Polyharmonic Green functions and nonlocal Bondi-Metzner-Sachs transformations of a free scalar field

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We express the nonlocal Bondi-Metzner-Sachs (BMS) charges of a free massless Klein-Gordon scalar field in $2+1$ in terms of the Green functions of the polyharmonic operators. Using the properties of these Green functions, we are able to discuss the asymptotic behavior of the fields that ensures the existence of the charges and prove that one obtains a realization of the $2+1$ BMS algebra in canonical phase space. We also discuss the transformations in configuration space and show that in this case the algebra closes only up to skew-symmetric combinations of the equations of motion. The formulation of the charges in terms of Green functions opens the way to the generalization of the formalism to other dimensions and systems.

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I. INTRODUCTION

The Bondi-Metzner-Sachs (BMS) group of symmetry transformations [1,2] has experienced a surge of interest during the last decade. It has been used, for instance, to deduce Weinberg's soft graviton theorems [3] as Ward identities of BMS supertranslations [4–7]. A pedagogical overview of the role of BMS symmetries in several problems in field theory and gravitation is presented in [8]. The BMS algebra in the case of the $2+1$ space-time has been studied in [9,10], and some applications can be found in [11–15].

The existence of the Noether charges that canonically generate these transformations implies an asymptotic behavior at spatial infinity for the scalar field. The asymptotic behavior of the scalar fields and the relation to soft theorems have been studied in [16], and the BMS asymptotic symmetries at spatial infinity using the Hamiltonian formalism has been considered in [17–19].

Following the ideas in [20], an explicit nonlocal realization of supertranslations for scalar free fields in $2+1$ Minkowski space-time was developed in [21]. In that paper the canonical momenta operators in Fourier space,

$$P^\mu = \int d\tilde{k}\, \bar{a}(\vec{k}) k^\mu a(\vec{k}), \qquad d\tilde{k} = \frac{\mathrm{d}^2 k}{2(2\pi)^2 \omega}, \qquad \omega = |\vec{k}|, \tag{1}$$

were extended to an infinite number of operators,

$$\mathcal{P}_\ell = \int d\tilde{k}\, \bar{a}(\vec{k}) w_\ell(\vec{k}) a(\vec{k}), \qquad \ell \in \mathbb{Z}, \tag{2}$$

using functions $w_\ell(\vec{k})$ generalizing the momentum space coordinates (ω, k_1, k_2), namely

$$w_\ell(\vec{k}) = \frac{(k_1 + i k_2)^\ell}{\omega^{\ell-1}}, \tag{3}$$

with the standard Poincaré operators obtained for $\ell = 0$ and from real combinations of the $\ell = \pm 1$ cases. The functions w_ℓ are the eigenfunctions with eigenvalue $+2$ of the operator

$$\nabla^2_{\text{massless}} = \omega^2 \partial_\omega^2 + 2\omega \partial_\omega. \tag{4}$$

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The rationale for this is the key observation in [20], which was in fact done for the $(3+1)$-dimensional case, that if one considers the mass-shell hyperboloid manifold of a massive particle, $\omega^2 - |\vec{k}|^2 = m^2$, and its associated Laplace-Beltrami operator, which for dimension $2+1$ is given by

$$\nabla^2 = \frac{1}{m^2}\left((z^2-1)\partial_z^2 + 2z\partial_z + \frac{1}{z^2-1}\partial_\theta^2\right), \tag{5}$$

with the hyperboloid parametrized by $z \in [1, +\infty)$, $\theta \in [0, 2\pi)$, then the functions k^μ are eigenfunctions of this operator, with eigenvalue $2/m^2$ in the case of $2+1$ space-time dimension. By computing all the eigenfunctions corresponding to this eigenvalue one can obtain the generalized momenta which finally yield the supertranslations in the massive case. The operator in (4) can be obtained by an appropriate $m \to 0$ limit of the above Laplace-Beltrami operator. It should also be remarked that $\nabla^2_{\text{massless}}$ is actually one of the two Casimirs of $SO(2,1)$, namely $-J^2 + K_1^2 + K_2^2$, with J the generator of rotations and K_1, K_2 those of the boosts.

The operators $\mathcal{P}_\ell$, together with the Lorentz transformations, form a representation of BMS_3. By inverting the Fourier modes in terms of the field ϕ and its canonical momentum π,

$$a(\vec{k}) = \int \mathrm{d}^2x e^{-ikx}(\omega\phi(t,\vec{x}) + i\pi(t,\vec{x})), \tag{6}$$

with $\bar{a}(\vec{k})$ given by the complex conjugate, one can obtain an expression of $\mathcal{P}_\ell$ in phase space, from which functional variations can be computed.

For a massless field [22], the above procedure yields a supertranslation transformation, as derived in [21], given by

$$\delta_\ell\phi(t,\vec{x}) = \int \mathrm{d}^2y[f_\ell(\vec{x}-\vec{y})\phi(t,\vec{y}) + g_\ell(\vec{x}-\vec{y})\pi(t,\vec{y})], \tag{7}$$

$$\delta_\ell\pi(t,\vec{x}) = \int \mathrm{d}^2y[h_\ell(\vec{x}-\vec{y})\phi(t,\vec{y}) + f_\ell(\vec{x}-\vec{y})\pi(t,\vec{y})], \tag{8}$$

where integration is all over two-dimensional space. The functions appearing in the above expressions are given by

$$f_\ell(\vec{x}) = 2\int d\tilde{k}\,\omega\,\omega_\ell(\vec{k})\sin(\vec{k}\cdot\vec{x}), \tag{9}$$

$$g_\ell(\vec{x}) = 2\int d\tilde{k}\,\omega_\ell(\vec{k})\cos(\vec{k}\cdot\vec{x}), \tag{10}$$

$$h_\ell(\vec{x}) = -2\int d\tilde{k}\,\omega^2\,\omega_\ell(\vec{k})\cos(\vec{k}\cdot\vec{x}), \tag{11}$$

with ω_ℓ given by (3) and the measure in momenta space as in (1). One can check that $\omega_{-\ell} = \omega_\ell^*$ and thus one can work with $\ell \geq 0$ and take the complex conjugate when negative indexes are needed. Furthermore, since $\omega_\ell(-\vec{k}) = (-1)^\ell\omega_\ell(\vec{k})$, one can see from (9)–(11) that

$$f_{2\ell}(\vec{x}) = 0, \qquad g_{2\ell+1}(\vec{x}) = 0, \qquad h_{2\ell+1}(\vec{x}) = 0, \qquad \ell \in \mathbb{Z}. \tag{12}$$

Notice that, in general, the transformations (7), (8) are nonlocal unless the functions f_ℓ, g_ℓ, h_ℓ are proportional to a delta function or a finite number of its derivatives. As we will see, this happens only for $\ell = 0, \pm 1$, which corresponds to ordinary space-time translations.

