Wilsonian approach to the interaction $\phi^2(i\phi)^\varepsilon$

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We study the renormalization of the non-Hermitian $\mathcal{PT}$-symmetric scalar field theory with the interaction $\phi^2(i\phi)^\varepsilon$ using the Wilsonian approach and without any expansion in ε. Specifically, we solve the Wetterich equation in the local potential approximation, both in the ultraviolet regime and with the loop expansion. We calculate the scale-dependent effective potential and its infrared limit. The theory is found to be renormalizable at the one-loop level only for integer values of ε, a result which is not yet established within the ε-expansion. Particular attention is therefore paid to the two interesting cases $\varepsilon = 1, 2$, and the one-loop beta functions for the coupling associated with the interaction $i\phi^3$ and $-\phi^4$ are computed. It is found that the $-\phi^4$ theory has asymptotic freedom in four-dimensional spacetime. Some general properties for the Euclidean partition function and n-point functions are also derived.

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I. INTRODUCTION

A real energy spectrum does not necessarily require the Hamiltonian to be Hermitian. Indeed, it was found by Bender and Boettcher [1] that there are a large variety of Hamiltonians with $\mathcal{PT}$ symmetry that can assure a real energy spectrum. Since then, quantum mechanics extended outside the Hermitian regime [2] has become an active subject [3–5]. In particular, non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians have found novel applications in condensed matter physics. See Refs. [6–8] for reviews. In recent years, $\mathcal{PT}$-symmetric Hamiltonians have also witnessed increasing interest in high-energy physics. The consideration of non-Hermitian Hamiltonians may provide new mechanisms for neutrino masses and oscillations [9–11], dark matter [12], Higgs decay [13], and the confinement/deconfinement phase transition in QCD [14]. The generalization of spontaneous symmetry breaking and the Goldstone theorem to non-Hermitian field theories has been carried out in Refs. [15–22]. Non-Hermitian Yukawa interactions with interesting phenomenological applications have been considered in Refs. [23–26]. Studies of the second quantization and inner product in Fock space are given for a $\mathcal{PT}$-symmetric scalar model in Ref. [27] and for a $\mathcal{PT}$-symmetric fermionic model in Ref. [28]. For some other studies, see, e.g., Refs. [29–41].

Non-Hermitian $\mathcal{PT}$-symmeric theories are mostly well understood in quantum mechanics, especially for the well-studied model

$$H = p^2 + \frac{1}{2}\mu^2 x^2 + \frac{1}{2}x^2(ix)^\varepsilon. \tag{1}$$

For $\varepsilon \geq 0$ the energy spectrum of the Hamiltonian was found to be real numerically [1]. For the massless case spectral reality was proved for $\varepsilon > 0$ by Dorey *et al.* using the methods of integrable systems [42]. The particular massless $\varepsilon = 2$ case can be mapped to a Hermitian Hamiltonian with the same spectrum [43,44]. These results are based on the Schrödinger equation directly.

For higher spacetime-dimensional quantum field theories, the Schrödinger equation is of functional type and very little information can be extracted from it. Therefore, alternative methods must be sought. A particularly useful tool is the path-integral formulation of quantum theories. Some earlier studies on $\mathcal{PT}$-symmetric theories using the path integral are found in Refs. [45,46]. Recently, in Ref. [47] based on the Euclidean path integral, a new perspective that relates a non-Hermitian $\mathcal{PT}$-symmetric theory to a Hermitian theory via analytic continuation is given. In this way, conclusions for non-Hermitian theories could be drawn from the corresponding Hermitian theories. The relation proposed in Ref. [47] assures that the Hamiltonian of form in Eq. (1) for $\varepsilon = 2$ has a real spectrum even for spacetime dimension greater than one.

Compared with quantum-mechanical models, a new feature of (continuum) quantum field theory models is

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the presence of divergences due to the infinite number of degrees of freedom. Therefore, when extending non-Hermitian $\mathcal{PT}$-symmetric quantum-mechanical models to field theory, one has to check that the theory is renormalizable, which is the topic of the present article. The theory we consider is the analog of Eq. (1), i.e., a scalar field theory with the bare potential at some cutoff scale Λ,

$$U_\Lambda(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{2}m^2\phi^2(i\phi/\phi_0)^\varepsilon, \qquad (2)$$

where ϕ_0 is some scale, $\varepsilon \geq 0$, and $\mu^2 > 0$. Recently, there have been attempts to renormalize this theory [36,45,48]. These attempts are based on an expansion in ε [49,50]. Although very interesting, these works have independently come to a puzzling conclusion that, at least within the ε-expansion, the theory seems to be trivial (at low orders in the expansion) for spacetime dimension $d = 2$ [36] and $d \geq 2$ [48]. Alternatively, these findings may indicate that the ε-expansion may not be valid for a systematic study of the renormalization of the theory. The present work aims to study the renormalization without an expansion in ε.

We base our study on the Wilsonian framework, which in principle allows a nonperturbative description of quantum fluctuations, via exact renormalization group (ERG) equations. The first ERG equation was derived by Wegner and Houghton [51]. Although very intuitive, this construction allows the evolution of the nonderivative part of the Wilsonian effective action only. An alternative ERG equation was proposed by Polchinski [52] who introduced a smooth cutoff function for Fourier modes, and therefore the derivation of flows for the whole running Wilsonian action is allowed. Wetterich proposed a third approach [53], which elegantly combines the concept of a smooth cutoff and the one-particle irreducible (1PI) technique, through the "average effective action." We will focus here on this third version of ERG equations. These equations always require some approximation to be solved, and in this article we focus on either the ultraviolet (UV) regime or the one-loop regime under the local potential approximation.

