhu, D. M. Li, E. Wang, L. S. Geng, and J. J. Xie, Theoretical study of the process $D_s^+ \to \pi^+ K_S^0 K_S^0$ and the isovector partner of $f_0(1710)$, Phys. Rev. D **105**, 116010 (2022).

[2] L. R. Dai, E. Oset, and L. S. Geng, The $D_s^+ \to \pi^+ K_S^0 K_S^0$ reaction and the $I = 1$ partner of the $f_0(1710)$ state, Eur. Phys. J. C **82**, 225 (2022).

[3] L. S. Geng and E. Oset, Vector meson-vector meson interaction in a hidden gauge unitary approach, Phys. Rev. D **79**, 074009 (2009).

[4] L. S. Geng, E. Oset, R. Molina, and D. Nicmorus, Dynamically generated resonances from vector meson-vector meson interaction based on a hidden-gauge unitary approach, *Proc. Sci.* EFT09 (2009) 040 [arXiv:0905.0419].

[5] M. L. Du, D. Gülmez, F. K. Guo, U. G. Meißner, and Q. Wang, Interactions between vector mesons and dynamically generated resonances, Eur. Phys. J. C **78**, 988 (2018).

[6] Z. L. Wang and B. S. Zou, Further study of $f_0(1710)$ with the coupled-channel approach and the hadron molecular picture, Phys. Rev. D **104**, 114001 (2021).

[7] Z. L. Wang and B. S. Zou, Two dynamical generated a_0 resonances by interactions between vector mesons, Eur. Phys. J. C **82**, 509 (2022).

[8] R. L. Workman (Particle Data Group), Review of particle physics, Prog. Theor. Exp. Phys. **2022**, 083C01 (2022).

[9] H. Nagahiro, L. Roca, E. Oset, and B. S. Zou, Role of meson loops in the $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ decays into $V\gamma$, Phys. Rev. D **78**, 014012 (2008).

[10] T. Branz, L. S. Geng, and E. Oset, Two-photon and one photon-one vector meson decay widths of the $f_0(1370)$, $f_2(1270)$, $f_0(1710)$, $f_2'(1525)$, and $K_2^*(1430)$, Phys. Rev. D **81**, 054037 (2010).

[11] L. S. Geng, F. K. Guo, C. Hanhart, R. Molina, E. Oset, and B. S. Zou, Study of the $f_2(1270)$, $f_2'(1525)$, $f_2(1370)$ and $f_0(1710)$ in the J/ψ radiative decays, Eur. Phys. J. A **44**, 305 (2010).

[12] W. L. Wang and Z. Y. Zhang, Dynamical coupled-channel study of $K^*\bar{K}^*$ and $\omega\phi$ states in a chiral quark model, Phys. Rev. C **84**, 054006 (2011).

[13] A. Martinez Torres, K. P. Khemchandani, F. S. Navarra, M. Nielsen, and E. Oset, The role of $f_0(1710)$ in the $\phi\omega$ threshold peak of $J/\Psi \to \gamma\phi\omega$, Phys. Lett. B **719**, 388 (2013).

[14] J. J. Xie and E. Oset, $\bar{B}^0$ and $\bar{B}_s^0$ decays into J/ψ and $f_0(1370)$, $f_0(1710)$, $f_2(1270)$, $f_2'(1525)$, $K_2^*(1430)$, Phys. Rev. D **90**, 094006 (2014).

[15] L. R. Dai, J. J. Xie, and E. Oset, Study of the $f_2(1270)$, $f_2(1525)$, $K_2^*(1430)$, $f_0(1370)$ and $f_0(1710)$ production from $\psi(nS)$ and $\Upsilon(nS)$ decays, Phys. Rev. D **91**, 094013 (2015).

[16] L. R. Dai, R. Pavao, S. Sakai, and E. Oset, $\tau^- \to \nu_\tau M_1 M_2$, with M_1, M_2 pseudoscalar or vector mesons, Eur. Phys. J. A **55**, 20 (2019).