Using standard equal-time Poisson brackets and the properties $f_\ell(-\vec{x}) = -f_\ell(\vec{x})$, $g_\ell(-\vec{x}) = g_\ell(\vec{x})$, $h_\ell(\vec{x}) = \nabla^2 g_\ell(\vec{x})$, it can be seen that the field transformations (7), (8) are generated by the supertranslation charges

$$\begin{aligned} Q_\ell(t) = \int \mathrm{d}^2x\mathrm{d}^2y\Big(& f_\ell(\vec{x}-\vec{y})\pi(t,\vec{x})\phi(t,\vec{y}) \\ &+ \frac{1}{2}g_\ell(\vec{x}-\vec{y})\pi(t,\vec{x})\pi(t,\vec{y}) \\ &- \frac{1}{2}h_\ell(\vec{x}-\vec{y})\phi(t,\vec{x})\phi(t,\vec{y})\Big). \end{aligned} \tag{13}$$

Once the asymptotic behavior of $\phi(t,\vec{x}), \pi(t,\vec{x})$ at spatial infinity is given, see Appendix A, and using standard equal-time Poisson brackets, it can be shown (see Appendix B) that these charges have zero Poisson bracket with the Hamiltonian of the massless scalar field

$$H(t) = \int \mathrm{d}^2x\left(\frac{1}{2}\pi^2(t,\vec{x}) + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{x}))^2\right), \tag{14}$$

and are thus conserved. The proof relies solely on the symmetry properties of the functions f_ℓ and g_ℓ and on the relation between g_ℓ and h_ℓ. The main goal of the paper is to show that the functions (9), (10) and (11) can be cast in terms of higher level objects, which turn out to be Green functions, and to use their properties to discuss some aspects of the transformations. The algebra of the transformations in terms only of $\phi(t,\vec{x})$ closes only up to an antisymmetric combination of the equations of motion.

The rest of the paper is organized as follows. Section II presents the expression of the supercharges in terms of Green functions of appropriate operators. Using these, the existence of the charges is discussed. Section III computes the Poisson brackets of the obtained charges, while Sec. IV considers the BMS transformations in configuration space. Finally, in Sec. V we summarize our results and suggest

that the formalism presented in this paper can be generalized to other dimensions and systems. Detailed calculations of all the results have been moved to the Appendices.

II. NONLOCAL TRANSFORMATION OF THE FIELDS IN TERMS OF POLYHARMONIC FUNCTIONS

Using the explicit expression for ω_ℓ, one can write g_ℓ as

$$g_\ell(\vec{x}) = \frac{1}{(2\pi)^2}\int d^2k\,\omega^{-\ell}(k^1+ik^2)^\ell\cos(\vec{k}\cdot\vec{x}). \tag{15}$$

For ℓ odd, the integrand is antisymmetric in $\vec{k}$ and the integral cancels out. For ℓ even and non-negative, one can write

$$\begin{aligned} g_{2\ell}(\vec{x}) &= \frac{1}{(2\pi)^2}\int d^2k\,\omega^{-2\ell}(k^1+ik^2)^{2\ell}\cos(\vec{k}\cdot\vec{x}), \\ &= (\partial_{x_1}+i\partial_{x_2})^{2\ell}(-1)^\ell\frac{1}{(2\pi)^2}\int d^2k\,\omega^{-2\ell}\cos(\vec{k}\cdot\vec{x}). \end{aligned} \tag{16}$$

Now, defining a distribution $G_\ell(\vec{x})$ such that

$$G_\ell(\vec{x}) = (-1)^\ell\frac{1}{(2\pi)^2}\int d^2k\,\omega^{-2\ell}\cos(\vec{k}\cdot\vec{x}), \tag{17}$$

one gets the equation for a polyharmonic Green function,

$$(\nabla^2_{\vec{x}})^\ell G_\ell(\vec{x}) = \frac{1}{(2\pi)^2}\int d^2k\,\cos(\vec{k}\cdot\vec{x}) = \delta(\vec{x}), \qquad \ell\geq 0. \tag{18}$$

One has that $G_0(\vec{x}) = \delta(\vec{x})$ and, for $\ell \geq 1$, the Green function is [26,27]

$$\begin{aligned} G_\ell(\vec{x},\vec{y}) &= G_\ell(\vec{x}-\vec{y}) \\ &= \frac{|\vec{x}-\vec{y}|^{2(\ell-1)}}{[(\ell-1)!]^2 2^{2\ell-1}\pi}(\log|\vec{x}-\vec{y}| - H_{\ell-1}), \end{aligned} \tag{19}$$

where $H_\ell = \sum_{i=1}^{\ell}\frac{1}{i}$ and $H_0 = 0$. It follows from (18) that

$$\nabla^2 G_\ell(\vec{x}-\vec{y}) = G_{\ell-1}(\vec{x}-\vec{y}), \qquad \ell\geq 1. \tag{20}$$

As shown in Appendix E, these functions also satisfy the convolution property

$$\int \mathrm{d}^2x G_\ell(\vec{y}-\vec{x})G_m(\vec{z}-\vec{x}) = G_{\ell+m}(\vec{y}-\vec{z}). \tag{21}$$

The expressions for f_ℓ and h_ℓ can then be directly obtained as a function of g_ℓ by observing that $f_{2\ell+1} = -(\partial_{x_1}+i\partial_{x_2})g_{2\ell}$ and $h_{2\ell} = \nabla^2 g_{2\ell}$, for $\ell\in\mathbb{N}$ and they are zero otherwise. Thus, these functions can be written in terms of the polyharmonic Green function $G_\ell(\vec{x})$ as

$$g_{2\ell}(\vec{x}) = (\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}), \tag{22}$$

$$f_{2\ell+1}(\vec{x}) = -(\partial_{x_1}+i\partial_{x_2})^{2\ell+1}G_\ell(\vec{x}), \tag{23}$$

$$h_{2\ell}(\vec{x}) = (\partial_{x_1}-i\partial_{x_2})(\partial_{x_1}+i\partial_{x_2})^{2\ell+1}G_\ell(\vec{x}), \tag{24}$$

for $\ell\geq 0$. For $\ell = 0$ one has $g_0(\vec{x}) = \delta(\vec{x})$, $f_1(\vec{x}) = -(\partial_{x_1}+i\partial_{x_2})\delta(\vec{x})$ and $h_0(\vec{x}) = \nabla^2\delta(\vec{x})$, which yield the standard space-time translations for the fields.

In terms of the G_ℓ the supertranslation charges (13) for $\ell\geq 0$ take the forms

$$\begin{aligned} Q_{2\ell}(t) = \int \mathrm{d}^2x\,\mathrm{d}^2y\Big(&\frac{1}{2}\pi(t,\vec{x})\pi(t,\vec{y}) + \frac{1}{2}\vec{\nabla}\phi(t,\vec{x}) \\ &\cdot\vec{\nabla}\phi(t,\vec{y})\Big)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}), \end{aligned} \tag{25}$$

$$\begin{aligned} Q_{2\ell+1}(t) &= \int \mathrm{d}^2x\,\mathrm{d}^2y(\partial_{x_1}+i\partial_{x_2})\pi(t,\vec{x})\phi(t,\vec{y}) \\ &\quad\times(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}), \end{aligned} \tag{26}$$

$$\begin{aligned} &= \int \mathrm{d}^2x\,\mathrm{d}^2y\phi(t,\vec{x})\pi(t,\vec{y}) \\ &\quad\times(\partial_{x_1}+i\partial_{x_2})^{2\ell+1}G_\ell(\vec{x}-\vec{y}), \end{aligned} \tag{27}$$

$$\begin{aligned} &= -\int \mathrm{d}^2x\,\mathrm{d}^2y(\partial_{x_1}+i\partial_{x_2})\phi(t,\vec{x})\pi(t,\vec{y}) \\ &\quad\times(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}). \end{aligned} \tag{28}$$