The outline of the paper is as follows. In the next section, we describe some generic properties of the Euclidean path integral for this model and construct the 1PI effective action. We also summarize the main features of the Wetterich average effective action. In Sec. III, we study the UV regime of the ERG equation, which can be mapped to a diffusion equation, and thus provides an intuitive understanding of how quantum fluctuations build up along the Wilsonian flow toward the infrared (IR). We show that the solution is analytical in the field for integer values of ε only, which indicates potential consistency problems for noninteger ε. We then focus on $\varepsilon = 1$ and discuss the beta function for the corresponding cubic coupling which, as expected, has the opposite sign compared to the Hermitian cubic interaction. Section IV focuses on the one-loop Wilsonian flow which, by construction, recovers the one-loop 1PI effective potential in the deep IR limit. The latter potential contains new interactions for noninteger ε, with diverging coefficients, which is not consistent with renormalizability. Only for integer ε can one absorb divergences in a redefinition of bare parameters, and we give the explicit one-loop renormalization for $\varepsilon = 1$. Section V is devoted to the special case $\varepsilon = 2$ in which a deformation of the integration contour is necessary to define a convergent path integral. We explain that the $\mathcal{PT}$ symmetry is respected if the deformed contour is invariant under the $\mathcal{PT}$-reflection. The construction of the 1PI effective action is however not modified and the results derived for a generic ε can then be used for $\varepsilon = 2$. We confirm that the interaction $-\phi^4$ is asymptotically free, unlike in the usual $+\phi^4$ theory. We conclude in Sec. VI.

II. PROPERTIES OF THE QUANTUM THEORY

A. Path integral convergence

We consider the Euclidean partition function

$$Z = \int \mathcal{D}[\phi] \exp\left(-\int \mathrm{d}^d x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + U_\Lambda(\phi)\right]\right), \qquad (3)$$

where $U_\Lambda(\phi)$ is given in Eq. (2). In the Hermitian case the scalar ϕ is supposed to be real. But in the non-Hermitian theory (3) the path integral is convergent (with a UV cutoff) only for ε restricted to specific intervals.[1] To see this, consider a real ϕ and write the self-interaction term as

$$m^2\frac{|\phi|^{2+\varepsilon}}{\phi_0^\varepsilon}(\cos(\pi\varepsilon/2 + \theta\varepsilon) + i\sin(\pi\varepsilon/2 + \theta\varepsilon)), \qquad (4)$$

where $\theta = 0$ if $\phi \geq 0$ and $\theta = \pi$ if $\phi < 0$. To make sure the real part is not negative for $|\phi| \to \infty$, we need to require

$$-1 + 4N \leq \varepsilon \leq 1 + 4N$$
$$\text{and} \quad \frac{1}{3}(-1 + 4N') \leq \varepsilon \leq \frac{1}{3}(1 + 4N'), \qquad (5)$$

where N and N' are integers. If we impose ε to be positive, the allowed values/intervals are $\varepsilon \in [0, 1/3]$; $\varepsilon = 1$; $\varepsilon \in [11/3, 13/3]$; etc. For real ϕ, the Lagrangian is invariant under the combined $\mathcal{PT}$ operation where

[1] In [36,45,48] the path integral is defined through a formal expansion in the parameter ε. Although formally each term in the ε expansion is calculated in terms of a convergent path integral, the properties and convergence of the series in ε are unknown. Truncating to low order in ε is in general an uncontrolled approximation.

$$\begin{aligned}\mathcal{P}:\ \ &\phi(t,\vec{x})\ \to\ -\phi(t,-\vec{x}),\\ \mathcal{T}:\ \ &\phi(t,\vec{x})\ \to\ \phi(-t,\vec{x})\quad\text{and}\quad i\to -i.\end{aligned}\tag{6}$$

For ε that does not fall into the regions given in Eq. (5), e.g., $\varepsilon=2$, ϕ necessarily takes values in the complex domain $\mathbb{C}$ to ensure the path integral to be convergent. In such a case, one may obtain the theory by analytically continuing ε from the regions given in Eq. (5) to the interested value. If one studies the theory directly with the path integral, one in principle should apply the Picard-Lefschetz theory [54–58] for the path integral with first complexifying the field configurations $\{\phi(x)\}\to\{\Phi(x)\}$ (where Φ takes values in $\mathbb{C}$) and then finding a middle-dimensional contour $\mathcal{C}_{\mathcal{PT}}$ in the path integral, with the requirements that the $\mathcal{PT}$ symmetry is still respected and the path integral is convergent. Therefore, one ends up with

$$Z=\int_{\mathcal{C}_{\mathcal{PT}}}\mathcal{D}[\Phi]\exp\bigg(-\frac{1}{2}\int \mathrm{d}^d x[\partial_\mu\Phi\partial^\mu\Phi+\mu^2\Phi^2+m^2\Phi^2(i\Phi/\phi_0)^\varepsilon]\bigg).\tag{7}$$