[17] R. Molina, L. R. Dai, L. S. Geng, and E. Oset, J/ψ decay into $\phi(\omega)$ and vector-vector molecular states, Eur. Phys. J. A **56**, 173 (2020).

[18] C. García-Recio, L. S. Geng, J. Nieves, L. L. Salcedo, E. Wang, and J. J. Xie, Low-lying even parity meson resonances and spin-flavor symmetry revisited, Phys. Rev. D **87**, 096006 (2013).

[19] F. E. Close and Q. Zhao, Production of $f_0(1710)$, $f_0(1500)$, and $f_0(1370)$ in J/ψ hadronic decays, Phys. Rev. D **71**, 094022 (2005).

[20] L. C. Gui, Y. Chen, G. , C. Liu, Y.-B. Liu, J.-P. Ma, Y.-B. Yang, and J.-B. Zhang (CLQCD Collaboration), Scalar Glueball in Radiative J/ψ Decay on the Lattice, Phys. Rev. Lett. **110**, 021601 (2013).

[21] S. Janowski, F. Giacosa, and D. H. Rischke, Is $f_0(1710)$ a glueball?, Phys. Rev. D **90**, 114005 (2014).

[22] A. H. Fariborz, A. Azizi, and A. Asrar, Proximity of $f_0(1500)$ and $f_0(1710)$ to the scalar glueball, Phys. Rev. D **92**, 113003 (2015).

[23] M. Ablikim *et al.* (BESIII Collaboration), Study of the decay $D_s^+ \to K_S^0 K_S^0 \pi^+$ and observation an isovector partner to $f_0(1710)$, Phys. Rev. D **105**, L051103 (2022).

[24] M. Ablikim *et al.* (BESIII Collaboration), Amplitude analysis and branching fraction measurement of $D_s^+ \to K^+K^-\pi^+$, Phys. Rev. D **104**, 012016 (2021).

[25] J. P. Lees *et al.* (*BABAR* Collaboration), Light meson spectroscopy from Dalitz plot analyses of η_c decays to $\eta' K^+K^-$, $\eta'\pi^+\pi^-$, and $\eta\pi^+\pi^-$ produced in two-photon interactions, Phys. Rev. D **104**, 072002 (2021).

[26] M. Ablikim *et al.* (BESIII Collaboration), Observation of $a_0(1710)^+ \to K_S^0 K^+$ in Study of the $D_s^+ \to K_S^0 K^+\pi^0$ Decay, Phys. Rev. Lett. **129**, 182001 (2022).

[27] D. Guo, W. Chen, H. X. Chen, X. Liu, and S. L. Zhu, Newly observed $a_0(1817)$ as the scaling point of constructing the scalar meson spectroscopy, Phys. Rev. D **105**, 114014 (2022).

[28] J. J. Xie, L. R. Dai, and E. Oset, The low lying scalar resonances in the D^0 decays into K_s^0 and $f_0(500)$, $f_0(980)$, $a_0(980)$, Phys. Lett. B **742**, 363 (2015).

[29] X. Z. Ling, M. Z. Liu, J. X. Lu, L. S. Geng, and J. J. Xie, Can the nature of $a_0(980)$ be tested in the $D_s^+ \to \pi^+\pi^0\eta$ decay?, Phys. Rev. D **103**, 116016 (2021).

[30] J. Y. Wang, M. Y. Duan, G. Y. Wang, D. M. Li, L. J. Liu, and E. Wang, The $a_0(980)$ and $f_0(980)$ in the process $D_s^+ \to K^+K^-\pi^+$, Phys. Lett. B **821**, 136617 (2021).

[31] R. Molina, J. J. Xie, W. H. Liang, L. S. Geng, and E. Oset, Theoretical interpretation of the $D_s^+ \to \pi^+\pi^0\eta$ decay and the nature of $a_0(980)$, Phys. Lett. B **803**, 135279 (2020).