Using that $\omega^*_\ell = \omega_{-\ell}$ one has that for $\ell<0$ the only difference is the appearance of $\partial_{x_1}-i\partial_{x_2}$ instead of $\partial_{x_1}+i\partial_{x_2}$, and thus

$$Q_{-2\ell}(t) = Q^*_{2\ell}(t), \qquad Q_{-(2\ell+1)}(t) = Q^*_{2\ell+1}(t). \tag{29}$$

The supertranslation transformations in terms of the G_ℓ are given by

$$\begin{aligned} \delta_{2\ell}\phi(t,\vec{x}) &= \{\phi(t,\vec{x}), Q_{2\ell}(t)\}, \\ &= \int \mathrm{d}^2y\,\pi(t,\vec{y})(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}), \end{aligned} \tag{30}$$

$$\begin{aligned}\delta_{2\ell+1}\phi(t,\vec{x}) &= \{\phi(t,\vec{x}), Q_{2\ell+1}(t)\},\\ &= \int \mathrm{d}^2 z\, \phi(t,\vec{z})(\partial_{z_1} + i\partial_{z_2})^{2\ell+1} G_\ell(\vec{z}-\vec{x}),\\ &= -\int \mathrm{d}^2 y\, \phi(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x}-\vec{y}),\end{aligned} \tag{31}$$

where we have used the Poisson bracket $\{\phi(t,\vec{x}),\pi(t,\vec{y})\} = \delta(\vec{x}-\vec{y})$.

In particular, using that $G_0(\vec{x}-\vec{y}) = \delta(\vec{x}-\vec{y})$, one can see that $Q_0(t) = H(t)$ is the generator of time translations and that

$$Q_{x_1}(t) = \frac{Q_1(t)+Q_{-1}(t)}{2} = -\int \mathrm{d}^2 x\, \pi(t,\vec{x})\partial_{x_1}\phi(t,\vec{x}), \tag{32}$$

$$Q_{x_2}(t) = \frac{Q_1(t)-Q_{-1}(t)}{2i} = -\int \mathrm{d}^2 x\, \pi(t,\vec{x})\partial_{x_2}\phi(t,\vec{x}), \tag{33}$$

generate the spatial translations.

We discuss next the asymptotic behavior of the fields that guarantees the existence of the supertranslation charges and of the symplectic form.

Let us first consider the kinetic term in the action, which eventually leads to a well-defined Poisson bracket,

$$\int \mathrm{d}^2 x\, \dot{\phi}(t,\vec{x})\pi(t,\vec{x}). \tag{34}$$

If we assume asymptotic expansions

$$\phi(t,\vec{x}) = \frac{\bar{\phi}_1}{|\vec{x}|} + \frac{\bar{\phi}_2}{|\vec{x}|^2} + \ldots, \tag{35}$$

$$\pi(t,\vec{x}) = \frac{\bar{\pi}_2}{|\vec{x}|^2} + \frac{\bar{\pi}_3}{|\vec{x}|^3} + \ldots, \tag{36}$$

where the $\bar{\phi}_1, \bar{\phi}_2, \bar{\pi}_1, \bar{\pi}_2, \ldots$, are functions depending on time and the angular variable, then

$$\int \mathrm{d}^2 x\, \dot{\phi}(t,\vec{x})\pi(t,\vec{x}) = \int \mathrm{d}\theta \int r\mathrm{d}r \left(\dot{\bar{\phi}}_1 \bar{\pi}_2 \frac{1}{r^3} + O(r^{-4})\right) \tag{37}$$

which makes the term well defined. It follows also from these conditions that the field configuration has then a finite energy, and in fact the conditions cannot be relaxed, by instance by assuming $\phi \sim \log r$ or $\pi \sim 1/r$, if one wants to have a finite energy. Notice that under these conditions no logarithmic divergence appears in (37), in contrast with the case in $3+1$ space-time discussed in [18].

The leading order behavior of $G_l(\vec{x}-\vec{y})$ for large $r = |\vec{x}-\vec{y}|$ is

$$G_\ell(r) \sim r^{2(\ell-1)} \log r. \tag{38}$$

As shown in Appendix A, the derivatives of order 2ℓ which appear in (25) and (28) behave as

$$(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(r) \sim \frac{1}{r^2} \qquad \forall\, \ell \geq 1. \tag{39}$$

Taking this into account and comparing (14) with (25) and (28), it follows that the supertranslation charges exist for field configurations behaving as in (35), (36).

In order to have the BMS algebra we will need, besides the generators of supertranslations, those of the Lorentz symmetries, given by

$$M^{12}(t) = -\int \mathrm{d}^2 x\, \pi(t,\vec{x})(x_1\partial_{x_2}\phi(t,\vec{x}) - x_2\partial_{x_1}\phi(t,\vec{x})), \tag{40}$$

$$M^{0i}(t) = -\int \mathrm{d}^2 x (t\pi(t,\vec{x})\partial_{x_i}\phi(t,\vec{x}) + x_i\mathcal{H}(t,\vec{x})), \tag{41}$$

for $i = 1, 2$, and with

$$\mathcal{H}(t,\vec{x}) = \frac{1}{2}(\pi^2(t,\vec{x}) + (\vec{\nabla}\phi(t,\vec{x}))^2) \tag{42}$$

the energy density of the scalar field.

III. THE BMS ALGEBRA IN PHASE SPACE

The abstract BMS algebra in $2+1$ is given by

$$[L_n, P_m] = (n-m)P_{m+n}, \tag{43}$$

$$[P_m, P_{m'}] = 0, \tag{44}$$

with $n \in \{-1, 0, 1\}$ and $m, m' \in \mathbb{Z}$, and with the L_n satisfying $[L_n, L_{n'}] = (n-n')L_{n+n'}$ and yielding the $2+1$ Lorentz algebra. The algebra can be extended to $n, n' \in \mathbb{Z}$ by introducing the superrotations L_n, $|n| > 1$ [28,29].

We will show that the above supertranslation charges (25), (26), (29) provide a realization of this algebra in terms of the standard equal-time Poisson brackets and with the following combinations of Lorentz generators:

$$L_0(t) = \frac{1}{2i}M^{12}(t), \tag{45}$$

$$L_1(t) = -M^{01}(t) - iM^{02}(t), \tag{46}$$

$$L_{-1}(t) = M^{01}(t) - iM^{02}(t) = -(L_1(t))^*. \tag{47}$$

The proof relies on the general symmetry properties of the Green functions G_ℓ and their derivatives, as well as on a key identity that is proved in Appendix D. The brackets between the supertranslation charges are discussed in Appendix B and here we will discuss only those involving the Lorentz generators.