Since $\Phi(x)$ takes in general complex values now, the Lagrangian in (7) is not $\mathcal{PT}$-symmetric anymore because under $\mathcal{T}$, one has to take in addition the complex conjugate of Φ. However, one may implement the $\mathcal{PT}$ symmetry for the path integral as a whole. Performing the $\mathcal{PT}$ operation for the path integral, one obtains

$$\tilde{Z}=\int_{\tilde{\mathcal{C}}_{\mathcal{PT}}}\mathcal{D}[\tilde{\Phi}]\exp\bigg(-\frac{1}{2}\int \mathrm{d}^d x[\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi}+\mu^2\tilde{\Phi}^2+m^2\tilde{\Phi}^2(-i\tilde{\Phi}/\phi_0)^\varepsilon]\bigg),\tag{8}$$

where $\tilde{\Phi}=-\Phi^*$ and $\tilde{\mathcal{C}}_{\mathcal{PT}}=\{\tilde{\Phi}(x){:}\,-\tilde{\Phi}^*(x)\in\mathcal{C}_{\mathcal{PT}}\}$. For $\varepsilon=2N$, the path integral is invariant under $\mathcal{PT}$ if

$$\tilde{\mathcal{C}}_{\mathcal{PT}}=\mathcal{C}_{\mathcal{PT}}.\tag{9}$$

For the zero-dimensional case, such a contour satisfy the so-called left-right symmetry in the complex plane. For other values of ε, a more delicate analysis of the contour is required. Note that doing the above procedure one is not adding more degrees of freedom to the theory because the middle-dimensional contour $\mathcal{C}_{\mathcal{PT}}$ has the same "dimension" as that of the original real configuration space.[2]

In this article, we study the functional renormalization of the theory (3). We are in particular interested in the region $\varepsilon\in[0,2]$. We carry out the analysis first for $\varepsilon\in[0,1/3]$ and $\varepsilon=1$ in which ϕ is kept real and then analytically continue the results to other values out of these regions. As we shall see below, one-loop divergences can be absorbed by counterterms for integer values of ε only.

B. One-particle-irreducible effective action

We assume here either $\varepsilon\in[0,1/3]$ or $\varepsilon=1$, in which case the path integration is done over real ϕ configurations. Given a (real) source J, we define the Euclidean partition function in such a way that the source term is invariant under $\mathcal{PT}$ symmetry

$$Z[J]=\int\mathcal{D}[\phi]\exp\bigg(-\frac{1}{2}\int \mathrm{d}^d x[\partial_\mu\phi\partial^\mu\phi+\mu^2\phi^2+m^2\phi^2(i\phi/\phi_0)^\varepsilon]-i\int \mathrm{d}^d xJ\phi\bigg).\tag{10}$$

The one-point function is defined from the connected generating functional $W[J]=-\ln Z[J]$ as

$$\varphi\equiv\langle\phi\rangle=\frac{\delta W}{i\delta J}=-\frac{1}{Z}\frac{\delta Z}{i\delta J},\tag{11}$$

where

$$\langle\cdots\rangle\equiv\frac{1}{Z}\int\mathcal{D}[\phi](\cdots)\exp\bigg(-\frac{1}{2}\int \mathrm{d}^d x[\partial_\mu\phi\partial^\mu\phi+\mu^2\phi^2+m^2\phi^2(i\phi/\phi_0)^\varepsilon]-i\int \mathrm{d}^d xJ\phi\bigg).\tag{12}$$

Taking the complex conjugate of $Z[J]$, one obtains

$$(Z[J])^*=\int\mathcal{D}[\phi]\exp\bigg(-\frac{1}{2}\int \mathrm{d}^d x[\partial_\mu\phi\partial^\mu\phi+\mu^2\phi^2+m^2\phi^2(-i\phi/\phi_0)^\varepsilon]+i\int \mathrm{d}^d xJ\phi\bigg),\tag{13}$$

and the change of variable, $\phi\to-\phi$ leads to

$$(Z[J])^*=Z[J].\tag{14}$$

Similarly, we have

$$(\varphi[J])^*=-\varphi[J],\tag{15}$$

such that the one-point function $\varphi[J]$ is purely imaginary, as long as the source J is real. Actually, one can extend this argument to arbitrary n-point functions. It can be seen that any $2N$-point correlation function (with N being a non-negative integer) is real and any $2N+1$-point correlation function is imaginary.

[2]Rigorously speaking, the dimension of the contour $\mathcal{C}_{\mathcal{PT}}$ is infinity.

We also have the usual relation

$$\frac{\delta^2 W}{i\delta J(x) i\delta J(y)} = \varphi(x)\varphi(y) - \langle\phi(x)\phi(y)\rangle, \tag{16}$$

which expresses the second functional derivative of W in terms of the variance of quantum fluctuations. One then inverts the relation $\varphi[J] \to J[\varphi]$ in order to define the Legendre transform

$$\Gamma[\varphi] = W[J[\varphi]] - i\int d^4x \varphi J[\varphi], \tag{17}$$

which represents the 1PI effective action, with functional derivatives

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = -iJ(x),$$
$$\frac{\delta^2\Gamma[\varphi]}{\delta\varphi(x)\delta\varphi(y)} = -\left(\frac{\delta^2 W}{i\delta J(x) i\delta J(y)}\right)^{-1}. \tag{18}$$

From the above equations we finally obtain

$$\frac{\delta^2\Gamma[\varphi]}{\delta\varphi(x)\delta\varphi(y)} = (\langle\phi(x)\phi(y)\rangle - \varphi(x)\varphi(y))^{-1}. \tag{19}$$

In summary, we find that the $\mathcal{PT}$-symmetric theory (10) has real $Z[J]$ as well as real 1PI effective action $\Gamma[\varphi]$. However, although ϕ is real, its one-point function φ is purely imaginary. Note that the physically relevant coupling constants are obtained from the derivatives $\delta^n\Gamma/\delta\varphi^n$ at $\varphi = 0$, and do not depend on φ being purely imaginary.

