

Spinor representation of the Hamiltonian constraint in 3D loop quantum gravity with a nonzero cosmological constant

Valentin Bonzom,^{1,*} Maité Dupuis,^{2,3,†} and Qiaoyin Pan^{4,2,3,‡}

¹*Université Sorbonne Paris Nord, LIPN, CNRS UMR 7030, F-93430 Villetaneuse, France*

²*Perimeter Institute, 31 Caroline St North, Waterloo, N2L 2Y5 Ontario, Canada*

³*Department of Applied Mathematics, University of Waterloo, Waterloo, N2L 3G1 Ontario, Canada*

⁴*Department of Physics, Florida Atlantic University, 777 Glades Road, Boca Raton, Florida 33431, USA*



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We develop in a companion paper the kinematics of three-dimensional loop quantum gravity in Euclidean signature and with a negative cosmological constant, focusing in particular on the spinorial representation that is well known at zero cosmological constant. In this paper, we put this formalism to the test by quantizing the Hamiltonian constraint on the dual of a triangulation. The Hamiltonian constraints are obtained by projecting the flatness constraints onto spinors, as done in the flat case by the first author and Livine. Quantization then relies on q -deformed spinors. The quantum Hamiltonian constraint acts in the q -deformed spin network basis as difference equations on physical states, which are thus the Wheeler-DeWitt equations in this framework. Moreover, we study how physical states transform under Pachner moves of the canonical surface. We find that those transformations are in fact q deformations of the transition amplitudes of the flat case as found by Noui and Perez. Our quantum Hamiltonian constraints, therefore, build a Turaev-Viro model at real q .

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I. INTRODUCTION

Three-dimensional gravity is often taken as a testing ground for new approaches to quantum gravity as it is much better understood compared to the four-dimensional case. Here we consider 3D gravity as a topological model (meaning the triad can degenerate) through the BF formulation. A criterion for the validity of novel approaches to their quantization is that they reproduce the results obtained via previous approaches, such as its topological invariance. In the absence of a cosmological constant, nonperturbative quantization in the canonical approach, i.e., the loop quantum gravity (LQG) based on the BF formulation of gravity [1,2], and the covariant approach, based on spin foams and more precisely the Ponzano-Regge model [3–7], give consistent results. In particular, it has been shown by Noui and Perez in [8] that the scalar products of physical states defined in LQG are given by the Ponzano-Regge amplitudes.

A more recent approach has emerged that aims at quantizing the Hamiltonian constraint instead of the flatness constraint derived from the BF formulation. In the case of vanishing cosmological constant (which we will often refer to as the flat case), the Hamiltonian constraint

translates in the spin network basis to difference equations on the coefficients of the physical states. These difference equations should really be seen as Wheeler-DeWitt equations for 3D LQG. In particular, they are solved by spin network evaluations, as expected from the Ponzano-Regge model [9,10].

When the cosmological constant is nonzero, the connection between LQG and spin foams is less clear. On the spin foam side, the Turaev-Viro model [11] is known to provide the partition function of 3D gravity in Euclidean signature with a positive cosmological constant [12]. It is a sum of states in $SU_q(2)$ representation, with q a root of unity encoding the cosmological constant. It is thus a q deformation of the Ponzano-Regge model, further providing a regularization through a natural cutoff on representations when replacing $SU(2)$ with $SU_q(2)$. The large spin limit of the $q - 6j$ symbol matches Regge calculus for curved tetrahedra [13]. The Turaev-Viro model thus provides an example of the interplay between the cosmological constant, curved geometries and the quantum group deformation of Lie groups.

On the LQG side, the Hamiltonian takes a more complicated form when the cosmological constant is nonzero, so much so that even how to discretize it has been unclear, and it seems to evade traditional LQG methods. It has nevertheless been conjectured for a long time that the quantum theory ought to be described by quantum groups, as expected from the spin foam

*bonzom@lipn.univ-paris13.fr

†mdupuis@perimeterinstitute.ca

‡qpan@fau.edu

model [14–18]. One (indirect) way to relate 3D LQG with a nonvanishing cosmological constant to the Turaev-Viro model is to take the Chern-Simons formulation of 3D gravity and consider the Witten-Reshetikhin-Turaev path integral $\mathcal{Z}_{WRT}(\mathcal{M})$ on a three-manifold \mathcal{M} with the Chern-Simons action with opposite levels, say k and $-k$. It has been well known that the Turaev-Viro state sum matches such path integral as $\mathcal{Z}_{TV}(\mathcal{M}) = |\mathcal{Z}_{WRT}(\mathcal{M})|^2$ [19,20].

A more direct approach for bridging the two quantum gravity approaches would be to work on the BF formulation with a cosmological constant term itself. Canonical analysis for the BF action written with the standard triad and connection variables leads to a torsion equation independent of the cosmological constant so that the kinematical Hilbert space upon quantization is spanned by the $SU(2)$ spin network as in the case with a zero cosmological constant. In this setting, one can expect, through the connection of cosmological constant and quantum group deformation, that the quantum group structure would only appear at the level of the physical Hilbert space since only the curvature equation depends on the cosmological constant. One proposal to realize this and thereby connect LQG directly to the Turaev-Viro model was given in [21–23]. There, a new curvature constraint was defined via a new (Poisson noncommutative) connection, leading to a redefinition of the physical scalar product and recovering the Turaev-Viro amplitude.

From the geometrical point of view, the nondeformed kinematical structures given by imposing the torsion-free, or Gauss, constraints represent discrete flat 2D geometries. Then the deformed dynamical structures are expected to describe the gluing of these flat 2D geometries to approximate the curved 3D geometries, as a deformed version of the case with a zero cosmological constant [9]. Indeed, one can approximate a curved 3D geometry by gluing flat 2D pieces and take the limit as the sizes of these pieces approach zero. It was moreover argued in [24–26] that the continuous symmetries survive at the discrete level when one uses curved 2D pieces instead of flat ones.

This suggests we rethink the definition of the kinematics and dynamics in the BF formulation with a nonvanishing cosmological constant. The kinematical phase space defined with the Gauss constraint can in fact be deformed so that its quantization naturally leads to a quantum group deformation. In particular, this kinematical phase space describes 2D curved geometries [27]. Then the physical phase space, defined by imposing the flatness constraint, describes the gluing of these 2D curved geometries into 3D curved geometries. The gap between the discrete, classical theory and the continuous action was further filled recently in [28].

This program was carried out in [27,29] to a large degree. There, the phase space is defined in terms of deformed

fluxes and holonomies and the Poisson structure is based on the Heisenberg double of $SU(2)$. In [29] we have investigated the quantization, using the same techniques as in [9], i.e., by building a Hamiltonian constraint out of the flatness constraints. It can be classically interpreted as generating displacements of the vertices of the triangulation [30]. At the quantum level, the Hamiltonian constraints give rise to difference equations, which can therefore be considered as the Wheeler-DeWitt equations in the spin network basis. We considered in [29] the (simple) case of the boundary of the tetrahedron and showed that the solution to those difference equations is the $q - 6j$ symbol.

Here we are interested in using the spinorial formalism for LQG instead of holonomies and fluxes, and in further extracting all building blocks for the transition amplitudes, i.e., to go beyond the case of the tetrahedron from [29]. In a companion paper [31], we revisit all kinematical aspects of this q -deformed LQG model in more detail, and in the spinor representation. (This was initiated in [32].) In particular, the quantization of the deformed spinors can be performed in terms of q bosons. We then use those q bosons to define the invariant operators that are needed for the quantization of the Hamiltonian constraint in spinor variables.

In this paper, we describe the dynamics using the deformed spinors and q bosons of [31]. The Hamiltonian constraint built from the deformed spinors is a direct generalization of the nondeformed version given in [10]. At the quantum level, the Hamiltonian constraints also give rise to difference equations which are direct q -deformed generalizations of those of [10]. Here we go further to provide the transformations of the physical states (in the spin network basis) under Pachner moves of the canonical surface. This is equivalent to finding the building blocks for spin foams, or for the transition amplitudes, as emphasized by Noui and Perez [8]. In particular, we find that those building blocks are exactly those of the Turaev-Viro model in a version with q real (note that this version suffers from the same finiteness issues as the Ponzano-Regge model).

This paper is organized as follows. In Sec. II, we concisely recall the discrete classical phase space in terms of the holonomies and (deformed) fluxes introduced in [27], as well as the Gauss constraints (used to define the kinematical phase space) and the flatness constraints (used to define the dynamical phase spaces). In Sec. III we move on to the construction of the deformed spinors, following [31], and of the Hamiltonian constraints. The quantization is performed in Sec. IV, again following the prescriptions of [31]. This is where in particular we find the difference equations encoding the Wheeler-DeWitt equations in the spin network basis. Then in Sec. V, we study how solutions to the difference equations are related under Pachner moves, thereby providing the building blocks for the transition amplitudes *à la* Noui-Perez.

II. CLASSICAL PHASE SPACE AND CONSTRAINTS

We start by recalling the main ingredients of the classical phase space for 3D loop gravity with a negative cosmological constant Λ in the Euclidean signature. More details for the mathematical setup can be found in [27,31]. The phase space is associated to a graph Γ , which is dual to a cellular decomposition of the canonical surface Σ . It has V vertices, E edges and F faces (the connected components of $\Sigma \setminus \Gamma$).

Here and throughout the paper, we use $\kappa := \frac{G\sqrt{-\Lambda}}{c}$, for a cosmological constant $\Lambda < 0$. It is a parameter that deforms the Poisson structure with respect to the case of vanishing cosmological constant.

A. Phase space for a single edge

We first consider a single edge and associate to it a phase space: the Heisenberg double $(\mathcal{D}(\text{SU}(2)), \pi_H)$ of $\text{SU}(2)$. It is the group $\mathcal{D}(\text{SU}(2)) = \text{SL}(2, \mathbb{C}) \cong \text{SU}(2) \bowtie \text{AN}(2)$ with Poisson structure π_H fully determined by a classical r matrix $r \in \mathfrak{sl}(2, \mathbb{C}) \otimes \mathfrak{sl}(2, \mathbb{C})$. The Poisson brackets can be compactly written as

$$\{d_1, d_2\} = -r_{21}d_1d_2 + d_1d_2r = rd_1d_2 - d_1d_2r_{21}, \quad \forall d \in \text{SL}(2, \mathbb{C}), \quad (1)$$

where $d_1 = d \otimes \mathbb{1}$, $d_2 = \mathbb{1} \otimes d$. The r matrix is chosen as

$$r = \frac{i\kappa}{4} \sum_{i=1}^3 \sigma_i \otimes \rho_i = \frac{i\kappa}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

Here $\sigma_{1,2,3}$ are the Pauli matrices while $\rho_i = \sigma_i + \frac{1}{2}[\sigma_3, \sigma_i]$. Finally, r_{21} is given by the permutation of the two vector space components of r (in the above 4×4 representation, r_{21} is simply the matrix transpose of r). The equality of the last two expressions in (1) is guaranteed by the property that $r_s := \frac{1}{2}(r + r_{21})$ is the Casimir thus $[r_s, d_1d_2] = 0$.

It is important for loop gravity to split an $\text{SL}(2, \mathbb{C})$ element via the Iwasawa decomposition into the product of an $\text{AN}(2)$ element and an $\text{SU}(2)$ element. One can write $d \in \text{SL}(2, \mathbb{C})$ in exactly two ways as

$$d = \ell u = \tilde{u} \tilde{\ell}, \quad \ell, \tilde{\ell} \in \text{AN}(2), \quad u, \tilde{u} \in \text{SU}(2). \quad (3)$$

This phase space can be seen as a deformation of the holonomy-flux phase space at $\Lambda = 0$ [27,33]. In the flat/nondeformed ($\Lambda = 0$) case, the phase space of an edge is described by $\text{ISU}(2)$, the holonomies are described by $\text{SU}(2)$ and the fluxes are described by \mathbb{R}^3 . Here, in the deformed phase space $\text{SL}(2, \mathbb{C})$, we also let the $\text{SU}(2)$

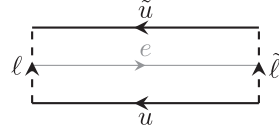


FIG. 1. A ribbon edge $R(e)$. The variables $\ell, u, \tilde{\ell}, \tilde{u}$ are assigned to the four sides of the ribbon edge and they are subject to the ribbon constraint $\ell u \tilde{\ell}^{-1} \tilde{u}^{-1}$ represented as the trivialization of the loop around $R(e)$. The positions of these variables are fixed such that the directions of u and \tilde{u} are opposite to that of the edge e (in gray).

subgroup describes the holonomies while the (deformed) fluxes correspond to an $\text{AN}(2)$ subgroup. That is, for each phase space variable d , we perform the Iwasawa decomposition (3) then u and \tilde{u} are holonomies, while ℓ and $\tilde{\ell}$ are fluxes.

We call the constraint

$$\mathcal{C} = \mathbb{1} \in \text{SL}(2, \mathbb{C}) \quad \text{for } \mathcal{C} := \ell u \tilde{\ell}^{-1} \tilde{u}^{-1} \quad (4)$$

the ribbon constraint, associated to every edge of Γ . It has six real components and forms a set of second-class constraints with respect to the Poisson brackets (1) (meaning that the brackets between the components do not close).

The ribbon constraint has a natural graphical interpretation. Since the edges of Γ are embedded in a surface, there is a natural clockwise walk around each of them. It goes (i) along the edge on one side, (ii) crosses it at its end, (iii) goes back along the edge on its other side, and (iv) finally crosses it again to close the walk. Equivalently, one thickens the edge by taking a tubular neighborhood in Σ , as in Fig. 1. The boundary has four pieces that naturally correspond to the four parts of the walk above.

If e is an edge in Γ , then we denote $R(e)$ its thickening, called the ‘‘ribbon edge.’’ The boundary pieces parallel to e will be called the ‘‘long edges’’ of $R(e)$ and the boundary pieces that cross e at its ends will be called the ‘‘short edges’’ of $R(e)$. We can orient the long and short edges clockwise around $R(e)$. The matrices $u, \ell, \tilde{u}^{-1}, \tilde{\ell}^{-1}$ are then assigned in this order and as pictured in Fig. 1. In particular, u and \tilde{u} are assigned to the long edges of $R(e)$, while ℓ and $\tilde{\ell}$ are assigned to its short edges. Equivalently, we can think of u and \tilde{u} as associated to e itself and ℓ and $\tilde{\ell}$ to each half edge, i.e., a pair of an edge and an incident vertex.

To fix the position of the variables around the ribbon, one can use the orientation on Γ and decide for instance that u is oriented opposite to e . This is the convention we will use. The ribbon constraint $\mathcal{C} = \ell u \tilde{\ell}^{-1} \tilde{u}^{-1} \cong \mathbb{1}$ is then a flatness constraint around $R(e)$.

The Poisson brackets (1) can be equivalently written as brackets between holonomies and fluxes

$$\begin{aligned}
 \{\ell_1, \ell_2\} &= -[r_{21}, \ell_1 \ell_2], & \{\ell_1, u_2\} &= -\ell_1 r_{21} u_2, \\
 \{u_1, \ell_2\} &= \ell_2 r u_1, & \{u_1, u_2\} &= -[r, u_1 u_2], \\
 \{\tilde{\ell}_1, \tilde{\ell}_2\} &= [r_{21}, \tilde{\ell}_1 \tilde{\ell}_2], & \{\tilde{\ell}_1, \tilde{u}_2\} &= -\tilde{u}_2 r_{21} \tilde{\ell}_1, \\
 \{\tilde{u}_1, \tilde{\ell}_2\} &= \tilde{u}_1 r \tilde{\ell}_2, & \{\tilde{u}_1, \tilde{u}_2\} &= [r, \tilde{u}_1 \tilde{u}_2].
 \end{aligned} \tag{5}$$

All other Poisson brackets, e.g., $\{\ell_1, \tilde{u}_2\}$ can also be obtained by combining (5) and (4).

The AN(2) elements ℓ and $\tilde{\ell}$ can be parametrized as 2×2 lower triangular matrices

$$\ell = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix}, \quad \tilde{\ell} = \begin{pmatrix} \tilde{\lambda} & 0 \\ \tilde{z} & \tilde{\lambda}^{-1} \end{pmatrix}, \quad \lambda, \tilde{\lambda} \in \mathbb{R}^+, \quad z, \tilde{z} \in \mathbb{C}. \tag{6}$$

By also writing the holonomies u and \tilde{u} in the fundamental representation, one can write down all the Poisson brackets between the matrix elements of u, \tilde{u} and $\lambda, \tilde{\lambda}, z, \tilde{z}, \bar{z}, \bar{\tilde{z}}$ (see Refs. [27,31] for details).

B. Ribbon graph phase space

We extend the phase space defined above for a single edge to the whole graph Γ , by taking the product of $\text{SL}(2, \mathbb{C})$ over the edges. Graphically, one thickens every edge of Γ as before. However, this is not enough since there will be interactions between the group elements of different edges which meet at vertices of Γ . An advantageous graphical representation is to thicken Γ itself. Each vertex v of valency d is fattened to a ribbon vertex $R(v)$, i.e., a d -gon whose boundary edges correspond to the edges incident to v , and are glued to the short edges of the ribbon edges [the boundary vertices of $R(v)$ correspond to the ‘‘corners’’ at v between adjacent edges]. In other words, a ribbon vertex is a face whose boundary sides are dressed with fluxes ℓ s and $\tilde{\ell}$ s.

Overall, the holonomies u_e, \tilde{u}_e are labeled by the edges of Γ and assigned to the long edges of the ribbon edges. There are two fluxes associated to every edge of Γ , denoted $\ell_e \in \text{AN}(2)$ if e is outgoing at v , and $\tilde{\ell}_e \in \text{AN}(2)$ if e is incoming at v . Since each is in fact assigned to a half edge (e, v) [equivalently a short edge of $R(v)$], we will use the generic notation ℓ_{ev} for either one of them.

Since there are also two holonomies along e , it is tempting to distinguish them in terms of half edges. This is possible using orientations. We denote u_{ev} the $\text{SU}(2)$ matrix, which points towards ℓ_{ev} (so that if $\ell_{ev} = \tilde{\ell}_e^{-1}$ then $u_{ev} = \tilde{u}_e^{-1}$ and else $\ell_{ev} = \ell_e$ and $u_{ev} = u_e$).

C. Gauss and flatness constraints

The phase space for Γ described above is constrained by the ribbon constraints $\mathcal{C}_e = \mathbb{1}$ on every edge. Gravity further imposes two additional sets of constraints, namely the Gauss and flatness constraints. Gauss constraints are

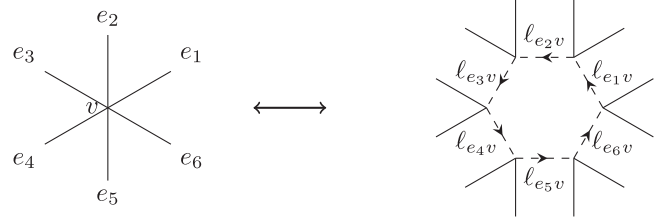


FIG. 2. A vertex v on which edges meet becomes a ribbon vertex $R(v)$ incident to ribbon edges. The ribbon vertex is here depicted with a dashed boundary. Due to the clockwise orientation of the short edges of every ribbon edge, the matrix ℓ_{ev} around $R(v)$ are all oriented counterclockwise.

associated to vertices and impose that the ordered products of the fluxes along the short edges of every $R(v)$ are trivial. Flatness constraints are associated to the faces and impose that the ordered products of the holonomies along the long edges that border every face is trivial. Those two sets of constraints are first class.

To write the Gauss constraints explicitly, choose (randomly) one edge of reference at each vertex of Γ and call it e_1 , then order the edges from 1 to n , i.e., e_1, \dots, e_n by going counterclockwise around v . Notice that all the AN(2) matrices $\ell_{e_i v}$ on the boundary of the ribbon vertex $R(v)$ are oriented counterclockwise, as shown in Fig. 2, for any choice of orientations of the edges incident to v . The Gauss constraint is then simply the flatness around $R(v)$. It reads

$$\ell_{e_n v} \cdots \ell_{e_1 v} = \mathbb{1}. \tag{7}$$

We repeat this construction on faces instead of vertices: choose a random edge of reference around each face f and denote it e_1 , then e_2, \dots, e_d are the edges encountered counterclockwise around f . For all possible orientations of the edges e_1, \dots, e_d on the boundary of f are, the $\text{SU}(2)$ matrices $u_{e_i v}$ are all counterclockwise. The flatness constraint on f reads

$$u_{e_d v_1} \cdots u_{e_2 v_3} u_{e_1 v_2} = \mathbb{1}, \tag{8}$$

as pictured in Fig. 3.

Gauss constraints generate local $\text{SU}(2)$ transformations through the Poisson brackets [27,31]. As usual in symplectic geometry, first-class constraints are not only imposed but one also needs to quotient out the phase space by the orbits they generate. This is called the symplectic quotient. Here, one obtains $\mathcal{P}_{\text{kin}} = \text{SL}(2, \mathbb{C})^E / \text{SU}(2)^V$, which is called the kinematical phase space, where E and V denote the number of edges and vertices in Γ .