Let us consider first the Poisson bracket $\{L_0(t), Q_{2\ell}(t)\}$, for $\ell \geq 0$. With the notation $\phi(x) = \phi(t, \vec{x})$ and so on, and defining

$$\mathcal{H}(x,y) = \frac{1}{2}(\pi(x)\pi(y) + \vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(y)), \tag{48}$$

one has

$$\{L_0(t), Q_{2\ell}(t)\} = -\frac{1}{2i}\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\{\pi(x)(x_1\partial_{x_2}\phi(x) - x_2\partial_{x_1}\phi(x)), \mathcal{H}(y,z)\}(\partial_{y_1} + i\partial_{y_2})^{2\ell} \times G_\ell(\vec{y}-\vec{z}).$$

This bracket is computed in Appendix C, and the result is [see Eq. (C3)]

$$\{L_0(t), Q_{2\ell}(t)\} = -2\ell Q_{2\ell}(t), \qquad \ell \geq 0. \tag{49}$$

Using similar steps to those in Appendix C, one can also obtain

$$\{L_0(t), Q_{2\ell+1}(t)\} = -(2\ell+1)Q_{2\ell+1}(t), \qquad \ell \geq 0. \tag{50}$$

The above computations are only valid for $\ell \geq 0$. For $\ell < 0$ one can take the complex conjugate of (49) and (50), and use (29) and also $L_0^*(t) = -L_0(t)$. In this way one obtains, for $\ell \geq 0$,

$$\{L_0(t), Q_{-2\ell}(t)\} = 2\ell Q_{-2\ell}(t) = -(-2\ell)Q_{-2\ell}(t), \tag{51}$$

$$\{L_0(t), Q_{-(2\ell+1)}(t)\} = (2\ell+1)Q_{-(2\ell+1)}(t), = -(-(2\ell+1))Q_{-(2\ell+1)}(t). \tag{52}$$

Relations (49)–(52) give the complete set of BMS algebra relations involving the rotation generator L_0.

Let us proceed now with the brackets involving the boost generators. Consider first

$$\{L_1(t), Q_{2\ell}(t)\} = \int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\{t\pi(x)(\partial_{x_1} + i\partial_{x_2})\phi(x) + (x_1 + ix_2)\mathcal{H}(x), \mathcal{H}(y,z)\} \times (\partial_{y_1} + i\partial_{y_2})^{2\ell}G_\ell(\vec{y}-\vec{z}).$$

As shown in Appendix C, this can be seen to be [see Eq. (C5)]

$$\{L_1(t), Q_{2\ell}(t)\} = (1-2\ell)Q_{2\ell+1}(t), \tag{53}$$

which is the correct action of L_1 on a supertranslation of order 2ℓ, $\ell > 0$, and can be extended to the trivial (Poincaré) case $\ell = 0$.

Using similar computations, together with $G_\ell = \vec{\nabla}^2 G_{\ell+1}$ one can show that, for $\ell > 0$,

$$\{L_1(t), Q_{2\ell+1}(t)\} = -2\ell\int \mathrm{d}^2x\mathrm{d}^2y\mathcal{H}(x,y) \times (\partial_{x_1} + i\partial_{x_2})^{2\ell+2}G_{\ell+1}(\vec{x}-\vec{y}). \tag{54}$$

Changing now $\ell \to \ell - 1$ one has

$$\{L_1(t), Q_{2\ell-1}(t)\} = \tag{55}$$

$$-2(\ell-1)\int \mathrm{d}^2x\mathrm{d}^2y\mathcal{H}(x,y)(\partial_{x_1} + i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}) = -2(\ell-1)Q_{2\ell}(t) = (1-(2\ell-1))Q_{2l}(t), \quad \ell > 1, \tag{56}$$

which is the correct relation, and which again can be extended to the Poincaré $\ell = 1$ case.

In order to get all the relations of the BMS algebra, an extra pair of brackets must be computed. The final results are

$$\{L_1(t), Q_{-2\ell}(t)\} = (2\ell+1)Q_{-2\ell+1}(t), = (1-(-2\ell))Q_{-2\ell+1}(t), \tag{57}$$

$$\{L_1(t), Q_{-(2\ell+1)}(t)\} = (2\ell+2)Q_{-2\ell}(t), = (1-(-(2\ell+1)))Q_{-2\ell}(t). \tag{58}$$

In these cases the computations involve slightly different manipulations but always using (D3).

The brackets involving L_{-1} can be computed from (53), (56), (57) and (58) by complex conjugation and using $L_1(t) = -L_{-1}^*(t)$, and one gets

$$\{L_{-1}(t), Q_{-2\ell}(t)\} = (-1+2\ell)Q_{-(2\ell+1)}(t), = (-1-(-2\ell))Q_{-(2\ell+1)}(t), \tag{59}$$

$$\{L_{-1}(t), Q_{2\ell}(t)\} = -(2\ell+1)Q_{2\ell-1}(t), = (-1-2\ell))Q_{2\ell-1}(t), \tag{60}$$

$$\{L_{-1}(t), Q_{2\ell+1}(t)\} = -(2\ell+2)Q_{2\ell}(t), = (-1-(2\ell+1))Q_{2\ell}(t), \tag{61}$$

$$\{L_{-1}(t), Q_{-2\ell-1}(t)\} = 2\ell Q_{-2(\ell+1)}(t), = (-1-(-2\ell-1))Q_{-2(\ell+1)}(t). \tag{62}$$

This completes the proof that the charges defined by (25), (26), and (29) provide, together with the Lorentz generators, a realization of the $2+1$ BMS algebra.

IV. THE BMS ALGEBRA IN CONFIGURATION SPACE

In the previous section we have shown that the BMS algebra is obtained in phase space using the expression of the supertranslation and Lorentz charges in terms of the fields ϕ, π. We will discuss now the transformations in configuration space, using ϕ and $\dot{\phi}$ as independent fields. The result is that one obtains a BMS algebra of transformations modulo trivial symmetry transformations, given by skew-symmetric combinations of the equations of motion of ϕ.

In order to show this, let us consider the specific case of the Lorentz transformation δ_1^B associated to L_1 and the transformation $\delta_{2\ell}$ given by the supertranslation charge $Q_{2\ell}$, $\ell \geq 0$. In configuration space we must substitute $\dot{\phi}$ for π, and the transformations are

$$\delta_1^B \phi(x) = t(\partial_{x_1} + i\partial_{x_2})\phi(x) + (x_1 + ix_2)\dot{\phi}(x), \tag{63}$$

$$\delta_{2\ell}\phi(x) = \int \mathrm{d}^2 y \dot{\phi}(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y}). \tag{64}$$

Since these are functional variations, the transformation of $\dot{\phi}$ is obtained by derivation, and one gets

$$\delta_1^B \dot{\phi}(x) = (\partial_{x_1} + i\partial_{x_2})\phi(x) + t(\partial_{x_1} + i\partial_{x_2})\dot{\phi}(x) + (x_1 + ix_2)\ddot{\phi}(x), \tag{65}$$

$$\delta_{2\ell}\dot{\phi}(x) = \int \mathrm{d}^2 y \ddot{\phi}(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y}). \tag{66}$$

Now we can compute the compositions of transformations

$$\begin{aligned}\delta_{2\ell}\delta_1^B\phi(x) &= t(\partial_{x_1} + i\partial_{x_2})\delta_{2\ell}\phi(x) + (x_1 + ix_2)\delta_{2\ell}\dot{\phi}(x),\\ &= t\int d^2 y \dot{\phi}(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x} - \vec{y})\\ &\quad + (x_1 + ix_2)\int d^2 y \ddot{\phi}(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell}\\ &\quad \times G_\ell(\vec{x} - \vec{y}),\end{aligned} \tag{67}$$

and

$$\begin{aligned}&\delta_1^B\delta_{2\ell}\phi(x)\\ &= \int d^2 y \delta_1^B \dot{\phi}(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y}),\\ &= \int d^2 y(\partial_{y_1} + i\partial_{y_2})\phi(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y})\\ &\quad + t\int d^2 y(\partial_{y_1} + i\partial_{y_2})\dot{\phi}(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y})\\ &\quad + \int d^2 y(y_1 + iy_2)\ddot{\phi}(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y}).\end{aligned} \tag{68}$$