C. Exact Wilsonian renormalization

The Wilsonian evolution of the Wetterich running action defined at some scale k is derived in the framework of the 1PI quantization, where a cutoff function is added to the bare action, in order to "freeze" infrared modes with momentum $|p| \lesssim k$ in the path integration [53]. This is achieved through the term

$$\frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \tilde{\phi}(p) R_k(p) \tilde{\phi}(-p), \tag{20}$$

where $\tilde{\phi}(p)$ is the Fourier transform of the field $\phi(x)$. The function $R_k(p)$ is not unique, but it vanishes for $k \to 0$, such that this limit reproduces the usual 1PI quantization. In the usual Hermitian context, the corresponding 1PI "average effective action" Γ_k satisfies the exact functional renormalization equation

$$\partial_k\Gamma_k = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \partial_k R_k(p) \left(\frac{\delta^2\Gamma_k}{\delta\tilde{\varphi}(p)\delta\tilde{\varphi}(-p)} + R_k(p)\right)^{-1}, \tag{21}$$

where $\tilde{\varphi}$ is the Fourier transform of the one-point function. We also note that, by construction, the IR limit $k \to 0$ of Γ_k reproduces the one-particle irreducible (1PI) effective action [59], which is independent of the blocking procedure in Fourier space.

In our situation, assuming either $\varepsilon \in [0, 1/3]$ or $\varepsilon = 1$, the change $J \to iJ$ doesn't modify the derivation of the equation (21), which therefore remains the same, but where $\varphi(x)$ is purely imaginary, such that $\tilde{\varphi}(-p) = -\tilde{\varphi}^*(p)$.

We choose the Litim cutoff function [60], and work in the local potential approximation, where the evolution of derivative terms are neglected and the running effective action takes the form

$$\Gamma_k[\varphi] \equiv \int d^d x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + U_k(\varphi)\right]. \tag{22}$$

The resulting exact renormalization equation (ERG) is

$$\partial_k U_k(\varphi) = \frac{\alpha_d k^{d+1}}{k^2 + U_k''(\varphi)}, \tag{23}$$

where

$$\alpha_d \equiv \frac{\hbar\Omega_d}{d(2\pi)^d}, \tag{24}$$

and $\Omega_d \equiv 2\pi^{d/2}/\Gamma[d/2]$ is the solid angle in dimension d. In Eq. (23), U_k is the running potential for $0 \le k \le \Lambda$ and a prime denotes a derivative with respect to φ. This equation ignores the renormalization of the derivative terms in the running action, but it provides a resummation of all quantum fluctuations in this approximation. This equation is a challenge to solve in the generic case but one can make the most of this resummation in some specific regimes, as we show in the next sections.

For integer ε, one can read the beta-functions of a theory from Eq. (23). Assume a running interaction of the form $\lambda_k\varphi^n/n!$ where $n \ge 3$ is an integer, then

$$\lambda_k = \frac{\partial^n}{\partial\varphi^n}(U_k(\varphi))_{\varphi=0}. \tag{25}$$

The mass dimension of λ_k is $[\lambda] = d - n(d/2 - 1)$, and one defines the dimensionless coupling by rescaling λ_k with the appropriate power of k

$$\tilde{\lambda}_k \equiv k^{-[\lambda]}\lambda_k. \tag{26}$$

The corresponding beta-function is then

$$\beta \equiv k\partial_k\tilde{\lambda}_k = -[\lambda]\tilde{\lambda}_k + k^{-[\lambda]}\frac{\partial^n}{\partial\varphi^n}(k\partial_k U_k(\varphi))_{\varphi=0}, \tag{27}$$

where $\partial_k U_k(\varphi)$ is obtained from Eq. (23). The first term on the right-hand side (rhs) of Eq. (27) corresponds to the trivial scaling law for λ_k, and the second term corresponds to the anomalous dimension, arising from quantum fluctuations.

III. ULTRAVIOLET REGIME

In this section, we focus on the UV behavior of the running potential. In the UV regime where $\Lambda^2 \geq k^2 \gg |U_k''(\phi_0)| \sim \mu^2 + m^2$, the ERG equation can then be written as

$$\partial_k U_k(\varphi) = \alpha_d k^{d-1}\sum_{n=0}^{\infty}(-1)^n k^{-2n}(U_k''(\varphi))^n. \tag{28}$$

A. Diffusion

If we introduce the notation

$$\hat{U}_k(\varphi) \equiv U_k(\varphi) - \frac{\alpha_d}{d}k^d, \tag{29}$$

and keep the dominant term $n = 1$ on the rhs of Eq. (28), we obtain the diffusion equation[3]

$$\partial_\tau\hat{U}_\tau(\varphi) = \partial_\varphi^2\hat{U}_\tau(\varphi), \tag{30}$$

where τ is defined as

$$\frac{\mathrm{d}\tau}{\mathrm{d}k} = -\alpha_d k^{d-3}. \tag{31}$$

In the present Wilsonian picture, the system gets dressed by quantum corrections as k decreases from Λ, or with the above parametrization, as τ increases from 0. Specifically,

$$\tau = \begin{cases} \alpha_2 \ln(\frac{\Lambda}{k}), & \mathrm{d} = 2, \\ \frac{\alpha_d}{d-2}(\Lambda^{d-2} - k^{d-2}), & \mathrm{d} \geq 3. \end{cases} \tag{32}$$

A solution that is analytical at $\tau = 0$ can be written formally as

$$\hat{U}_\tau(\varphi) = \exp(\tau\partial_\varphi^2)\hat{U}_{\tau=0}(\varphi), \tag{33}$$

such that

$$\hat{U}_k(\varphi) = \mu^2\tau + \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}m^2\varphi^2(i\varphi/\phi_0)^\varepsilon\sum_{n=0}^{\infty}\frac{1}{n!}\frac{f_n\tau^n}{\varphi^{2n}}, \tag{34}$$

where $f_0 = 1$ and for $n \geq 1$

$$f_n = \prod_{p=0}^{n-1}(\varepsilon + 2 - 2p)(\varepsilon + 1 - 2p). \tag{35}$$

Hence for any noninteger ε, the sum (34) is infinite and is not analytical at $\varphi = 0$. However, for any integer ε the sum (34) is finite and analytical at $\varphi = 0$. Also, one can check that the quadratic potential for $\varepsilon = 0$ does not get quantum corrections:

$$\begin{aligned}\hat{U}_k(\varphi) &= \mu^2\tau + \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}m^2\varphi^2\left(1 + \frac{2\tau}{\varphi^2}\right) \\ &= \frac{1}{2}(\mu^2 + m^2)\varphi^2 + \varphi\text{-independent terms},\end{aligned} \tag{36}$$

which is expected.