It was shown in [27] that the Gauss constraint for a trivalent vertex geometrically represents the hyperbolic cosine law, implying that the kinematical phase space describes hyperbolic geometry (at least hyperbolic triangles in that case).

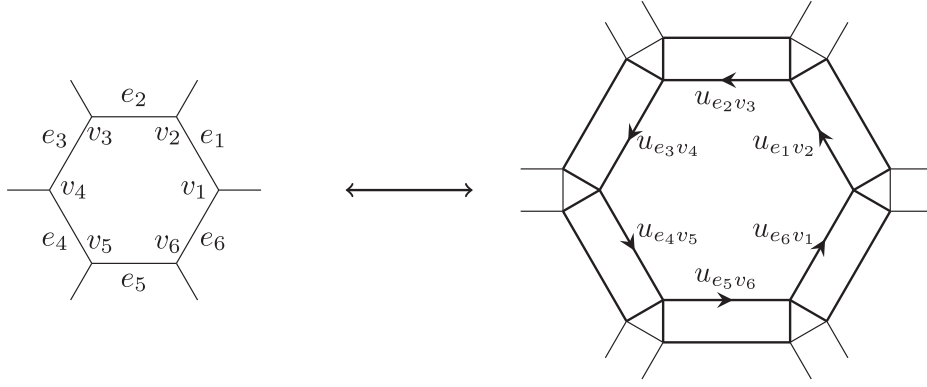


FIG. 3. When considering ribbon edges and ribbon vertices, the face on the left bounded by edges e_1, \dots, e_6 becomes bounded by long edges on the right. All matrices $u_{e_i v_{i+1}}, i = 1, \dots, 6$ are oriented counterclockwise around the face.

On the other hand, flatness constraints generate (deformed) translations [27]. The physical phase space is then obtained via the symplectic quotient of the kinematical phase space by the flatness constraints, $\mathcal{P}_{\text{phys}} = \mathcal{P}_{\text{kin}} // \text{AN}(2)^F$ with F as the number of the faces in Γ .

In the $\Lambda \rightarrow 0$ limit, one recovers the Poincaré phase space structure of first-order 3D gravity. In particular, the flatness constraints generate an \mathbb{R}^3 action, i.e., translations. Geometrically, those translations simply move the vertices of the triangulation (dual to the faces of Γ) around (three directions for the three components of the constraints). The flatness constraints also enforce the dihedral angles to be functions of the angles within triangles as in flat, Euclidean geometry (recall that dihedral angles measure the extrinsic curvature at the discrete level) [9]. This geometric picture arises when the constraints are written on the basis determined by the fluxes themselves. It is also possible to describe them on a spinor basis [10], which is what we will focus on in this paper.

III. SPINORIAL REPRESENTATION

In this section, we rewrite the q -deformed loop gravity phase space structure described above in the spinor representation. In particular, we define the deformed

spinors that can be naturally associated to the ribbon graph and reproduce the $\text{SU}(2)$ holonomies and the $\text{AN}(2)$ fluxes. We also define the scalar products of these deformed spinors, living at the corners of the ribbon graph, which are $\text{SU}(2)$ -invariant quantities hence live in the kinematical phase space. These scalar products are especially useful in constructing the Hamiltonian.

A. Deformed spinors

Here we describe the κ -deformed spinors that can be used to describe the phase space and the constraints in place of the variables u_{ev}, ℓ_{ev} . We will only give the main ingredients needed to construct the Hamiltonian constraint. The fully detailed construction appears in [31].

The building blocks are two independent pairs of κ -deformed spinor variables $(\zeta_0^\kappa, \zeta_1^\kappa), (\tilde{\zeta}_0^\kappa, \tilde{\zeta}_1^\kappa)$ and their complex conjugates $(\bar{\zeta}_0^\kappa, \bar{\zeta}_1^\kappa), (\bar{\tilde{\zeta}}_0^\kappa, \bar{\tilde{\zeta}}_1^\kappa)$. The norms of these κ -deformed spinor variables are $\bar{\zeta}_A^\kappa \zeta_A^\kappa = \frac{2}{\kappa} \sinh\left(\frac{\kappa N_A}{2}\right)$, $\bar{\tilde{\zeta}}_A^\kappa \tilde{\zeta}_A^\kappa = \frac{2}{\kappa} \sinh\left(\frac{\kappa \tilde{N}_A}{2}\right)$, $A = 0, 1$ where N_A and \tilde{N}_A are real functions of the κ -deformed spinor variables.¹ They satisfy the Poisson brackets

$$\begin{aligned} \{\zeta_A^\kappa, \bar{\zeta}_B^\kappa\} &= -i\delta_{AB} \cosh\left(\frac{\kappa N_A}{2}\right), \quad \{N_A, \zeta_B^\kappa\} = i\delta_{AB} \zeta_A^\kappa, \quad \{N_A, \bar{\zeta}_B^\kappa\} = -i\delta_{AB} \bar{\zeta}_A^\kappa, \\ \{\tilde{\zeta}_A^\kappa, \bar{\tilde{\zeta}}_B^\kappa\} &= -i\delta_{AB} \cosh\left(\frac{\kappa \tilde{N}_A}{2}\right), \quad \{\tilde{N}_A, \tilde{\zeta}_B^\kappa\} = i\delta_{AB} \tilde{\zeta}_A^\kappa, \quad \{\tilde{N}_A, \bar{\tilde{\zeta}}_B^\kappa\} = -i\delta_{AB} \bar{\tilde{\zeta}}_A^\kappa, \quad A, B = 0, 1, \end{aligned} \quad (9)$$

and all other Poisson brackets vanish.

Let $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. These κ -deformed spinor variables can be used to define two types of deformed spinors:

¹ N_A ($A = 0, 1$) is the norm of the κ spinors at $\kappa \rightarrow 0$ thus the norm of the standard spinor variables and likewise for \tilde{N}_A .

- (i) The SU(2)-covariant spinors, transforming under SU(2) gauge transformations in a covariant way. We denote them as $|t\rangle$ and $|\tilde{t}\rangle$, and their duals as $\langle t|$ and $\langle \tilde{t}|$,

$$\begin{aligned} |t\rangle &= \begin{pmatrix} t_- \\ t_+ \end{pmatrix} = \begin{pmatrix} e^{\frac{\kappa N_1}{4}} \zeta_0^\kappa \\ e^{-\frac{\kappa N_0}{4}} \zeta_1^\kappa \end{pmatrix}, & \langle t| &= \epsilon |\tilde{t}\rangle = \begin{pmatrix} -\tilde{t}_+ \\ \tilde{t}_- \end{pmatrix} = \begin{pmatrix} -e^{-\frac{\kappa N_0}{4}} \tilde{\zeta}_1^\kappa \\ e^{\frac{\kappa N_1}{4}} \tilde{\zeta}_0^\kappa \end{pmatrix}, \\ |\tilde{t}\rangle &= \begin{pmatrix} \tilde{t}_- \\ \tilde{t}_+ \end{pmatrix} = \begin{pmatrix} e^{\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \\ e^{-\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \end{pmatrix}, & \langle \tilde{t}| &= \epsilon |\tilde{\tilde{t}}\rangle = \begin{pmatrix} -\tilde{\tilde{t}}_+ \\ \tilde{\tilde{t}}_- \end{pmatrix} = \begin{pmatrix} -e^{-\frac{\kappa \tilde{N}_0}{4}} \tilde{\tilde{\zeta}}_1^\kappa \\ e^{\frac{\kappa \tilde{N}_1}{4}} \tilde{\tilde{\zeta}}_0^\kappa \end{pmatrix}. \end{aligned} \quad (10)$$

- (ii) The braided-covariant spinors, transforming in a braided-covariant way [31]. We denote them as $|\tau\rangle$ and $|\tilde{\tau}\rangle$ and their duals as $\langle \tau|$ and $\langle \tilde{\tau}|$,

$$\begin{aligned} |\tau\rangle &= \begin{pmatrix} \tau_- \\ \tau_+ \end{pmatrix} = \begin{pmatrix} e^{-\frac{\kappa N_1}{4}} \zeta_0^\kappa \\ e^{\frac{\kappa N_0}{4}} \zeta_1^\kappa \end{pmatrix}, & \langle \tau| &= \epsilon |\tilde{\tau}\rangle = \begin{pmatrix} -\tilde{\tau}_+ \\ \tilde{\tau}_- \end{pmatrix} = \begin{pmatrix} -e^{-\frac{\kappa N_0}{4}} \tilde{\zeta}_1^\kappa \\ e^{-\frac{\kappa N_1}{4}} \tilde{\zeta}_0^\kappa \end{pmatrix}, \\ |\tilde{\tau}\rangle &= \begin{pmatrix} \tilde{\tau}_- \\ \tilde{\tau}_+ \end{pmatrix} = \begin{pmatrix} e^{-\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \\ e^{\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \end{pmatrix}, & \langle \tilde{\tau}| &= \epsilon |\tilde{\tilde{\tau}}\rangle = \begin{pmatrix} -\tilde{\tilde{\tau}}_+ \\ \tilde{\tilde{\tau}}_- \end{pmatrix} = \begin{pmatrix} -e^{-\frac{\kappa \tilde{N}_0}{4}} \tilde{\tilde{\zeta}}_1^\kappa \\ e^{-\frac{\kappa \tilde{N}_1}{4}} \tilde{\tilde{\zeta}}_0^\kappa \end{pmatrix}. \end{aligned} \quad (11)$$

The norms are $\langle t|t\rangle = \langle \tau|\tau\rangle = \frac{2}{\kappa} \sinh \frac{\kappa}{2} (N_0 + N_1)$, and similarly $\langle \tilde{t}|\tilde{t}\rangle = \langle \tilde{\tau}|\tilde{\tau}\rangle = \frac{2}{\kappa} \sinh \frac{\kappa}{2} (\tilde{N}_0 + \tilde{N}_1)$ (the norm of a dual is the same since $\epsilon^\dagger = \epsilon^{-1}$). They match if the so-called norm matching condition holds, which is just $N_0 + N_1 \cong \tilde{N}_0 + \tilde{N}_1$.

Holonomies and fluxes can be reconstructed as follows:

$$\begin{aligned} \ell &= \begin{pmatrix} \exp(\frac{\kappa}{4}(N_1 - N_0)) & 0 \\ -\kappa \tilde{\zeta}_0^\kappa \tilde{\zeta}_1^\kappa & \exp(\frac{\kappa}{4}(N_0 - N_1)) \end{pmatrix}, & \tilde{\ell} &= \begin{pmatrix} \exp(\frac{\kappa}{4}(\tilde{N}_0 - \tilde{N}_1)) & 0 \\ \kappa \tilde{\tilde{\zeta}}_0^\kappa \tilde{\tilde{\zeta}}_1^\kappa & \exp(\frac{\kappa}{4}(\tilde{N}_1 - \tilde{N}_0)) \end{pmatrix}, \\ u &= \frac{|\tau\rangle \langle \tilde{t}| - |\tilde{\tau}\rangle \langle t|}{\sqrt{\langle \tau|\tau\rangle \langle \tilde{t}|\tilde{t}\rangle}}, & \tilde{u} &= \frac{|\tilde{t}\rangle \langle \tilde{\tau}| - |\tilde{\tilde{\tau}}\rangle \langle \tilde{t}\rangle}{\sqrt{\langle \tilde{t}|\tilde{t}\rangle \langle \tilde{\tilde{\tau}}|\tilde{\tilde{\tau}}\rangle}}, \end{aligned} \quad (12)$$

with $N_0 + N_1 = \tilde{N}_0 + \tilde{N}_1$. It is straightforward to check that the deformed spinors are related to one another by parallel transport via fluxes and holonomies.

$$|\tau\rangle = e^{-\frac{\kappa(N_0+N_1)}{4}} \ell^{-1} |t\rangle, \quad |\tilde{\tau}\rangle = e^{\frac{\kappa(\tilde{N}_0+\tilde{N}_1)}{4}} \tilde{\ell} |\tilde{t}\rangle, \quad u |\tilde{t}\rangle = |\tau\rangle, \quad \tilde{u} |\tilde{\tau}\rangle = |\tilde{t}\rangle. \quad (13)$$

Those relations have a natural graphical interpretation: the spinors can be assigned to the corners of the ribbon edge. For instance, $|\tilde{t}\rangle$ is at the source end of the long edge carrying u and $|\tau\rangle$ is at its target end. This is depicted in Fig. 4 (we do not include the factors $e^{-\frac{\kappa(N_0+N_1)}{4}}$ and $e^{\frac{\kappa(\tilde{N}_0+\tilde{N}_1)}{4}}$ in the graphical representation).

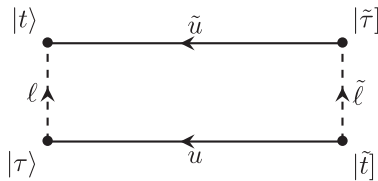


FIG. 4. The ribbon edge with the holonomies on its long edges, fluxes on its short edges and spinors on its corners.

To avoid writing separate equations for $|t\rangle$ and $|\tilde{t}\rangle$, we introduce the following notations:

$$\begin{aligned} t_A^- &:= (-1)^{\frac{1}{2}+A} t_A, & \tilde{t}_A^- &:= (-1)^{\frac{1}{2}+A} \tilde{t}_A, \\ t_A^+ &:= \tilde{t}_{-A}, & \tilde{t}_A^+ &:= \tilde{\tilde{t}}_{-A}, \end{aligned} \quad (14)$$

for $A = \pm 1/2$. Similar but exchanged notations are used for τ and $\tilde{\tau}$,

$$\begin{aligned} \tau_A^- &:= \tau_A, & \tilde{\tau}_A^- &:= \tilde{\tau}_A, \\ \tau_A^+ &:= (-1)^{\frac{1}{2}-A} \tau_{-A}, & \tilde{\tau}_A^+ &:= (-1)^{\frac{1}{2}-A} \tilde{\tau}_{-A}. \end{aligned} \quad (15)$$

For reference, we explicitly write the spinors and dual spinors with those notations in a footnote.² The norms read

$$\begin{aligned}\langle t|t\rangle &= \frac{1}{2} \sum_{\epsilon=\pm} \sum_{A=\pm\frac{1}{2}} \epsilon(-1)^{\frac{1}{2}+A} t_A^\epsilon t_{-A}^{-\epsilon}, & \langle \tilde{t}|\tilde{t}\rangle &= \frac{1}{2} \sum_{\epsilon=\pm} \sum_{A=\pm\frac{1}{2}} \epsilon(-1)^{\frac{1}{2}+A} \tilde{t}_A^\epsilon \tilde{t}_{-A}^{-\epsilon}, \\ \langle \tau|\tau\rangle &= \frac{1}{2} \sum_{\epsilon=\pm} \sum_{A=\pm\frac{1}{2}} \epsilon(-1)^{\frac{1}{2}+A} \tau_A^\epsilon \tau_{-A}^{-\epsilon}, & \langle \tilde{\tau}|\tilde{\tau}\rangle &= \frac{1}{2} \sum_{\epsilon=\pm} \sum_{A=\pm\frac{1}{2}} \epsilon(-1)^{\frac{1}{2}+A} \tilde{\tau}_A^\epsilon \tilde{\tau}_{-A}^{-\epsilon},\end{aligned}\quad (17)$$

and the holonomies

$$u_{AB} = -\frac{1}{\sqrt{\langle \tau|\tau\rangle \langle \tilde{t}|\tilde{t}\rangle}} \sum_{\epsilon=\pm} \epsilon \tau_A^\epsilon \tilde{t}_{-B}^{-\epsilon}, \quad \tilde{u}_{AB}^{-1} = \frac{1}{\sqrt{\langle \tilde{\tau}|\tilde{\tau}\rangle \langle t|t\rangle}} \sum_{\epsilon=\pm} \epsilon \tilde{\tau}_A^\epsilon t_{-B}^{-\epsilon}.\quad (18)$$

B. Gauge invariant quantities

We have described the spinors associated to an edge. Consider now two edges e_1, e_2 meeting at a vertex v and incident to the same corner of Γ . The ribbon edges $R(e_1)$ and $R(e_2)$ share a corner where we have a spinor of e_1 and a spinor of e_2 . Their scalar product is gauge invariant. Given fixed orientations of the edges, there are four possible products (each spinor and its dual). There are moreover four configurations of orientations, shown in Fig. 5. For instance, the four scalar products for the bottom-right configuration are $\langle t_2|t_1\rangle$, $\langle t_2|\tau_1\rangle$, $[t_2|\tau_1]$, and $[t_2|t_1]$.

Using the notations (14) and (15), we can give a uniform expression to the four scalar products at fixed orientations. For example, when both edges are outgoing,

$$E_{e_2 e_1}^{\epsilon_2, \epsilon_1} := \sum_{A=\pm 1/2} \tau_{2,-A}^{\epsilon_2} t_{1,A}^{\epsilon_1} = \begin{cases} \langle t_2|t_1\rangle & \text{for } \epsilon_1 = +, \epsilon_2 = + \\ \langle t_2|\tau_1\rangle & \text{for } \epsilon_1 = -, \epsilon_2 = + \\ [t_2|\tau_1] & \text{for } \epsilon_1 = +, \epsilon_2 = - \\ [t_2|t_1] & \text{for } \epsilon_1 = -, \epsilon_2 = - \end{cases}.\quad (19)$$

²

$$\begin{aligned}|t\rangle &= \begin{pmatrix} t_-^- \\ -t_+^- \end{pmatrix}, & |t\rangle &= \begin{pmatrix} -t_+^+ \\ t_+^+ \end{pmatrix}, & |\tilde{t}\rangle &= \begin{pmatrix} \tilde{t}_-^- \\ -\tilde{t}_+^- \end{pmatrix}, & |\tilde{t}\rangle &= \begin{pmatrix} -\tilde{t}_+^+ \\ \tilde{t}_+^+ \end{pmatrix}, \\ \langle t| &= (t_+^+, t_+^-), & [t| &= (t_+^-, t_+^-), & \langle \tilde{t}| &= (\tilde{t}_+^+, \tilde{t}_+^-), & [\tilde{t}| &= (\tilde{t}_+^-, \tilde{t}_+^-), \\ |\tau\rangle &= \begin{pmatrix} \tau_-^- \\ \tau_+^- \end{pmatrix}, & |\tau\rangle &= \begin{pmatrix} \tau_+^+ \\ \tau_+^- \end{pmatrix}, & |\tilde{\tau}\rangle &= \begin{pmatrix} \tilde{\tau}_-^- \\ \tilde{\tau}_+^- \end{pmatrix}, & |\tilde{\tau}\rangle &= \begin{pmatrix} \tilde{\tau}_+^+ \\ \tilde{\tau}_+^- \end{pmatrix}, \\ \langle \tau| &= (\tau_+^+, -\tau_+^-), & [\tau| &= (-\tau_+^-, \tau_+^-), & \langle \tilde{\tau}| &= (\tilde{\tau}_+^+, -\tilde{\tau}_+^-), & [\tilde{\tau}| &= (-\tilde{\tau}_+^-, \tilde{\tau}_+^-),\end{aligned}\quad (16)$$

where the subscripts $A = \pm \frac{1}{2}$ have been notated as $A = \pm$ for simplicity.

The other orientations are obtained by changing τ_1 to $\tilde{\tau}_1$ and t_2 to \tilde{t}_2 and the invariant is still denoted $E_{e_2, e_1}^{\epsilon_2, \epsilon_1}$.

It will be convenient to encode all the orientations and have a fully uniform way of writing the invariant. We orient the corner between e_1 and e_2 counterclockwise. We say that the orientation o_i of e_i for $i = 1, 2$ is positive if it matches that of the corner, and negative otherwise. We denote the spinors meeting there as $t_{e_1 v}$ and $t_{e_2 v}$ according to

$$\begin{array}{c|c} & t_{e_1 v} \\ \hline o_1 = 1 & \tilde{\tau}_1 \\ \hline o_1 = -1 & \tau_1 \end{array} \quad \begin{array}{c|c} & t_{e_2 v} \\ \hline o_2 = 1 & t_2 \\ \hline o_2 = -1 & \tilde{t}_2 \end{array} \quad (20)$$

so that

$$E_{e_2 e_1}^{\epsilon_2, \epsilon_1} = \begin{cases} \langle t_{e_2 v}|t_{e_1 v}\rangle & \text{for } \epsilon_1 = +, \epsilon_2 = + \\ \langle t_{e_2 v}|t_{e_1 v}\rangle & \text{for } \epsilon_1 = -, \epsilon_2 = + \\ [t_{e_2 v}|t_{e_1 v}] & \text{for } \epsilon_1 = +, \epsilon_2 = - \\ [t_{e_2 v}|t_{e_1 v}] & \text{for } \epsilon_1 = -, \epsilon_2 = - \end{cases}.\quad (21)$$

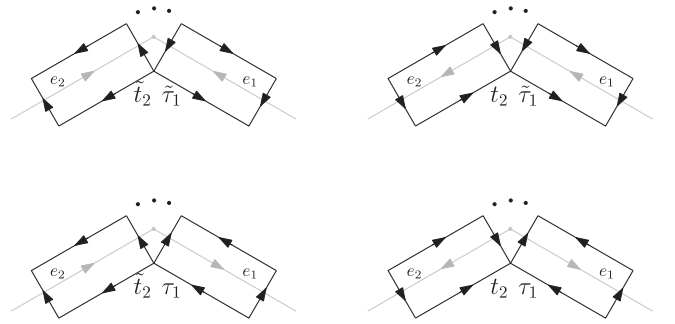


FIG. 5. Two edges meet at a vertex and share a corner. There are four configurations of orientations and we indicate the spinors at the common corner.

