Penrose inequality in holography

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(Received 19 August 2022; accepted 23 December 2022; published 19 January 2023)

The recent holographic deduction of the Penrose inequality only assumes the null energy condition, while a weak or dominant energy condition is required in the usual geometric proof. We take a step toward filling the gap between these two approaches. For planar or spherically symmetric asymptotically Schwarzschild anti-de Sitter (AdS) black holes, we give a purely geometric proof for the Penrose inequality by assuming the null energy condition. We also point out that two naive generalizations of the charged Penrose inequality are generally not true, and we propose two new candidates. When the spacetime is asymptotically AdS but not Schwarzschild-AdS, the total mass is defined according to holographic renormalization and depends on the scheme of quantization. In this case, the holographic argument implies that the Penrose inequality should still be valid, but we use a concrete example to show that whether the Penrose inequality holds or not will depend on what kind of quantization scheme we employ.

DOI: 10.1103/PhysRevD.107.026013

I. INTRODUCTION

In general relativity, there are many well-known universal inequalities, such as the Penrose inequality [1,2], the positive mass theorem [3,4], the second law of black holes [5,6], and so on. As a theoretical test, the Penrose inequality is related to the establishment of cosmic censorship. Specifically, given the ADM mass or energy M of a four-dimensional asymptotically flat spacetime which contains a black hole as the initial data and denoting A to be the minimal area of the surface enclosing the apparent horizon σ, the Penrose inequality states that the spacetime's total mass M should be at least $\sqrt{A/16\pi}$ and the saturation appears only when the exterior is Schwarzschild. As pointed out by Ref. [7], the apparent horizon area, in general, may not satisfy the Penrose inequality. Penrose's argument [8] is as follows: If we wait a very long time, the black hole will eventually settle down to a Kerr solution. In a Kerr solution, the relationship between the black hole's mass $M_{\rm kerr}$ and the area of event horizon $A_{\rm ev}$ is $M_{\rm kerr} \geq \sqrt{A_{\rm ev}/16\pi}$. Under this evolution, the black hole's mass, which is described by the Bondi mass, cannot increase. Assuming cosmic censorship and appropriate energy conditions, the apparent horizon either lies within or coincides with the event horizon. Combining with the second law of black holes that states the area of the event horizon cannot decrease, Penrose found his inequality immediately:

$$M \geq \sqrt{\frac{A}{16\pi}}. \tag{1}$$

It is worth noting that the above argument is based on a lot of mathematical or physical assumptions. Although mathematicians have proven that the Penrose inequality is true in certain cases, there is no general proof for the Penrose inequality(see, e.g., Refs. [9,10]).

Taking the same argument, we can also conjecture the Penrose inequality for four-dimensional asymptotically AdS spacetime [11]:

$$M \geq \left(\frac{A}{16\pi}\right)^{\frac{1}{2}} + \frac{1}{2\ell_{\rm AdS}^2}\left(\frac{A}{4\pi}\right)^{\frac{3}{2}}, \tag{2}$$

where M is the total mass or energy defined according to holographic renormalization and A is the minimal area to enclose the apparent horizon σ for this asymptotically AdS spacetime. Here we have absorbed the Casimir energy [12] into the definition of total mass. When the black hole is described by the Schwarzschild-AdS solution, the inequality takes an equal sign. Another way to phrase this conjecture is as follows: Given the same mass, the minimal area to enclose the apparent horizon is bounded by the area of the AdS Schwarzschild black hole's horizon. In holography, the bulk's geometry is dual to the two asymptotical

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boundary's QFT state [13,14]. For each boundary's reduced density matrix, its holographic entropy is proportional to the area of the black hole's apparent horizon [15,16]. Moreover, given the same total mass M, Ref. [17] shows via holography that the boundary's QFT state dual to the Schwarzschild-AdS black hole has the maximum entropy. Consequently, from the above basic holography's argument, the AdS Penrose inequality can be

$$A \le \max A = A_{\mathrm{sch}}. \tag{3}$$

Here A_{sch} stands for the horizon area of the Schwarzschild-AdS black hole with the same total energy. This idea was recently used by Ref. [18] to argue the Penrose inequality in asymptotically AdS spacetime. We note that Refs. [17,18], though, discussed the AdS black hole in which the cross section of the event horizon has spherical topology, regardless of the topology of the event horizon. Thus, if one follows their discussions, one obtains generalized Penrose inequalities of asymptotically AdS black holes with planar or hyperbolic topologies.

For charged black holes, one might wonder if the charged generalization for the Penrose inequality can be argued with Penrose's original idea. However, this does not work for the charged cases. Given the initial data mass M and charge Q, the relation between the initial total mass M, the final black hole mass M_{RN}, and the final event horizon area A_{ev}, before using the second law of black holes, is

$$M \ge M_{\mathrm{RN}} = \left(\frac{A_{\mathrm{ev}}}{16\pi}\right)^{\frac{1}{2}} + \frac{1}{2\ell_{\mathrm{AdS}}^2}\left(\frac{A_{\mathrm{ev}}}{4\pi}\right)^{\frac{3}{2}} + \frac{Q^2}{2}\sqrt{\frac{4\pi}{A_{\mathrm{ev}}}}. \tag{4}$$

Here we assume that no charge can be radiated away. Assuming cosmic censorship and appropriate energy conditions, the event horizon area A_{ev} is larger than the minimal area A of the surface enclosing the apparent horizon according to the second law of black holes, i.e., $A_{\mathrm{ev}} \ge A$. If the right-hand side of the inequality (4) is a monotonically increasing function of area, then we would obtain the charged generalization proposed by Ref. [11],

$$M \ge \left(\frac{A}{16\pi}\right)^{\frac{1}{2}} + \frac{1}{2\ell_{\mathrm{AdS}}^2}\left(\frac{A}{4\pi}\right)^{\frac{3}{2}} + \frac{Q^2}{2}\sqrt{\frac{4\pi}{A}}. \tag{5}$$

Unfortunately, such monotonicity is not so obvious. Thus, one should not be surprised if the charged generalization (5) is broken in some cases. In fact, counterexamples of (5) have been reported by Refs. [19,20] for the case $\ell_{\mathrm{AdS}} \to \infty$. Though the original idea of Penrose's is invalid, the logic of the holographic argument proposed by Ref. [18] still works. If such a holographic argument is really true, the minimal area A would be bounded by the event horizon area of Reissner-Nordström (RN) black holes if fixing the total mass M and charge Q,

$$A(M,Q) \le A_{\mathrm{RN}}(M,Q). \tag{6}$$

This is a different generalization of the Penrose inequality, which was holographically argued to be true by Ref. [18].

Although the recent holographic argument for the Penrose inequality does not need to assume cosmic censorship, it requires matter to satisfy the null energy condition in the bulk[1] [18]. However, in recent years the Penrose inequality has been proven in certain cases, including the asymptotically AdS spacetimes, which require that a dominant or weak energy condition [11,21,22]. Both dominant and weak energy conditions are stronger than the null energy condition. This forms a gap between the holographic argument and current geometric proofs on the Penrose inequality in asymptotically AdS spacetime. If matter decays rapidly enough near the AdS boundary, the total mass M and minimal area A are geometrically well defined. The Penrose inequality in this case becomes a purely geometric inequality. Since the argument of Ref. [18] uses the conjecture of holography, if its conclusion is true, then it is necessary to ask the following: Is it possible to find a purely geometric proof for the AdS Penrose inequality under the null energy condition without referring to the unproved conjecture of the holographic principle?

According to the holographic argument for charged black holes, RN black holes will have the maximum entropy, so the charged generalization is given by Eq. (6). If setting the AdS radius ℓ_{AdS} to infinity, the charged generalization (6) will follow the generalized Penrose inequality in asymptotically flat spacetime,

$$\left(\frac{A}{16\pi}\right)^{1/2} \le \frac{1}{2}\left[M + \sqrt{M^2 - Q^2}\right]. \tag{7}$$

However, there are also counterexamples [19,20] for such a charged generalization (7). Thus, we believe that the charged generalization (6) from the holographic argument is not generally true under the null energy condition. What is the correct generalization for the charged case? Such counterexamples also give us enough motivation and necessity to seek purely geometric checks for the conclusions obtained from the holographic principle.

So far we have assumed that matter decays rapidly enough near the AdS boundary and that black holes are in fact asymptotically Schwarzschild-AdS black holes.[2] Both total mass[3] M and the area of the minimal

[1]The Penrose inequality focuses on the black hole's horizon and its exterior. More precisely, the matter should satisfy the null energy condition in the black hole's exterior.

[2]"Asymptotically Schwarzschild-AdS" is stronger than "asymptotically AdS", see Ref. [23].

[3]In the following proof, we will not distinguish between ADM mass and Bondi mass since the ADM mass is equal to Bondi mass for static asymptotically Schwarzschild-AdS black holes.

surface[4] A are determined by the bulk's geometry in this case [24]. If matter does not decay rapidly enough, the spacetime may still be asymptotically AdS but not asymptotically Schwarzschild-AdS. In this case, the situation is complicated. For instance, the existence of matter on the AdS boundary will contribute to the total mass for asymptotically AdS black holes according to holographic renormalization; see, e.g., Refs. [25,26]. Since the total mass obtained from the holographic renormalization is not determined by bulk geometry, can the null energy condition in the bulk still guarantee the Penrose inequality?

This paper aims to answer the question above (at least partially). For static asymptotically Schwarzschild-AdS black holes, we prove that the null energy condition can guarantee the Penrose inequality only for planar or spherical horizon geometry cases; however, to guarantee the inequality for the hyperbolically symmetric case, we have to assume a weak energy condition. A concrete counterexample is given to show that the Penrose inequality is broken for the hyperbolic horizon geometry under the null energy condition. This implies that the conclusions of Refs. [17,18] implicitly depend on the topology of the event horizon, though a more detailed reason is still unclear to us. For a charged black hole, as we have explained, the naive generalization (5) and holographic version (6) are both incorrect. We then propose two kinds of charged generalizations of the Penrose inequality. As we mentioned before, the total mass is very subtle for asymptotically AdS black holes. This paper follows the standard holographic renormalization procedure [27,28], and we obtain the holographic mass as the total mass. Without loss of generality, we construct an asymptotically AdS black hole coupled to a scalar field to check the Penrose inequality in holography. When the source of the scalar field is nonzero, the spacetime is asymptotically AdS but not asymptotically Schwarzschild-AdS. In this case, we find that the null energy condition is not enough to guarantee the Penrose inequality. More precisely, whether the inequality holds or not in this case depends on what kind of quantization scheme we employ.