B. Beta-function for the interaction $i\phi^3$

For $\varepsilon = 1$, the cubic coupling is defined as

$$i\lambda_k \equiv \left(\frac{\partial^3 U_k(\varphi)}{\partial\varphi^3}\right)_{\varphi=0}, \tag{37}$$

and has the bare value $\lambda_\Lambda = 3m^2/\phi_0$. The dimensionless coupling is $\tilde{\lambda}_k = k^{d/2-3}\lambda_k$ with the corresponding beta-function

$$\beta \equiv k\partial_k\tilde{\lambda}_k = (d/2 - 3)\tilde{\lambda}_k - ik^{d/2-3}\frac{\partial^3}{\partial\varphi^3}(k\partial_k U_k(\varphi))_{\varphi=0}. \tag{38}$$

From the result (34) for $\varepsilon = 1$, the UV running potential is

$$\hat{U}_k(\varphi) = \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}m^2\varphi^2(i\varphi/\phi_0) + \tau(\mu^2 + 3m^2(i\varphi/\phi_0)),$$

where we note that the third derivative with respect to φ does not depend on k, such that naively the beta-function vanishes. This is due to the truncation in Eq. (28), and we need to reintroduce the resummation before calculating the beta-function. In terms of the original potential $U_k(\varphi)$, we obtain

[3] A related approach for $d = 1$ was studied in [61].

$$\begin{aligned}\partial_k U_k(\varphi) &= \alpha_d k^{d-1} + \frac{d\tau}{dk}(\mu^2 + 3m^2(i\varphi/\phi_0)) \\ &= \alpha_d k^{d-1} - \alpha_d k^{d-3}(\mu^2 + 3m^2(i\varphi/\phi_0)) \\ &\simeq \frac{\alpha_d k^{d+1}}{k^2 + \mu^2 + 3m^2(i\varphi/\phi_0)},\end{aligned} \tag{39}$$

which reproduces the one-loop evolution equation (details are given in the next section). As a consequence, the UV beta function coincides with the one-loop beta-function provided one performs the above resummation. It is interesting to note that the UV regime contains the same information as the one-loop result, which shows the equivalence between the small-coupling regime and the UV regime, where quantum fluctuations are perturbative.

From Eq. (39), one then obtains

$$-i\left(\frac{\partial^3[k\partial_k U_k(\varphi)]}{\partial\varphi^3}\right)_{\phi=0} = 162\alpha_d \frac{m^6}{\phi_0^3}\frac{k^{d+2}}{(k^2+\mu^2)^4}. \tag{40}$$

Substituting the above into Eq. (38), one finally arrives at the beta-function in the UV regime,

$$\beta^{(\mathrm{UV})} = (d/2-3)\tilde{\lambda}_k + \frac{6\alpha_d(\tilde{\lambda}_\Lambda)^3}{(1+(\tilde{\mu})^2)^4}, \tag{41}$$

where $\tilde{\mu} = \mu/k$. It is interesting to look at the specific case $d = 6$, where one can replace the bare coupling by the dressed coupling on the rhs of Eq. (41), since the difference is of order $\hbar^2$. The beta function in the UV regime $\tilde{\mu} \ll 1$ is then

$$\beta^{(\mathrm{UV})} \simeq 6\alpha_6\lambda_k^3 = \frac{\hbar\lambda_k^3}{64\pi^3} \quad \text{for } d = 6, \tag{42}$$

and is positive, unlike the corresponding beta-function for the real cubic interaction.

C. Corrections to the diffusion equation

Coming back to Eq. (28), we can write

$$\partial_\tau \hat{U}_\tau(\varphi) = \sum_{n=0}^{\infty}(-1)^n k^{-2n}(\hat{U}''_\tau(\varphi))^{n+1}. \tag{43}$$

Through the relation given in Eq. (32), one can perform a systematic perturbative expansion of the solution in powers of Λ^{-2}. For example, for $d = 4$, we have

$$\tau = \frac{\alpha_4}{2}(\Lambda^2 - k^2), \tag{44}$$

such that the first-order correction to the diffusion equation reads, in the UV regime $\tau \ll \Lambda^2$,

$$\partial_\tau \hat{U}_\tau(\varphi) = \hat{U}''_\tau(\varphi) - \frac{1}{\Lambda^2}(\hat{U}''_\tau(\varphi))^2 + \mathcal{O}(1/\Lambda^4). \tag{45}$$

The solution can then be expanded as

$$\hat{U}_\tau(\varphi) = \hat{U}^{(0)}_\tau(\varphi) + \frac{1}{\Lambda^2}\hat{U}^{(1)}_\tau(\varphi) + \mathcal{O}(1/\Lambda^4), \tag{46}$$

where $\hat{U}^{(0)}_\tau$ satisfies the diffusion equation, and $\hat{U}^{(1)}_\tau$ satisfies

$$\partial_\tau \hat{U}^{(1)}_\tau(\varphi) = \partial_\varphi^2 \hat{U}^{(1)}_\tau(\varphi) - (\partial_\varphi^2 \hat{U}^{(0)}_\tau(\varphi))^2. \tag{47}$$

We are interested in a particular solution only, which satisfies the appropriate boundary conditions, because the homogeneous equation is the diffusion equation, with a solution proportional to $\hat{U}^{(0)}_\tau(\varphi)$. The latter equation can in principle be solved, but in what follows we turn to the one-loop approximation, without assuming k to be large.