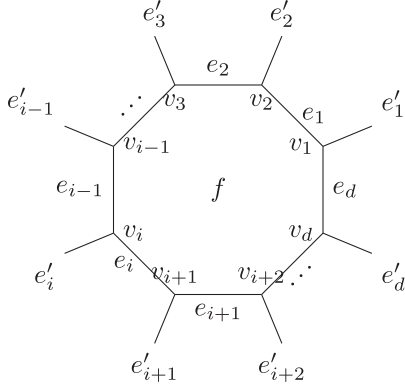


FIG. 6. A sunny graph with edges e_1, \dots, e_d counterclockwise oriented around the face f . Each triple of edges (e_i, e_{i-1}, e'_i) are incident to a vertex v_i .

C. Hamiltonian constraint

By plugging u and \tilde{u} from (12) into the flatness constraint (8), one obtains a spinorial expression of the constraint. By then taking the matrix elements of the constraints between different spinors, we get some scalar constraints that we call the Hamiltonian constraints. They are the κ -deformed versions of [10].

We first write the Hamiltonian constraints generally, i.e., on faces of arbitrary lengths, then specialize them to the case of faces of length 3.

1. The Hamiltonian on a face of arbitrary degree

Let f be a face of length d . We will introduce a constraint, derived from the flatness constraint, for every pair of edges (e, e') around f . Label the edges counterclockwise around f as e_1, \dots, e_d . Without loss of generality, we set the pair (e, e') , which labels our function to (e_1, e_k) for $k \in \{2, \dots, d\}$. Label the vertices around f as v_1, \dots, v_d counterclockwise, such that e_i is incident to v_i and v_{i+1} , for $i = 1, \dots, d \bmod d$, as shown in Fig. 6. We assume that f visits each vertex and edge exactly once (as when Γ is dual to a simplicial complex), so that all e_i s and v_i s are distinct.

By convention, we denote the orientation $o_i = 1$ if e_i is counterclockwise and $o_i = -1$ otherwise (this is the relative orientation of the edge with respect to the counterclockwise orientation of f). With the notation u_{e_v} introduced earlier, the flatness constraint reads $u_{e_d v_1} \cdots u_{e_2 v_3} u_{e_1 v_2} = \mathbb{1}$

in $SU(2)$. In order to simplify the notations a bit, we will use

$$u_{e_i f} := u_{e_i v_{i+1}} = \begin{cases} \tilde{u}_{e_i}^{-1}, & \text{if } o_i = 1 \\ u_{e_i}, & \text{if } o_i = -1 \end{cases} \quad (22)$$

Furthermore we denote $t_{e_i v_i}$ the spinor along the long edge of $R(e_i)$, which is incident to both f and v_i . It is determined by the orientation of e_i ,

$$\begin{aligned} o_i = 1 &\Rightarrow t_{e_i v_i} = t_{e_i} \quad \text{and} \quad t_{e_i v_{i+1}} = \tilde{\tau}_{e_i}, \\ o_i = -1 &\Rightarrow t_{e_i v_i} = \tilde{\tau}_{e_i} \quad \text{and} \quad t_{e_i v_{i+1}} = \tau_{e_i}. \end{aligned} \quad (23)$$

Notice that we can combine the parallel transport relations (13) with the notations (14), (15) to relate the spinors which are on both ends of the long edge of e_i incident to f ,

$$\begin{aligned} t_{e_i v_i, -A}^{\epsilon_i} &= -o_i \sum_{B=\pm 1/2} t_{e_i v_{i+1}, -B}^{-\epsilon_i} (-1)^{\frac{1}{2}+B} u_{e_i f, BA}, \\ t_{e_i v_{i+1}, A}^{\epsilon_i} &= o_i \sum_{B=\pm 1/2} u_{e_i f, AB} (-1)^{\frac{1}{2}+B} t_{e_i v_i, B}^{-\epsilon_i}. \end{aligned} \quad (24)$$

The flatness constraint on f is thus $u_{e_d f} \cdots u_{e_1 f} = \mathbb{1}$. Assume momentarily that all edges are counterclockwise. Then, $\tilde{u}_{e_d}^{-1} \cdots \tilde{u}_{e_1}^{-1} = \mathbb{1}$ implies for all k

$$\begin{aligned} [t_{e_k} | \tilde{u}_{e_{k-1}}^{-1} \cdots \tilde{u}_{e_2}^{-1} | \tilde{\tau}_{e_1}] &= [t_{e_k} | \tilde{u}_{e_k} \tilde{u}_{e_{k+1}} \cdots \tilde{u}_{e_d} \tilde{u}_{e_1} | \tilde{\tau}_{e_1}] \\ &= \langle \tilde{\tau}_{e_k} | \tilde{u}_{e_{k+1}} \cdots \tilde{u}_{e_d} | t_{e_1} \rangle. \end{aligned} \quad (25)$$

In the first equality, we have used the constraint itself, while in the second equality we have used the parallel transport relations on the edges e_1 and e_k . Then, by rewriting $\tilde{u}_1, \dots, \tilde{u}_d$ with (12), one obtains the following result: a constraint written as a sum of products of scalar invariants living on the corners around the face. Obviously, one can change $[t_{e_k}]$ to $\langle t_{e_k} |$ and $|\tilde{\tau}_{e_1}]$ to $|\tilde{\tau}_{e_1}]$ without changing that result (qualitatively). Similarly, one should be able to write this function for arbitrarily chosen edge orientations. The notations we have introduced will help us write it in the most generic way.

Going back to arbitrary edge orientations around f , we consider

$$E_{e_1 \rightarrow e_k}^{\epsilon_1, \epsilon_k} = \sum_{A, B=\pm 1/2} t_{e_k v_k, -A}^{\epsilon_k} (u_{e_{k-1} f} \cdots u_{e_2 f})_{AB} t_{e_1 v_2, B}^{\epsilon_1} \quad (26)$$

as the generalization of the left-hand side of (25). Using the parallel transport relations (24), it reads

$$E_{e_1 \rightarrow e_k}^{\epsilon_1, \epsilon_k} = -o_1 o_k \sum_{C, D=\pm 1/2} t_{e_k v_{k+1}, -C}^{-\epsilon_k} (-1)^{\frac{1}{2}-C} (u_{e_k f} u_{e_{k-1} f} \cdots u_{e_2 f} u_{e_1 f})_{CD} (-1)^{\frac{1}{2}-D} t_{e_1 v_1, D}^{-\epsilon_1}. \quad (27)$$

If the flatness constraint holds, the holonomy going counterclockwise from e_1 to e_k can then be replaced with the holonomy the other way around f , i.e., clockwise. We thus define

$$E_{e_1 \leftarrow e_k}^{\epsilon_1, \epsilon_k} = \sum_{A, B = \pm 1/2} t_{e_1 v_1, -A}^{-\epsilon_1} (u_{e_d f} \cdots u_{e_{k+1} f})_{AB} t_{e_k v_{k+1}, B}^{-\epsilon_k}. \quad (28)$$

So if the flatness constraint holds, then

$$E_{e_1 \rightarrow e_k}^{\epsilon_1, \epsilon_k} + o_1 o_k E_{e_1 \leftarrow e_k}^{\epsilon_1, \epsilon_k} = 0. \quad (29)$$

Indeed, using the flatness constraint in (27) we get

$$E_{e_1 \rightarrow e_k}^{\epsilon_1, \epsilon_k} = -o_1 o_k \sum_{C, D = \pm 1/2} t_{e_k v_{k+1}, -C}^{-\epsilon_k} (-1)^{\frac{1}{2}-C} (u_{e_d f} \cdots u_{e_{k+1} f})_{CD}^{-1} (-1)^{\frac{1}{2}-D} t_{e_1 v_1, D}^{-\epsilon_1}. \quad (30)$$

For any matrix $g \in \text{SU}(2)$, the matrix elements of the inverse can be written $g_{CD}^{-1} = (-1)^{\frac{1}{2}-D} g_{-D-C} (-1)^{\frac{1}{2}-C}$. This can be used to transform the above expression into $o_1 o_k E_{e_1 \leftarrow e_k}^{\epsilon_1, \epsilon_k}$. The last step to define our Hamiltonian constraints is to rewrite $E_{e_1 \rightarrow e_k}^{\epsilon_1, \epsilon_k}$ and $E_{e_1 \leftarrow e_k}^{\epsilon_1, \epsilon_k}$ in terms of scalars like (21). The matrix elements of the holonomies are indeed

$$u_{e_{if}, A_{i+1} A_i} = o_i \frac{1}{N_{e_i}} \sum_{\epsilon_i = \pm} \epsilon_i t_{e_i v_{i+1}, A_{i+1}}^{\epsilon_i} t_{e_i v_i, -A_i}^{\epsilon_i}, \quad (31)$$

$$\text{with } N_{e_i} = \frac{1}{2} \sqrt{\sum_{\epsilon_i, \epsilon'_i = \pm} \sum_{A, B = \pm 1/2} \epsilon \epsilon' (-1)^{\frac{1}{2}-A} (-1)^{\frac{1}{2}-B} t_{e_i v_{i+1}, A}^{\epsilon_i} t_{e_i v_{i+1}, -A}^{-\epsilon_i} t_{e_i v_i, B}^{\epsilon'_i} t_{e_i v_i, -B}^{-\epsilon'_i}}, \quad (32)$$

so that one can reorganize the products over the vertices instead of edges,

$$E_{e_1 \rightarrow e_k}^{\epsilon_1, \epsilon_k} = \sum_{\substack{\epsilon_2, \dots, \epsilon_{k-1} = \pm \\ A_2, \dots, A_{k-1} = \pm 1/2}} \left(\prod_{i=2}^{k-1} \frac{O_i \epsilon_i}{N_{e_i}} \right) \left(\prod_{i=2}^k t_{e_i v_i, -A_i}^{\epsilon_i} t_{e_{i-1} v_i, A_i}^{\epsilon_{i-1}} \right), \quad (33)$$

$$E_{e_1 \leftarrow e_k}^{\epsilon_1, \epsilon_k} = (-1)^{d-k} \sum_{\substack{\epsilon_{k+1}, \dots, \epsilon_d = \pm \\ A_{k+1}, \dots, A_{d+1} = \pm 1/2}} \left(\prod_{i=k+1}^d \frac{O_i \epsilon_i}{N_{e_i}} \right) \left(\prod_{i=k+1}^{d+1} t_{e_i v_i, -A_i}^{-\epsilon_i} t_{e_{i-1} v_i, A_i}^{-\epsilon_{i-1}} \right). \quad (34)$$

We can now use the quadratic invariants defined in (21), $E_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}} = \sum_{A = \pm 1/2} t_{e_i v_i, -A}^{\epsilon_i} t_{e_{i-1} v_i, A}^{\epsilon_{i-1}}$, which encodes all four scalar products of the two spinors meeting at v_i , i.e.,

$$E_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}} = \begin{cases} \langle t_{e_i v_i} | t_{e_{i-1} v_i} \rangle & \text{for } \epsilon_i = +, \epsilon_{i-1} = + \\ \langle t_{e_i v_i} | t_{e_{i-1} v_i} \rangle & \text{for } \epsilon_i = +, \epsilon_{i-1} = - \\ [t_{e_i v_i} | t_{e_{i-1} v_i}] & \text{for } \epsilon_i = -, \epsilon_{i-1} = + \\ [t_{e_i v_i} | t_{e_{i-1} v_i}] & \text{for } \epsilon_i = -, \epsilon_{i-1} = - \end{cases}, \quad (35)$$

where the spinors $t_{e_i v_i}$ and $t_{e_{i-1} v_{i-1}}$ are given by the rule (23) according to the orientations. This leads us to the following definition of the Hamiltonian constraints.

Definition 3.1. Let f be a face of length d , with edges labeled by e_1, \dots, e_d counterclockwise around f . A Hamiltonian is associated to f and a pair of edges along f with a sign attached to each of them. Without loss of generality, the pair can be chosen to be (e_1, e_k) with signs $(\epsilon_1, \epsilon_k) \in \{+, -\}^2$, for $k \in \{2, \dots, d\}$, and the Hamiltonian is

$$h_{f, e_1, e_k}^{\epsilon_1, \epsilon_k} = \sum_{\epsilon_2, \dots, \epsilon_{k-1} = \pm} \left(\prod_{i=2}^k \frac{O_i \epsilon_i}{N_{e_i}} E_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}} \right) + (-1)^{d-k} \epsilon_1 \epsilon_k \frac{N_{e_1}}{N_{e_k}} \sum_{\epsilon_{k+1}, \dots, \epsilon_d = \pm} \left(\prod_{i=k+1}^{d+1} \frac{O_i \epsilon_i}{N_{e_i}} E_{e_i e_{i-1}}^{-\epsilon_i, -\epsilon_{i-1}} \right). \quad (36)$$

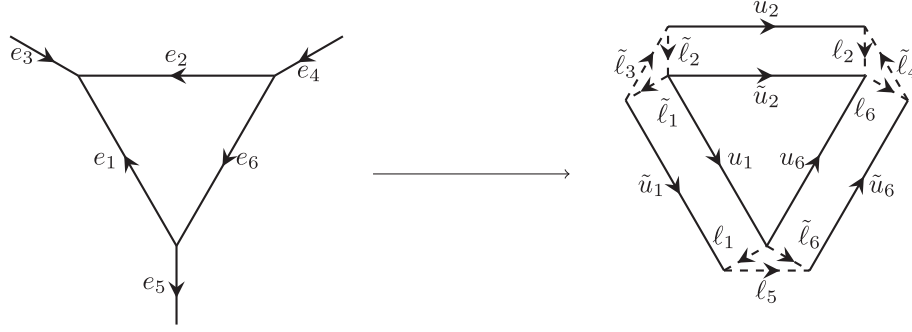


FIG. 7. On the left, a triangular face with its adjacent edges. On the right, the ribbon graph it gives rise to.

The Hamiltonian constraint (36) captures the flatness constraint completely with all choices of pairs (e_1, e_k) and of signs (ϵ_1, ϵ_k) . The proof is the same as in the vector case at $\kappa = 0$, see Ref. [9].

2. Application to faces of degree three

Let us discuss more explicitly the case of triangular faces. We use the notations and orientations of Fig. 7 as an example. In particular $o_1 = o_6 = -1$ and $o_2 = 1$. Here there are three choices of pairs of edges [which label the Hamiltonians (36)], which correspond to the three corners of the face.

On a corner, say between the edges e_2 and e_6 , there are four invariant quantities quadratic in the spinors, $\langle t_2 | \tau_6 \rangle, \langle t_2 | \tau_6 \rangle, [t_2 | \tau_6], [t_2 | \tau_6]$, which are encoded in the scalar product (21)

$$E_{26}^{\epsilon_2, \epsilon_6} = \sum_{A=\pm 1/2} t_{2,-A}^{\epsilon_2} \tau_{6,A}^{\epsilon_6} = \begin{cases} \langle t_2 | \tau_6 \rangle & \text{for } \epsilon_2 = \epsilon_6 = + \\ \langle t_2 | \tau_6 \rangle & \text{for } \epsilon_2 = -\epsilon_6 = + \\ [t_2 | \tau_6] & \text{for } \epsilon_2 = -\epsilon_6 = - \\ [t_2 | \tau_6] & \text{for } \epsilon_2 = \epsilon_6 = - \end{cases}. \quad (37)$$

Similarly at the corners between e_1, e_2 and e_6, e_1 ,

$$E_{12}^{\epsilon_1, \epsilon_2} = \sum_{A=\pm 1/2} \tilde{t}_{1,-A}^{\epsilon_1} \tilde{\tau}_{2,A}^{\epsilon_2}, \quad E_{61}^{\epsilon_6, \epsilon_1} = \sum_{A=\pm 1/2} \tilde{t}_{6,-A}^{\epsilon_6} \tilde{\tau}_{1,A}^{\epsilon_1}. \quad (38)$$

The flatness constraint $\tilde{u}_2 u_1^{-1} u_6^{-1} = \mathbb{I}$ implies that if $|\tau_6\rangle$ and $|\tau_6\rangle$ are transported around the face via $\tilde{u}_2 u_1^{-1} u_6^{-1}$, the above quadratic quantities are left unchanged, that is

$$\begin{aligned} \langle t_2 | \tilde{u}_2 u_1^{-1} u_6^{-1} | \tau_6 \rangle &= \langle t_2 | \tau_6 \rangle, & \langle t_2 | \tilde{u}_2 u_1^{-1} u_6^{-1} | \tau_6 \rangle &= \langle t_2 | \tau_6 \rangle, \\ [t_2 | \tilde{u}_2 u_1^{-1} u_6^{-1} | \tau_6 \rangle &= [t_2 | \tau_6 \rangle, & [t_2 | \tilde{u}_2 u_1^{-1} u_6^{-1} | \tau_6 \rangle &= [t_2 | \tau_6 \rangle. \end{aligned} \quad (39)$$

Similarly at the corners where e_1, e_2 and e_6, e_1 meet, respectively,

$$\begin{aligned} \langle \tilde{t}_1 | u_1^{-1} u_6^{-1} \tilde{u}_2 | \tilde{\tau}_2 \rangle &= \langle \tilde{t}_1 | \tilde{\tau}_2 \rangle, & \langle \tilde{t}_6 | u_6^{-1} \tilde{u}_2 u_1^{-1} | \tau_1 \rangle &= \langle \tilde{t}_6 | \tau_1 \rangle, \\ \langle \tilde{t}_1 | u_1^{-1} u_6^{-1} \tilde{u}_2 | \tilde{\tau}_2 \rangle &= \langle \tilde{t}_1 | \tilde{\tau}_2 \rangle, & \langle \tilde{t}_6 | u_6^{-1} \tilde{u}_2 u_1^{-1} | \tau_1 \rangle &= \langle \tilde{t}_6 | \tau_1 \rangle, \\ [\tilde{t}_1 | u_1^{-1} u_6^{-1} \tilde{u}_2 | \tilde{\tau}_2 \rangle &= [\tilde{t}_1 | \tilde{\tau}_2 \rangle, & [\tilde{t}_6 | u_6^{-1} \tilde{u}_2 u_1^{-1} | \tau_1 \rangle &= [\tilde{t}_6 | \tau_1 \rangle, \\ [\tilde{t}_1 | u_1^{-1} u_6^{-1} \tilde{u}_2 | \tilde{\tau}_2 \rangle &= [\tilde{t}_1 | \tilde{\tau}_2 \rangle, & [\tilde{t}_6 | u_6^{-1} \tilde{u}_2 u_1^{-1} | \tau_1 \rangle &= [\tilde{t}_6 | \tau_1 \rangle. \end{aligned} \quad (40)$$

In fact, this set of constraints simply amounts to rewriting the constraint $\tilde{u}_2 u_1^{-1} u_6^{-1} = \mathbb{I}$ in the coherent state basis. Therefore, as long as those vectors are generic (hence linearly independent), this whole set is equivalent to $\tilde{u}_2 u_1^{-1} u_6^{-1} = \mathbb{I}$.

Let us consider the constraint $\langle t_2 | \tilde{u}_2 u_1^{-1} u_6^{-1} | \tau_6 \rangle - \langle t_2 | \tau_6 \rangle$ and rewrite it like in (36). Use the parallel transport relations $u_6^{-1} | \tau_6 \rangle = -| \tilde{t}_6 \rangle$ and $\langle t_2 | \tilde{u}_2 = \langle \tilde{\tau}_2 |$, which gives $\langle t_2 | \tilde{u}_2 u_1^{-1} u_6^{-1} | \tau_6 \rangle = -\langle \tilde{\tau}_2 | u_1^{-1} | \tilde{t}_6 \rangle = [\tilde{t}_6 | u_1 | \tilde{\tau}_2 \rangle$. We then use $u_1 = \frac{1}{N_{e_1}} (|\tau_1\rangle \langle \tilde{t}_1| - |\tau_1\rangle \langle \tilde{t}_1|)$ so that the constraint becomes

$$\begin{aligned} \langle t_2 | \tilde{u}_2 u_1^{-1} u_6^{-1} | \tau_6 \rangle - \langle t_2 | \tau_6 \rangle &= \frac{1}{N_{e_1}} (\langle \tilde{t}_6 | \tau_1 \rangle \langle \tilde{t}_1 | \tilde{\tau}_2 \rangle - \langle \tilde{t}_6 | \tau_1 \rangle \langle \tilde{t}_1 | \tilde{\tau}_2 \rangle) - \langle t_2 | \tau_6 \rangle, \\ &= -\frac{1}{N_{e_1}} \sum_{\epsilon=\pm} \epsilon E_{61}^{\epsilon} E_{12}^{\epsilon, -} - E_{26}^{+, +}, \end{aligned} \quad (41)$$

which is exactly the specialization of (36) to $d = 3, k = 2, o_1 = o_6 = -o_2 = -1$, and $\epsilon_2 = \epsilon_6 = +$,

$$h_{e_2 e_6}^{\epsilon_2, \epsilon_6} = E_{26}^{\epsilon_2, \epsilon_6} + \frac{1}{N_{e_1}} \sum_{\epsilon_1=\pm} \epsilon_1 E_{12}^{\epsilon_1, -\epsilon_2} E_{61}^{-\epsilon_6, \epsilon_1}, \quad (42)$$

where we recall that $N_{e_1} = \sqrt{\langle \tau_1 | \tau_1 \rangle \langle \tilde{t}_1 | \tilde{t}_1 \rangle}$.