The organization of this paper is as follows. In Sec. II, given the metric ansatz for static $(d+1)$-dimensional asymptotically Schwarzschild-AdS black holes, we find that the null energy condition guarantees the Penrose inequality only for spherically and planar symmetric black holes. In Sec. III, we propose two types of charged generalizations for the Penrose inequality and prove them in the static planar and spherically symmetric cases. In Sec. IV, we construct a four-dimensional Einstein-scalar gravity and numerically check the Penrose inequality with two different quantization schemes for the scalar field sector.

II. PROOF OF PENROSE INEQUALITY WITH NULL ENERGY CONDITION

In this section, we propose a general version of the Penrose inequality in $(d+1)$-dimensional asymptotically AdS spacetime with the null energy condition and prove it under spherically, planar, or hyperbolic symmetric cases. But before the general proof, we first revisit the Penrose inequality in Eqs. (1) and (2) in four-dimensional spacetime and give some comments. In this paper, we consider three kinds of topologies for the event horizon, which are denoted by the parameter k. For asymptotically flat spacetime, only the spherical topology $(k=+1)$ can exist in a black hole solution. However, in asymptotically AdS spacetime, the black holes can have three topologies for the cross section of the event horizon, i.e., the spherical $(k=1)$, planar $(k=0)$, and hyperbolic $(k=-1)$ topologies. The Penrose inequality should then be generalized into

$$M \geq \left(\frac{A}{16\pi}\right)^{\frac{1}{2}} k + \frac{1}{2\ell_{\mathrm{AdS}}^2}\left(\frac{A}{4\pi}\right)^{\frac{3}{2}}. \tag{8}$$

As pointed out by Ref. [11], there is nonzero Casimir energy [12] for spherical and hyperbolic topologies. Here we have absorbed the Casimir energy into the definition of total mass to simplify our notations. For the hyperbolic and planar geometries, the volume of the cross section will be infinite, which will leave the inequality (8) meaningless. However, for a static asymptotically AdS spacetime, we can always choose the coordinate gauge so that the leading term of the metric near the AdS boundary has the following form:

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\Sigma_{k,d-1}^2. \tag{9}$$

Here $d\Sigma_{k,d-1}$ is the transverse metric of the unit sphere, planar, or hyperboloid defined by Eq. (15). We denote $\Omega_{k,d-1} \coloneqq \int d\Sigma_{k,d-1}$. For the event horizon, we can always define an "effective" radius r_h according to the equation

$$A = \Omega_{k,d-1} r_h^{d-1}. \tag{10}$$

Similarly, we can introduce a "mass density parameter" f_0 according to

$$M = \frac{(d-1)\Omega_{k,d-1}}{16\pi} f_0^d. \tag{11}$$

Although both the total mass M and area A are infinite in the planar and hyperbolic cases, we notice that the mass parameter f_0^d and the horizon radius r_h are always finite.

[4]For stationary solutions, the apparent horizon will coincide with the event horizon. So the minimal area to enclose apparent horizon A is just the area of event horizon.

The Penrose inequality (8) can then be reorganized in terms of the following inequality (see, e.g., Refs. [11,21,22]),

$$\frac{1}{\ell_{\mathrm{AdS}}^2}+\frac{k}{r_h^2}-\frac{f_0^d}{r_h^d}\le 0, \tag{12}$$

for general dimensional and all three different topologies of the horizon. We then propose the following conjecture for static asymptotically Schwarzschild-AdS black holes.

Conjecture 1. For a static asymptotically Schwarzschild-AdS black hole, if (1) the Einstein equation is satisfied, (2) matter's energy momentum tensor $T_{\mu\nu}$ satisfies the null energy condition, and (3) the cross section of the event horizon has spherical or planar topology, then the inequality (12) is true, and the saturation appears only if the exterior of the event horizon is Schwarzschild-AdS.

Note that the parameters f_0 and r_h in asymptotically Schwarzschild-AdS black holes will be determined completely by the bulk geometry, so we expect that there should be a geometrical proof without referring to the conjecture of AdS/CFT. If we recall the inequality (12), conjecture 1 then implies

$$f_0^d\ge r_h^d\left(\frac{1}{\ell_{\mathrm{AdS}}^2}+\frac{k}{r_h^2}\right). \tag{13}$$

Combining this with the definition for the total mass (11), we can see that the Penrose inequality is the stronger version of the positive energy theorem if the cross section of the event horizon has planar or spherical topology. This is interesting and seemingly surprising since the local energy density could be negative under the null energy condition. In the following, we give such a geometrical proof in spherically and planar symmetric static cases. We also give a detailed counterexample to show that the inequality (12) can be broken in hyperbolic topology if we impose only the null energy condition.

A. Einstein equation in spherically, planar, and hyperbolically symmetric cases

For spherical, planar, and hyperbolic symmetric geometries, the metric ansatz for asymptotically $(d+1)$-dimensional black holes is given by

$$ds^2=-f(r)e^{-\chi(r)}dt^2+\frac{dr^2}{f(r)}+r^2d\Sigma_{k,d-1}^2. \tag{14}$$

Here $k=0,\pm1$ represents different symmetric cases:

$$d\Sigma_{k,d-1}^2=\begin{cases} d\Omega_{d-1}^2=d\theta^2+\sin^2\theta d\Omega_{d-2}^2 & \text{for } k=+1\\ d\ell_{d-1}^2=\sum_{i=1}^{d-1}dx_i^2 & \text{for } k=0\\ d\Xi_{d-1}^2=d\theta^2+\sinh^2\theta d\Omega_{d-2}^2 & \text{for } k=-1.\end{cases} \tag{15}$$

There is an event horizon[5] for the black hole at $r=r_h$ which is the largest root of $f(r)=0$. The outermost horizon condition will lead to $f'(r_h)\ge 0$; i.e., the derivative of the blackening factor $f(r)$ with respect to r at the horizon r_h is non-negative. From the perspective of the thermal ensemble, the temperature of the black hole is non-negative because the temperature is given by

$$T=\frac{e^{-\chi(r_h)/2}}{4\pi}f'(r_h)\ge 0. \tag{16}$$

In the following proof and following sections, we set $\ell_{\mathrm{AdS}}=1$ for convenience. The energy momentum tensor $T^\mu{}_\nu$ has the form

$$8\pi T^\mu{}_\nu=\mathrm{diag}[-\rho(r),p_r(r),p_T(r),p_T(r),\dots,p_T(r)]. \tag{17}$$

The Einstein equation shows the following three independent equations:

$$f'=\frac{d-2}{r}k-\frac{2}{d-1}r\hat\rho-\frac{(d-2)f}{r}, \tag{18a}$$

$$\chi'=-\frac{2r}{(d-1)f}(\rho+p_r), \tag{18b}$$

$$p_r'=\frac{(d-2)\hat\rho+2(d-1)\hat p_T-d\hat p_r}{2r}-\frac{(\hat p_r+\hat\rho)[\hat p_r r^2+(d-2)(d-1)k/2]}{(d-1)rf}. \tag{18c}$$

Here we define some auxiliary variables $\{\hat\rho,\hat p_r,\hat p_T\}$ to include the effects of the cosmological constants,

$$\hat\rho=\rho-\frac{d(d-1)}{2},\qquad \hat p_r=p_r+\frac{d(d-1)}{2},$$
$$\hat p_T=p_T+\frac{d(d-1)}{2}. \tag{19}$$

The extra factor $\frac{d(d-1)}{2}$ is contributed by the cosmological constant term $\Lambda g_{\mu\nu}$ in the Einstein equation. As is well known, in order to match the asymptotically AdS boundary condition, the two functions $f(r)$ and $\chi(r)$ must follow the asymptotically behaviors as

$$\lim_{r\to\infty}\frac{f(r)}{r^2}=1,\qquad \lim_{r\to\infty}\chi(r)=0. \tag{20}$$

However, the "asymptotically AdS" boundary condition is not enough for our proof. Moreover, we need the matter to decay rapidly near the AdS boundary $r\to\infty$ so that

[5] For a static solution, the outermost horizon will coincide with the event horizon [29,30].

the functions $f(r)$ and $\chi(r)$ satisfy the following asymptotically Schwarzschild-AdS boundary conditions:

$$\lim_{r\to\infty}\frac{f(r)}{r^2} = 1 + \frac{k}{r^2} - f_0^d/r^d + \cdots,$$
$$\lim_{r\to\infty}\chi(r) = \chi_0/r^{d+\alpha} + \cdots, \quad \alpha > 0. \tag{21}$$

Here f_0^d is the mass density parameter. The richness of the asymptotically Schwarzschild-AdS black holes is reflected in the dots of the above boundary condition. However, no matter what the form of higher order terms, they will all share the same ADM mass. By virtue of the asymptotic behavior for $f(r)$ and $\chi(r)$, we can define a "quasilocal mass" $m(r)$ for an equal-r surface:

$$m(r) = \frac{k}{d}[r^{d-2} + X(r)] + \frac{r^{d+1}e^{\chi/2}}{2d}\left(\frac{fe^{-\chi}}{r^2}\right)', \tag{22}$$

where $X(r)$ is an auxiliary function

$$X(r) \equiv (d-2)\int_r^\infty [1 - e^{-\chi(x)/2}]x^{d-3}dx. \tag{23}$$

One can check that[6]

$$m(\infty) = f_0^d/2. \tag{24}$$

We can use the energy density ρ and transverse pressure density p_T to express $m'(r)$. One can verify

$$m'(r) = \frac{r^{d-1}\mathrm{e}^{-\chi/2}}{d}(\hat{\rho} + \hat{p}_T) = \frac{r^{d-1}\mathrm{e}^{-\chi/2}}{d}(\rho + p_T). \tag{25}$$

If the matter satisfies the null energy condition, combined with Eqs. (18b) and (25), we can directly conclude that

$$\chi' \le 0, \qquad m' \ge 0 \tag{26}$$

and vice versa. The boundary condition (21) also implies $\chi(r) \ge 0$ and $X(r) \ge 0$ outside the horizon, which we will use in the following proof. Ultimately, $m' \ge 0$ and $m(\infty) = f_0^d/2$ imply that

$$m(r) \le m(\infty) = f_0^d/2. \tag{27}$$

At the horizon we have $f'(r_h) \ge 0$, so Eq. (22) implies

$$m(r_h) \ge \frac{k}{d}\left[r_h^{d-2} + X(r_h)\right]. \tag{28}$$

We can conclude that the quasilocal mass $m(r)$ is a monotonically increasing function outside the black hole, which takes the minimum value $\frac{k}{d}[r_h^{d-2} + X(r_h)]$ at the horizon and the maximum value $f_0^d/2$ on the AdS boundary. Let us discuss three different horizon topologies.