IV. ONE-LOOP APPROXIMATION

In this section, we truncate the ERG equation (23) to one loop.

A. Coupling constant

If we introduce the notation $\xi \equiv \varphi/\phi_0$, the bare potential can be written

$$U_\Lambda(\xi) = \frac{1}{2}\mu^2\phi_0^2\xi^2 + \frac{1}{2}m^2\phi_0^2\xi^2(i\xi)^\epsilon, \tag{48}$$

and because the ERG equation (23) is not linear, it induces a phase change in the running potential, unlike what happens in the UV regime. The running potential can be parametrized as

$$U_k(\xi) = \frac{1}{2}m^2\phi_0^2 V_k(\xi), \tag{49}$$

such that in terms of $V_t(\xi)$ the ERG equation (23) reads

$$\partial_t V_t(\xi) = \frac{g t^{d+1}}{t^2 + \partial_\xi^2 V_t(\xi)}, \tag{50}$$

where

$$g \equiv \frac{\alpha_d m^{d-2}}{2^{d/2-1}\phi_0^2} \quad \text{and} \quad t = \frac{\sqrt{2}k}{m}. \tag{51}$$

One can see that the dimensionless parameter g plays the role of a coupling constant, and the evolution in t of the running potential is proportional to g. As a consequence, if $g \ll 1$, the ERG equation can be expanded in this coupling constant

$$\partial_t V_t(\xi) = \frac{g t^{d+1}}{t^2 + \partial_\xi^2 V_T(\xi)} + \mathcal{O}(g^2), \tag{52}$$

where $T = \sqrt{2}\Lambda/m$ and

$$\partial_\xi^2 V_T(\xi) = 2\mu^2/m^2 + (2+\varepsilon)(1+\varepsilon)(i\xi)^\varepsilon. \tag{53}$$

The evolution equation for the one-loop running potential $V_t^{(1)}(\xi)$ is finally

$$\partial_t V_t^{(1)}(\xi) = \frac{g t^{d+1}}{t^2 + \partial_\xi^2 V_T(\xi)}. \tag{54}$$

This equation can be solved for arbitrary d. But below we will focus on the cases with $d = 4$ and $d = 2$. The calculation can of course be easily generalized to other cases with different values of d, e.g., $d = 3$.

B. One-loop effective potential ($d = 4$)

The integration of Eq. (54) is straightforward, and for $d = 4$ it leads to

$$\begin{aligned} V_t^{(1)}(\xi) &= V_T(\xi) + g\int_T^t \frac{u^5 du}{u^2 + \partial_\xi^2 V_T(\xi)} \\ &= V_T(\xi) + \frac{g}{2}(T^2 - t^2)\partial_\xi^2 V_T(\xi) \\ &\quad - \frac{g}{2}(\partial_\xi^2 V_T(\xi))^2 \ln\left(\frac{T^2 + \partial_\xi^2 V_T(\xi)}{t^2 + \partial_\xi^2 V_T(\xi)}\right), \end{aligned} \tag{55}$$

where a field-independent term is disregarded. In terms of the original variables, we obtain then

$$\begin{aligned} U_k^{(1)}(\varphi) &= \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}m^2\varphi^2(i\varphi/\phi_0)^\varepsilon + \frac{\hbar}{64\pi^2}(\Lambda^2 - k^2) \\ &\quad \times [\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon/2] \\ &\quad - \frac{\hbar}{64\pi^2}[\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon/2]^2 \\ &\quad \times \ln\left(\frac{2\Lambda^2 + 2\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon}{2k^2 + 2\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon}\right). \end{aligned} \tag{56}$$

Finally, the one-loop Wilsonian effective potential is obtained in the limit $k \to 0$. If we ignore terms vanishing in the limit $m/\Lambda \to 0$, it reads

$$\begin{aligned} U_{\text{eff}}^{(1)}(\varphi) &= \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}m^2\varphi^2(i\varphi/\phi_0)^\varepsilon + \frac{\hbar}{64\pi^2}\Lambda^2 \\ &\quad \times [\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon/2] \\ &\quad - \frac{\hbar}{64\pi^2}[\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon/2]^2 \\ &\quad \times \left[\ln\left(\frac{\Lambda^2}{m^2}\right) - \ln\left(\frac{\mu^2}{m^2} + \frac{1}{2}(2+\varepsilon)(1+\varepsilon)\right.\right. \\ &\quad \left.\left. \times (i\varphi/\phi_0)^\varepsilon\right)\right], \end{aligned} \tag{57}$$

and is identical to the one-loop 1PI effective potential.

We can note an important property: for a generic non-integer ε, one-loop corrections generate the new interactions $(i\varphi/\phi_0)^\varepsilon$ and $(i\varphi/\phi_0)^{2\varepsilon}$, with quadratic and logarithmic divergent coefficients respectively. Hence one cannot define counterterms without changing the dynamics of the problem; the theory is not renormalizable for a generic non-integer ε.