This way, the Hamiltonian constraint does not involve holonomy variables anymore like in (39), but only the quadratic invariants of spinors.

IV. QUANTUM HAMILTONIAN CONSTRAINT

We now proceed to the quantization of the system. The aim is to quantize the Hamiltonian constraints (36) and

solve them at the quantum level. This requires quantizing the quadratic invariant $E_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$. It has been constructed consistently with the quantization of holonomies and fluxes in the companion paper [31]. Therefore, we start with recalling the main ingredients before proceeding to the construction of the quantum Hamiltonian.

A. Quantum deformed spinors

Let $q := e^{\hbar\kappa}$ and denote $[n] := \frac{q^n - q^{-n}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ the q numbers. The κ -deformed spinors are quantized as q bosons (in the same way they are at $\kappa = 0$ [34]). Consider two independent pairs of q -boson operators (a, a^\dagger) and (b, b^\dagger) satisfying the relations

$$\begin{aligned} aa^\dagger - q^{\mp\frac{1}{2}}a^\dagger a &= q^{\pm\frac{N_a}{2}}, & a^\dagger a - q^{\pm\frac{1}{2}}aa^\dagger &= -q^{\pm\frac{N_a+1}{2}}, & [N_a, a^\dagger] &= a^\dagger, & [N_a, a] &= -a, \\ bb^\dagger - q^{\mp\frac{1}{2}}b^\dagger b &= q^{\pm\frac{N_b}{2}}, & b^\dagger b - q^{\pm\frac{1}{2}}bb^\dagger &= -q^{\pm\frac{N_b+1}{2}}, & [N_b, b^\dagger] &= b^\dagger, & [N_b, b] &= -b, \end{aligned} \quad (43)$$

with $a^\dagger a = [N_a]$, $aa^\dagger = [N_a + 1]$, $b^\dagger b = [N_b]$, $bb^\dagger = [N_b + 1]$. We furthermore introduce two other independent pairs of q bosons denoted $(\tilde{a}, \tilde{a}^\dagger)$ and $(\tilde{b}, \tilde{b}^\dagger)$, and satisfying the same relations as above. The tilde and nontilde operators are chosen to commute with each other.

We will then use the following quantization map

$$\begin{aligned} (\zeta_0^\kappa, \zeta_1^\kappa) &\rightarrow (a, b), & (\bar{\zeta}_0^\kappa, \bar{\zeta}_1^\kappa) &\rightarrow (a^\dagger, b^\dagger), & (N_0, N_1) &\rightarrow (N_a, N_b), \\ (\tilde{\zeta}_0^\kappa, \tilde{\zeta}_1^\kappa) &\rightarrow (\tilde{a}, \tilde{b}), & (\tilde{\bar{\zeta}}_0^\kappa, \tilde{\bar{\zeta}}_1^\kappa) &\rightarrow (\tilde{a}^\dagger, \tilde{b}^\dagger), & (\tilde{N}_0, \tilde{N}_1) &\rightarrow (\tilde{N}_a, \tilde{N}_b). \end{aligned} \quad (44)$$

In previous works by the first and second authors, the fluxes ℓ and $\tilde{\ell}$ had been quantized in terms of the quantum algebra $\mathcal{U}_q(\mathfrak{su}(2))$. This can be replicated in a manner which is consistent with the q bosons. Indeed, the Jordan map builds generators of $\mathcal{U}_q(\mathfrak{su}(2))$ out of the above q bosons,

$$\begin{aligned} J_+ &= a^\dagger b, & J_- &= ab^\dagger, & K &= q^{\frac{J_z}{2}} = q^{\frac{N_a - N_b}{4}}, \\ \tilde{J}_+ &= \tilde{a}^\dagger \tilde{b}, & \tilde{J}_- &= \tilde{a} \tilde{b}^\dagger, & \tilde{K} &= q^{\frac{\tilde{J}_z}{2}} = q^{\frac{\tilde{N}_a - \tilde{N}_b}{4}}, \end{aligned} \quad (45)$$

where $J_\pm, K \equiv q^{\frac{J_z}{2}}$ and $\tilde{J}_\pm, \tilde{K} \equiv q^{\frac{\tilde{J}_z}{2}}$ are two independent copies of the $\mathcal{U}_q(\mathfrak{su}(2))$ generators satisfying the relations

$$KJ_\pm K^{-1} = q^{\pm\frac{1}{2}}J_\pm, \quad [J_+, J_-] = [2J_z], \quad \tilde{K}\tilde{J}_\pm\tilde{K}^{-1} = q^{\pm\frac{1}{2}}\tilde{J}_\pm, \quad [\tilde{J}_+, \tilde{J}_-] = [2\tilde{J}_z], \quad (46)$$

and others vanish. Performing the quantization as follows,

$$\begin{aligned} \ell &= \begin{pmatrix} \exp(\frac{\kappa}{4}(N_1 - N_0)) & 0 \\ -\kappa \bar{\zeta}_0^\kappa \zeta_1^\kappa & \exp(\frac{\kappa}{4}(N_0 - N_1)) \end{pmatrix} \longrightarrow L = \begin{pmatrix} K^{-1} & 0 \\ -q^{\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_+ & K \end{pmatrix}, \\ \tilde{\ell} &= \begin{pmatrix} \exp(\frac{\kappa}{4}(\tilde{N}_0 - \tilde{N}_1)) & 0 \\ \kappa \tilde{\bar{\zeta}}_0^\kappa \tilde{\zeta}_1^\kappa & \exp(\frac{\kappa}{4}(\tilde{N}_1 - \tilde{N}_0)) \end{pmatrix} \longrightarrow \tilde{L} = \begin{pmatrix} \tilde{K} & 0 \\ q^{-\frac{1}{4}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\tilde{J}_+ & \tilde{K}^{-1} \end{pmatrix}, \end{aligned} \quad (47)$$

one finds, as in [31], that $L \in F_{q^{-1}}(\text{AN}(2)) \cong \mathcal{U}_{q^{-1}}(\mathfrak{su}(2))$ and $\tilde{L} \in F_q(\text{AN}(2)) \cong \mathcal{U}_q(\mathfrak{su}(2))$ [35].

The map (44) quantizes the κ -deformed spinors (11) as follows:

$$\begin{aligned}
|t\rangle &= \begin{pmatrix} e^{\frac{\kappa N_1}{4}} \zeta_0^\kappa \\ e^{-\frac{\kappa N_0}{4}} \zeta_1^\kappa \end{pmatrix} \rightarrow \mathbf{t}^- = \begin{pmatrix} \mathbf{t}_+^- \\ \mathbf{t}_-^- \end{pmatrix} = \begin{pmatrix} q^{\frac{N_b}{4}} a \\ q^{-\frac{N_a}{4}} b \end{pmatrix}, & |t\rangle &= \begin{pmatrix} -e^{-\frac{\kappa N_0}{4}} \bar{\zeta}_1^\kappa \\ e^{\frac{\kappa N_1}{4}} \bar{\zeta}_0^\kappa \end{pmatrix} \rightarrow \mathbf{t}^+ = \begin{pmatrix} \mathbf{t}_+^+ \\ \mathbf{t}_-^+ \end{pmatrix} = \begin{pmatrix} -b^\dagger q^{\frac{N_{a+1}}{4}} \\ a^\dagger q^{\frac{N_{b+1}}{4}} \end{pmatrix}, \\
|\tau\rangle &= \begin{pmatrix} e^{-\frac{\kappa N_1}{4}} \zeta_0^\kappa \\ e^{\frac{\kappa N_0}{4}} \zeta_1^\kappa \end{pmatrix} \rightarrow \boldsymbol{\tau}^- = \begin{pmatrix} \boldsymbol{\tau}_+^- \\ \boldsymbol{\tau}_-^- \end{pmatrix} = \begin{pmatrix} q^{-\frac{N_b}{4}} a \\ q^{\frac{N_a}{4}} b \end{pmatrix}, & |\tau\rangle &= \begin{pmatrix} -e^{\frac{\kappa N_0}{4}} \bar{\zeta}_1^\kappa \\ e^{-\frac{\kappa N_1}{4}} \bar{\zeta}_0^\kappa \end{pmatrix} \rightarrow \boldsymbol{\tau}^+ = \begin{pmatrix} \boldsymbol{\tau}_+^+ \\ \boldsymbol{\tau}_-^+ \end{pmatrix} = \begin{pmatrix} -b^\dagger q^{\frac{N_{a+1}}{4}} \\ a^\dagger q^{\frac{N_{b+1}}{4}} \end{pmatrix}, \\
|\tilde{t}\rangle &= \begin{pmatrix} e^{\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \\ e^{-\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \end{pmatrix} \rightarrow \tilde{\mathbf{t}}^- = \begin{pmatrix} \tilde{\mathbf{t}}_+^- \\ \tilde{\mathbf{t}}_-^- \end{pmatrix} = \begin{pmatrix} q^{\frac{\tilde{N}_b}{4}} \tilde{a} \\ q^{-\frac{\tilde{N}_a}{4}} \tilde{b} \end{pmatrix}, & |\tilde{t}\rangle &= \begin{pmatrix} -e^{-\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \\ e^{\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \end{pmatrix} \rightarrow \tilde{\mathbf{t}}^+ = \begin{pmatrix} \tilde{\mathbf{t}}_+^+ \\ \tilde{\mathbf{t}}_-^+ \end{pmatrix} = \begin{pmatrix} -\tilde{b}^\dagger q^{\frac{\tilde{N}_{a+1}}{4}} \\ \tilde{a}^\dagger q^{\frac{\tilde{N}_{b+1}}{4}} \end{pmatrix}, \\
|\tilde{\tau}\rangle &= \begin{pmatrix} e^{-\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \\ e^{\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \end{pmatrix} \rightarrow \tilde{\boldsymbol{\tau}}^- = \begin{pmatrix} \tilde{\boldsymbol{\tau}}_+^- \\ \tilde{\boldsymbol{\tau}}_-^- \end{pmatrix} = \begin{pmatrix} q^{-\frac{\tilde{N}_b}{4}} \tilde{a} \\ q^{\frac{\tilde{N}_a}{4}} \tilde{b} \end{pmatrix}, & |\tilde{\tau}\rangle &= \begin{pmatrix} -e^{\frac{\kappa \tilde{N}_0}{4}} \tilde{\zeta}_1^\kappa \\ e^{-\frac{\kappa \tilde{N}_1}{4}} \tilde{\zeta}_0^\kappa \end{pmatrix} \rightarrow \tilde{\boldsymbol{\tau}}^+ = \begin{pmatrix} \tilde{\boldsymbol{\tau}}_+^+ \\ \tilde{\boldsymbol{\tau}}_-^+ \end{pmatrix} = \begin{pmatrix} -\tilde{b}^\dagger q^{\frac{\tilde{N}_{a+1}}{4}} \\ \tilde{a}^\dagger q^{\frac{\tilde{N}_{b+1}}{4}} \end{pmatrix}. \tag{48}
\end{aligned}$$

These objects are in fact really spinors for some $\mathcal{U}_q(\mathfrak{su}(2))$ actions: \mathbf{t}^ϵ and $\tilde{\mathbf{t}}^\epsilon$ are spinors under the $\mathcal{U}_q(\mathfrak{su}(2))$ adjoint right action, while $\boldsymbol{\tau}^\epsilon$ and $\tilde{\boldsymbol{\tau}}^\epsilon$ are spinors under the $\mathcal{U}_{q^{-1}}(\mathfrak{su}(2))$ adjoint right action [31].

Notice that the map $q \rightarrow q^{-1}$ exchanges \mathbf{t}^ϵ with $\boldsymbol{\tau}^\epsilon$ and $\tilde{\mathbf{t}}^\epsilon$ with $\tilde{\boldsymbol{\tau}}^\epsilon$ [the operators $a, b, \tilde{a}, \tilde{b}$ are invariant under $q \rightarrow q^{-1}$ (43)].

B. Kinematical Hilbert space

The kinematical state space was defined in [31]. We here describe the corresponding spin network basis. Each edge e of Γ carries an irreducible representation V_{j_e} of $\mathcal{U}_q(\mathfrak{su}(2))$, characterized by its spin $j_e \in \mathbb{N}/2$. The Gauss constraint then projects the tensor product of the incident representations at each vertex onto the invariant subspace.

We consider the usual magnetic basis $\{|j, m\rangle\}_{m=-j, \dots, j}$ on each V_j , which diagonalizes K , i.e., $K|j, m\rangle = q^m|j, m\rangle$. The q bosons act on V_j as

$$\begin{aligned}
a^\dagger|j, m\rangle &= \sqrt{[j+m+1]}|j + \frac{1}{2}, m + \frac{1}{2}\rangle, & a|j, m\rangle &= \sqrt{[j+m]}|j - \frac{1}{2}, m - \frac{1}{2}\rangle, \\
b^\dagger|j, m\rangle &= \sqrt{[j-m+1]}|j + \frac{1}{2}, m - \frac{1}{2}\rangle, & b|j, m\rangle &= \sqrt{[j-m]}|j - \frac{1}{2}, m + \frac{1}{2}\rangle, \\
N_a|j, m\rangle &= (j+m)|j, m\rangle, & N_b|j, m\rangle &= (j-m)|j, m\rangle. \tag{49}
\end{aligned}$$

In particular a^\dagger, b^\dagger (a, b) map V_j to $V_{j+1/2}$ (to $V_{j-1/2}$). The tilde q bosons $\tilde{a}, \tilde{a}^\dagger, \tilde{b}, \tilde{b}^\dagger$ act on these basis in the same way by definition. It leads to the Wigner-Eckart theorem for the quantum spinors (48):

$$\langle j_1, m_1 | \mathbf{t}_m^\epsilon | j_2, m_2 \rangle = \delta_{j_1, j_2 + \epsilon/2} \sqrt{[d_{j_1}]}_q C_{m_1 - m}^{j_1 \frac{1}{2} j_2 m_2}, \tag{50a}$$

$$\langle j_1, m_1 | \boldsymbol{\tau}_m^\epsilon | j_2, m_2 \rangle = \delta_{j_1, j_2 + \epsilon/2} \sqrt{[d_{j_1}]}_{q^{-1}} C_{m_1 - m}^{j_1 \frac{1}{2} j_2 m_2}, \tag{50b}$$

$$\langle j_1, m_1 | \tilde{\mathbf{t}}_m^\epsilon | j_2, m_2 \rangle = \delta_{j_1, j_2 + \epsilon/2} \sqrt{[d_{j_1}]}_q C_{m_1 - m}^{j_1 \frac{1}{2} j_2 m_2}, \tag{50c}$$

$$\langle j_1, m_1 | \tilde{\boldsymbol{\tau}}_m^\epsilon | j_2, m_2 \rangle = \delta_{j_1, j_2 + \epsilon/2} \sqrt{[d_{j_1}]}_{q^{-1}} C_{m_1 - m}^{j_1 \frac{1}{2} j_2 m_2}, \tag{50d}$$

where ${}_q C_{m_1 m_2 m_3}^{j_1 j_2 j_3}$ is the Clebsch-Gordan coefficient for $\mathcal{U}_q(\mathfrak{su}(2))$.

Before enforcing the Gauss constraints, the space of states is $\bigoplus_{\{j_e\}} \otimes_e V_{j_e} \otimes V_{j_e}^*$, where V_{j_e} is associated to the target end of e and $V_{j_e}^*$ to its source. At each vertex v , the Gauss constraint enforces a projection of the tensor product of the vectors meeting at v onto the trivial representation. If the edges meeting at v are denoted e_1, \dots, e_n , we further denote $\text{Inv}(j_{e_1 v} \otimes \dots \otimes j_{e_n v})$ the space of intertwiners, i.e., the invariant subspace of the tensor product $V_{j_{e_1}} \otimes \dots \otimes V_{j_{e_n}}$ if all e_i s are incoming at v , and we dualize to $V_{j_{e_i}}^*$ if e_i is outgoing at v . Therefore, the kinematical Hilbert space is given by

$$\mathcal{H}_{\text{kin}} = \bigoplus_{\{j_e \in \mathbb{N}/2\}} \bigotimes_v \text{Inv}(j_{e_1 v} \otimes \cdots \otimes j_{e_n v}). \quad (51)$$

A basis is obtained at fixed spins $\{j_e\}$ by specifying a basis of $\text{Inv}(j_{e_1 v} \otimes \cdots \otimes j_{e_n v})$ for all v . We denote an element of this space as $i_{j_{e_1 v} \cdots j_{e_n(v)v}}$ (the letter i referring to intertwiner).

A kinematical state $|\psi\rangle$ thus admits the expansion

$$|\psi\rangle = \sum_{\{j_e\}} \sum_{\{i_v\}} \psi(\{j_e, i_v\}) |\{j_e, i_v\}\rangle, \quad (52)$$

with

$$|\{j_e, i_v\}\rangle = \bigotimes_v i_{j_{e_1 v} \cdots j_{e_n(v)v}}, \quad (53)$$

where the sum over each i_v runs over a basis of the invariant space at v . The state $|\{j_e, i_v\}\rangle$ is called a spin network state. They form a basis of \mathcal{H}_{kin} . In the case of trivalent vertices, the invariant space $\text{Inv}(j_{e_1 v} \otimes j_{e_2 v} \otimes j_{e_3 v})$ is one dimensional. This is the case we are most interested in and will be considered when constructing the quantum Hamiltonian. Let us now give more details on the intertwiners in this case.

1. The q intertwiner for a three-valent vertex

As shown in [29], the order of the $\mathcal{U}_q(\mathfrak{su}(2))$ -invariant spaces on different nodes is irrelevant (only the linear order of the links incident to each node matters). In the basis $|j_1, m_1\rangle \otimes |j_2, m_2\rangle \otimes |j_3, m_3\rangle$, which diagonalizes the $\mathcal{U}_q(\mathfrak{su}(2))$ generator K for each particle, the components of the state (up to normalization) are the q -deformed Clebsch-Gordan coefficients [29,31]. Explicitly, the intertwiner for a vertex with three incoming edges incident reads

$$i_{j_1 j_2 j_3} = \sum_{m_i} \frac{(-1)^{j_3+m_3}}{\sqrt{[d_{j_3}]}} q^{-\frac{m_3}{2}} {}_q C_{m_1, m_2, -m_3}^{j_1, j_2, j_3} |j_1, m_1\rangle \otimes |j_2, m_2\rangle \otimes |j_3, m_3\rangle. \quad (54)$$

It solves the quantum Gauss constraint equation $\hat{\mathcal{G}} i_{j_1 j_2 j_3} := \Delta^{(2)} \tilde{L} i_{j_1 j_2 j_3} \equiv \tilde{L} \otimes \tilde{L} \otimes \tilde{L} i_{j_1 j_2 j_3} = i_{j_1 j_2 j_3}$. Changing the orientation of each edge, say e_i , leads to the flipping of the vector space V_{j_i} to the dual vector space $V_{j_i}^*$. To write down the expression of the corresponding intertwiner, we make use of the $\mathcal{U}_q(\mathfrak{su}(2))$ -invariant bilinear form, $\mathcal{B}_q: V^j \otimes V^j \rightarrow \mathbb{C}$, which is defined with the q -WCG coefficient projected on the trivial representation [36]. Explicitly, for two given vectors $w = \sum_m w_m |j, m\rangle$, $r = \sum_n r_n |j, n\rangle \in V^j$,

$$\mathcal{B}_q(w, r) = \sum_m {}_q C_{-m, m, 0}^j w_{-m} r_m = \sum_m (-1)^{j+m} q^{-\frac{m}{2}} w_{-m} r_m. \quad (55)$$

One can thus define the dual vector w^* of w as

$$w^* \equiv \sum_m \langle j, m | w_m^* := \sum_m \langle j, m | q^{-\frac{m}{2}} (-1)^{j+m} w_{-m} \\ \Rightarrow w_m^* = q^{-\frac{m}{2}} (-1)^{j+m} w_{-m}. \quad (56)$$

Apparently, this dual operation is not an involution.³

When the edge e_2 is outgoing and e_1, e_3 incoming, one needs to dualize the vector on e_2 , which is to change ${}_q C_{m_1, m_2, -m_3}^{j_1, j_2, j_3} |j_2, m_2\rangle \rightarrow {}_q C_{m_1, -m_2, -m_3}^{j_1, j_2, j_3} \langle j_2, m_2|$ and add $(-1)^{j_2+m_2} q^{-\frac{m_2}{2}}$ according to (56). Thus the correspondent intertwiner is

$$i_{j_1 j_2^* j_3} = \sum_{m_i} \frac{(-1)^{j_3+m_3}}{\sqrt{[d_{j_3}]}} q^{-\frac{m_3+m_2}{2}} (-1)^{j_2+m_2} {}_q C_{m_1, -m_2, -m_3}^{j_1, j_2, j_3} |j_1, m_1\rangle \otimes \langle j_2, m_2| \otimes |j_3, m_3\rangle, \quad (57)$$

which can be checked to be the eigenstate for the quantum Gauss constraint $\hat{\mathcal{G}} = \tilde{L} \otimes L \otimes \tilde{L}$.