[6]For instance, $k = +1$, $d = 3$, and the total mass M is equal to $m(\infty) = f_0^d/2$. This is why we call $m(r)$ the quasilocal mass.

B. Planar geometry

For the planar horizon case $k = 0$, the expression of the quasilocal mass $m(r)$ is reduced to

$$m(r) = \frac{r^{d+1}e^{\chi/2}}{2d}\left(\frac{fe^{-\chi}}{r^2}\right)'. \tag{29}$$

Solving $fe^{-\chi}/r^2$ in terms of $m(r)$ and $\chi(r)$, we obtain

$$\frac{fe^{-\chi}}{r^2} = 2d\int_{r_h}^r \frac{m(x)e^{-\chi(x)/2}}{x^{d+1}}dx. \tag{30}$$

When $r \to \infty$, the boundary condition indicates that

$$1 = 2d\int_{r_h}^\infty \frac{m(x)e^{-\chi(x)/2}}{x^{d+1}}dx. \tag{31}$$

If the null energy condition is satisfied, then we have $0 \le e^{-\chi(r)/2} \le 1$. Combined with $0 \le m(r) \le f_0^d/2$ in the planar case, the above equation becomes an inequality:

$$1 = 2d\int_{r_h}^\infty \frac{m(x)e^{-\chi(x)/2}}{x^{d+1}}dx \le d\int_{r_h}^\infty \frac{f_0^d}{x^{d+1}}dx = f_0^d/r_h^d, \tag{32}$$

which is the Penrose inequality (12) for the planar symmetric case.

The inequality is saturated only if $\chi = 0$ and $m = f_0^d/2$. From Eq. (30), we can solve $f(r)$ in terms of $\chi(r)$ and $m(r)$, which are both constant in this case. The solution is

$$f(r) = r^2(1 - f_0^d/r^d), \qquad \rho = p_r = p_T = 0, \tag{33}$$

which is exactly the metric of the Schwarzschild-AdS black hole. Thus, we conclude that, for planar, symmetric, static, asymptotically Schwarzschild-AdS black holes, if the null energy condition is satisfied, then the Penrose inequality is true and its saturation appears only if the black hole is a Schwarzschild-AdS black hole.

C. Spherical geometry

In this subsection, we consider the spherical symmetric ($k = 1$) case. We first solve the function $f(r)$ in terms of $m(r)$ and $\chi(r)$ and obtain

$$\frac{f(r)e^{-\chi(r)}}{r^2} = 2\int_{r_h}^r \frac{[dm(y) - X(y) - y^{d-2}]e^{-\chi(y)/2}}{y^{d+1}}dy. \tag{34}$$

When r evolves to ∞, the left-hand side of Eq. (34) becomes unit one,

$$1 = 2\int_{r_h}^{\infty} \frac{[dm(x) - X(x) - x^{d-2}]e^{-\chi(x)/2}}{x^{d+1}} dx. \quad (35)$$

Similar to the planar symmetric case, we focus on the integral on the right-hand side of Eq. (34). From Eq. (28), we find $m(r_h) \geq 0$ because $X(r) \geq 0$. Combining this with $m(r) \leq m(\infty) = f_0^d/2$, we then obtain

$$1 \leq 2\int_{r_h}^{\infty} \frac{(df_0^d/2 - x^{d-2})e^{-\chi(x)/2}}{x^{d+1}} dx. \quad (36)$$

Let r_0 be the root of $df_0^d/2 - x^{d-2} = 0$; the condition $\chi' \leq 0$ ensures

$$(df_0^d/2 - r^{d-2})[e^{-\chi(r)/2} - e^{-\chi(r_0)/2}] \leq 0 \quad (37)$$

which leads to

$$\int_{r_h}^{\infty} \frac{(df_0^d/2 - x^{d-2})e^{-\chi(r)/2}}{x^{d+1}} dx \leq e^{-\chi(r_0)/2} \int_{r_h}^{\infty} \frac{df_0^d/2 - x^{d-2}}{x^{d+1}} dx. \quad (38)$$

Combining this result with Eq. (36) then yields

$$1 \leq e^{\chi(r_0)/2} \leq \int_{r_h}^{\infty} \frac{df_0^d - 2x^{d-2}}{x^{d+1}} dx = \frac{f_0^d}{r_h^d} - \frac{1}{r_h^2}, \quad (39)$$

and the Penrose inequality (12) follows. The inequality is saturated only if $\chi = 0$ and $m = f_0^d/2$, which leads to

$$f(r) = r^2(1 + 1/r^2 - f_0^d/r^d), \qquad \rho = p_r = p_T = 0. \quad (40)$$

Thus, we conclude that for a static asymptotically Schwarzschild-AdS black hole with spherical symmetry, if the null energy condition is satisfied, then the Penrose inequality is true, and its saturation appears only if the black hole is a Schwarzschild-AdS black hole. This proves conjecture 1 in the spherically and planar symmetrical cases.

D. Broken case: Hyperbolic geometry

For the hyperbolic symmetric case, the null energy condition cannot guarantee the Penrose inequality (12). To verify this conclusion, we give a concrete counterexample.[7] We note that Eq. (34) becomes

[7]In view of [31], the validity of the Penrose inequality in general hyperbolic cases seems to be rather unlikely.

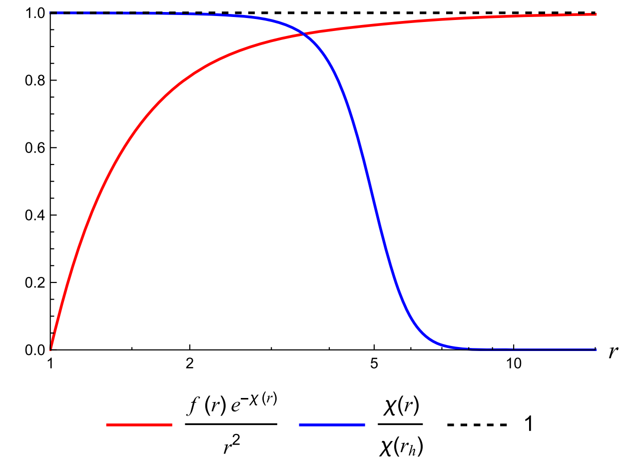

FIG. 1. Values of $f(r)e^{-\chi(r)}/r^2$ and $\chi(r)/\chi(r_h)$. We see that $f(r)e^{-\chi(r)}/r^2$ and $\chi(r)$ have the expected asymptotic behavior, but the inequality is still broken.

$$\frac{f(r)e^{-\chi(r)}}{r^2} = 2\int_{r_h}^{r} \frac{[dm(y) + X(y) + y^{d-2}]e^{-\chi(y)/2}}{y^{d+1}} dy. \quad (41)$$

Now let us take $r_h = 1, d = 3$ and

$$e^{-\chi/2} = \frac{4 + \tanh(r-5)}{5}, \qquad m(r) = f_0^d/2 \approx -0.15105. \quad (42)$$

[8]Then, we can get the expression of $X(r)$,

$$X(r) = \frac{1}{5}[\ln\cosh(r-5) - r + \ln 2] + 1, \quad (43)$$

and two functions $f(r)e^{-\chi(r)}/r^2$ and $\chi(r)$, which are shown in Fig. 1. Since $\chi' < 0$ and $m' = 0$, the null energy condition is guaranteed. However, let us check the sign of $[1 - \frac{1}{r_h^2} - \frac{f_0^d}{r_h^3}]$; we find that

$$1 - \frac{1}{r_h^2} - \frac{f_0^d}{r_h^3} = -\frac{f_0^d}{r_h^3} > 0. \quad (44)$$

Thus, we conclude that the Penrose inequality (12) is broken when $k = -1$.

The null energy condition does not require the energy density ρ to be non-negative, but the sum of the energy density and pressure density needs to be non-negative. If matter satisfies the weak energy condition, we can prove that the inequality is true, and its saturation appears only in the Schwarzschild-AdS black hole, at least for the

[8]For hyperbolic black holes, the mass parameter f_0^d can take a negative value.

maximally symmetric and static cases. See Appendix A for details.

III. CHARGED BLACK HOLES

Before we discuss the charged generalization of the AdS Penrose inequality, we first consider the asymptotically charged flat case, which can be regarded as the limit of $\ell_{\mathrm{AdS}} \to \infty$. With the total mass M and charge Q as initial data, there are two naive charged generalizations in four-dimensional spacetime,

$$M \geq \sqrt{\frac{A}{16\pi}} + Q^2\sqrt{\frac{\pi}{A}}, \tag{45}$$

and a weaker version,

$$\left(\frac{A}{16\pi}\right)^{1/2} \leq \frac{1}{2}\left[M + \sqrt{M^2 - Q^2}\right]. \tag{46}$$

Here the saturation appears only if the black holes are RN black holes. When we introduce the charge Q as initial data, as well as M, the definition of Q is vague. For general charged black holes, $Q(r)$ is defined as

$$\frac{1}{2\Omega_{k,d-1}} \int_{S_r} F_{\mu\nu} dS^{\mu\nu} = Q(r), \tag{47}$$

where S_r is an equal-r surface and $\Omega_{k,d-1}$ denotes the dimensionless volume of the relevant horizon geometry (15). For RN black holes, the charge $Q(r)$ is a constant outside the black holes. This is because there is no charge outside the RN black holes. For general cases, $Q(r)$ is dependent on r because matter outside the black hole usually also carries charge. If interpreting Q^2 in inequality (46) as the square of the total charge, i.e., $Q^2(\infty)$, we find that the inequality (46) is not always true. See Refs. [19,20] for a counterexample. This reminds us that the charged generalization for the Penrose inequality needs to be treated carefully, and naive generalizations (5) and (6) are both incorrect, in general. In this section, we conjecture two different types of charged generalizations for the Penrose inequality.