The first term in (68),

$$\begin{aligned}&\int d^2 y(\partial_{y_1} + i\partial_{y_2})\phi(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y})\\ &= \int d^2 y \phi(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x} - \vec{y}),\end{aligned} \tag{69}$$

is just $-\delta_{2\ell+1}\phi(x)$ and, assembling the remaining terms, the commutator of the two transformations turns out to be

$$\begin{aligned}[\delta_1^B, \delta_{2\ell}]\phi(x) = -\delta_{2\ell+1}\phi(x) - \int d^2 y((x_1 - y_1)\\ + i(x_2 - y_2))\ddot{\phi}(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y}).\end{aligned} \tag{70}$$

Now we add and subtract $\vec{\nabla}^2\phi$ and obtain

$$\begin{aligned}[\delta_1^B, \delta_{2\ell}]\phi(x) &= -\delta_{2\ell+1}\phi(x) - \int \mathrm{d}^2 y((x_1 - y_1) + i(x_2 - y_2))\vec{\nabla}_y^2\phi(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y})\\ &\quad - \int \mathrm{d}^2 y((x_1 - y_1) + i(x_2 - y_2))(\ddot{\phi}(y) - \vec{\nabla}_y^2\phi(y))(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y})\\ &= -\delta_{2\ell+1}\phi(x) - \int \mathrm{d}^2 y((x_1 - y_1) + i(x_2 - y_2)\phi(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell}(\partial_{x_1} - i\partial_{x_2}) G_\ell(\vec{x} - \vec{y})\\ &\quad - 2\int \mathrm{d}^2 y \phi(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x} - \vec{y}) - \int \mathrm{d}^2 y F_\ell(x, y)(\ddot{\phi}(y) - \vec{\nabla}_y^2\phi(y)),\end{aligned} \tag{71}$$

where we have integrated twice by parts the term $\vec{\nabla}^2\phi$ and defined

$$F_\ell(x, y) = ((x_1 - y_1) + i(x_2 - y_2))(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y}). \tag{72}$$

The third term in (71) is $2\delta_{2\ell+1}\phi(x)$, while the second term, following the same steps that led to (C4), becomes $2(\ell-1)\delta_{2\ell+1}\phi(x)$. Putting everything together one has

$$[\delta_1^B, \delta_{2\ell}]\phi(x) = (2\ell-1)\delta_{2\ell+1}\phi(x) - \int d^2y F_\ell(x,y)(\ddot{\phi}(y) - \vec{\nabla}_y^2\phi(y)). \quad (73)$$

Taking into account that, for any transformations generated by charges A, B, one has that $[\delta_A, \delta_B]\phi = -\delta_{\{A,B\}}\phi$, the first term in (73) is the one expected from the BMS algebra and, because $F_\ell(x,y) = -F_\ell(y,x)$, the extra term is a skew-symmetric linear combination of the equations of motion,

$$\delta_{\ell,\text{trivial}}\phi(x) = \int d^2y\, F_\ell(x,y)(\ddot{\phi}(y) - \vec{\nabla}_y^2\phi(y)), \quad (74)$$

which is a trivial symmetry transformation of any system. Notice that, for $\ell = 0$, $F_\ell(x,y) = 0$ and, as it must be, the extra term is not present for the standard commutator of a time translation and a Lorentz boost.

Similar results are obtained for the other commutators of transformations, and hence the algebra closes on shell in a consistent way.

V. DISCUSSION AND OUTLOOK

We have obtained an explicit expression for the BMS supertranslation charges of free massless scalar real scalar field in $2+1$ space-time, in terms of the Green functions of the polyharmonic operator.

We work first in phase space, and discuss the asymptotic behavior of the fields that ensures the existence of the charges, as well as that of the symplectic form associated to the Poisson brackets.

The conservation of the charges only depends on general symmetry properties of the involved functions, but the commutative character of the algebra satisfied by these charges relies on a convolution property of the Green functions. Finally, the algebra relations with the Lorentz generators is obtained by using more specific properties of the polyharmonic Green functions.

We also discuss the closure of the transformations in configuration space, and it turns out that the correct algebra is obtained modulo transformations given by skew-symmetric combinations of the equations of motion, which are trivial symmetry transformations of any system.

The form of the supertranslation charges presented in this paper, in terms of Green functions of an appropriate operator, opens the way to the generalization to other cases and/or dimensions. For instance, one could consider the extension of the results to the $3+1$ space-time dimension case, or to the case of the generators of superrotations which were also constructed in [21].

Another subject worth of study is whether the supertranslation transformations of the field ϕ can be interpreted as a base transformation of the coordinates in Minkowski space, or if additional coordinates, each associated to a supertranslation, must be introduced.

Asymptotic symmetries for massless scalar fields have recently received some attention in the literature [16,18,30,31], with approaches and techniques different from ours.

Our discussion of the asymptotic behavior of the fields in order to guarantee the existence of the conserved charges and the symplectic form follows that of [18]. In that work, the formulation of the theory in terms of a dual 2-form field allows the introduction of an infinite number of symmetries that can be interpreted as large gauge transformations for extra surface degrees of freedom at spatial infinity. These, however, cannot be realized in terms of the original scalar field. Asymptotic BMS symmetries for massless particles of any spin are discussed in [30] (see also [31]).

In a different approach, the existence of asymptotic symmetries for scalars fields is also treated in [16], and they are cast in the framework of Ward identities and the associated factorization soft theorems for interacting theories. The charges thus obtained are written in terms of the asymptotic fields, and have a linear and a quadratic part. They are discussed for both massless and massive fields. Asymptotic BMS symmetries for massless particles of any spin are discussed in [30] (see also [31]).

When trying to connect our results to those of the above papers, one encounters the difficulty that the transformations and charges presented in this paper are defined in the bulk, not just on the appropriate manifold at infinity. Even if the transformation for $\phi(t,\vec{x})$ is asymptotically approximated for the corresponding limit of $t, \vec{x}$, the fact remains that it is still given by an integral in the bulk.

As discussed above, one also has the problem that our nonlocal transformations cannot be easily interpreted as given by vector fields on the standard Minkowski manifold. In contrast, the presentation of BMS symmetries as asymptotic symmetries in asymptotically flat space-times can be formulated in terms of vector fields defined on the celestial sphere. Further work will be necessary to elucidate these relations.