When edge e_1 is outgoing and e_2, e_3 incoming, the intertwiner is obtained using the same dualization as in (57) but for j_1 and m_1 . The last case of keeping the orientation of e_3 unchanged is to switch both e_1 and e_2 to be outgoing, then the same dualization should be applied to both (j_1, m_1) and (j_2, m_2) .

What needs special care is when one switches the orientation of e_3 , i.e., when e_3 is outgoing and e_1, e_2 incoming. In this case, one needs to dualize the vector on e_3 with a different rule. This is because the q -WCG coefficient ${}_q C_{m_1, m_2, m_3}^{j_1, j_2, j_3} = \langle j_1, m_1; j_2, m_2 | (j_1 j_2) j_3, m_3 \rangle$ can be viewed as the coefficient w_{m_1} (respectively, w_{m_2}) of a vector in V^{j_1} (respectively, V^{j_2}) or the coefficient $w_{m_3}^*$ of a dual vector in V^{j_3} in the sense of the decomposition (56).

Note that the factor $(-1)^{j_3+m_3} q^{-\frac{m_3}{2}}$ in $i_{j_1 j_2 j_3}$ is the transformation factor from the coefficient w_m of a vector w to the coefficient w_m^* of a dual vector w^* as shown in (56), thus one needs to change $(-1)^{j_3+m_3} q^{-\frac{m_3}{2}} {}_q C_{m_1, m_2, -m_3}^{j_1, j_2, j_3} |j_3, m_3\rangle \rightarrow (-1)^{j_3-m_3} q^{\frac{m_3}{2}} {}_q C_{m_1, m_2, m_3}^{j_1, j_2, j_3} \langle j_3, m_3|$ and add $(-1)^{j_3-m_3} q^{-\frac{m_3}{2}}$, which is the factor of the inverse transformation of w^* . This leads to the intertwiner

$$i_{j_1 j_2 j_3^*} = \sum_{m_i} \frac{1}{\sqrt{[d_{j_3}]}} {}_q C_{m_1, m_2, m_3}^{j_1, j_2, j_3} |j_1, m_1\rangle \otimes |j_2, m_2\rangle \otimes \langle j_3, m_3|, \quad (58)$$

³One can also define the dual vector with the $\mathcal{U}_{q^{-1}}(\mathfrak{su}(2))$ -invariant bilinear form $\mathcal{B}_{q^{-1}}$, which is to replace q with q^{-1} in (56).

which is exactly the eigenstate for the quantum Gauss constraint $\hat{G} = \tilde{L} \otimes \tilde{L} \otimes L$. Equation (58) can also be used to define the q -WCG coefficient

$${}_q C_{m_1 m_2 m_3}^{j_1 j_2 j_3} := \langle j_1, m_1 | \otimes \langle j_2, m_2 | i_{j_1 j_2 j_3^*} | j_3, m_3 \rangle. \quad (59)$$

Indeed, when we change the orientation of e_3 again, we recover the original intertwiner $i_{j_1 j_2 j_3}$ by adding the regular factor $(-1)^{j_3+m_3} q^{-\frac{m_3}{2}}$ as in obtaining $i_{j_1 j_2 j_3^*}$ from $i_{j_1 j_2 j_3}$.

Given the explicit expressions of the intertwiners for a three-valent vertex, our goal now is to construct the scalar operators in terms of the quantum spinors (48), which act on the intertwiner in a uniform way regardless of the orientations of all the incident edges. This will largely simplify the construction of the quantum Hamiltonian constraint since we do not need to consider different orientations of relevant edges separately.⁴

C. Scalar operators

We proceed to the quantization of the quadratic invariant (21), $E_{e_2 e_1}^{\epsilon_2, \epsilon_1}$. The quantization of the spinors $t_{e_1 v}, t_{e_2 v}$ themselves is given by (48). As can be seen from IV B, the operators $\mathbf{t}^\epsilon, \boldsymbol{\tau}^\epsilon, \tilde{\mathbf{t}}^\epsilon, \tilde{\boldsymbol{\tau}}^\epsilon$ transform as spinors under $\mathcal{U}_q(\mathfrak{su}(2))$ or $\mathcal{U}_{q^{-1}}(\mathfrak{su}(2))$. Therefore to ensure that $E_{e_2 e_1}^{\epsilon_2, \epsilon_1}$

is quantized as a quantum group invariant, one needs to contract the two spinor operators via some q -Clebsch-Gordan coefficients, ${}_q C_{AB0}^{\frac{110}{}0} = (-1)^{\frac{1}{2}-A} q^{\frac{A}{2}} \delta_{B,-A}$ or ${}_{q^{-1}} C_{AB0}^{\frac{110}{}0} = (-1)^{\frac{1}{2}-A} q^{-\frac{A}{2}} \delta_{B,-A}$.

Since changing the orientation of an edge exchanges \mathbf{t}^ϵ with $\tilde{\mathbf{t}}^\epsilon$, and $\boldsymbol{\tau}^\epsilon$ with $\tilde{\boldsymbol{\tau}}^\epsilon$, and since \mathbf{t}^ϵ and $\tilde{\mathbf{t}}^\epsilon$ are in fact the same operator (and also $\boldsymbol{\tau}^\epsilon$ and $\tilde{\boldsymbol{\tau}}^\epsilon$), one would expect the quantum operator for $E_{e_2 e_1}^{\epsilon_2, \epsilon_1}$ to be independent of the orientations of e_1 and e_2 . It is entirely possible to proceed this way.

We will however not do so. Our motivation is that while $E_{e_2 e_1}^{\epsilon_2, \epsilon_1}$ would be independent of orientations, the vector space on which it acts does depend on orientations (V_j^* versus V_j). Therefore the action on an intertwiner would in fact depend explicitly on the orientations. Instead, we decide to perform the quantization so that its action on intertwiners is independent of orientations.

This requires changing the spinor operator to its q^{-1} version when flipping the orientation. Obviously, this exchanges the \mathbf{t} s with the $\boldsymbol{\tau}$ s. However, we prefer to keep the same letter for the spinor operator because we think exchanging \mathbf{t} s with $\boldsymbol{\tau}$ s could be confusing in the ribbon picture. We therefore define $\tilde{\mathbf{t}}^\epsilon := \boldsymbol{\tau}^\epsilon$ and same with the tildes, and eventually⁵

$$\begin{aligned} \mathbf{E}_{e_2 e_1}^{\epsilon_2, \epsilon_1} &= -o_1 \sqrt{[2]} (-1)^{\frac{1-o_1+e_1}{2}} (-1)^{\frac{1-o_2+e_2}{2}} \sum_{A=\pm\frac{1}{2}} q_{o_1} C_{A-A0}^{\frac{110}{}0} \mathbf{T}_{e_1 v, A}^{o_1 \epsilon_1} \otimes \mathbf{T}_{e_2 v, -A}^{-o_2 \epsilon_2} \\ &= \begin{cases} \sum_{A=\pm\frac{1}{2}} (-1)^{\frac{1}{2}-A} q^{\frac{A}{2}} \epsilon_2 \tilde{\mathbf{t}}_A^{\epsilon_1} \otimes \tilde{\boldsymbol{\tau}}_{-A}^{\epsilon_2} & \text{for } -o_1 = o_2 = -1 \\ \sum_{A=\pm\frac{1}{2}} (-1)^{\frac{1}{2}+A} q^{\frac{A}{2}} \tilde{\mathbf{t}}_A^{\epsilon_1} \otimes \overline{\boldsymbol{\tau}}_{-A}^{-\epsilon_2} & \text{for } o_1 = o_2 = 1 \\ \sum_{A=\pm\frac{1}{2}} (-1)^{\frac{1}{2}-A} q^{-\frac{A}{2}} \epsilon_1 \epsilon_2 \tilde{\mathbf{t}}_A^{-\epsilon_1} \otimes \tilde{\boldsymbol{\tau}}_{-A}^{\epsilon_2} & \text{for } -o_1 = -o_2 = 1 \\ \sum_{A=\pm\frac{1}{2}} (-1)^{\frac{1}{2}+A} q^{-\frac{A}{2}} \epsilon_1 \overline{\boldsymbol{\tau}}_A^{-\epsilon_1} \otimes \overline{\boldsymbol{\tau}}_{-A}^{-\epsilon_2} & \text{for } -o_1 = o_2 = 1 \end{cases}, \quad (60) \end{aligned}$$

where $\mathbf{T}_{e_2 v, A}^{-o_2 \epsilon_2} = \mathbf{t}_{e_2 v, A}^{\epsilon_2}$ if $o_2 = -1$ while $\mathbf{T}_{e_2 v, A}^{-o_2 \epsilon_2} = \overline{\boldsymbol{\tau}}_{e_2 v, A}^{-\epsilon_2}$ if $o_2 = 1$, and similarly for $\mathbf{T}_{e_1 v, A}^{o_1 \epsilon_1}$. We then extend this definition to the space $\text{Inv}(j_{e_1 v} \otimes \cdots \otimes j_{e_n v})$ of invariant vectors at v by tensoring with the identity as necessary. It comes

$$\mathbf{E}_{e_2 e_1}^{\epsilon_2, \epsilon_1} i_{j_1 j_2 k} = \sqrt{[d_{j_1}][d_{j_2}][d_{l_1}][d_{l_2}] \delta_{l_1, j_1 + \frac{\epsilon_1}{2}} \delta_{l_2, j_2 + \frac{\epsilon_2}{2}}} \left\{ \begin{matrix} l_1 & j_1 & \frac{1}{2} \\ j_2 & l_2 & k \end{matrix} \right\}_q (-1)^{l_1 + l_2 + k} i_{l_1 l_2 k}. \quad (61)$$

It thus maps the intertwiner space $\text{Inv}(j_1 \otimes j_2 \otimes k)$ to $\text{Inv}(l_1 \otimes l_2 \otimes k)$.

⁴However, in [31], we define the scalar operators differently so that the algebras they form have the same expression. The different forms of the scalar operators in this paper and in [31] should be viewed as the same object represented in different bases.

⁵Note that the definition of o_1 is opposite to that in [31] which leads to a slight difference for the definition of the scalar operator (60) compared to that in [31]. This is because, in [31], o_1 and o_2 are considered to be the orientation of edges relative to vertex v and $+1$ (respectively, -1) denotes outgoing (respectively, incoming). Here, in contrast, o_1 and o_2 are considered to be the orientation of edges relative to the orientation of the face f . As an example, when edges e_2 and e_1 are both outgoing to v , e_2 is counterclockwise while e_1 is clockwise relative to f .

This definition also works for two edges e_i, e_{i+1} sharing a corner in $\text{Inv}(j_{e_1 v} \otimes \cdots \otimes j_{e_n v})$, for $i = 1, \dots, n-1$. In the trivalent case, this gives $E_{e_3 e_2}^{\epsilon_3, \epsilon_2} i_{j_1 j_2 j_3}$ exactly as in (61) with $e_1 \rightarrow e_2, e_2 \rightarrow e_3, e_3 \rightarrow e_1$

$$\mathbf{E}_{e_3 e_2}^{\epsilon_3, \epsilon_2} i_{j_1 j_2 j_3} = \sqrt{[d_{j_2}][d_{j_3}][d_{l_2}][d_{l_3}] \delta_{l_2, j_2 + \frac{\epsilon_2}{2}} \delta_{l_3, j_3 + \frac{\epsilon_3}{2}}} \times \begin{Bmatrix} l_2 & j_2 & \frac{1}{2} \\ j_3 & l_3 & j_1 \end{Bmatrix}_q (-1)^{l_2 + l_3 + j_1} i_{j_1 l_2 l_3}. \quad (62)$$

For the case $i = n$, i.e., $E_{e_1 e_n}^{\epsilon_1, \epsilon_n}$, the definition has to be amended to obtain an invariant operator [31] and eventually one finds the same expression for $E_{e_1 e_3}^{\epsilon_1, \epsilon_3} i_{j_1 j_2 j_3}$ as (61) with the appropriate permutation of the indices, i.e.,

$$\mathbf{E}_{e_1 e_3}^{\epsilon_1, \epsilon_3} i_{j_1 j_2 j_3} = \sqrt{[d_{j_3}][d_{j_1}][d_{l_2}][d_{l_1}] \delta_{l_3, j_3 + \frac{\epsilon_3}{2}} \delta_{l_1, j_1 + \frac{\epsilon_1}{2}}} \times \begin{Bmatrix} l_3 & j_3 & \frac{1}{2} \\ j_1 & l_1 & j_2 \end{Bmatrix}_q (-1)^{l_3 + l_1 + j_2} i_{l_1 j_2 l_3}. \quad (63)$$

D. Quantum Hamiltonian constraint

We now need to quantize the classical Hamiltonian (36) as a well-defined operator on \mathcal{H}_{kin} [defined in (51)]. The first step is obviously to use the quantization map described in the previous section to turn the observables $E_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$ into operators $\mathbf{E}_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$. The second step is concerned with quantization ambiguities. Indeed, factors N_{e_i} appear in (36) and they are expected to be diagonal on the spin network basis, as a function of j_{e_i} only in fact. Notice however that the operator $E_{e_2 e_1}^{\epsilon_2, \epsilon_1}$ changes the spins of the edges e_1, e_2 by $\epsilon_1/2$ and $\epsilon_2/2$. There are therefore ordering ambiguities. The results differ according to whether N_{e_i} is before or after some operators E , which changes j_{e_i} . We found an ordering, see below, that ultimately leads to a topological model, which would presumably not be true for other orderings.

Let us introduce

$$\mathbf{h}_{f, e_1, e_p}^{\epsilon_1, \epsilon_p} = \frac{1}{N_{e_1 v_2}} \left(\sum_{\epsilon_2, \dots, \epsilon_{p-1} = \pm} \prod_{i=2}^p \mathbf{E}_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}} \frac{O_i \epsilon_i}{N_{e_i v_i}} \right) + (-1)^{d-p} \epsilon_1 \epsilon_p \frac{1}{N_{e_p v_{p+1}}} \times \left(\sum_{\epsilon_{p+1}, \dots, \epsilon_d = \pm} \prod_{i=p+1}^{d+1} \mathbf{E}_{e_i e_{i-1}}^{-\epsilon_i, -\epsilon_{i-1}} \frac{O_i \epsilon_i}{N_{e_i v_i}} \right), \quad (64)$$

where $N_{e_i v_i}$ is diagonal on $V_{j_{e_i}}$ (or its dual), $N_{e_i v_i} |j_{e_i}, m_{e_i}\rangle = [d_{j_{e_i}}] |j_{e_i}, m_{e_i}\rangle$. We include the vertex v_i in the notation

because here $N_{e_i v_i}$ only acts on the space of intertwiners at v_i , where e_i and e_{i-1} meet. As already discussed, the ordering is important because $[N_{e_i v_i}, E_{e_{i-1} e_i}^{\epsilon_{i-1}, \epsilon_i}] \neq 0$. However $[N_{e_i v_i}, E_{e_i e_{i+1}}^{\epsilon_i, \epsilon_{i+1}}] = 0$ by definition, so that the operators $\mathbf{E}_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}} \frac{O_i \epsilon_i}{N_{e_i v_i}}$ that act on the space of intertwiners at v_i commute with one another. Here $N_{e_i v_i}$ is placed to the right of $\mathbf{E}_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$, which is also the case if one reconstructs the quantum holonomies (i.e., the quantization of u and \tilde{u}^{-1}) from the quantum spinors [31].

However, the operator $\mathbf{h}_{f, e_1, e_p}^{\epsilon_1, \epsilon_p}$ as such is not defined on \mathcal{H}_{kin} . Indeed, a state in \mathcal{H}_{kin} is a superposition of spin network states that assigns a spin to each edge along with the space V_j to the target end and V_j^* to the source end. Say the edge e_1 gets the spin j_1 . Then the first term of the above operator acts on e_1 with $\mathbf{E}_{e_2 e_1}^{\epsilon_2, \epsilon_1}$ which shifts the spin j_1 to $j_1 + \epsilon_1/2$, on the intertwiner that sits at the vertex where e_1 and e_2 meet. It thus maps V_{j_1} to $V_{j_1 + \epsilon_1/2}$ or $V_{j_1}^*$ to $V_{j_1 + \epsilon_1/2}^*$ depending on orientations but not both; i.e., it does not shift j_1 at the vertex where e_d and e_1 meet. Therefore the operator brings the state out of \mathcal{H}_{kin} .

Similarly, the second term of $\mathbf{h}_{f, e_1, e_p}^{\epsilon_1, \epsilon_p}$ acts on e_1 through $\mathbf{E}_{e_1 e_d}^{-\epsilon_1, -\epsilon_d}$. This shifts j_1 to $j_1 - \epsilon_1/2$ at the vertex where e_d and e_1 meet. If $\mathbf{E}_{e_2 e_1}^{\epsilon_2, \epsilon_1}$ in the first term acted on V_{j_1} , then this operator acts on $V_{j_1}^*$ (or the other way around).

We thus turn $\mathbf{h}_{f, e_1, e_p}^{\epsilon_1, \epsilon_p}$ into a well-defined operator on \mathcal{H}_{kin} by multiplying it by a product of operators $\mathbf{E}_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$ so that the intertwiners of both ends of the same edge have the same spin. Notice that the first term in (64) only contains the shift operators for $i = 2, \dots, p$, one can add $\mathbf{E}_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$ for all the remaining vertices, i.e., $i = p+1, \dots, d+1$, so that the change of spins for both ends of each edge are the same. For the second term in (64), adding these shift operators also shift all the spins $j_i - \epsilon_i/2$ to j_i thus drags the state back in \mathcal{H}_{kin} . This is the method which was already used in [10] to construct the quantum Hamiltonian in the spinor representation in the flat case.

Definition 4.1. We define the quantum Hamiltonian on the face f , labeled by the pair of edges (e_1, e_p) , to be

$$\mathbf{H}_{f, e_1, e_p}^{\epsilon_1, \epsilon_p, \epsilon_{p+1}, \dots, \epsilon_d} = \left[\prod_{i=p+1}^{d+1} \mathbf{E}_{e_i e_{i-1}}^{\epsilon_i, \epsilon_{i-1}} \right] \mathbf{h}_{f, e_1, e_p}^{\epsilon_1, \epsilon_p}. \quad (65)$$

Compared to the operator (64), the quantum Hamiltonian defined as such not only depends on ϵ_1 and ϵ_p but also $\epsilon_{p+1}, \dots, \epsilon_{d+1}$. The physical Hilbert space is spanned by the physical states that are solutions to the quantum Hamiltonian. In the spin representation, the coefficients of these physical spin network states satisfy a set of difference equations, which is stated in the following theorem.