A. First type of generalization

We separate the energy momentum tensor $T_{\mu\nu}$ into two parts,

$$T_{\mu\nu} = T^{(M)}_{\mu\nu} + T^{(o)}_{\mu\nu} \tag{48}$$

where the energy momentum tensor $T^{(M)}_{\mu\nu}$ for the Maxwell field is defined as

$$T^{(M)}_{\mu\nu} = 2\left(F_\mu{}^\sigma F_{\nu\sigma} - \frac{g_{\mu\nu}}{4} F_{\sigma\tau}F^{\sigma\tau}\right), \tag{49}$$

and $T^{(o)}_{\mu\nu}$ stands for the other parts in $T_{\mu\nu}$. To present our generalized Penrose inequality in static charged black holes, we need a little more preparation.

We denote γ to be a codimension-2 spacelike surface and l^μ to be a future-directed, infalling, null geodesic, vector field normal to γ and satisfying $\xi^\mu l_\mu = -1$, where ξ^μ is the Killing vector standing for static symmetry and normalized at infinity by $\xi^\mu \xi_\mu = -1$. We denote the expansion $\theta_{(l)}$ for l^μ. Then we define Q_m^2 to be

$$Q_m^2 = \inf_S \left[\frac{(1-d)Q^2(\gamma)S(\gamma)}{r_s \int_\gamma \theta_{(l)} dS}\right]. \tag{50}$$

Here r_s is an "effective" radius which satisfies

$$r_s^{d-1}\Omega_{k,d-1} = S(\gamma), \tag{51}$$

where $S(\gamma)$ denotes the area of γ. We now propose the first type of charged generalization:

Conjecture 2. For an asymptotically Schwarzschild-AdS black hole, if (1) the Einstein equation is satisfied, (2) $T^{(o)}_{\mu\nu}$ satisfies the null energy condition, and (3) the cross section of the event horizon has spherical or planar topology, then the charged generalization of the Penrose inequality reads

$$1 + \frac{k}{r_h^2} - \frac{f_0^d}{r_h^d} + \frac{2Q_m^2}{(d-1)(d-2)r_h^{2d-2}} \leq 0. \tag{52}$$

The saturation appears only in RN black holes.

This is very similar to the generalization proposed by Ref. [11]; however, in general, Q_m will be different from the total charge.

To support this generalization, we give the proof for the spherically and planar symmetric cases. Under the coordinate gauge (14), the expansion $\theta_{(l)}$ for l^μ is given by

$$\theta_{(l)} = (1-d)\frac{e^{\chi/2}}{r} \tag{53}$$

and the Maxwell field strength tensor has the form

$$F_{\mu\nu} = -\frac{Q(r)e^{-\chi/2}}{r^{d-1}}(dt)_\mu \wedge (dr)_\nu. \tag{54}$$

The nonvanishing components of the surface element $dS_{\mu\nu}$ read

$$dS_{01} = -dS_{10} = e^{-\chi/2}r^{d-1}d\Sigma_{k,d-1}. \tag{55}$$

Substituting this result into the definition (50), we can obtain the expression of Q_m^2,

$$Q_m^2 = \min\left[Q^2(r)e^{-\frac{\chi(r)}{2}}\right], \quad \text{for } r \geq r_h. \tag{56}$$

In order to prove this inequality (52), the key step is to separate the energy density ρ and pressure density $\{p_r, p_T\}$ into two parts:

$$\rho = \frac{Q^2}{r^{2d-2}} + \rho^{(o)}, \qquad p_r = -\frac{Q^2}{r^{2d-2}} + p_r^{(o)},$$
$$p_T = \frac{Q^2}{r^{2d-2}} + p_T^{(o)}. \tag{57}$$

Since we require that $\rho^{(o)} + p_T^{(o)} \geq 0$ and $\rho^{(o)} + p_r^{(o)} \geq 0$, then we obtain

$$\rho + p_r \geq 0, \qquad \rho + p_T \geq \frac{2Q^2}{r^{2d-2}}. \tag{58}$$

To prove the inequality (52), we introduce a new quasilocal mass $\tilde{m}(r)$,

$$\tilde{m}(r) = m(r) + \int_r^\infty \frac{2Q^2 e^{-\frac{\chi}{2}}}{dy^{d-1}} dy, \tag{59}$$

so the derivative of $\tilde{m}(r)$ is always non-negative,

$$\tilde{m}'(r) = \frac{r^{d-1}e^{-\chi/2}}{d}\left(\rho + p_T - \frac{2Q^2}{r^{2d-2}}\right) \geq 0. \tag{60}$$

When $r \to \infty$,

$$\tilde{m}(\infty) = m(\infty) = df_0^d/2. \tag{61}$$

This implies

$$\tilde{m}(r) \leq df_0^d/2. \tag{62}$$

Recall the definition of the quasilocal mass in Eq. (22). We substitute the expression of $m(r)$ into our new quasilocal mass in Eq. (59):

$$\tilde{m}(r) = \frac{k}{d}[r^{d-2} + X(r)] + \frac{r^{d+1}e^{\chi/2}}{2d}\left(\frac{fe^{-\chi}}{r^2}\right)' + \int_r^\infty \frac{2Q^2 e^{-\frac{\chi}{2}}}{dy^{d-1}} dy. \tag{63}$$

Here $k = 0$ and 1, which stand for planar and spherically symmetry, respectively. Following the standard procedure in Sec. II, we solve $fe^{-\chi}/r^2$ in terms of $m(r)$ and $\chi(r)$,

$$2\int_{r_h}^r \left[d\tilde{m}(x) - \int_x^\infty \frac{2Q^2 e^{-\frac{\chi}{2}}}{y^{d-1}} dy - kX(x) - kx^{d-2}\right] \times e^{-\chi(x)/2}x^{-(d+1)}dx = \frac{f(r)e^{-\chi(r)}}{r^2}. \tag{64}$$

Recall that $f(r_h)' \geq 0$ because the surface of $r = r_h$ is the outermost horizon, so we can obtain

$$\frac{\left[d\tilde{m}(r_h) - \int_{r_h}^\infty \frac{2Q^2 e^{-\frac{\chi}{2}}}{y^{d-1}} dy - kX(r_h) - kr_h^{d-2}\right]e^{-\chi(r_h)/2}}{r_h^{d+1}} \geq 0. \tag{65}$$

Because of $X(r) \geq 0$ and $e^{-\chi(r_h)/2} \geq 0$, the above inequality (65) becomes

$$\frac{d\tilde{m}(r_h) - \int_{r_h}^\infty \frac{2Q^2 e^{-\frac{\chi}{2}}}{y^{d-1}} dy - kr_h^{d-2}}{r_h^{d+1}} \geq 0. \tag{66}$$

We see that the above inequality, which is defined at the horizon r_h, plays a decisive role in the following proof. In particular, the inequality (66) restricts the evaluation relationship between total mass M and Q_m^2. For the convenience of our proof, we define an auxiliary function $W(r)$,

$$W(r) = df_0^d/2 - \frac{2Q_m^2}{(d-2)r^{d-2}} - kr^{d-2}. \tag{67}$$

We combine this with Eqs. (56), (62), and (66) to obtain

$$\frac{W(r)}{r^{d+1}} \geq \frac{d\tilde{m}(r) - \int_r^\infty \frac{2Q^2 e^{-\frac{\chi}{2}}}{y^{d-1}} dy - kr^{d-2}}{r^{d+1}}. \tag{68}$$

Particularly, at the horizon $r = r_h$ we have

$$\frac{W(r_h)}{r_h^{d+1}} \geq \frac{d\tilde{m}(r_h) - \int_{r_h}^\infty \frac{2Q^2 e^{-\frac{\chi}{2}}}{y^{d-1}} dy - kr_h^{d-2}}{r_h^{d+1}} \geq 0, \tag{69}$$

due to inequality (66). Thus the horizon r_h is limited by the value of function $W(r)$.

Returning to Eq. (64), the left-hand side will become unit one when r evolves to ∞, which is the boundary condition of asymptotically AdS spacetime,

$$1 = 2\int_{r_h}^\infty \left[d\tilde{m}(x) - \int_x^\infty \frac{2Q^2 e^{-\frac{\chi}{2}}}{y^{d-1}} dy - kX(x) - kx^{d-2}\right] \times e^{-\chi(x)/2}x^{-(d+1)}dx. \tag{70}$$

Through Eqs. (28) and (59), we find $\tilde{m}(r_h) \geq 0$. Combining this with $\tilde{m}(r) \leq \tilde{m}(\infty) = f_0^d/2$ and $X(r) \geq 0$, we then obtain

$$1 \leq 2\int_{r_h}^\infty \frac{\left[df_0^d/2 - \int_x^\infty \frac{2Q^2 e^{-\frac{\chi}{2}}}{y^{d-1}} dy - kx^{d-2}\right]e^{-\chi(x)/2}}{x^{d+1}} dx. \tag{71}$$

Using the inequality (68), we see that the above inequality becomes

$$1 \leq 2\int_{r_h}^{\infty}\frac{W(x)e^{-\chi(x)/2}}{x^{d+1}}dx. \tag{72}$$

The above inequality implies that the maximum value of $W(r)$ must be positive,

$$\max W(r) = \max\left[df_0^d/2 - \frac{2Q_m^2}{(d-2)r^{d-2}} - kr^{d-2}\right] > 0. \tag{73}$$

Otherwise the above integration in Eq. (72) will be negative.