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APPENDIX A: ASYMPTOTIC BEHAVIOR OF THE POLYHARMONIC GREEN FUNCTIONS

The polyharmonic Green function G_ℓ has the form

$$G_\ell(x) = A_\ell^{(0)} |\vec{x}|^{2\ell-2} \log|\vec{x}| + B_\ell^{(0)} |\vec{x}|^{2\ell-2}, \tag{A1}$$

where the constants $A_\ell^{(0)}$ and $B_\ell^{(0)}$ can be read from (19). Successive applications of $\partial_{x_1} + i\partial_{x_2}$ yield expressions of the same form, with decreasing powers of $|\vec{x}|$, until one reaches

$$\begin{aligned}&(\partial_{x_1} + i\partial_{x_2})^{\ell-1} G_\ell(\vec{x}) \\ &= (x_1 + ix_2)^{\ell-1} \left(A_\ell^{(\ell-1)} \log|\vec{x}| + B_\ell^{(\ell-1)} \right).\end{aligned} \tag{A2}$$

From this point the derivatives cease to content the log term and one can see that the derivative of order 2ℓ is of the form

$$(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}) = C^{(2\ell)} (x_1 + ix_2)^{2\ell} \frac{1}{|\vec{x}|^{2\ell+2}}, \tag{A3}$$

with a constant $C^{(2\ell)}$. This is a rational function of x_1, x_2 with asymptotic behavior

$$(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}) \sim \frac{1}{|\vec{x}|^2} \quad \forall\, \ell \geq 1 \quad \text{for } |\vec{x}| \to \infty, \tag{A4}$$

which is independent of $\ell \geq 1$. This allows us to study the conditions that must be imposed on the fields so that the supertranslation charges are finite. A general supertranslation charge Q_ℓ, $\ell \geq 1$, has integrals of the form

$$Q_\ell(t) = \int \mathrm{d}^2x \mathrm{d}^2y F(t,\vec{x}) G(t,\vec{y}) (\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y}), \tag{A5}$$

where F and G are either π or first order derivatives of ϕ. Performing a change of variables $\vec{x} = \vec{y} + \vec{r}$ and using the asymptotic behavior (A4), the existence of the charge reduces to the existence of the integral

$$\int \mathrm{d}^2r \mathrm{d}^2y F(\vec{y} + \vec{r}) G(\vec{y}) \frac{1}{r^2}. \tag{A6}$$

Let us assume now that the fields F and G behave, for a large argument, as

$$F(\vec{y} + \vec{r}) \sim \frac{\bar{F}}{|\vec{y} + \vec{r}|^\alpha}, \qquad G(\vec{y}) \sim \frac{\bar{G}}{|\vec{y}|^\beta}, \tag{A7}$$

with $\bar{F}$, $\bar{G}$ depending on the angular variable and time. This leads to the study of the integral

$$\int \mathrm{d}^2\Omega \bar{F} \bar{G} \int r\mathrm{d}r\, y\mathrm{d}y \frac{1}{|\vec{y} + \vec{r}|^\alpha |\vec{y}|^\beta r^2}, \tag{A8}$$

where $\mathrm{d}^2\Omega$ is the angular measure. Performing a change to polar coordinates in $\mathbb{R}^2_+$ for the $\mathrm{d}r\mathrm{d}y$ measure, one finally gets

$$\int \mathrm{d}^3\Omega \bar{F} \bar{G} \int \rho \mathrm{d}\rho \frac{1}{\rho^{\alpha+\beta}}, \tag{A9}$$

with $\mathrm{d}^3\Omega$ including the additional integration over the angular coordinate of the polar change of variables. For this integral to converge it is necessary that

$$\alpha + \beta > 2. \tag{A10}$$

Considering the forms of F and G for the different supertranslation charges, one concludes that the asymptotic behavior of the fields which guarantees the existence of all the Q_ℓ is the one given in (35) and (36).

APPENDIX B: BRACKETS BETWEEN THE SUPERTRANSLATION CHARGES

Consider two arbitrary supertranslation charges $Q_\ell(t)$ and $Q_m(t)$. Using standard Poisson brackets, one gets

$$\{Q_\ell(t), Q_m(t)\} = \tag{B1}$$

$$\begin{aligned}&\int \mathrm{d}^2x \mathrm{d}^2y \mathrm{d}^2z (f_\ell(\vec{x} - \vec{y}) f_m(\vec{y} - \vec{z}) \pi(x) \phi(z) \\ &- f_\ell(\vec{x} - \vec{y}) f_m(\vec{z} - \vec{x}) \phi(y) \pi(z) \\ &+ f_\ell(\vec{x} - \vec{y}) g_m(\vec{y} - \vec{z}) \pi(x) \pi(z) \\ &+ f_\ell(\vec{x} - \vec{y}) h_m(\vec{x} - \vec{z}) \phi(y) \phi(z) \\ &- g_\ell(\vec{x} - \vec{y}) f_m(\vec{z} - \vec{x}) \pi(y) \pi(z) \\ &+ g_\ell(\vec{x} - \vec{y}) h_m(\vec{x} - \vec{z}) \pi(y) \phi(z) \\ &- h_\ell(\vec{x} - \vec{y}) f_m(\vec{x} - \vec{z}) \phi(y) \phi(z) \\ &- h_\ell(\vec{x} - \vec{y}) g_m(\vec{x} - \vec{z}) \phi(y) \pi(z)).\end{aligned} \tag{B2}$$

Let us consider first the case $m = 0$, that is $Q_0(t) = H(t)$, for which $f_0(\vec{x} - \vec{y}) = 0$, $g_0(\vec{x} - \vec{y}) = \delta(\vec{x} - \vec{y})$ and $h_0(\vec{x} - \vec{y}) = \vec{\nabla}_x^2 \delta(\vec{x} - \vec{y})$. Integration by parts yields

$$\dot{Q}_\ell(t) = \{Q_\ell(t), H(t)\} = \tag{B3}$$

$$\begin{aligned}&= \int \mathrm{d}^2x \mathrm{d}^2y (f_\ell(\vec{x} - \vec{y}) \pi(x) \pi(y) + \vec{\nabla}_x^2 f_\ell(\vec{x} - \vec{y}) \phi(y) \phi(x) \\ &+ \vec{\nabla}_x^2 g_\ell(\vec{x} - \vec{y}) \pi(y) \phi(x) - h_\ell(\vec{x} - \vec{y}) \phi(y) \pi(x)).\end{aligned} \tag{B4}$$

The first two terms are zero, each by itself, due to the skew symmetry of f_ℓ and its even-order derivatives, while the two last terms cancel each other after using $\vec{\nabla}^2(x)g_\ell(\vec{x}-\vec{y}) = h_\ell(\vec{x}-\vec{y})$ and the symmetry of h_ℓ. This shows that the supertranslation charges are conserved by virtue of the symmetry properties of f_ℓ, g_ℓ and h_ℓ, and the relation between g_ℓ and h_ℓ, without using the explicit form of these functions in terms of the polyharmonic Green functions.

For general ℓ and m one must consider the different cases separately.

(1) ℓ and m odd. In this case $g_\ell = h_\ell = g_m = h_m = 0$ and, after renaming the variables of integration in the first nonzero contribution,

$$\{Q_\ell(t), Q_m(t)\} = \int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z(f_\ell(\vec{z}-\vec{x})f_m(\vec{x}-\vec{y}) - f_\ell(\vec{x}-\vec{y})f_m(\vec{z}-\vec{x}))\phi(y)\pi(z). \tag{B5}$$

(2) ℓ even and m odd. Now $f_\ell = 0$ and $g_m = h_m = 0$, and the result can be written as

$$\{Q_\ell(t), Q_m(t)\} = -\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z g_\ell(\vec{x}-\vec{y})f_m(\vec{z}-\vec{x})\pi(y)\pi(z) - \int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z h_\ell(\vec{x}-\vec{y})f_m(\vec{x}-\vec{z})\phi(y)\phi(z). \tag{B6}$$

(3) ℓ and m even. We have $f_\ell = f_m = 0$ and (B2) boils down to

$$\{Q_\ell(t), Q_m(t)\} = \int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z(g_\ell(\vec{x}-\vec{z})h_m(\vec{x}-\vec{y}) - h_\ell(\vec{x}-\vec{y})g_m(\vec{x}-\vec{z}))\phi(y)\pi(z). \tag{B7}$$

The elementary symmetry properties used up to now are not enough to show that the above expressions are actually zero. In order to do so, one must use the fact that the functions f_ℓ, g_ℓ and h_ℓ can be written in terms of Green functions that obey the convolution property (21).