Theorem 4.2. The constraint

$$\forall k_e \quad \langle \{k_e\} | \mathbf{H}_{f, e_1, e_p}^{\epsilon_1, \epsilon_p, \epsilon_{p+1}, \dots, \epsilon_d} | \psi \rangle = 0, \quad (66)$$

is equivalent to the following set of difference equations on the spin network coefficients $\psi(k_1, k_2, \dots, k_d, \{k_e\}_{e \notin df})$ of $|\psi\rangle$,

$$\begin{aligned} & \sum_{\tilde{\epsilon}_2, \dots, \tilde{\epsilon}_{p-1} = \pm} \left(\prod_{i=2}^p A_{o_i}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}}(k_i, k_{i-1}, l_i) \right) \psi \left(k_1 - \frac{\epsilon_1}{2}, k_2 - \frac{\tilde{\epsilon}_2}{2}, \dots, k_{p-1} - \frac{\tilde{\epsilon}_{p-1}}{2}, k_p - \frac{\epsilon_p}{2}, \dots, k_d - \frac{\epsilon_d}{2}, \{k_e\}_{e \notin df} \right) \\ & + (-1)^{d-p} \alpha^{\epsilon_1, \epsilon_p}(k_1, k_p) \sum_{\tilde{\epsilon}_{p+1}, \dots, \tilde{\epsilon}_d = \pm} \left(\prod_{i=p+1}^{d+1} B_{o_i}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}} \left(k_i - \frac{\epsilon_i}{2}, k_{i-1} - \frac{\tilde{\epsilon}_{i-1}}{2}, l_i \right) \right) \\ & \psi \left(k_1, \dots, k_p, k_{p+1} - \frac{\epsilon_{p+1}}{2} + \frac{\tilde{\epsilon}_{p+1}}{2}, \dots, k_d - \frac{\epsilon_d}{2} + \frac{\tilde{\epsilon}_d}{2}, \{k_e\}_{e \notin df} \right) = 0. \end{aligned} \quad (67)$$

Here

- (i) l_1, \dots, l_d are the spins carried by the edges e'_1, \dots, e'_d incident to f , see Fig. 6.
- (ii) By definition, $\tilde{\epsilon}_1 = \epsilon_1, \tilde{\epsilon}_p = \epsilon_p$, while $\epsilon_{p+1}, \dots, \epsilon_d$ are fixed.
- (iii) The coefficients are

$$A_{o_i}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}}(k_i, k_{i-1}, l_i) = o_i \tilde{\epsilon}_i [d_{k_i}] (-1)^{k_i + k_{i-1} + l_i} \begin{Bmatrix} k_i & k_i - \frac{\tilde{\epsilon}_i}{2} & \frac{1}{2} \\ k_{i-1} - \frac{\tilde{\epsilon}_{i-1}}{2} & k_{i-1} & l_i \end{Bmatrix}_q, \quad (68)$$

$$B_{o_i}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}}(k_i, k_{i-1}, l_i) = o_i \tilde{\epsilon}_i [d_{k_i}] (-1)^{k_i + k_{i-1} + l_i} \begin{Bmatrix} k_i & k_i + \frac{\tilde{\epsilon}_i}{2} & \frac{1}{2} \\ k_{i-1} + \frac{\tilde{\epsilon}_{i-1}}{2} & k_{i-1} & l_i \end{Bmatrix}_q, \quad (69)$$

$$\alpha^{\epsilon_1, \epsilon_p}(k_1, k_p) = \epsilon_1 \epsilon_p \frac{[d_{k_p}]}{[d_{k_1 - \frac{\epsilon_1}{2}}]}. \quad (70)$$

Those constraints are recursions on the physical states. They generalize the one found in [10] for a triangular face. Improving on [9], the differences are shifts of the spins by $1/2$ instead of 1. Moreover, edge orientations are kept arbitrary.

Those constraints have two types of contributions: the A terms and the B terms. Notice that $\mathbf{h}_{f, e_1, e_p}^{\epsilon_1, \epsilon_p}$ contains all the operators $\mathbf{E}_{e_i, e_{i-1}}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}}$ exactly once, for $i = 1, \dots, d$. Whether an operator $\mathbf{E}_{e_i, e_{i-1}}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}}$ gives rise to an A term or a B term depends on the choice of the reference edges e_1 and e_p around f . It is important that the coefficients $A_{o_i}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}}(k_i, k_{i-1}, l_i)$ and $B_{o_i}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}}(k_i, k_{i-1}, l_i)$ are local: they only depend on the spins incident to the vertex and are determined by the choice of a corner on that vertex. As a consequence, for example, if one considers another constraint on the same face with $e_q, q < p$, choosing e_1 as reference edge, then the coefficients $A_{o_i}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}}(k_i, k_{i-1}, l_i)$ for $i = 1, \dots, q$ would be the same as those appearing above, and similarly for the B terms. The structure of the constraint is schematically pictured in Fig. 8.

By exchanging the role of e_1 with e_p , the A terms become the B terms and vice versa. The constraint obtained this way is equivalent to (67), as we now show. First, evaluate (67) on $k_1 + \epsilon_1/2, \dots, k_d + \epsilon_d/2$, and then flip the signs of all the ϵ_i and $\tilde{\epsilon}_i$. That gives the constraint

$$\begin{aligned} & \sum_{\tilde{\epsilon}_2, \dots, \tilde{\epsilon}_{p-1} = \pm} \left(\prod_{i=2}^p A_{o_i}^{-\tilde{\epsilon}_i, -\tilde{\epsilon}_{i-1}} \left(k_i - \frac{\epsilon_i}{2}, k_{i-1} - \frac{\tilde{\epsilon}_{i-1}}{2}, l_i \right) \right) \psi \left(k_1, k_2 - \frac{\epsilon_2}{2} + \frac{\tilde{\epsilon}_2}{2}, \dots, k_{p-1} - \frac{\epsilon_{p-1}}{2} + \frac{\tilde{\epsilon}_{p-1}}{2}, k_p, \dots, k_d, \{k_e\}_{e \notin df} \right) \\ & + (-1)^{d-p} \alpha^{\epsilon_1, \epsilon_p} \left(k_1 - \frac{\epsilon_1}{2}, k_p - \frac{\epsilon_p}{2} \right) \sum_{\tilde{\epsilon}_{p+1}, \dots, \tilde{\epsilon}_d = \pm} \left(\prod_{i=p+1}^{d+1} B_{o_i}^{-\tilde{\epsilon}_i, -\tilde{\epsilon}_{i-1}}(k_i, k_{i-1}, l_i) \right) \\ & \psi \left(k_1 - \frac{\epsilon_1}{2}, \dots, k_p - \frac{\epsilon_p}{2}, k_{p+1} - \frac{\tilde{\epsilon}_{p+1}}{2}, \dots, k_d - \frac{\tilde{\epsilon}_d}{2}, \{k_e\}_{e \notin df} \right) = 0. \end{aligned} \quad (71)$$

We then use the key relation between the coefficients A and B ,

$$B_{o_i}^{\tilde{\epsilon}_i, \tilde{\epsilon}_{i-1}}(k_i, k_{i-1}, l_i) = -A_{o_i}^{-\tilde{\epsilon}_i, -\tilde{\epsilon}_{i-1}}(k_i, k_{i-1}, l_i) \quad (72)$$

to get

$$\begin{aligned}
 & (-1)^p \sum_{\tilde{e}_2, \dots, \tilde{e}_{p-1} = \pm} \left(\prod_{i=2}^p B_{o_i \tilde{e}_{i-1}}^{\tilde{e}_i, \tilde{e}_{i-1}} \left(k_i - \frac{\epsilon_i}{2}, k_{i-1} - \frac{\epsilon_{i-1}}{2}, l_i \right) \right) \psi \left(k_1, k_2 - \frac{\epsilon_2}{2} + \frac{\tilde{\epsilon}_2}{2}, \dots, k_{p-1} - \frac{\epsilon_{p-1}}{2} + \frac{\tilde{\epsilon}_{p-1}}{2}, k_p, \dots, k_d, \{k_e\}_{e \notin \partial f} \right) \\
 & + \alpha^{\epsilon_1, \epsilon_p} \left(k_1 - \frac{\epsilon_1}{2}, k_p - \frac{\epsilon_p}{2} \right) \sum_{\tilde{e}_{p+1}, \dots, \tilde{e}_d = \pm} \left(\prod_{i=p+1}^{d+1} A_{o_i \tilde{e}_{i-1}}^{\tilde{e}_i, \tilde{e}_{i-1}} (k_i, k_{i-1}, l_i) \right) \\
 & \psi \left(k_1 - \frac{\epsilon_1}{2}, \dots, k_p - \frac{\epsilon_p}{2}, k_{p+1} - \frac{\tilde{\epsilon}_{p+1}}{2}, \dots, k_d - \frac{\tilde{\epsilon}_d}{2}, \{k_e\}_{e \notin \partial f} \right) = 0, \tag{73}
 \end{aligned}$$

where we recognize the matrix element $\langle \{k_e\} | \mathbf{H}_{f, e_p, e_1}^{\epsilon_1, \epsilon_2, \dots, \epsilon_p} | \psi \rangle$ and have shown the equivalence

$$\langle \{k_e\} | \mathbf{H}_{f, e_p, e_1}^{\epsilon_1, \epsilon_2, \dots, \epsilon_{p-1}} | \psi \rangle = 0 \Leftrightarrow \langle \{k_e\} | \mathbf{H}_{f, e_1, e_p}^{\epsilon_1, \epsilon_p, \epsilon_{p+1}, \dots, \epsilon_d} | \psi \rangle = 0. \tag{74}$$

Proof of Theorem 4.2.—There are two types of terms in (65), whose action on spin network states is now presented. First,

$$\begin{aligned}
 \prod_{i=p+1}^{d+1} \mathbf{E}_{e_i, e_{i-1}}^{\epsilon_i, \epsilon_{i-1}} \frac{1}{\mathbf{N}_{e_1 v_2}} \left(\prod_{i=2}^p \mathbf{E}_{e_i, e_{i-1}}^{\epsilon_i, \epsilon_{i-1}} \frac{o_i \epsilon_i}{\mathbf{N}_{e_i v_i}} \right) | \{j_e\} \rangle &= \frac{1}{[d_{k_1}]} \prod_{i=2}^p \frac{o_i \epsilon_i}{[d_{j_i}]} \prod_{i=2}^{d+1} \delta_{k_i, j_i + \frac{\epsilon_i}{2}} [d_{k_i}] [d_{j_i}] (-1)^{k_i + k_{i-1} + l_i} \\
 &\times \left\{ \begin{array}{ccc} k_i & k_i - \frac{\epsilon_i}{2} & \frac{1}{2} \\ k_{i-1} - \frac{\epsilon_{i-1}}{2} & k_{i-1} & l_i \end{array} \right\}_q | \{k_i\}_{i=1, \dots, d}, \{j_e\}_{e \notin \partial f} \rangle, \tag{75}
 \end{aligned}$$

where we have applied the action (61) of $\mathbf{E}_{e_i, e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$ on the intertwiner $i_{j_{i-1} j_i l_i}$ at the vertex where e_{i-1} , e_i , and e'_i meet for all $i = 1, \dots, d$. Each operator $1/\mathbf{N}_{e_i v_i}$ acts before the shift operator $\mathbf{E}_{e_i, e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$; thus the result picks up a factor $1/[d_{j_i}]$. For $i = 1, \dots, d$, the spin j_i is shifted to $j_i + \frac{\epsilon_i}{2}$ after the action of $\mathbf{E}_{e_i, e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$. The spins l_i 's of the edges e'_i 's not on the boundary of the face f remain unchanged. In addition, $\frac{1}{\mathbf{N}_{e_1 v_2}}$ acts after $\mathbf{E}_{e_2, e_1}^{\epsilon_2, \epsilon_1}$ thus the result picks up the factor $1/[d_{k_1}]$. As each edge is incident to two vertices, the assigned spin shows up in two intertwiners; thus the term $\sqrt{[d_{k_i}][d_{j_i}]}$ appears twice in the result, which gives the factor $[d_{k_i}][d_{j_i}]$. The $q-6j$ symbols and the sign factors naturally follow from (61).

Second,

$$\begin{aligned}
 \prod_{i=p+1}^{d+1} \mathbf{E}_{e_i, e_{i-1}}^{\epsilon_i, \epsilon_{i-1}} \frac{1}{\mathbf{N}_{e_p v_{p+1}}} \left(\prod_{i=p+1}^{d+1} \mathbf{E}_{e_i, e_{i-1}}^{-\tilde{e}_i, -\tilde{e}_{i-1}} \frac{o_i \epsilon_i}{\mathbf{N}_{e_i v_i}} \right) | \{j_e\} \rangle &= \frac{1}{[d_{j_p - \frac{\tilde{\epsilon}_p}{2}}]} \prod_{i=p+1}^{d+1} \frac{o_i \epsilon_i}{[d_{j_i}]} \delta_{k_i, j_i - \frac{\tilde{\epsilon}_i}{2} + \frac{\epsilon_i}{2}} (-1)^{k_i - \frac{\epsilon_i}{2} + k_{i-1} - \frac{\epsilon_{i-1}}{2} + l_i} (-1)^{k_i + k_{i-1} + l_i} [d_{j_i - \frac{\tilde{\epsilon}_i}{2}}] [d_{j_{i-1} - \frac{\tilde{\epsilon}_{i-1}}{2}}] \\
 &\times \sqrt{[d_{j_i}][d_{k_i}][d_{j_{i-1}}][d_{k_{i-1}}]} \left\{ \begin{array}{ccc} k_i - \frac{\epsilon_i}{2} & j_i & \frac{1}{2} \\ j_{i-1} & k_{i-1} - \frac{\epsilon_{i-1}}{2} & l_i \end{array} \right\}_q \left\{ \begin{array}{ccc} k_i - \frac{\epsilon_i}{2} & k_i & \frac{1}{2} \\ k_{i-1} & k_{i-1} - \frac{\epsilon_{i-1}}{2} & l_i \end{array} \right\}_q | \{j_i\}_{i=1, \dots, p}, \{k_i\}_{i=p+1, \dots, d}, \{j_e\}_{e \notin \partial f} \rangle, \\
 = \frac{[d_{k_p}]}{[d_{k_1}][d_{k_1 - \frac{\epsilon_1}{2}}]} \prod_{i=p+1}^{d+1} o_i \epsilon_i \delta_{k_i, j_i - \frac{\tilde{\epsilon}_i}{2} + \frac{\epsilon_i}{2}} (-1)^{k_i - \frac{\epsilon_i}{2} + k_{i-1} - \frac{\epsilon_{i-1}}{2} + l_i} (-1)^{k_i + k_{i-1} + l_i} [d_{k_i - \frac{\epsilon_i}{2}}]^2 [d_{k_i}] & \\
 \times \left\{ \begin{array}{ccc} k_i - \frac{\epsilon_i}{2} & j_i & \frac{1}{2} \\ j_{i-1} & k_{i-1} - \frac{\epsilon_{i-1}}{2} & l_i \end{array} \right\}_q \left\{ \begin{array}{ccc} k_i - \frac{\epsilon_i}{2} & k_i & \frac{1}{2} \\ k_{i-1} & k_{i-1} - \frac{\epsilon_{i-1}}{2} & l_i \end{array} \right\}_q | \{j_i\}_{i=1, \dots, p}, \{k_i\}_{i=p+1, \dots, d}, \{j_e\}_{e \notin \partial f} \rangle. & \tag{76}
 \end{aligned}$$

Here, two shift operators act on each site for $i = p+1, \dots, d+1$ and we denote $k_i = j_i - \frac{\tilde{\epsilon}_i}{2} + \frac{\epsilon_i}{2}$. The first shift operator $\mathbf{E}_{e_i, e_{i-1}}^{-\tilde{e}_i, -\tilde{e}_{i-1}}$ (in the bracket) acts on the spin network state and shifts j_i and j_{i-1} to $j_i - \tilde{e}_i/2$ and $j_{i-1} - \tilde{e}_{i-1}/2$, respectively. It also gives the first $q-6j$ symbol in the third line and the term $(-1)^{k_i - \frac{\epsilon_i}{2} + k_{i-1} - \frac{\epsilon_{i-1}}{2} + l_i} \sqrt{[d_{j_i}][d_{j_{i-1}}][d_{j_i - \frac{\tilde{\epsilon}_i}{2}}][d_{j_{i-1} - \frac{\tilde{\epsilon}_{i-1}}{2}}]}$. The result picks up a factor $1/[d_{j_i}]$ by the action of $1/\mathbf{N}_{e_i v_i}$ before the shift operator. In addition, $1/\mathbf{N}_{e_p v_{p+1}}$ acts on the spin network state after $\mathbf{E}_{e_{p+1}, e_p}^{-\tilde{\epsilon}_{p+1}, -\tilde{\epsilon}_p}$ and thus brings a factor $1/[d_{j_p - \frac{\tilde{\epsilon}_p}{2}}]$. The action of the second shift operator $\mathbf{E}_{e_i, e_{i-1}}^{\epsilon_i, \epsilon_{i-1}}$ shifts the spins

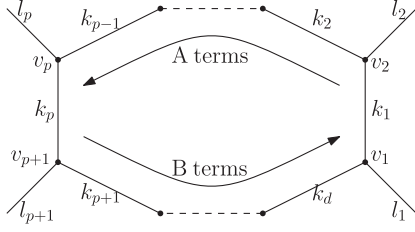


FIG. 8. The schematic representation of the quantum constraint (67) with its A terms and B terms associated to the corners around the face.

$j_i - \tilde{\epsilon}_i/2$ and $j_{i-1} - \tilde{\epsilon}_{i-1}/2$ to k_i and k_{i-1} , respectively, and brings the second $q - 6j$ symbol in the third line as well as the term $(-1)^{k_i+k_{i-1}+l_i} \sqrt{[d_{k_i}][d_{k_{i-1}}][d_{k_i-\frac{\epsilon_i}{2}}][d_{k_{i-1}-\frac{\epsilon_{i-1}}{2}}]}$. Note that the spin $j_1 = k_1$ and $j_p = k_p$ are kept unchanged in the result as $\epsilon_1 = \tilde{\epsilon}_1$ and $\epsilon_p = \tilde{\epsilon}_p$. The last equality is the rearrangement of the result.

Putting them together, using the orthogonality of the spin network states, $\langle \{k_e\} | \{j_e\} \rangle \propto \prod_e \delta_{k_e, j_e}$ and eliminating the common terms $\frac{1}{[d_{k_i}]} \prod_{i=p+1}^{d+1} [d_{k_i-\frac{\epsilon_i}{2}}][d_{k_i}] (-1)^{k_i+k_{i-1}+l_i} \left\{ \begin{matrix} k_i & k_i - \frac{\epsilon_i}{2} & \frac{1}{2} \\ k_{i-1} - \frac{\epsilon_{i-1}}{2} & k_{i-1} & l_i \end{matrix} \right\}_q$ leads to the expected difference equations. ■

The dependence of ψ on the orientations is given by the following lemma.

Lemma 4.3. If $|\psi\rangle$, with spin network coefficients $\psi(\{j_e\})$, satisfies all the constraints (67) for given edge orientations $\{o_e\}$, then $(-1)^{2j^*} \psi(\{j_e\})$ satisfies all the constraints on the same graph with reversed orientation $-o_{e^*}$ on the edge e^* . ■

Proof.—Consider the constraint (67) on the fixed face f . If $e^* \notin \partial f$, then multiplication by $(-1)^{2j^*}$ does not change anything. If $e^* \equiv e_s \in \{e_2, \dots, e_{p-1}\}$, then the coefficient $A_{o_s, \tilde{\epsilon}_s-1}(k_s, k_{s-1}, l_s)$ changes sign. Moreover, it is the only one that depends on o_s . The state coefficient on the first line of (67) changes from $\psi(k_1 - \epsilon_1, \dots, k_{e_p} - \tilde{\epsilon}_p, \dots)$ to $(-1)^{2k_s+1} \psi(k_1 - \epsilon_1, \dots, k_p - \tilde{\epsilon}_p, \dots)$ since $(-1)^{\tilde{\epsilon}_s} = -1$. Moreover, the coefficients B_s are independent of the orientation o_s and the state coefficient on the second line changes from $\psi(k_1, \dots, k_p, \dots)$ to $(-1)^{2k_s} \psi(k_1, \dots, k_p, \dots)$. Factorizing $(-1)^{2k_s}$ from the equation reveals that only the first line is modified, by $-o_s \times (-1) = o_s$. The constraint therefore still holds. If $e^* \equiv e_s \in \{e_{p+1}, \dots, e_{d+1}\}$, then the coefficient $B_{o_s, \tilde{\epsilon}_s-1}(k_s - \frac{\epsilon_s}{2}, k_{s-1} - \frac{\epsilon_{s-1}}{2}, l_s)$ changes sign while the coefficients A_s remain unchanged. The same analysis leads to the same conclusion.

The argument is the same for all edges in the boundary of f , since the orientation of any of those edges appears in a single coefficient of the equation. ■

In this section, we have quantized the four deformed spinors on each ribbon to q -deformed quantum spinors as given explicitly in (48) and constructed the quantum Hamiltonian constraint of the q -deformed LQG model purely in terms of (the scalar products of) these q -deformed quantum spinors. Section IV 2 is the main result of the current paper. It gives rise to the difference equations that the physical states satisfy. To verify that the Hamiltonian constraint we construct is the correct one, one can justify the topological invariance of the solutions to the constraint. That is, the solutions to the Hamiltonian constraints for graphs related by a series of Pachner moves are the same (up to normalization). This is what we will illustrate in the next section. The difference equations we derived in Sec. IV 2 will turn out to play a key role in the analysis.

V. PACHNER MOVES

We now show how to relate the physical states on triangulations which are related by Pachner moves. This is an extension of [8] to q real (using Hamiltonian constraints instead of projection on flat connections). In two dimensions, there are two types of Pachner moves, the 3-1 moves and the 2-2 moves (as well as their inverses). In this section, we will first analyze the case of the 2-2 moves. The 3-1 moves result naturally and follow the analysis of removing an edge of a triangle since, in this case, two out of the three vertices of the triangle are removed due to gauge invariance on the bivalent vertices.