For $k=1$, it is obvious that when $r^{d-2} = r_\Delta^{d-2} = \sqrt{\frac{2Q_m^2}{d-2}}$, $W(r)$ takes the maximum value $df_0^d/2 - 2\sqrt{\frac{2Q_m^2}{d-2}}$,

$$\max W(r) = W(r_\Delta) = df_0^d/2 - 2\sqrt{\frac{2Q_m^2}{d-2}} > 0. \tag{74}$$

The two points r_1, r_2 are the roots of $W(r)=0$,

$$r_1^{d-2} = df_0^d/4 - \sqrt{(df_0^d/4)^2 - \frac{2Q_m^2}{d-2}},$$
$$r_2^{d-2} = df_0^d/4 + \sqrt{(df_0^d/4)^2 - \frac{2Q_m^2}{d-2}}. \tag{75}$$

As we can see, $r_1 \leq r_h < r_2$ from Fig. 2 because $W(r_h) \geq 0$.[9] We separate the interval $[r_h, \infty)$ into two parts: $[r_h, r_2)$ and $[r_2, r_\infty)$. There are two different situations.

(i) $r_h \leq r < r_2$:
Here $\chi' \leq 0$ ensures $e^{-\chi(r)/2} - e^{-\chi(r_2)/2} \leq 0$ and $W(r) \geq 0$. We then have

$$W(r)[e^{-\chi(r)/2} - e^{-\chi(r_2)/2}] \leq 0. \tag{76}$$

(ii) $r_2 \leq r$:
We see that $e^{-\chi(r)/2} - e^{-\chi(r_2)/2} \geq 0$ and $W(r) \leq 0$, so we still have the inequality (76).

The inequality (76) is true for all $r \in [r_h, \infty)$. This leads to

$$1 \leq 2\int_{r_h}^{\infty}\frac{W(r)e^{-\chi(r)/2}}{r^{d+1}}dr \leq 2e^{-\chi(r_2)/2}\int_{r_h}^{\infty}\frac{W(r)}{r^{d+1}}dr. \tag{77}$$

Multiplying $e^{-\chi(r_2)/2}$ by the above inequality, we finally obtain

[9]The value of r_h cannot be equal to or greater than r_2; otherwise the integration of Eq. (72) will be negative.

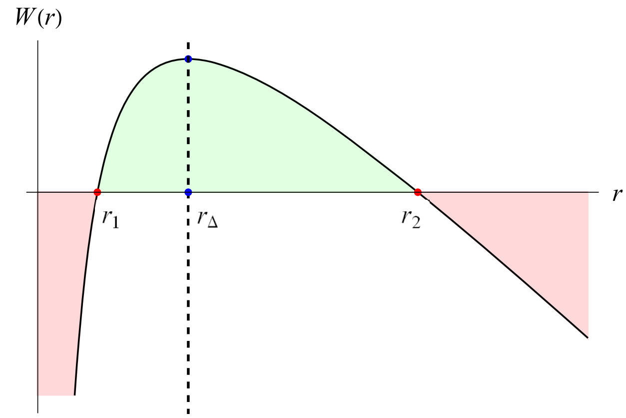

FIG. 2. Schematic diagram of the function $W(r)$ for $k=1$. The value of the function $W(r)$ at the horizon r_h must be non-negative.

$$1 \leq e^{\chi(r_2)/2} \leq 2\int_{r_h}^{\infty}\frac{W(r)}{x^{d+1}}dx$$
$$= \frac{f_0^d}{r_h^d} - \frac{1}{r_h^2} - \frac{2Q_m^2}{(d-1)(d-2)r_h^{2d-2}}. \tag{78}$$

For $k=0$, $W(r)$ is a monotonically increasing function, so

$$\max W(r) = W(r\to\infty) = df_0^d/2 > 0. \tag{79}$$

The point r_0 satisfies $W(r_0)=0$,

$$r_0^{d-2} = \frac{4Q_m^2}{d(d-2)f_0^d}. \tag{80}$$

Combining $W(r_h) \geq 0$ with $W'(r) \geq 0$ for the planar case, we obtain that $W(r) \geq 0$ for $r \geq r_h$. The inequality (72) becomes

$$1 \leq 2\int_{r_h}^{\infty}\frac{W(x)e^{-\chi(x)/2}}{x^{d+1}}dx \leq 2\int_{r_h}^{\infty}\frac{W(x)}{x^{d+1}}dx$$
$$= \frac{f_0^d}{r_h^d} - \frac{2Q_m^2}{(d-1)(d-2)r_h^{2d-2}}. \tag{81}$$

Combining two symmetric cases, the first type of charged generalization (52) for the Penrose inequality is used.

Recall the whole proof; the saturation for the charged inequality appears if $\chi(r)=0, \tilde{m}(r)=f_0^d/2$ and $Q(r)=Q_m$. This implies charged density $j(r)=0$, and the exterior is a RN black hole.

B. Second type of generalization

The inequality (52) is not expressed in term of the boundary quantities of asymptotically AdS spacetime. It would be more satisfactory if we could use boundary quantities to express the charged generalization of the Penrose inequality since such versions of the Penrose inequality can be interpreted as the inequality of dual boundary field theory according to holography. In order to alleviate the contradiction between charged generalization for the Penrose inequality and the basic idea from holography, we propose the second type of charged Penrose inequality. Consider the Maxwell equation with source J_ν,

$$\nabla^\mu F_{\mu\nu} = 4\pi J_\nu. \tag{82}$$

We introduce the gauge potential which satisfies

$$F_{\mu\nu} = (dA)_{\mu\nu}. \tag{83}$$

To find the generalization in the general case, we separate the energy momentum tensor $T^{(o)}_{\mu\nu}$ as follows:

$$T^{(o)}_{\mu\nu} = \tilde{T}^{(o)}_{\mu\nu} - 8\pi\left(g_{\mu\nu}J_\rho A^\rho - \frac{1}{2}J_\mu A_\nu - \frac{1}{2}J_\nu A_\mu\right), \tag{84}$$

so the total energy momentum tensor reads

$$T_{\mu\nu} = T^{(M)}_{\mu\nu} + \tilde{T}^{(o)}_{\mu\nu} - 8\pi\left(g_{\mu\nu}J_\rho A^\rho - \frac{1}{2}J_\mu A_\nu - \frac{1}{2}J_\nu A_\mu\right). \tag{85}$$

In the static case, the gauge potential A_μ and charge density J_μ have the following form:

$$A_\mu \propto \xi_\mu, \qquad J_\mu \propto \xi_\mu. \tag{86}$$

Here the potential is $\Phi \coloneqq A_\mu \xi^\mu$, and the charged density is $j \coloneqq J_\mu \xi^\mu$. In holography, the potential Φ_∞, which is defined on the AdS boundary,[10] is interpreted as the chemical potential. Given the initial data $f_0^d/2$ and Φ_∞ and taking the gauge $\Phi = 0$ at the event horizon, we have the following conjecture.

Conjecture 3. For an asymptotically Schwarzschild-AdS black hole, if (1) the Einstein equation is satisfied, (2) $\tilde{T}^{(o)}_{\mu\nu}$ and $T_{\mu\nu}$ both satisfy the null energy condition, (3) the charge of the black hole and charge density j have the same sign, and (4) the cross section of the event horizon has spherical or planar topology, then the charged generation of the Penrose inequality reads

$$1 \le \frac{f_0^d}{r_h^d} - \frac{k}{r_h^2} - \frac{2(d-2)\Phi_\infty^2}{(d-1)r_h^2}, \tag{87}$$

and the saturation appears only if the exterior of the event horizon is AdS-RN.

[10] In this paper, we abbreviate $\Phi(\infty)$ as Φ_∞.

In contrast to the naive generalizations (5) and (6), here we use the chemical potential to replace the charge. Like previous sections, to support this conjecture, we prove it under the spherically or planar symmetric spacetime.

Under the same coordinates gauge (14), the gauge potential A and charge density J_μ have the following form:

$$A_\mu = \Phi(r)(dt)_\mu, \qquad J_\mu = j(r)(dt)_\mu. \tag{88}$$

Here $\Phi(r)$ and $j(r)$ are the potential and current density. According to Eq. (54), we obtain

$$\frac{Qe^{-\chi/2}}{r^{d-1}} = \Phi'. \tag{89}$$

The Maxwell equation reads

$$(\Phi' e^{\chi/2} r^{d-1})' = \frac{4\pi j e^{\chi/2} r^{d-1}}{f}. \tag{90}$$

The energy density ρ and pressure density $\{p_r, p_T\}$ are now replaced by

$$\begin{aligned} \rho &= \Phi'^2 e^\chi + \tilde{\rho}^{(o)}, \\ p_r &= -\Phi'^2 e^\chi + \frac{8\pi\Phi j e^\chi}{f} + \tilde{p}_r^{(o)}, \\ p_T &= \Phi'^2 e^\chi + \frac{8\pi\Phi j e^\chi}{f} + \tilde{p}_T^{(o)}. \end{aligned} \tag{91}$$

The null energy condition requires $\rho + p_r \ge 0, \rho + p_T \ge 0$; then we obtain

$$\begin{aligned} \frac{8\pi\Phi j e^\chi}{f} + \tilde{\rho}^{(o)} + \tilde{p}_r^{(o)} &\ge 0, \\ 2\Phi'^2 + \frac{8\pi\Phi j e^\chi}{f} + \tilde{\rho}^{(o)} + \tilde{p}_T^{(o)} &\ge 0. \end{aligned} \tag{92}$$

A new quasilocal mass $\tilde{m}(r)$ is defined as

$$\tilde{m}(r) = m(r) - \frac{2}{d}(\Phi Q), \tag{93}$$

so the derivative of $\tilde{m}(r)$ is

$$\begin{aligned} \tilde{m}'(r) &= m'(r) - \frac{2}{d}(\Phi\Phi' e^{\chi/2} r^{d-1})' \\ &= m'(r) - \frac{2}{d} e^{\chi/2} r^{d-1}\Phi'^2 - \frac{2}{d}\Phi(e^{\chi/2} r^{d-1}\Phi')'. \end{aligned} \tag{94}$$

Substituting Eq. (25), (90), and (92) into the above equation, we obtain

$$\tilde{m}'(r) = \frac{r^{d-1}\mathrm{e}^{-\chi/2}}{d}(\tilde{\rho}^{(o)} + \tilde{p}_T^{(o)}) \ge 0. \tag{95}$$