One-loop renormalizability can be achieved for $\varepsilon = 1$ though, where

$$\begin{aligned} U_{\text{eff}}^{(1)}(\varphi) &= \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}m^2\varphi^2(i\varphi/\phi_0) \\ &\quad + \frac{\hbar}{64\pi^2}\Lambda^2[\mu^2 + 3m^2 i\varphi/\phi_0] \\ &\quad - \frac{\hbar}{64\pi^2}(\mu^2 + 3m^2 i\varphi/\phi_0)^2 \\ &\quad \times \left[\ln\left(\frac{\Lambda^2}{m^2}\right) - \ln\left(\frac{\mu^2}{m^2} + 3i\varphi/\phi_0\right)\right], \end{aligned} \tag{58}$$

and contains tadpole terms (linear in φ) which can be eliminated with counterterms with no modification of the dynamics. The φ-independent divergent terms have no physical effect and can be discarded also. The logarithmically divergent quadratic term can be absorbed in a redefinition of μ^2 through the renormalized mass squared

$$\mu_{\text{R}}^2 = \mu^2 + \frac{9\hbar m^4}{32\pi^2\phi_0^2}\ln\left(\frac{\Lambda^2}{m^2}\right), \tag{59}$$

and the remaining terms are finite. The renormalized one-loop effective potential is then

$$\begin{aligned} U_{\text{R}}^{(1)}(\varphi) &= \frac{1}{2}\mu_{\text{R}}^2\varphi^2 + \frac{1}{2}m^2\varphi^2(i\varphi/\phi_0) \\ &\quad + \frac{\hbar}{64\pi^2}(\mu_{\text{R}}^2 + 3m^2 i\varphi/\phi_0)^2 \ln(\mu_{\text{R}}^2/m^2 + 3i\varphi/\phi_0). \end{aligned} \tag{60}$$

Note that one cannot consistently set $\mu^2 = 0$, since mass term corrections are generated for $\varepsilon = 1$, with divergences that have to be absorbed in the bare mass term.

C. One-loop effective potential ($d = 2$)

For $d = 2$ the integration of Eq. (54) leads to, up to a φ-independent term,

$$V_t^{(1)}(\xi) = V_T(\xi) + \frac{g}{2}\partial_\xi^2 V_T(\xi)\ln\left(\frac{T^2 + \partial_\xi^2 V_T(\xi)}{t^2 + \partial_\xi^2 V_T(\xi)}\right), \quad (61)$$

which gives

$$U_k^{(1)}(\varphi) = \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}m^2\varphi^2(i\varphi/\phi_0)^\varepsilon + \frac{\hbar}{8\pi}[\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon/2] \times \ln\left(\frac{2\Lambda^2 + 2\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon}{2k^2 + 2\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon}\right). \quad (62)$$

Taking $k \to 0$ and ignoring terms vanishing in the limit $\Lambda \to \infty$ leads to

$$U_{\rm eff}^{(1)}(\varphi) = \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}m^2\varphi^2(i\varphi/\phi_0)^\varepsilon + \frac{\hbar}{8\pi}(\mu^2 + m^2(2+\varepsilon)(1+\varepsilon)(i\varphi/\phi_0)^\varepsilon/2) \times \left[\ln\left(\frac{\Lambda^2}{m^2}\right) - \ln\left(\frac{\mu^2}{m^2} + \frac{1}{2}(2+\varepsilon)(1+\varepsilon) \times (i\varphi/\phi_0)^\varepsilon\right)\right]. \quad (63)$$

As in the situation where $d = 4$, this one-loop potential is renormalizable for $\varepsilon = 1$ only, since only a tadpole appears to be divergent, which can be removed with a counterterm. The divergent φ-independent term is not physical and can be discarded. The resulting renormalized effective potential has new interactions, which are however all finite

$$U_{\rm R}^{(1)}(\varphi) = \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}m^2\varphi^2(i\varphi/\phi_0) - \frac{\hbar}{8\pi}(\mu^2 + 3m^2(i\varphi/\phi_0)) \times \ln\left(\frac{\mu^2}{m^2} + 3(i\varphi/\phi_0)\right). \quad (64)$$

V. ANALYTICAL CONTINUATION FOR $\varepsilon = 2$

In this section, we extend the analysis to $\varepsilon = 2$. In this case, the integral (3) is not convergent for real ϕ and therefore one needs to consider a contour different from the space of real configurations as discussed in Eq. (7). In order for the path integral to respect the $\mathcal{PT}$ symmetry, it is sufficient to enforce the condition (9) on the contour $\mathcal{C}_{\mathcal{PT}}$. Note that although seemly the potential is unbounded from below for $\varepsilon = 2$, the theory in the $\mathcal{PT}$-symmetric framework actually has a stable ground state at least for $d = 1$. The conjectural relation proposed in Ref. [47] indicates that this is also the case in higher-dimensional spacetime.

A. One-particle-irreducible effective action

Introducing a source term, we have

$$Z[J] = \int_{\mathcal{C}_{\mathcal{PT}}} \mathcal{D}[\Phi]\exp\left(-\frac{1}{2}\int \mathrm{d}^d x[\partial_\mu\Phi\partial^\mu\Phi + \mu^2\Phi^2 - (m^2/\phi_0^2)\Phi^4] - i\int \mathrm{d}^d x J\Phi\right), \quad (65)$$

where we again assume that J is real. Taking the complex conjugate of the above equation, one obtains

$$(Z[J])^* = \int_{\mathcal{C}^*_{\mathcal{PT}}} \mathcal{D}[\Phi^*]\exp\left(-\frac{1}{2}\int \mathrm{d}^d x[\partial_\mu\Phi^*\partial^\mu\Phi^* + \mu^2\Phi^{*2} - (m^2/\phi_0^2)\Phi^{*4}] + i\int \mathrm{d}^d x J\Phi^*\right), \quad (66)$$