[32] M. Y. Duan, J. Y. Wang, G. Y. Wang, E. Wang, and D. M. Li, Role of scalar $a_0(980)$ in the single Cabibbo suppressed process $D^+ \to \pi^+\pi^0\eta$, Eur. Phys. J. C **80**, 1041 (2020).

[33] Y. K. Hsiao, Y. Yu, and B. C. Ke, Resonant $a_0(980)$ state in triangle rescattering $D_s^+ \to \pi^+\pi^0\eta$ decays, Eur. Phys. J. C **80**, 895 (2020).

[34] A. Martinez Torres, L. S. Geng, L. R. Dai, B. Sun, Xi, E. Oset, and B. S. Zou, Study of the $J/\psi \to \phi(\omega) f_2(1270)$, $J/\psi \to \phi(\omega) f_2'(1525)$ and $J/\psi \to K^{*0}(892)\bar{K}_2^{*0}(1430)$ decays, Phys. Lett. B **680**, 310 (2009).

[35] A. Bramon, A. Grau, and G. Pancheri, Intermediate vector meson contributions to $V^0 \to P^0P^0\gamma$ decays, Phys. Lett. B **283**, 416 (1992).

[36] W. H. Liang and E. Oset, B^0 and B_s^0 decays into $J/\psi f_0(980)$ and $J/\psi f_0(500)$ and the nature of the scalar resonances, Phys. Lett. B **737**, 70 (2014).

[37] E. Oset, W. H. Liang, M. Bayar, J. J. Xie, L. R. Dai, M. Albaladejo, M. Nielsen, T. Sekihara, F. Navarra, L. Roca *et al.*, Weak decays of heavy hadrons into dynamically generated resonances, Int. J. Mod. Phys. E **25**, 1630001 (2016).

[38] J. J. Xie and L. S. Geng, The $a_0(980)$ and $\Lambda(1670)$ in the $\Lambda_c^+ \to \pi^+\eta\Lambda$ decay, Eur. Phys. J. C **76**, 496 (2016).

[39] G. Y. Wang, N. C. Wei, H. M. Yang, E. Wang, L. S. Geng, and J. J. Xie, Roles of $a_0(980)$, $\Lambda(1670)$, and $\Sigma(1385)$ in the $\Lambda_c^+ \to \eta\Lambda\pi^+$ decay, Phys. Rev. D **106**, 056001 (2022).

[40] L. R. Dai, J. J. Xie, and E. Oset, $B^0 \to D^0\bar{D}^0K^0$, $B^+ \to D^0\bar{D}^0K^+$, and the scalar $D\bar{D}$ bound state, Eur. Phys. J. C **76**, 121 (2016).

[41] J. J. Xie and G. Li, The decays of $\bar{B}^0$, $\bar{B}_s^0$ and B^- into η_c plus a scalar or vector meson, Eur. Phys. J. C **78**, 861 (2018).

[42] W. Liang, C. W. Xiao, and E. Oset, Study of $\eta K\bar{K}$ and $\eta' K\bar{K}$ with the fixed center approximation to Faddeev equations, Phys. Rev. D **88**, 114024 (2013).

[43] J. A. Oller and E. Oset, Chiral symmetry amplitudes in the S wave isoscalar and isovector channels and the σ, $f_0(980)$, $a_0(980)$ scalar mesons, Nucl. Phys. **A620**, 438 (1997); **A652**, 407(E) (1999).

[44] J. A. Oller, E. Oset, and J. R. Pelaez, Meson meson interaction in a nonperturbative chiral approach, Phys. Rev. D **59**, 074001 (1999); **60**, 099906(E) (1999); **75**, 099903(E) (2007).

[45] J. A. Oller, E. Oset, and J. R. Pelaez, Nonperturbative Approach to Effective Chiral Lagrangians and Meson Interactions, Phys. Rev. Lett. **80**, 3452 (1998).