Thus, we obtain $\tilde{m}(r) \leq \tilde{m}(\infty) = f_0^d/2 - \frac{2}{d}(\Phi_\infty Q_\infty)$. Let us rephrase the Maxwell equation:

$$Q' = \frac{4\pi j e^{\chi/2} r^{d-1}}{f}. \tag{96}$$

Because $Q(r_h)$ and j have the same sign, we can take $Q(r_h) \geq 0$ and $j \geq 0$ without losing generality. so the charge $Q(r)$ will always be non-negative:

$$Q(r) \geq 0. \tag{97}$$

According to the relationship between Q and Φ in Eq. (89), we can obtain

$$\Phi' \geq 0. \tag{98}$$

In holography, we generally set the value of the potential Φ at the horizon equal to zero as a gauge fixing,

$$\Phi(r_h) = 0. \tag{99}$$

After the gauge fixing, we can directly obtain the chemical potential Φ_∞ on the AdS boundary. In order to compare with RN black holes, we rephrase the Maxwell equation as

$$(\Phi' r^{d-1})' = \left(\frac{4\pi j}{f} - \frac{\chi'\Phi'}{2}\right) r^{d-1}. \tag{100}$$

Let us denote $\Phi_{\rm RN}(r)$ as the gauge potential of RN black holes with the same horizon and chemical potential, i.e., $\Phi_{\rm RN}(r)$ satisfies $\Phi_{\rm RN}(r_h) = 0$, $\Phi_{\rm RN}(\infty) = \Phi_\infty$, and

$$(\Phi'_{\rm RN} r^{d-1})' = 0. \tag{101}$$

We define $\Delta\Phi = \Phi - \Phi_{\rm RN}$,

$$(\Delta\Phi' r^{d-1})' = \left(\frac{4\pi j}{f} - \frac{\chi'\Phi'}{2}\right) r^{d-1} \geq 0. \tag{102}$$

Since $j \geq 0$, then $\chi' \leq 0$ and $\Phi' \leq 0$. Thus, the "maximal principle" shows that the maximum of $\Delta\Phi$ can only be attained at endpoints. Thus, we have

$$\max \Delta\Phi = \Delta\Phi(r_h) = \Delta\Phi(\infty) = 0. \tag{103}$$

Thus, the relationship between Φ and $\Phi_{\rm RN}$ is

$$0 \leq \Phi \leq \Phi_{\rm RN}. \tag{104}$$

As usual, we solve the function $f(r)$ in terms of $m(r)$ and $\chi(r)$,

$$\begin{aligned}\frac{f(r)e^{-\chi(r)}}{r^2} &\\ = 2\int_{r_h}^{r} &\frac{[d\tilde{m}(x) + 2(\Phi Q) - kX(x) - kx^{d-2}]e^{-\chi(x)/2}}{x^{d+1}} dx,\end{aligned} \tag{105}$$

and $f(r_h)' \geq 0$ leads to

$$\frac{[d\tilde{m}(r_h) + 2\Phi(r_h)Q(r_h) - kX(r_h) - kr_h^{d-2}]e^{-\chi(r_h)/2}}{r_h^{d+1}} \geq 0. \tag{106}$$

Because $X(r) \geq 0$, $e^{-\chi(r_h)/2} \geq 0$, and $\Phi(r_h) = 0$, the above inequality becomes

$$\frac{d\tilde{m}(r_h) - kr_h^{d-2}}{r_h^{d+1}} \geq 0. \tag{107}$$

Since $\tilde{m}'(r) \geq 0$, we can obtain

$$\begin{aligned}\frac{df_0^d/2 - 2(\Phi_\infty Q_\infty) - kr_h^{d-2}}{r_h^{d+1}} &= \frac{d\tilde{m}(\infty) - kr_h^{d-2}}{r_h^{d+1}}\\ &\geq \frac{d\tilde{m}(r_h) - kr_h^{d-2}}{r_h^{d+1}}\\ &\geq 0.\end{aligned} \tag{108}$$

Just like the proof in the first type of generalization, we define an auxiliary function $\tilde{W}(r)$,

$$\begin{aligned}\tilde{W}(r) &= df_0^d/2 - 2(\Phi_\infty Q_\infty) + 2(\Phi_{\rm RN} Q_\infty) - kr^{d-2}\\ &= df_0^d/2 - 2\frac{\Phi_\infty Q_\infty r_h^{d-2}}{r^{d-2}} - kr^{d-2}.\end{aligned} \tag{109}$$

Here the $\Phi_{\rm RN}$ is given by Eq. (101), for which the solution reads

$$\Phi_{\rm RN} = \Phi_\infty - \frac{\Phi_\infty r_h^{d-2}}{r^{d-2}}. \tag{110}$$

One can verify that

$$\tilde{W}(r_h) = df_0^d/2 - 2(\Phi_\infty Q_\infty) - kr_h^{d-2}. \tag{111}$$

Combining this with (108), we obtain

$$\tilde{W}(r_h) \geq 0. \tag{112}$$

Thus, the horizon r_h is limited by the value of the function $W(r)$. When $r \to \infty$, the inequality becomes

$$1 \leq 2\int_{r_h}^{\infty}[df_0^d/2 - 2(\Phi_\infty Q_\infty) + 2(\Phi Q) - kx^{d-2}] \times e^{-\chi(x)/2}x^{-(d+1)}dx, \tag{113}$$

which is due to $d\tilde{m}(r) \leq df_0^d/2 - 2(\Phi_\infty Q_\infty)$ and $X(r) \geq 0$. Combining this with $\Phi_{\rm RN} \geq \Phi \geq 0$, we obtain

$$1 \leq 2\int_{r_h}^{\infty}[df_0^d/2 - 2(\Phi_\infty Q_\infty) + 2(\Phi_{\rm RN} Q) - kx^{d-2}] \times e^{-\chi(x)/2}x^{-(d+1)}dx. \tag{114}$$

We require $j \geq 0$, and the derivative of $Q(r)$ is greater than zero, $Q'(r) \geq 0$. Combined with $Q \geq 0$, we finally get

$$1 \leq 2\int_{r_h}^{\infty}[df_0^d/2 - 2(\Phi_\infty Q_\infty) + 2(\Phi_{\rm RN} Q_\infty) - kx^{d-2}] \times e^{-\chi(x)/2}x^{-(d+1)}dx. \tag{115}$$

We substitute (110) into the above inequality:

$$1 \leq 2\int_{r_h}^{\infty}\frac{\tilde{W}(x)e^{-\chi(x)/2}}{x^{d+1}}dx, \tag{116}$$

which is very similar to the proof of the first type of charged generalization. The only difference is the coefficient of $1/r^{d-2}$ in the auxiliary function. Thus, we can use the discussion in Sec. III A to obtain[11]

$$1 \leq 2\int_{r_h}^{\infty}\frac{\tilde{W}(x)e^{-\chi(x)/2}}{x^{d+1}}dx \leq 2e^{-\chi(r_2)/2}\int_{r_h}^{\infty}\frac{\tilde{W}(x)}{x^{d+1}}dx, \tag{117}$$

which yields

$$\frac{f_0^d}{r_h^d} - \frac{k}{r_h^2} - \frac{2\Phi_\infty Q_\infty}{(d-1)r_h^d} = 2\int_{r_h}^{\infty}\frac{\tilde{W}(x)}{x^{d+1}}dx \geq e^{\chi(r_2)/2} \geq 1. \tag{118}$$

The next step is to find the relation between Φ_∞ and Q_∞. Note that the infinity sign of the subscript represents the value of the potential and charge on the AdS boundary. Near infinity we have the following asymptotic expansions for Φ,

$$\Phi = \Phi_\infty - \frac{Q_\infty}{(d-2)r^{d-2}} + \ldots \tag{119}$$

[11]Like the first type of charged generalization, if $\max W(r) \leq 0$, the inequality will be broken. One can verify that the mass parameter f_0^d has an inequality relation: $df_0^d/2 - 2\sqrt{2\Phi_\infty Q_\infty r_h^{d-2}} \geq 0$ for $k = 1$ and $df_0^d/2 \geq 0$ for $k = 0$.

We already know that $\Phi \leq \Phi_{\rm RN}$ for all $r \geq r_h$; then, Eqs. (B5) and (110) imply

$$\frac{Q_\infty}{d-2} \geq \Phi_\infty r_h^{d-2}. \tag{120}$$

We finally reach the expected charged generalization:

$$1 \leq \frac{f_0^d}{r_h^d} - \frac{k}{r_h^2} - \frac{2\Phi_\infty Q_\infty}{(d-1)r_h^d} \leq \frac{f_0^d}{r_h^d} - \frac{k}{r_h^2} - \frac{2(d-2)\Phi_\infty^2}{(d-1)r_h^2}. \tag{121}$$

To saturate this inequality, we see from Eqs. (95), (113), and (115) that $j(r) = \chi(r) = 0$ and $\tilde{m}(r) = f_0^d/2$. The zero charge density and $\chi = 0$ show that $\Phi(r) = \Phi_{\rm RN}(r)$. This leads to

$$f(r) = r^2\left[1 + \frac{k}{r^2} - \frac{f_0^d}{r^d} + \frac{2Q_\infty^2}{(d-1)(d-2)r^{2d-2}}\right], \tag{122}$$

so the bulk geometry is a RN black hole.

IV. PENROSE INEQUALITY AND SCHEME OF QUANTIZATION

In the previous sections, we assume that the bulk geometry is asymptotically Schwarzschild-AdS so that all the quantities, especially the total mass, are defined only by bulk geometry. In holography, when the dual field theory has a nonzero external source, the total mass cannot be read directly from the bulk metric. Instead, we have to use the so-called "holographic renormalization" approach to find the total mass. In this case, our previous proof is invalid. It is interesting to ask whether we can still obtain the Penrose inequality in such a case if the bulk matter satisfies the null energy condition. In this section, we consider the asymptotically AdS black hole with scalar field ϕ as a concrete example.