Let us prove, for instance, that the first term in (B6) is zero. Changing $\ell \to 2\ell$ and $m \to 2m+1$ one has, assuming $l > 0$, $m > 0$,

$$\begin{aligned}
&-\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z g_\ell(\vec{x}-\vec{y})f_m(\vec{z}-\vec{x})\pi(y)\pi(z) \to \\
&-\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y})(\partial_{x_1}+i\partial_{x_2})^{2m+1}G_m(\vec{x}-\vec{z})\pi(y)\pi(z) \\
&= -\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z G_\ell(\vec{x}-\vec{y})G_m(\vec{x}-\vec{z})(\partial_{y_1}+i\partial_{y_2})^{2\ell}\pi(y)(\partial_{z_1}+i\partial_{z_2})^{2m+1}\pi(z) \\
&\overset{(21)}{=} -\int \mathrm{d}^2y\mathrm{d}^2z G_{\ell+m}(\vec{y}-\vec{z})(\partial_{y_1}+i\partial_{y_2})^{2\ell}\pi(y)(\partial_{z_1}+i\partial_{z_2})^{2m+1}\pi(z) \\
&= -\int \mathrm{d}^2y\mathrm{d}^2z(\partial_{y_1}+i\partial_{y_2})^{2\ell+2m+1}G_{\ell+m}(\vec{y}-\vec{z})\pi(y)\pi(z) = 0
\end{aligned}$$

due to the skew symmetry of

$$(\partial_{y_1}+i\partial_{y_2})^{2\ell+2m+1}G_{\ell+m}(\vec{y}-\vec{z}),$$

and the second term of (B6) can also be shown to be zero using the same manipulations. Notice that the same reasoning can be used for ℓ and/or m negative, since this amounts to change some $\partial_{x_1}+i\partial_{x_2}$ to $\partial_{x_1}-i\partial_{x_2}$ and the result, which only depends on the number of derivatives, is the same.

Using the same techniques and the convolution property, the two terms which appear in (B5) or (B7) can be shown to be the same and hence that the corresponding brackets are zero. This completes the proof that the supertranslation charges yield a commutative algebra under the Poisson brackets.

APPENDIX C: DETAILED COMPUTATION OF SOME POISSON BRACKETS

Consider first

$$\{L_0(t), Q_{2\ell}(t)\} = -\frac{1}{2i}\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\{\pi(x)(x_1\partial_{x_2}\phi(x) - x_2\partial_{x_1}\phi(x)), \mathcal{H}(y,z)\}(\partial_{y_1}+i\partial_{y_2})^{2\ell}G_\ell(\vec{y}-\vec{z}).$$

Using that even-order derivatives of an even-symmetric function are even, and the equal-time Poisson brackets $\{\phi(x), \pi(y)\} = \delta(\vec{x} - \vec{y})$, the above expression equals

$$\begin{aligned}\{L_0(t), Q_{2\ell}(t)\} &= -\frac{1}{2i}\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z((x_1\partial_{x_2}\phi(x) - x_2\partial_{x_1}\phi(x))\vec{\nabla}\phi(z)\cdot\vec{\nabla}_y(-\delta(\vec{x}-\vec{y})) \\ &\quad + \pi(x)\pi(z)(x_1\partial_{x_2} - x - 2\partial_{x_1})\delta(\vec{x}-\vec{y}))(\partial_{y_1} + i\partial_{y_2})^{2\ell}G_\ell(\vec{y}-\vec{z}), \\ &= -\frac{1}{2i}\int \mathrm{d}^2x\mathrm{d}^2z(2\mathcal{H}(x,z))(x_1\partial_{x_2} - x_2\partial_{x_1})(\partial_{x_1} + i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{z}), \end{aligned} \tag{C1}$$

where several integrations by parts, assuming the appropriate asymptotic behavior for the fields, have been performed, and the relation $\vec{\nabla}_x G_\ell(\vec{x}-\vec{y}) = -\vec{\nabla}_y G_\ell(\vec{x}-\vec{y})$ has been used. Next, we use the commutator $[x_1\partial_{x_2} - x_2\partial_{x_1}, (\partial_{x_1} + i\partial_{x_2})^n] = \mathrm{i}n(\partial_{x_1} + i\partial_{x_2})^n, n = 0, 1, \ldots$ to write (C1) as

$$\{L_0(t), Q_{2\ell}(t)\} = -\frac{1}{i}\int \mathrm{d}^2x\mathrm{d}^2z\mathcal{H}(x,z)((\partial_{x_1} + i\partial_{x_2})^{2\ell}(x_1\partial_{x_2} - x_2\partial_{x_1}) + i2\ell(\partial_{x_1} + i\partial_{x_2})^{2\ell})G_\ell(\vec{x}-\vec{z}).$$

One has that

$$(x_1\partial_{x_2} - x_2\partial_{x_1})G_\ell(\vec{x}-\vec{z}) = G'_\ell(|\vec{x}-\vec{z}|)\frac{-x_1z_2 + x_2z_1}{|\vec{x}-\vec{z}|} \tag{C2}$$

is an odd function under $\vec{x} \leftrightarrow \vec{z}$, and hence its derivatives of even order are also odd. The product with $\mathcal{H}(x,z)$ is also odd and the term vanishes under integration in x and z. Thus one is left only with the term from the commutator and

$$\{L_0(t), Q_{2\ell}(t)\} = -2\ell\int \mathrm{d}^2x\mathrm{d}^2z\mathcal{H}(x,z)(\partial_{x_1} + i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{z}) = -2\ell Q_{2\ell}(t), \qquad \ell \geq 0. \tag{C3}$$

Let us compute now the Poisson bracket of the boost generator $L_1(t)$ with an even-order supertranslation charge,

$$\{L_1(t), Q_{2\ell}(t)\} = \int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\{t\pi(x)(\partial_{x_1} + i\partial_{x_2})\phi(x) + (x_1 + ix_2)\mathcal{H}(x), \mathcal{H}(y,z)\}(\partial_{y_1} + i\partial_{y_2})^{2\ell}G_\ell(\vec{y}-\vec{z}).$$

After computing the Poisson brackets and using integration by parts and the symmetry of G_ℓ and its derivatives, the terms containing t can be written as

$$2t\int \mathrm{d}^2x\mathrm{d}^2z\mathcal{H}(x,z)(\partial_{x_1} + i\partial_{x_2})G_\ell(\vec{x}-\vec{z}),$$

which is zero due to the skew symmetry of $(\partial_{x_1} + i\partial_{x_2})G_\ell(\vec{x}-\vec{z})$ under $\vec{x} \leftrightarrow \vec{z}$. Performing the same manipulations, the remaining terms can be written as

$$\begin{aligned}\{L_1(t), Q_{2\ell}(t)\} = \int \mathrm{d}^2x\mathrm{d}^2z((x_1 + ix_2)\pi(x)\vec{\nabla}\phi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell}\cdot\vec{\nabla}_x G_\ell(\vec{x}-\vec{z}) \\ + (x_1 + ix_2)\vec{\nabla}\phi(x)\pi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell}\cdot\vec{\nabla}_x G_\ell(\vec{x}-\vec{z})).\end{aligned}$$