A. 2-2 Pachner move

The 2-2 Pachner move changes a portion of the graph into another one as follows:



We denote the initial graph which contains the left-hand side as Γ_i , and the final graph which contains the right-hand side as Γ_f . The orientations of all edges are left arbitrary.

Theorem 5.1. Let $|\psi_f\rangle$ on Γ_f be defined in the spin network basis by

$$\psi_f(j_1, j_2, j_3, j_4, j_0, \dots) = (-1)^{j_1+j_2+j_3+j_4} [d_{j_0}] \sum_{j_5} (-1)^{(1-o_s)j_5+(1-o_0)j_0} \left\{ \begin{matrix} j_1 & j_2 & j_0 \\ j_3 & j_4 & j_5 \end{matrix} \right\}_q \psi_i(j_1, j_2, j_3, j_4, j_5, \dots), \quad (78)$$

where the ellipses denote spins which are the same on both sides (for edges that are not affected by the move). Then $|\psi_i\rangle$ is a state which satisfies all the constraints on Γ_i if and only if $|\psi_f\rangle$ satisfies all the constraints on Γ_f .

Since the 2-2 move is its own inverse, there is symmetry between both sides of the move. This must translate into a symmetry that exchanges the role of $|\psi_i\rangle$ and $|\psi_f\rangle$ in (78). This is indeed true thanks to the orthonormality of the $q-6j$ symbols,

$$\sum_{j_5} [d_{j_5}] [d_{j_0}] \left\{ \begin{matrix} j_1 & j_2 & j_0 \\ j_3 & j_4 & j_5 \end{matrix} \right\}_q \left\{ \begin{matrix} j_1 & j_2 & j'_0 \\ j_3 & j_4 & j_5 \end{matrix} \right\}_q = \delta_{j_0, j'_0}, \quad (79)$$

which transforms (78) into

$$\psi_i(j_1, j_2, j_3, j_4, j_5, \dots) = (-1)^{j_1+j_2+j_3+j_4} [d_{j_5}] \sum_{j'_0} (-1)^{(1-o_5)j_5+(1-o_0)j'_0} \left\{ \begin{matrix} j_1 & j_2 & j'_0 \\ j_3 & j_4 & j_5 \end{matrix} \right\}_q \psi_f(j_1, j_2, j_3, j_4, j'_0, \dots). \quad (80)$$

Proof of Theorem 5.1.—There are four faces involved in the move on each side. Clearly, $|\psi_i\rangle$ and $|\psi_f\rangle$ satisfy the same constraints associated to faces that are not among those four. Therefore, we can focus on the four faces involved in the move, and for symmetry reasons, we can simply look at the constraints on two faces: the face f_{12} , which has e_1, e_2 in its boundary, and the face f_{14} , which has e_1, e_4 in its boundary.

1. Face f_{12}

It has a different boundary on Γ_f and Γ_i , due to the disappearance of e_5 . On Γ_i , there are constraints where $\mathbf{E}_{e_5 e_1}^{\epsilon_5, \epsilon_1}$ and $\mathbf{E}_{e_2 e_5}^{\epsilon_2, \epsilon_5}$ are both among the A terms of the constraint (67). Let us denote the two reference edges [e_1 and e_k in (67)] as e and e' , which may be e_1 and/or e_2 . Then the difference equations (67) read

$$\begin{aligned} & \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right) \sum_{\tilde{e}_5 = \pm} A_{o_5, \tilde{e}_1}^{\tilde{e}_5, \tilde{e}_1}(k_5, k_1, k_4) A_{o_2, \tilde{e}_5}^{\tilde{e}_2, \tilde{e}_5}(k_2, k_5, k_3) \psi_i \left(k_1 - \frac{\tilde{e}_1}{2}, k_2 - \frac{\tilde{e}_2}{2}, k_3, k_4, k_5 - \frac{\tilde{e}_5}{2}, \dots \right) \\ & + (-1)^{d_{12,i} - d_{e e'}} \alpha^{e, e'}(k_e, k_{e'}) \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right) \psi_i(k_1, k_2, k_3, k_4, k_5, \dots) = 0, \end{aligned} \quad (81)$$

where $d_{12,i}$ denotes the number of boundary edges of f_{12} in Γ_i and $d_{e e'}$ the number of edges from e to e' counterclockwise. Notice that \tilde{e}_1 (respectively, \tilde{e}_2) is fixed if $e = e_1$ (respectively, if $e' = e_2$) and summed over otherwise. We have indicated in ψ_i only the spins which are involved in the move.

We have written $\sum_{\{\tilde{e}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right)$ and $\sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right)$ schematically the coefficients of the equation which are associated to corners not involved in the move. Here $\sum_{\{\tilde{e}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right)$ is the product of the A terms over the corners from e to e' going counterclockwise, except for the two corners with e_5 , whose A terms are distinguished. Then $\sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right)$ is the product of the B terms over the corners from e' to e counterclockwise. This is depicted in the Fig. 9.

On the other hand, a state on Γ_f must also satisfy a constraint along the face f_{12} with the two reference edges e and e' . It reads

$$\begin{aligned} & \langle \{k_e\}_{\Gamma_f} | \mathbf{H}_{f_{12}, e, e'} | \psi_f \rangle \propto \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right) A_{o_2, \epsilon_1}^{\epsilon_2, \epsilon_1}(k_2, k_1, k_0) \psi_f \left(k_1 - \frac{\tilde{e}_1}{2}, k_2 - \frac{\tilde{e}_2}{2}, k_3, k_4, k_0, \dots \right) \\ & + (-1)^{d_{12,i} - d_{e e'} + 1} \alpha^{e, e'}(k_e, k_{e'}) \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right) \psi_f(k_1, k_2, k_3, k_4, k_0, \dots). \end{aligned} \quad (82)$$

There is also a constraint where $\mathbf{E}_{e_1 e_2}^{\tilde{e}_1, \tilde{e}_2}$ gives rise to a B term, but as we have shown this is equivalent to the above constraint. Here it is important that the products of the A terms and B terms over all corners except the one where e_1 and e_2 meet are the same as in (81). The reason is obviously that those terms are local and the 2-2 move does not involve their corners. As in (81), \tilde{e}_1 and \tilde{e}_2 may be fixed or summed over.

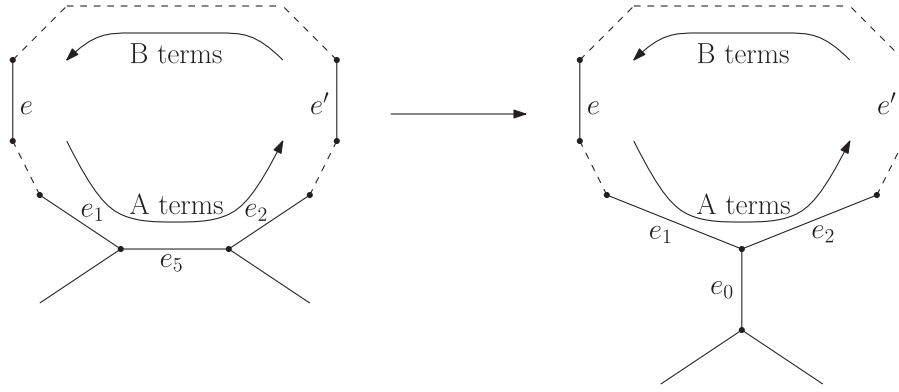


FIG. 9. A graphical representation of Eq. (81) on the lhs and (82) on the rhs.

We now plug (78) into (82) to check that it vanishes, provided the constraint (81) holds. First compute, with $j_{1,2} := k_{1,2} - \frac{\tilde{\epsilon}_{1,2}}{2}$,

$$A_{o_2}^{\tilde{\epsilon}_2, \tilde{\epsilon}_1}(k_2, k_1, k_0) \psi_f(j_1, j_2, k_3, k_4, k_0, \dots) = o_2 \tilde{\epsilon}_2 [d_{k_2}] [d_{k_0}] \sum_{j_5} (-1)^{(1-o_5)j_5 + (1-o_0)k_0} (-1)^{j_1 + j_2 + k_3 + k_4} (-1)^{k_0 + k_1 + k_2} \times \left\{ \begin{matrix} k_1 & j_1 & \frac{1}{2} \\ j_2 & k_2 & k_0 \end{matrix} \right\}_q \left\{ \begin{matrix} j_1 & j_2 & k_0 \\ k_3 & k_4 & j_5 \end{matrix} \right\}_q \psi_i(j_1, j_2, k_3, k_4, j_5, \dots). \quad (83)$$

The Biedenharn-Elliott identity on $q-6j$ symbols gives precisely

$$(-1)^{j_1 + j_2 + k_3 + k_4} (-1)^{k_0 + k_1 + k_2} \left\{ \begin{matrix} k_1 & j_1 & \frac{1}{2} \\ j_2 & k_2 & k_0 \end{matrix} \right\}_q \left\{ \begin{matrix} j_1 & j_2 & k_0 \\ k_3 & k_4 & j_5 \end{matrix} \right\}_q = \sum_{k_5} [d_{k_5}] (-1)^{k_5 + j_5 + \frac{1}{2}} \left\{ \begin{matrix} k_1 & k_2 & k_0 \\ k_3 & k_4 & k_5 \end{matrix} \right\}_q \left\{ \begin{matrix} k_1 & j_1 & \frac{1}{2} \\ j_5 & k_5 & k_4 \end{matrix} \right\}_q \left\{ \begin{matrix} k_5 & j_5 & \frac{1}{2} \\ j_2 & k_2 & k_3 \end{matrix} \right\}_q. \quad (84)$$

Setting $j_5 = k_5 - \frac{\tilde{\epsilon}_5}{2}$ to change the summation over j_5 to one over $\tilde{\epsilon}_5$ (there are no other values of j_5 allowed by the triangular inequalities on the $q-6j$ symbol), we get

$$A_{o_2}^{\tilde{\epsilon}_2, \tilde{\epsilon}_1}(k_2, k_1, k_0) \psi_f(j_1, j_2, k_3, k_4, k_0, \dots) = o_2 \epsilon_2 [d_{k_2}] [d_{k_0}] \sum_{k_5, \tilde{\epsilon}_5} (-1)^{(1-o_5)j_5 + (1-o_0)k_0} (-1)^{k_5 + j_5 + \frac{1}{2}} \times [d_{k_5}] \left\{ \begin{matrix} k_1 & k_2 & k_0 \\ k_3 & k_4 & k_5 \end{matrix} \right\}_q \left\{ \begin{matrix} k_1 & j_1 & \frac{1}{2} \\ j_5 & k_5 & k_4 \end{matrix} \right\}_q \left\{ \begin{matrix} k_5 & j_5 & \frac{1}{2} \\ j_2 & k_2 & k_3 \end{matrix} \right\}_q \psi_i(j_1, j_2, k_3, k_4, j_5, \dots). \quad (85)$$

Using $\epsilon_5 = (-1)^{\frac{1}{2} + j_5 - k_5}$, we find $(-1)^{k_5 + j_5 + \frac{1}{2}} = \epsilon_5 (-1)^{2k_5}$. We also use $(-1)^{(1-o_5)j_5} = o_5 (-1)^{(1-o_5)k_5}$ and notice that a $q-6j$ symbol can be factored. Thus,

$$\begin{aligned}
\langle \{k_e\}_{\Gamma_f} | \mathbf{H}_{f_{12}, e, e'} | \psi_f \rangle &\propto \sum_{k_5, \tilde{e}_5} (-1)^{(1-o_5)k_5 + (1-o_0)k_0} (-1)^{k_1+k_2+k_3+k_4} \left\{ \begin{matrix} k_1 & k_2 & k_0 \\ k_3 & k_4 & k_5 \end{matrix} \right\}_q [d_{k_0}] \\
&\times \left(\sum_{\{\tilde{e}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right) o_5 \epsilon_5 [d_{k_5}] (-1)^{2k_5} (-1)^{k_1+k_2+k_3+k_4} o_2 \epsilon_2 [d_{k_2}] \right. \\
&\times \left\{ \begin{matrix} k_1 & j_1 & \frac{1}{2} \\ j_5 & k_5 & k_4 \end{matrix} \right\}_q \left\{ \begin{matrix} k_5 & j_5 & \frac{1}{2} \\ j_2 & k_2 & k_3 \end{matrix} \right\}_q \psi_i(j_1, j_2, k_3, k_4, j_5, \dots) \\
&\left. + (-1)^{d_{12,i} - d_{e,e'}} \alpha^{e, e'}(k_e, k_{e'}) \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right) \psi_i(k_1, k_2, k_3, k_4, k_5, \dots) \right). \quad (86)
\end{aligned}$$

We now recognize the coefficients in (81),

$$\begin{aligned}
\langle \{k_e\}_{\Gamma_f} | \mathbf{H}_{f_{12}, e, e'} | \psi_f \rangle &\propto \sum_{k_5} (-1)^{(1-o_5)k_5 + (1-o_0)k_0} (-1)^{k_1+k_2+k_3+k_4} \left\{ \begin{matrix} k_1 & k_2 & k_0 \\ k_3 & k_4 & k_5 \end{matrix} \right\}_q [d_{k_0}] \\
&\times \left(\sum_{\{\tilde{e}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right) \sum_{\tilde{e}_5 = \pm} A_{o_5}^{\tilde{e}_5, \tilde{e}_1}(k_5, k_1, k_4) A_{o_2}^{\tilde{e}_2, \tilde{e}_5}(k_2, k_5, k_3) \psi_i(j_1, j_2, k_3, k_4, j_5, \dots) \right. \\
&\left. + (-1)^{d_{12,i} - d_{e,e'}} \alpha^{e, e'}(k_e, k_{e'}) \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right) \psi_i(k_1, k_2, k_3, k_4, k_5, \dots) \right), \quad (87)
\end{aligned}$$

and conclude that (82) vanished provided (81) and (78).

2. Face f_{14}

We now perform the same analysis on the constraints which act on the face f_{14} . We use the same notation as for the face f_{12} , i.e., let e and e' be two reference edges around f_{14} and consider the Hamiltonian constraints associated to them on Γ_i and Γ_f . On Γ_i , the Hamiltonians contain the operator $\mathbf{E}_{e_1 e_4}^{e_1, e_4}$, which, without loss of generality, can be considered to give rise to an A term. The constraints on the spin network coefficients of $|\psi_i\rangle$ read

$$\sum_{\{\tilde{e}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right) A_{o_1}^{\tilde{e}_1, \tilde{e}_4}(k_1, k_4, k_5) \psi_i(j_1, k_2, k_3, j_4, k_5, \dots) + (-1)^{d_{14,i} - d_{e,e'}} \alpha^{e, e'}(k_e, k_{e'}) \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right) \psi_i(k_1, k_2, k_3, k_4, k_5, \dots) = 0, \quad (88)$$

with $j_1 = k_1 - \tilde{e}_1/2$, $j_4 = k_4 - \tilde{e}_4/2$. The sign \tilde{e}_1 (respectively, \tilde{e}_4) is fixed if $e = e_1$ (respectively, if $e' = e_4$) and summed over otherwise. Here, $\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A$ is the product of the A terms from e to e' counterclockwise, except for the one on the corner of e_1, e_4 which has been singled out. As for $\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B$, it is the product of the B terms going counterclockwise from e' to e .

On Γ_f , we need to look at two types of constraints. Either the operators $\mathbf{E}_{e_0 e_4}^{\tilde{e}_0, \tilde{e}_4}$ and $\mathbf{E}_{e_1 e_0}^{\tilde{e}_1, \tilde{e}_0}$, which enter $\mathbf{H}_{f_{14}, e, e'}$ on Γ_f , both contribute to A terms of the constraint (or both to B terms but this is the same), or one gives rise to an A -term and the other one to a B term.

In the case that they both give rise to A terms, we are in the same situation as in our previous analysis on the face f_{12} , with the role of Γ_i and Γ_f exchanged. Since the relation (78) between $|\psi_f\rangle$ and $|\psi_i\rangle$ can be inverted with the same form, we have nothing to prove.

If $\mathbf{E}_{e_0 e_4}^{\tilde{e}_0, \tilde{e}_4}$ contributes to a B term, and $\mathbf{E}_{e_1 e_0}^{\tilde{e}_1, \tilde{e}_0}$ contributes to an A term, then this means that $e_0 = e$ is a reference edge chosen for the constraint. The Hamiltonians of this type on Γ_f are $\mathbf{H}_{f_{14}, e_0, e'}^{e_0, e', \dots, e_4}$ and they are labeled by signs for all the edges from e' to e_0 counterclockwise. The matrix elements read

$$\begin{aligned}
 \langle \{k_e\}_{\Gamma_f} | \mathbf{H}_{f_{14, e_0, e'}}^{\epsilon_0, \epsilon', \dots, \epsilon_4} | \psi_f \rangle &\propto \sum_{\substack{\{\bar{e}\} \\ \text{c.c.}}} \left(\prod A \right) A_{\partial_1}^{\bar{e}_1, \epsilon_0}(k_1, k_0, k_2) \psi_f(j_1, k_2, k_3, j_4, j_0, \dots) \\
 &+ (-1)^{d_{14, f} - d_{e_0, e'}} \alpha^{\epsilon_0, \epsilon'}(k_0, k_{e'}) \sum_{\substack{\{\bar{e}\} \\ \text{c.c.}}} \left(\prod B \right) B_{\partial_0}^{\epsilon_0, \bar{e}_4}(j_0, j_4, k_3) \psi_f(k_1, k_2, k_3, l_4, k_0, \dots), \quad (89)
 \end{aligned}$$

where ϵ_0 is fixed (but \bar{e}_4 only is if $e' = e_4$) and $j_{0,4} = k_{0,4} - \epsilon_{0,4}/2$, and $l_4 = j_4 + \bar{e}_4/2$.

We now plug (78) into the above matrix elements. We first look at the A term,

$$\begin{aligned}
 A_{\partial_1}^{\epsilon_1, \epsilon_0}(k_1, k_0, k_2) \psi_f(j_1, k_2, k_3, j_4, j_0, \dots) &= [d_{j_0}] \sum_{k_5} o_1 \epsilon_1 [d_{k_1}] (-1)^{k_0 + k_1 + k_2} (-1)^{(1-o_5)k_5 + (1-o_0)j_0} (-1)^{j_1 + k_2 + k_3 + j_4} \\
 &\times \left\{ \begin{matrix} j_1 & k_2 & j_0 \\ k_3 & j_4 & k_5 \end{matrix} \right\}_q \left\{ \begin{matrix} k_1 & j_1 & \frac{1}{2} \\ j_0 & k_0 & k_2 \end{matrix} \right\}_q \psi_i(j_1, k_2, k_3, j_4, k_5, \dots). \quad (90)
 \end{aligned}$$

The relevant Biedenharn-Elliott identity is

$$\begin{aligned}
 &(-1)^{k_0 + k_1 + k_2 + j_0 + j_1 + k_3 + k_5} \left\{ \begin{matrix} j_1 & k_2 & j_0 \\ k_3 & j_4 & k_5 \end{matrix} \right\}_q \left\{ \begin{matrix} k_1 & j_1 & \frac{1}{2} \\ j_0 & k_0 & k_2 \end{matrix} \right\}_q \\
 &= \sum_{l_4} [d_{l_4}] (-1)^{j_4 + l_4 + \frac{1}{2}} \left\{ \begin{matrix} j_4 & l_4 & \frac{1}{2} \\ k_1 & j_1 & k_5 \end{matrix} \right\}_q \left\{ \begin{matrix} j_4 & l_4 & \frac{1}{2} \\ k_0 & j_0 & k_3 \end{matrix} \right\}_q \left\{ \begin{matrix} k_1 & k_2 & k_0 \\ k_3 & l_4 & k_5 \end{matrix} \right\}_q. \quad (91)
 \end{aligned}$$

As for the B term,

$$\begin{aligned}
 \sum_{l_4} B_{\partial_0}^{\epsilon_0, \bar{e}_4}(j_0, j_4, k_3) \psi_f(k_1, k_2, k_3, l_4, k_0, \dots) &= \sum_{k_5, l_4} o_0 \epsilon_0 [d_{j_0}] [d_{k_0}] (-1)^{j_0 + k_3 + j_4} (-1)^{(1-o_5)k_5 + (1-o_0)k_0} (-1)^{k_1 + k_2 + k_3 + l_4} \\
 &\times \left\{ \begin{matrix} j_4 & l_4 & \frac{1}{2} \\ k_0 & j_0 & k_3 \end{matrix} \right\}_q \left\{ \begin{matrix} k_1 & k_2 & k_0 \\ k_3 & l_4 & k_5 \end{matrix} \right\}_q \psi_i(k_1, k_2, k_3, l_4, k_5, \dots). \quad (92)
 \end{aligned}$$