A. Model

For $(d+1)$-dimensional Einstein-scalar gravity, the action reads

$$S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\left[\mathcal{R} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right]. \tag{123}$$

Here g denotes the determinant of the spacetime metric $g_{\mu\nu}$, and $\mathcal{R}$ is the Ricci scalar. For the bulk scalar field ϕ, $V(\phi)$ is some potential function dependent on ϕ. Considering the static asymptotically AdS black hole with spherical, planar, or hyperbolic horizon geometry, the ansatz is the same as in Eq. (14),

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{k,d-1}^2. \tag{124}$$

In order to satisfy the asymptotically AdS boundary condition, the function $f(r)$ and $\chi(r)$ must satisfy the following conditions at the AdS boundary $r \to \infty$:

$$f(r) = r^2 + \cdots, \qquad \chi(r) = \chi_0/r^\alpha + \cdots \alpha > 0. \tag{125}$$

If $\phi(r) = 0$ at any r, the scalar potential $V(\phi)$ will return to $-d(d-1)$ so that the theory (123) is pure AdS gravity. Without loss of generality, assuming $\phi(r \to \infty) \to 0$, we choose the potential function as

$$V(\phi) = -d(d-1) + \frac{1}{2}m^2\phi^2 + \mathcal{O}(\phi^3) \tag{126}$$

near the boundary.[12] The parameter m is the mass of the scalar field. In holography, the mass squared of the scalar field can be negative, but above the Breitenlohner-Freedman bound $m^2_{\rm BF}$,[13]

$$m^2 > m^2_{\rm BF} = -\frac{d^2}{4}. \tag{127}$$

According to the action (123), the equations of motion are

$$\nabla_\mu \nabla^\mu \phi - \partial_\phi V = 0, \tag{128}$$

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\left(-\frac{1}{2}\nabla_\rho\phi\nabla^\rho\phi - V(\phi)\right)g_{\mu\nu}. \tag{129}$$

One can check that the scalar hairy black hole solution satisfies the null energy condition. Substituting the ansatz (124) into the above equations, we obtain

$$\phi'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{d-1}{r}\right)\phi' - \frac{1}{f}\partial_\phi V = 0, \tag{130a}$$

$$\frac{\chi'}{r} + \frac{1}{d-1}\phi'^2 = 0, \tag{130b}$$

$$\frac{2}{r}\frac{f'}{f} - \frac{\chi'}{r} + \frac{2}{d-1}\frac{V}{f} + \frac{2(d-2)(f-k)}{r^2 f} = 0. \tag{130c}$$

Near the AdS boundary, the scalar field has an asymptotic form,

$$\phi(r) = \frac{\phi_s}{r^{d-\Delta}}(1 + \cdots) + \frac{\phi_v}{r^\Delta}(1 + \cdots), \tag{131}$$

where ϕ_s and ϕ_v are coefficients of the leading terms and Δ is the conformal dimension of the dual operator. We see the usual relationship [13,34] between Δ and m^2,

$$\Delta = (d + \sqrt{d^2 + 4m^2})/2. \tag{132}$$

In order to get the value of every expansion coefficient, we expand the metric (124) at large r and substitute the expansion of both the scalar field ϕ and metric into the equations of motion (130). Then, given the boundary condition, we can solve these coefficients order by order. However, it depends on the specific form of the potential $V(\phi)$ in such a process. With loss of generality, we consider the specific model in four-dimensional spacetime with planar horizon geometry ($k = 0$) to illustrate the key feature. We take the scalar potential function [35] as

$$V(\phi) = -6 - \frac{4}{\delta^2}\sinh\left[\frac{\delta\phi}{2}\right]^2 \tag{133}$$

where δ is a constant. One can check the boundary's asymptotic form of the potential,

$$V(\phi) = -6 - \phi^2 + \mathcal{O}(\phi^4). \tag{134}$$

Here $\Delta = 2$ and $m^2 = -2$, which satisfies the Breitenlohner-Freedman bound (127). Then, near the AdS boundary, we expand the metric that is determined by functions $f(r)$ and $\chi(r)$:

$$f(r) = r^2\left[1 + \frac{\phi_s^2}{4r^2} - \frac{f_0^3}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\right], \tag{135a}$$

$$\chi(r) = \frac{\phi_s^2}{4r^2} + \frac{2\phi_s\phi_v}{3r^3} + \mathcal{O}\left(\frac{1}{r^4}\right). \tag{135b}$$

B. Numerical check on the Penrose inequality

If we want to obtain the right holographic stress tensor $T^\mu{}_\nu$ through the well-defined variational principle, the Gibbons-Hawking-York boundary term [36,37] should be added to the action (123),

$$S_{\rm GHY} = \lim_{r\to\infty}\frac{1}{8\pi G}\int d^3x\sqrt{-h}K. \tag{136}$$

Here h_{ij} is the induced metric on the AdS boundary and h denotes the determinant of h_{ij}. Note that K is the trace of the second fundamental form K_{ij},

$$K_{ij} \equiv -\frac{1}{2}\mathcal{L}_n h_{ij} = -\frac{1}{2}\nabla_i n_j - \frac{1}{2}\nabla_j n_i, \qquad K = h^{ij}K_{ij}, \tag{137}$$

where n^μ is the outward-pointing unit vector normal to the AdS boundary. Because the action is still divergent in the

[12]Recall that we have taken the AdS radius $\ell_{\rm AdS}$ equal to unity in the beginning of this paper.

[13]It was first derived in Refs. [32,33]. Loosely speaking, the negative mass squared below the Breitenlohner-Freedman bound $m^2 < m^2_{\rm BF}$ will lead to an instability.

AdS boundary after adding the Gibbons-Hawking-York boundary term, we should introduce a boundary counterterm to regulate infinity. Following the standard holographic renormalization scheme [12,27,38], the counterterm for the gravitational sector in this case is given by

$$S_{\mathrm{c.t.}} = \lim_{r\to\infty} \frac{1}{16\pi G} \int d^3x\sqrt{-h}(-4). \tag{138}$$

Since the mass satisfies $-\frac{d^2}{4} < m^2 < 1 - \frac{d^2}{4}$, there are two different renormalization schemes [28,39,40] for the scalar field $\phi(r)$ sector. For instance, if we treat ϕ_s as the source, we must fix the value of ϕ_s on the AdS boundary which is referred to as the standard quantization for $\phi(r)$. Then, we should add the following counterterm:

$$S_{\phi_s} = \lim_{r\to\infty} \frac{1}{16\pi G} \int d^3x\sqrt{-h}\left(-\frac{1}{2}\phi^2\right). \tag{139}$$

However, if we fix the value of ϕ_v on the boundary, the counterterm we need to add is different from the previous one:

$$S_{\phi_v} = \lim_{r\to\infty} \frac{1}{16\pi G} \int d^3x\sqrt{-h}\left[\phi(n^\mu\partial_\mu\phi) + \frac{1}{2}\phi^2\right]. \tag{140}$$

Then, we obtain the regulated action $\tilde{S}$,

$$\tilde{S} = S_{\mathrm{GHY}} + S_{\mathrm{c.t.}} + S_{\phi_{s,v}}. \tag{141}$$

So far, $\tilde{S}$ is finite when $r\to\infty$. Thus, we can obtain the holographic stress tensor

$$T_{\mu\nu} = \frac{1}{16\pi G}\lim_{r\to\infty} r\left[2(Kh_{\mu\nu} - K_{\mu\nu} - 2h_{\mu\nu}) + h_{\mu\nu}\times\begin{cases} -\frac{1}{2}\phi^2 \\ \phi(n^\mu\partial_\mu\phi) + \frac{1}{2}\phi^2 \end{cases}\right]. \tag{142}$$

Substituting the asymptotic expansion into $T_{\mu\nu}$, we can obtain the value of the tt component,

$$16\pi G T_{tt} = \begin{cases} 2f_0^3 + \phi_s\phi_v & \text{fix } \phi_s \\ 2f_0^3 + 2\phi_s\phi_v & \text{fix } \phi_v. \end{cases} \tag{143}$$

The new mass parameter $\tilde{f}_0^3$ which is defined by the holographic mass or energy is relevant to the value of T_{tt}. In this case, $\tilde{f}_0^3$ is given by

$$\tilde{f}_0^3/2 = 4\pi G T_{tt} = \begin{cases} f_0^3/2 + \phi_s\phi_v/4 & \text{fix } \phi_s \\ f_0^3/2 + \phi_s\phi_v/2 & \text{fix } \phi_v. \end{cases} \tag{144}$$

Fixing $r_h = 1$ and $\delta = 1$, we can solve the equations of motion (130) numerically and then read the data (f_0^3, ϕ_s, ϕ_v) from the asymptotic form of Eq. (135) on the boundary. In Fig. 3, the independent variable is the value of $\phi(r_h)$, which is the value of ϕ at the horizon. Under the same $\phi(r_h)$, it is obvious that the value of the holographic mass M is different while employing two kinds of quantization schemes. In this case, the Penrose inequality is given by

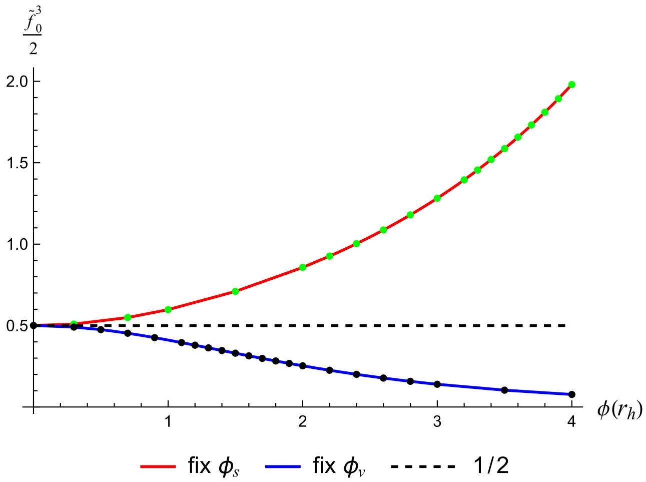

FIG. 3. Mass parameter $\tilde{f}_0^3/2$ with two renormalization schemes.

$$\frac{4\pi M}{\Omega_{0,2}} = \frac{\tilde{f}_0^3}{2} \geq \frac{r_h}{2} = \frac{1}{2}. \tag{145}$$

The inequality is guaranteed as $\phi(r_h)$ increases if we fix ϕ_s on the boundary. Otherwise, the inequality will be broken if we fix ϕ_v on the boundary. This means that the Penrose inequality is generally not true if we use alternative quantization in holography.