The $\vec{\nabla}\phi(z)$ in the first term can be integrated by parts, yielding one term, while the integration by parts of $\vec{\nabla}\phi(x)$ yields two. After a change of variables and using $(\partial_{z_1} + i\partial_{z_2})^{2\ell}\vec{\nabla}_z^2 G_l(\vec{x}-\vec{z}) = (\partial_{x_1} + i\partial_{x_2})^{2\ell}\vec{\nabla}_x^2 G_l(\vec{x}-\vec{z})$, the two terms that are similar can be combined, and the result is

$$\begin{aligned}\{L_1(t), Q_{2\ell}(t)\} = \int \mathrm{d}^2x\mathrm{d}^2z(((x_1 - z_1) + i(x_2 - z_2))\pi(x)\phi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell}\cdot\vec{\nabla}_x^2 G_\ell(\vec{x}-\vec{z}) \\ - \phi(x)\pi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell}(\partial_{x_1} + i\partial_{x_2})G_\ell(\vec{x}-\vec{z})).\end{aligned}$$

The second term is just $-Q_{2\ell+1}(t)$ and the integrand in the first one can be rewritten, using that $[x_1+ix_2,\partial_{x_1}+i\partial_{x_2}]=0$ and $\vec{\nabla}_x^2=(\partial_{x_1}+i\partial_{x_2})(\partial_{x_1}-i\partial_{x_2})$, as

$$\{L_1(t),Q_{2\ell}(t)\}=-Q_{2\ell+1}(t)+\int \mathrm{d}^2x\mathrm{d}^2z(\pi(x)\phi(z)(\partial_{x_1}+i\partial_{x_2})^{2\ell+1}[(x_1-z_1)+i(x_2-z_2)](\partial_{x_1}-i\partial_{x_2})G_\ell(\vec{x}-\vec{z})). \quad \text{(C4)}$$

Since we are considering $\ell\geq 1$ (the case $\ell=0$ correspond to the standard Poincaré algebra) and $2\ell+1>2(\ell-1)$, we can use relation (D3) to rewrite (C4) as

$$\begin{aligned}\{L_1(t),Q_{2\ell}(t)\}&=-Q_{2\ell+1}(t)+2(\ell-1)\int \mathrm{d}^2x\mathrm{d}^2z\pi(x)\phi(z)(\partial_{x_1}+i\partial_{x_2})^{2\ell+1}G_\ell(\vec{x}-\vec{z})\\&=-Q_{2\ell+1}(t)-2(\ell-1)\int \mathrm{d}^2x\mathrm{d}^2z\phi(x)\pi(z)(\partial_{x_1}+i\partial_{x_2})^{2\ell+1}G_\ell(\vec{x}-\vec{z})\\&=-Q_{2\ell+1}(t)-2(\ell-1)Q_{2\ell+1}(t)\\&=(1-2\ell)Q_{2\ell+1}(t). \quad \text{(C5)}\end{aligned}$$

APPENDIX D: SOME IDENTITIES SATISFIED BY THE POLYHARMONIC GREEN FUNCTIONS

Assuming $\ell>1$ and using $H_{\ell-1}=H_{\ell-2}+1/(\ell-1)$ one can show from (19) that

$$(\partial_{x_1}-i\partial_{x_2})G_\ell(\vec{x}-\vec{y})=\frac{1}{2(\ell-1)}((x_1-y_1)-i(x_2-y_2))G_{\ell-1}(\vec{x}-\vec{y})-\frac{|\vec{x}-\vec{y}|^{2(\ell-2)}}{[(\ell-1)!]^2 2^{2\ell-1}\pi}((x_1-y_1)-i(x_2-y_2)),$$

which is a recurrence relation valid for $\ell>1$. Multiplying by $(x_1-y_1)+i(x_2-y_2)$ one gets

$$((x_1-y_1)+i(x_2-y_2))(\partial_{x_1}-i\partial_{x_2})G_\ell(\vec{x}-\vec{y})=\frac{1}{2(\ell-1)}|\vec{x}-\vec{y}|^2G_{\ell-1}(\vec{x}-\vec{y})-\frac{|\vec{x}-\vec{y}|^{2(\ell-1)}}{[(\ell-1)!]^2 2^{2\ell-1}\pi}, \quad \text{(D1)}$$

which, except for polynomial terms, has the functional dependence of G_ℓ. Indeed, using again the relation between $H_{\ell-1}$ and $H_{\ell-2}$, one obtains

$$((x_1-y_1)+i(x_2-y_2))(\partial_{x_1}-i\partial_{x_2})G_\ell(\vec{x}-\vec{y})=2(\ell-1)G_\ell(\vec{x}-\vec{y})+\frac{|\vec{x}-\vec{y}|^{2(\ell-1)}}{[(\ell-1)!]^2 2^{2\ell-1}\pi}. \quad \text{(D2)}$$

Although (D2) has been obtained under the assumption that $\ell>1$ it can be checked by direct computation that it is also valid for $\ell=1$.

In the computations in Sec. III the left-hand side of this identity appears with derivatives $\partial_{x_1}+i\partial_{x_2}$ acting on it. Since the second term in (D2) is a polynomial of order $2(\ell-1)$ in the components of $\vec{x}$, it turns out that, for $\ell\geq 1$ and $n>2(\ell-1)$,

$$(\partial_{x_1}+i\partial_{x_2})^n((x_1-y_1)+i(x_2-y_2))(\partial_{x_1}-i\partial_{x_2})G_\ell(\vec{x}-\vec{y})=2(\ell-1)(\partial_{x_1}+i\partial_{x_2})^nG_\ell(\vec{x}-\vec{y}). \quad \text{(D3)}$$

APPENDIX E: CONVOLUTION PROPERTY OF THE POLYHARMONIC GREEN FUNCTIONS

One has

$$\begin{aligned}\vec{\nabla}_y^{2(\ell+m)}\int \mathrm{d}^2x G_\ell(\vec{y}-\vec{x})G_m(\vec{z}-\vec{x}) &= \int \mathrm{d}^2x \vec{\nabla}_y^{2(\ell+m)} G_\ell(\vec{y}-\vec{x})G_m(\vec{z}-\vec{x}),\\ &= \int \mathrm{d}^2x \vec{\nabla}_y^{2\ell}\vec{\nabla}_x^{2m} G_\ell(\vec{y}-\vec{x})G_m(\vec{z}-\vec{x}),\\ &= \int \mathrm{d}^2x \vec{\nabla}_y^{2\ell} G_\ell \vec{y}-\vec{x}\vec{\nabla}_x^{2m} G_m(\vec{z}-\vec{x}),\\ &= \int \mathrm{d}^2x \delta(\vec{y}-\vec{x})\delta(\vec{z}-\vec{x}) = \delta(\vec{y}-\vec{z}). \end{aligned} \tag{E1}$$

Under standard regularity conditions, the homogeneous polyharmonic problem has only the trivial solution [32], and the above computation proves (21).

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