We recognize the two same $q - 6j$ symbols as in the Biedenharn-Elliott identity above. We can thus factor them out, so that the matrix elements of the Hamiltonian are proportional to

$$\begin{aligned}
 &\langle \{k_e\}_{\Gamma_f} | \mathbf{H}_{f_{14, e_0, e'}}^{\epsilon_0, \epsilon', \dots, \epsilon_4} | \psi_f \rangle \\
 &\propto \sum_{k_5, l_4} (-1)^{j_0 + k_3 + j_4} (-1)^{(1-o_5)k_5 + (1-o_0)k_0} (-1)^{k_1 + k_2 + k_3 + l_4} \left\{ \begin{matrix} j_4 & l_4 & \frac{1}{2} \\ k_0 & j_0 & k_3 \end{matrix} \right\}_q \left\{ \begin{matrix} k_1 & k_2 & k_0 \\ k_3 & l_4 & k_5 \end{matrix} \right\}_q [d_{j_0}] \\
 &\times \left(\sum_{\substack{\{\bar{e}\} \\ \text{c.c.}}} \left(\prod A \right) o_0 o_1 \epsilon_1 [d_{k_1}] [d_{l_4}] (-1)^{k_2 + j_4 - j_0 - k_5} (-1)^{j_4 + l_4 + \frac{1}{2}} (-1)^{j_0 + k_3 + j_4} (-1)^{k_1 + k_2 + k_3 + l_4} \left\{ \begin{matrix} j_4 & l_4 & \frac{1}{2} \\ k_1 & j_1 & k_5 \end{matrix} \right\}_q \right. \\
 &\left. \times \psi_i(j_1, k_2, k_3, j_4, k_5, \dots) + (-1)^{d_{14, f} - d_{e_0, e'}} \alpha^{\epsilon_0, \epsilon'}(k_0, k_{e'}) \sum_{\substack{\{\bar{e}\} \\ \text{c.c.}}} \left(\prod B \right) o_0 \epsilon_0 [d_{k_0}] \psi_i(k_1, k_2, k_3, l_4, k_5, \dots) \right). \quad (93)
 \end{aligned}$$

It now suffices us to show that the expression into brackets vanishes thanks to (88). Let us take care of the following signs:

$$(-1)^{k_2 + j_4 - j_0 - k_5} (-1)^{j_4 + l_4 + \frac{1}{2}} (-1)^{j_0 + k_3 + j_4} (-1)^{k_1 + k_2 + k_3 + l_4} = (-1)^{2(k_2 - k_5 + k_3)} (-1)^{4j_4} (-1)^{\frac{1 + \bar{e}_4}{2}} (-1)^{k_1 + l_4 + k_5} = -\bar{e}_4 (-1)^{k_1 + l_4 + k_5}.$$

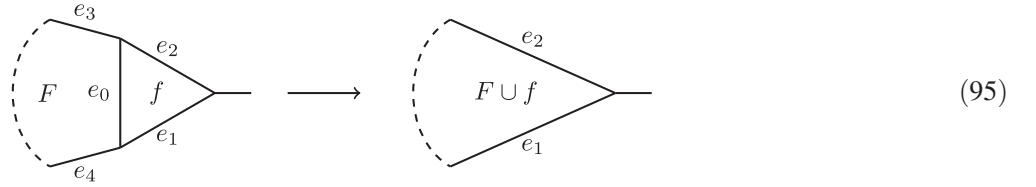
Replacing the sign factor in the bracket, we get

$$\begin{aligned}
 & - \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e_1 \rightarrow e' \\ \text{c.c.}}} A \right) o_0 o_1 \epsilon_1 [d_{k_1}] [d_{l_4}] (-1)^{j_1 - k_1 + j_4 - l_4} (-1)^{j_1 + l_4 + k_5 + \frac{1}{2}} \left\{ \begin{matrix} j_4 & l_4 & \frac{1}{2} \\ k_1 & j_1 & k_5 \end{matrix} \right\}_q \psi_i(j_1, k_2, k_3, j_4, k_5, \dots) \\
 & + (-1)^{d_{14, f} - d_{e_0 e'}} \alpha^{\epsilon_0, \epsilon'}(k_0, k_{e'}) \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e_4 \\ \text{c.c.}}} B \right) o_0 \epsilon_0 [d_{k_0}] \psi_i(k_1, k_2, k_3, l_4, k_5, \dots) \\
 & = -o_0 \tilde{e}_4 [d_{l_4}] \left(\sum_{\{\tilde{e}\}} \left(\prod_{\substack{e_1 \rightarrow e' \\ \text{c.c.}}} A \right) A_{o_1}^{\epsilon_1, \tilde{e}_4}(k_1, l_4, k_5) \psi_i(j_1, k_2, k_3, j_4, k_5, \dots) \right. \\
 & \left. + (-1)^{d_{14, i} - d_{e_4 e'}} \alpha^{\tilde{e}_4, \epsilon'}(l_4, k_{e'}) \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e_4 \\ \text{c.c.}}} B \right) \psi_i(k_1, k_2, k_3, l_4, k_5, \dots) \right). \tag{94}
 \end{aligned}$$

The expression into brackets on the rhs is exactly the constraint (88) on Γ_i with the choice $e = e_4$ of reference edge and arbitrary \tilde{e}_4 fixed.

B. Removing an edge

Consider two adjacent faces F and f , separated by an edge e_0 . We consider the move which consists in removing e_0 (as well as its two end vertices). By performing a series of 2-2 Pachner moves (described in Sec. VA), we can always assume that f is triangular,



If $|\psi_i\rangle$ is a state that satisfies all the constraints before the edge removal, then we want to describe how it transforms through the move.

Theorem 5.2. $|\psi_f\rangle$ with spin network coefficients

$$\psi_f(j_1, j_2, \dots) = (-1)^{(1+o_1)j_1 + (1+o_2)j_2} \sqrt{[d_{j_1}][d_{j_2}]} \psi_i(0, j_1, j_2, j_2, j_1, \dots) \tag{96}$$

is a solution of the constraints on the graph after the edge removal. Here o_1, o_2 are the orientations of the edges e_1, e_2 with respect to f (counterclockwise oriented) and $\psi(j_0, j_1, j_2, j_3, j_4, \dots)$ is the spin network coefficient of $|\psi_i\rangle$.

In other words, $|\psi_i\rangle$ gives rise to a solution of the constraints on Γ_f , obtained by keeping only its $j_0 = 0$ components. We will use this relation to study the 3-1 Pachner move.

Proof.—Consider two reference edges e, e' in $F \cup f$, and the associated constraint such that $\mathbf{E}_{e_1 e_2}^{\tilde{e}_1, \tilde{e}_2}$ is an A term (without loss of generality since A and B terms can be exchanged). Its matrix elements $\langle \{k_e\} | \mathbf{H}_{F \cup f, e, e'} | \psi_f \rangle$ read

$$\langle \{k_e\} | \mathbf{H}_{F \cup f, e, e'} | \psi_f \rangle \propto \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right) A_{o_2}^{\tilde{e}_2, \tilde{e}_1}(k_2, k_1, l_2) \psi_f(j_1, j_2, \dots) + (-1)^{d_f - d_{e e'}} \alpha^{\epsilon_e, \epsilon'}(k_e, k_{e'}) \sum_{\{\tilde{e}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right) \psi_f(k_1, k_2, \dots), \tag{97}$$

with $j_{1,2} = k_{1,2} - \tilde{e}_{1,2}/2$, and d_f denotes the number of boundary edges surrounding $F \cup f$. Here, $\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A$ is the product of the A terms from e to e' counterclockwise, except for the one on the corner of e_1, e_2 , which has been singled out. We will show those matrix elements vanish as soon as the constraints on f and on F are both satisfied on $|\psi_i\rangle$, given (96).

On f , we have the constraint, for fixed ϵ_1, ϵ_2 , and $j_{1,2} = k_{1,2} - \epsilon_{1,2}/2$,

$$B_{o_2}^{-\epsilon_2, -\epsilon_1}(k_2, k_1, l_2) \psi_i(k_0, j_1, j_2, k_3, k_4, \dots) = \alpha^{\epsilon_1, \epsilon_2}(k_1, k_2) \sum_{\epsilon_0 = \pm} A_{o_{0,f}}^{\epsilon_0, -\epsilon_2}(k_0, j_2, k_3) A_{o_1}^{-\epsilon_1, \epsilon_0}(j_1, k_0, k_4) \times \psi_i\left(k_0 - \frac{\epsilon_0}{2}, k_1, k_2, k_3, k_4, \dots\right), \quad (98)$$

where $o_{0,f}$ is the orientation of e_0 relative to f . On F there is a constraint similar to (97), from the Hamiltonians $\mathbf{H}_{F,e,e'}$ with the same signs ϵ s. It reads,

$$\sum_{\{\tilde{\epsilon}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right) \sum_{\tilde{\epsilon}_0 = \pm} A_{o_{0,F}}^{\tilde{\epsilon}_0, \tilde{\epsilon}_4} \left(k_0, k_4 + \frac{\tilde{\epsilon}_4}{2}, k_1 \right) A_{o_3, \tilde{\epsilon}_0}^{\tilde{\epsilon}_3, \tilde{\epsilon}_0} \left(k_3 + \frac{\tilde{\epsilon}_3}{2}, k_0, k_2 \right) \psi_i \left(k_0 - \frac{\tilde{\epsilon}_0}{2}, k_1, k_2, k_3 - \frac{\tilde{\epsilon}_3}{2}, k_4 - \frac{\tilde{\epsilon}_4}{2}, \dots \right) + (-1)^{d_f - d_{e,e'}} \alpha^{\epsilon_e, \epsilon_{e'}}(k_e, k_{e'}) \sum_{\{\tilde{\epsilon}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right) \psi_i(k_0, k_1, k_2, k_3, k_4, \dots) = 0, \quad (99)$$

where $o_{0,F}$ is the orientation of e_0 as the boundary of F , which is opposite to $o_{0,f}$. Here $\tilde{\epsilon}_4$ (respectively, $\tilde{\epsilon}_3$) is fixed if $e = e_4$ (respectively, if $e' = e_3$) and summed over otherwise.

We now specialize (98) and (99) to $k_0 = 0$, where they simplify a lot. First, that enforces $\epsilon_0 = -$ in (98) and $\tilde{\epsilon}_0 = -$ in (99), so that those sums reduce to a single term. In (99) we further take $k_3 = k_2$ and $k_4 = k_1$. All $q - 6j$ symbols with a spin equal to 0 can be evaluated as $\left\{ \begin{matrix} j_1 & k_1 & \frac{1}{2} \\ \frac{1}{2} & 0 & k_4 \end{matrix} \right\}_q = \delta_{j_1, k_4} (-1)^{j_1 + k_1 + \frac{1}{2}} / \sqrt{[2][d_{j_1}]}$.

As a consequence, (98) gives

$$B_{o_2}^{-\epsilon_2, -\epsilon_1}(k_2, k_1, l_2) \psi_i(0, j_1, j_2, j_2, j_1, \dots) = -\alpha^{\epsilon_1, \epsilon_2}(k_1, k_2) o_{0,f} o_1 \epsilon_2 \frac{[d_{j_1}]}{[2] \sqrt{[d_{j_1}][d_{j_2}]}} \psi_i\left(\frac{1}{2}, k_1, k_2, j_2, j_1, \dots\right), \quad (100)$$

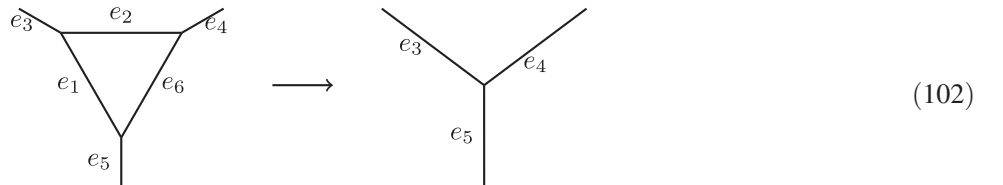
where $k_3 = j_2$ and $k_4 = j_1$ on the last term are enforced by the special evaluations of the $q - 6j$ symbols with a spin 0. Equation (99) gives

$$\sum_{\{\tilde{\epsilon}\}} \left(\prod_{\substack{e \rightarrow e' \\ \text{c.c.}}} A \right) o_{0,f} o_3 \epsilon_1 \frac{[d_{k_2}]}{[2] \sqrt{[d_{k_1}][d_{k_2}]}} \psi_i\left(\frac{1}{2}, k_1, k_2, j_2, j_1, \dots\right) + (-1)^{d_f - d_{e,e'}} \alpha^{\epsilon_e, \epsilon_{e'}}(k_e, k_{e'}) \sum_{\{\tilde{\epsilon}\}} \left(\prod_{\substack{e' \rightarrow e \\ \text{c.c.}}} B \right) \psi_i(0, k_1, k_2, k_2, k_1, \dots) = 0. \quad (101)$$

The term $\psi_i(\frac{1}{2}, k_1, k_2, j_2, j_1, \dots)$ can be eliminated using (100). Moreover we turn the B coefficient of this equation into an A coefficient using $-A_{o_2}^{\epsilon_2, \epsilon_1}(k_2, k_1, l_2) = B_{o_2}^{-\epsilon_2, -\epsilon_1}(k_2, k_1, l_2)$. It is then enough to recognize ψ_f as given in (96) to obtain that (97) vanishes. ■

C. 3-1 Pachner move

The 3-1 Pachner move removes a triangular face from the graph and replaces it with a vertex. The edges incident to the face become incident to the vertex,



(102)

The orientations of all the edges are left arbitrary.

Theorem 5.3. If $|\psi_i\rangle$ is a state on the initial graph Γ_i that satisfies all the constraints, then its spin network coefficients can be written

$$\psi_i(j_1, j_2, j_3, j_4, j_5, j_6, \dots) = (-1)^{(1+o_1)j_1+(1+o_2)j_2+(1+o_6)j_6} (-1)^{j_3+j_4+j_5} \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}_q \psi_f(j_3, j_4, j_5, \dots), \quad (103)$$

where $\psi_f(j_3, j_4, j_5, \dots)$ are the spin network coefficients of a state $|\psi_f\rangle$, which satisfies all the constraints on the final graph Γ_f .

Proof.—Let us write the constraints on the triangular face. There is one constraint for each pair of edges of the boundary. For the pair (e_2, e_6) , for instance, one gets

$$\sum_{\epsilon_1=\pm} A_{o_1}^{\epsilon_1, \epsilon_2}(k_1, k_2, k_3) A_{o_6}^{\epsilon_6, \epsilon_1}(k_6, k_1, k_5) \psi_i(j_1, j_2, k_3, k_4, k_5, j_6, \dots) + \alpha^{\epsilon_2, \epsilon_6}(k_2, k_6) B_{o_2}^{\epsilon_2, \epsilon_6}(j_2, j_6, k_4) \psi_i(k_1, k_2, k_3, k_4, k_5, k_6, \dots) = 0. \quad (104)$$

Here $j_i = k_i - \epsilon_i/2$, for $i = 1, 2, 6$. The coefficients are

$$\begin{aligned} A_{o_1}^{\epsilon_1, \epsilon_2}(k_1, k_2, k_3) &= o_1 \epsilon_1 [d_{k_1}] (-1)^{k_1+k_2+k_3} \left\{ \begin{matrix} k_1 & j_1 & \frac{1}{2} \\ j_2 & k_2 & k_3 \end{matrix} \right\}_q, \\ A_{o_6}^{\epsilon_6, \epsilon_1}(k_6, k_1, k_5) &= o_6 \epsilon_6 [d_{k_6}] (-1)^{k_1+k_5+k_6} \left\{ \begin{matrix} k_6 & j_6 & \frac{1}{2} \\ j_1 & k_1 & k_5 \end{matrix} \right\}_q, \\ B_{o_2}^{\epsilon_2, \epsilon_6}(j_2, j_6, k_4) &= o_2 \epsilon_2 [d_{j_2}] (-1)^{j_2+k_4+j_6} \left\{ \begin{matrix} k_6 & j_6 & \frac{1}{2} \\ j_2 & k_2 & k_4 \end{matrix} \right\}_q. \end{aligned} \quad (105)$$

We thus have the recursion

$$\begin{aligned} \sum_{\epsilon_1=\pm} o_1 o_2 o_6 [d_{k_1}] (-1)^{2k_1+k_2+k_3+k_5+k_6+\frac{1-\epsilon_1}{2}+\frac{\epsilon_2}{2}+\frac{\epsilon_6}{2}} \left\{ \begin{matrix} k_1 & j_1 & \frac{1}{2} \\ j_2 & k_2 & k_3 \end{matrix} \right\}_q \left\{ \begin{matrix} k_6 & j_6 & \frac{1}{2} \\ j_1 & k_1 & k_5 \end{matrix} \right\}_q \psi_i(j_1, j_2, k_3, k_4, k_5, j_6, \dots) \\ + (-1)^{k_2+k_4+k_6} \left\{ \begin{matrix} k_6 & j_6 & \frac{1}{2} \\ j_2 & k_2 & k_4 \end{matrix} \right\}_q \psi_i(k_1, k_2, k_3, k_4, k_5, k_6, \dots) = 0, \end{aligned} \quad (106)$$

and similarly for the pairs (e_1, e_2) , (e_6, e_1) . A similar result for the flat case was found in [10], where q is set to 1. Those recursions determine the dependence of ψ_i on j_1, j_2, j_6 up to a single initial condition. As the recursion involves three terms, it may seem like several initial conditions are required. However, at $k_1 = 0$, only two terms are left in the recursion, as shown in Eq. (100). This means that from the initial condition $\psi_i(0, k_3, k_3, k_4, k_5, k_5)$, one gets $\psi_i(\frac{1}{2}, k_3 - \epsilon_2/2, k_3, k_4, k_5, k_5 - \epsilon_6/2)$. Then this determines ψ_i for arbitrary k_1, k_2, k_6 . The result is known to be

$$\psi_i(k_1, k_2, k_3, k_4, k_5, k_6, \dots) = \left\{ \begin{matrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \end{matrix} \right\}_q \phi(k_3, k_4, k_5, \dots), \quad (107)$$

where $\phi(k_3, k_4, k_5, \dots)$ is independent of k_1, k_2, k_6 . To determine ϕ , we set $k_1 = 0$,

$$\phi(k_3, k_4, k_5, \dots) = \left\{ \begin{matrix} 0 & k_3 & k_3 \\ k_4 & k_5 & k_5 \end{matrix} \right\}_q^{-1} \psi_i(0, k_3, k_3, k_4, k_5, k_5, \dots) = (-1)^{k_3+k_4+k_5} \sqrt{[d_{k_3}][d_{k_5}]} \psi_i(0, k_3, k_3, k_4, k_5, k_5, \dots). \quad (108)$$

We conclude with Theorem 5.2. ■

The relation between the physical states before and after the 3-1 Pachner move provides a way to relate the q -deformed LQG to the Turaev-Viro model with q real. Consider the graphs on two adjacent time slices in a spin foam different by a $3 - 1$ move. This part of the spin foam gives a Turaev-Viro vertex amplitude, which is simply a $q - 6j$ symbol. We have

reproduced this vertex amplitude in (107) by relating the coefficients of the physical states before and after the Pachner move. This is also consistent with the method to relate LQG to the spin foam model by considering the physical scalar product of states introduced in [8].

In this section, we have proved that the physical states for graphs related by a Pachner move, either 2-2 move or 3-1 move, are equivalent hence the physical states are topological states. The equivalence is shown by the exact relation between the coefficients of the spin network basis for physical states before and after the Pachner move as shown in (78) and (103). This also justifies the validity of the Hamiltonian expression (65) from the direct quantization of the scalar products of deformed spinors in the classical Hamiltonian (36).

VI. CONCLUSION

In this paper, we have given a realization of the interplay between the cosmological constant, curved geometries and quantum group structure in the 3D loop quantum gravity framework in Euclidean signature with a negative cosmological constant, which we call the q -deformed LQG model. In particular, the deformed constraints at the classical level represent discrete hyperbolic geometries, as shown in [27]. Upon the standard quantization procedure, these deformed constraints become quantum constraints with a quantum group structure.

We have focused on the Hamiltonian constraints, obtained from the flatness constraints. We have written them with the deformed spinors and performed the quantization following the companion paper [31]. The result is a generalization of the quantum Hamiltonian constraints derived in [10] for flat space. By studying the way the solutions to the quantum constraints change under Pachner moves, we provide a generalization of the Noui-Perez transition amplitudes [8] to $q \neq 1$ real: the transition amplitudes are the coefficients relating the physical states

in the spin network basis under Pachner moves. Here, they clearly lead to a Turaev-Viro model for q real. It is a topological model (with the same finiteness issues as the $q = 1$ version, the Ponzano-Regge model).

Our method is radically different from [8], however, and maybe more in the spirit of LQG. On its way to linking q -deformed LQG to spin foams, our method derives the Wheeler-DeWitt equations as difference equations on the spin network coefficients of the states, see Eq. (67). In the flat case, the Hamiltonian constraint can be interpreted as displacements of the vertices of the triangulation [30]. Our difference equations (67) are quantum implementations of those symmetries.

Although our constraints are in fact derived from the flatness constraints, we believe that this approach is promising to study both how to incorporate the cosmological constant in 4D and how to write interesting dynamics for curved 4D geometries. A first step in the continuous theory has been initiated in [37].

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