V. SUMMARY

The recent holographic deduction of the Penrose inequality only assumes the null energy condition, while the weak or dominant energy condition is required in the usual geometric proof. Here, we take a step toward filling the gap between these two approaches. We first discussed the AdS Penrose inequality and null energy condition from the viewpoint of pure geometry. For an asymptotically Schwarzschild-AdS black hole, the matter decays fast enough so that we can read the total mass directly from the asymptotical expansion of the bulk metric near the AdS boundary. By virtue of this property, we defined a quasi-local mass (22) which satisfies $m(r) = M$ on the AdS boundary. Additionally, due to its particular form, the derivative of $m(r)$ is non-negative in Eq. (25), which is guaranteed by the null energy condition. Our proof indicates that the null energy condition guarantees the Penrose inequality for black holes with planar or spherical symmetries, as expected from the holographic argument of Ref. [18]. This argument also implies that the null energy condition could guarantee the Penrose inequality for hyperbolically symmetric black holes; however, we find a counterexample (42) to show that this is not true. These results inspired us to conjecture that the null energy condition can guarantee the Penrose inequality in asymptotically Schwarzschild-AdS black holes only when the cross section of the horizon has planar or spherical topology.

Next we proposed two kinds of charged generalizations for the inequality for charged black holes. The holographic argument of Ref. [18] implies that the naive generalization (6) provided the null energy condition. However, counterexamples in the static spherically symmetric case have been found independently in Refs. [19,20] for such a naive generalization. After reexamining the charge Q in the inequality (46), we proposed the first type of generalization, which interprets Q in the inequality as Q_m in (50). However, Q_m is not defined at the boundary, so it cannot be interpreted as a physical quantity of dual boundary field theory according to AdS/CFT correspondence. Thus, we proposed the second version (87) of the charged Penrose inequality, in which the charge Q is replaced by the chemical potential Φ_∞. We then gave the proofs for two such generalizations in the spherically and planar symmetric cases.

Furthermore, we found that the null energy condition is not enough to guarantee the inequality in holography if the bulk geometry is asymptotically AdS but not asymptotically Schwarzschild-AdS. In order to make this argument more explicit, we constructed the asymptotically AdS black holes coupled to a scalar field. Following the holographic renormalization, we found that different quantizations for $\phi(r)$ will lead to different values of holographic mass (144). We gave strong numerical evidence to show that whether the Penrose inequality holds or not will depend on the quantization scheme. We note that the arguments of Refs. [17,18] are valid regardless of the topologies of the horizon and the quantization scheme. Thus, such holographic arguments would lead to similar conclusions for different topologies and quantization schemes. However, our geometric proofs and concrete examples show that, if we only impose the null energy condition, whether the generalized Penrose inequality in asymptotically AdS spacetime is true or not will strongly depend on the topologies of the horizon and the quantization schemes.

In the present paper, though we proposed the conjectures for the general static case, we can only give the proofs in the spherically and planar symmetric cases. It is worth examining our conjectures in inhomogeneous cases. In addition, it is well known that in general relativity there are many other definitions of mass, such as Kamor mass, ADM mass, and so on. In this paper, we used the holographic renormalization to define the total mass in the Penrose inequality. The question is how much the mass of different definitions influences the structure of the Penrose inequality. Nevertheless, this paper only considered the scalar hairy black holes. It would also be interesting to consider other types of black holes, such as vector hairy black holes.

ACKNOWLEDGMENTS

This work is supported by the Natural Science Foundation of China under Grant No. 12005155.

APPENDIX A: PROOF OF PENROSE INEQUALITY WITH THE WEAK ENERGY CONDITION

In Sec. II, we considered the Penrose inequality by assuming the null energy condition. The null energy condition does not require the energy density ρ to be non-negative, but the sum of the energy density and pressure density needs to be non-negative. If matter satisfies the weak energy condition, we find that the Penrose inequality follows directly since ρ is always non-negative. Just like the previous procedure, let us define a quasilocal mass, which is known as the Hawking mass [41,42],

$$m(r) = \frac{r^{d-2}(r^2 + k - f)}{2}. \quad \text{(A1)}$$

When $r \to \infty$, one can check that $m(r)$ is equal to one-half of the mass parameter f_0^d,

$$m(\infty) = f_0^d/2. \quad \text{(A2)}$$

Taking the derivative of $m(r)$ with respect to r, we obtain

$$m'(r) = \frac{(d-2)k + dr^2 - (d-2)f - rf'}{r^{3-d}}. \quad \text{(A3)}$$

Integrating the left and right sides of Eq. (A3) from the horizon r_h to ∞, we find

$$m(\infty) - m(r_h) = \int_{r_h}^{\infty} \frac{(d-2)k + dr^2 - (d-2)f - rf'}{r^{3-d}} dr. \quad \text{(A4)}$$

From Eq. (18a), the expression of ρ is

$$\rho = \frac{(d-1)[(d-2)k + dr^2 - (d-2)f - rf']}{2r^2} \geq 0. \quad \text{(A5)}$$

Combining this with the weak energy condition, we obtain

$$m(\infty) - m(r_h) = f_0^d/2 - \frac{r_h^d + kr_h^{d-2}}{2} \geq 0, \quad \text{(A6)}$$

so the Penrose inequality (12) follows. In order to see why the weak energy condition guarantees the Penrose inequality with no difficulty, we express the integral (A4) in terms of the energy density ρ,

$$m(\infty) - m(r_h) = \int_{r_h}^{\infty} \frac{2\rho}{d-1} r^{d-1} dr. \tag{A7}$$

If we interpret ρ as the mass density for the quasilocal mass $m(r)$, the $m(\infty)$ must greater than or equal to $m(r_h)$ due to $\rho \geq 0$. By virtue of construction of the quasilocal mass in (A1), we can see more clearly that the Penrose inequality is a stronger version of the positive energy theorem for planar and spherical cases.

We conclude that the inequality is saturated only if $\chi(r) = 0$ and $m(r) = m(\infty) = f_0^d/2$, which leads to

$$f(r) = r^2(1 + k/r^2 - f_0^d/r_h^d), \qquad \rho = p_r = p_T = 0. \tag{A8}$$

We can see that, if the weak energy condition is satisfied, then the Penrose inequality in all three topologies is true. The saturation appears only if the black hole is a Schwarzschild-AdS black hole.

APPENDIX B: COMMENTS ABOUT ASYMPTOTICALLY SCHWARZSCHILD-AdS SPACETIME

In this appendix, we discuss asymptotically Schwarzschild-AdS spacetime. According to the Fefferman-Graham construction [43], any asymptotically AdS geometry can be described by a metric such as

$$ds^2 = \frac{L^2}{z^2}[dz^2 + g_{ij}(x,z)dx^i dx^j]. \tag{B1}$$

Here L denotes the AdS radius, x^i denotes the boundary coordinates that can be extended to the bulk in some way, and z is the emergent radial coordinate in the bulk with the AdS boundary located at $z = 0$. Considering the $(d+1)$-dimensional asymptotically AdS spacetime with d boundary dimensions, we can expand the boundary metric $g_{ij}(x,z)$ with z^2 near the AdS boundary $z = 0$:

$$\begin{aligned} g(x,z) = g_0 + z^2 g_1 + z^4 g_2 + \cdots + z^d g_{d/2} \\ + z^d \ln(z/L) f + \cdots \end{aligned} \tag{B2}$$

Here $g_0(x)$ is the boundary metric. The logarithmic term $z^d \ln(z/L) f_{ij}(x)$ arises for even d. It should be noted that, only for $n < \frac{d}{2}$, the expansion coefficients $g_n(x)$ are determined by the boundary metric g_0 through the Einstein equation. More specifically, these coefficients can be solved order by order; see [28] for details. However, starting from $n = d/2$, i.e., $g_{d/2}$, there is another set of linearly independent solutions that cannot be fixed by the boundary metric g_0. In order to fix these independent solutions, we need to know the stress energy tensor in the boundary. In asymptotically Schwarzschild-AdS spacetime, the coefficient of the z^d term is determined by the ADM mass of spacetime (see the relation between the mass density parameter f_0^d and the total mass M). Thus, for asymptotically Schwarzschild-AdS spacetimes, we do not require additional information from the boundary matter field's stress energy tensor to fix the coefficient of $n \leq d/2$ terms.

However, when the matter field does not decay fast enough, like the scalar field in our paper, we need more information from other matter fields to fix the coefficient of the $n = d/2$ term. In Sec. IV, we expanded the metric of some Einstein-scalar theory. Let us rephrase the result with z equal to the inverse of r:

$$\begin{aligned} f(z) &= 1 + \frac{\phi_s^2}{4} z^2 - f_0^3 z^3 + \mathcal{O}(z^4), \\ \chi(z) &= \frac{\phi_s^2}{4} z^2 + \frac{2\phi_s \phi_v}{3} z^3 + \mathcal{O}(z^4). \end{aligned} \tag{B3}$$

Then, we combine the above expansions as

$$f(z)e^{-\chi(z)} = 1 - \left(f_0^3 + \frac{2\phi_s\phi_v}{3}\right) z^3 + \mathcal{O}(z^4). \tag{B4}$$

We can clearly see that giving only the ADM mass of spacetime is not enough to specify the bulk geometry. As we claimed in our paper, the system's total mass should identify as the holographic mass. The metric of asymptotically Schwarzschild-AdS spacetime needs to meet

$$\begin{aligned} f(z) &= 1 + kz^2 - f_0^d z^d + \cdots, \\ \chi(z) &= \chi_0 z^{d+\alpha} + \cdots, \quad \alpha > 0, \end{aligned} \tag{B5}$$

at the AdS boundary. Therefore, the richness of this spacetime is reflected in the higher order term denoted by the dots in Eq. (B5). Still considering Einstein-scalar theory here as an example, we can fix some terms in the dots determined by the boundary stress energy tensor from the scalar sector. Then, we can conversely construct the corresponding scalar potential function through the equation of motion. A rich family of potential functions satisfies our requirements for asymptotic behavior of metric (B5). One may wonder if solutions for Einstein-scalar theory generated this way could be nonphysical. However, we note that the solutions for the action

$$S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\left[\mathcal{R} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right]$$

do not violate the null energy condition.

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