where $\mathcal{C}^*_{\mathcal{PT}}$ is obtained from $\mathcal{C}_{\mathcal{PT}}$ by taking complex conjugate of all its elements. The change of functional variable $\Phi^* \to -\Phi^*$, together with the condition (9), finally lead to

$$(Z[J])^* = Z[J]. \quad (67)$$

Of course, the reality of the partition function is a consequence of its $\mathcal{PT}$-invariance. Similarly, for the one-point function

$$\varphi[J] \equiv \langle\Phi\rangle \equiv \frac{1}{Z[J]}\int_{\mathcal{C}_{\mathcal{PT}}} \mathcal{D}[\Phi]\Phi\exp\left(-\frac{1}{2}\int \mathrm{d}^d x[\partial_\mu\Phi\partial^\mu\Phi + \mu^2\Phi^2 - (m^2/\phi_0^2)\Phi^4] - i\int \mathrm{d}^d x J\Phi\right), \quad (68)$$

we have $(\varphi[J])^* = -\varphi[J]$ so that the one-point function is purely imaginary. Again, one can extend this argument to arbitrary n-point functions to show that any $2N$-point correlation function is real and any $2N+1$-point correlation function is imaginary. The 1PI effective action,

$$\Gamma[\varphi] = -\ln Z[J[\varphi]] - i\int \mathrm{d}^d x\varphi J[\varphi], \quad (69)$$

is thus real. Using the above 1PI effective action, one can still carry out a derivation of the exact functional renormalization equation, ending up with the same Eq. (21).

B. One-loop effective potential ($d = 4$)

Given the above properties, the one-loop expression (57) can be used for $\varepsilon = 2$ to give

$$U_{\text{eff}}^{(1)}(\varphi)=\frac{1}{2}\mu^2\varphi^2-\frac{\lambda}{24}\varphi^4-\frac{\hbar}{64\pi^2}\Lambda^2(\mu^2-\lambda\varphi^2/2)-\frac{\hbar}{64\pi^2}(\mu^2-\lambda\varphi^2/2)^2\times\left[\ln\left(\frac{\Lambda^2}{m^2}\right)-\ln\left(\frac{\mu^2-\lambda\varphi^2/2}{m^2}\right)\right], \tag{70}$$

where $\lambda\equiv 12m^2/\phi_0^2$. The logarithmic and quadratic divergences can be absorbed by the introduction of the renormalized parameters

$$\mu_{\text{R}}^2=\mu^2-\frac{\hbar\lambda\Lambda^2}{64\pi^2}+\frac{\hbar\lambda\mu^2}{16\pi^2}\ln\left(\frac{\Lambda}{m}\right) \tag{71a}$$

$$\lambda_{\text{R}}=\lambda+\frac{3\hbar\lambda^2}{16\pi^2}\ln\left(\frac{\Lambda}{m}\right), \tag{71b}$$

and the renormalized one-loop effective potential is, after ignoring the $\mathcal{O}(\hbar^2)$ and divergent but φ-independent terms,

$$U_{\text{R}}^{(1)}(\varphi)=\frac{1}{2}\mu_{\text{R}}^2\varphi^2-\frac{\lambda_{\text{R}}}{24}\varphi^4+\frac{\hbar}{64\pi^2}(\mu_{\text{R}}^2-\lambda_{\text{R}}\varphi^2/2)^2\times\ln\left(\frac{\mu_{\text{R}}^2-\lambda_{\text{R}}\varphi^2/2}{m^2}\right). \tag{72}$$

Note that, since φ is purely imaginary, the argument of the logarithm is always positive for $\mu_{\text{R}}^2>0$, and the effective potential (72) is always real. The possibility to define quantum corrections for the potential $-\phi^4$ is consistent with the conjecture that the effective theory does have a ground state at $\varphi=0$.

Also, quartic and logarithmic corrections come with the opposite sign compared to the usual $+\phi^4$ theory. As a consequence, the interaction is asymptotically free; for a fixed renormalized coupling λ_{R}, the bare coupling can be written as

$$\begin{aligned}\lambda&=\lambda_{\text{R}}-\frac{3\hbar\lambda^2}{16\pi^2}\ln\left(\frac{\Lambda}{m}\right)\\&=\lambda_{\text{R}}-\frac{3\hbar\lambda_{\text{R}}^2}{16\pi^2}\ln\left(\frac{\Lambda}{m}\right)+\mathcal{O}(\hbar^2)\\&=\frac{\lambda_{\text{R}}}{1+\frac{3\hbar\lambda_{\text{R}}}{16\pi^2}\ln(\frac{\Lambda}{m})}+\mathcal{O}(\hbar^2),\end{aligned} \tag{73}$$

which, based on this one-loop result, leads to asymptotic freedom. Alternatively, the one-loop beta-function of the model is negative

$$\beta^{(1)}\equiv\Lambda\partial_\Lambda\lambda=-\frac{3\hbar\lambda^2}{16\pi^2}<0, \tag{74}$$

and should be considered with the boundary condition $\lambda(m)=\lambda_{\text{R}}$.

C. One-loop effective potential ($d=2$)

Substituting $\varepsilon=2$ into Eq. (63), we obtain

$$U_{\text{eff}}^{(1)}(\varphi)=\frac{1}{2}\mu^2\varphi^2-\frac{\lambda}{24}\varphi^4+\frac{\hbar}{8\pi}(\mu^2-\lambda\varphi^2/2)\times\left[\ln\left(\frac{\Lambda^2}{m^2}\right)-\ln\left(\frac{\mu^2-\lambda\varphi^2/2}{m^2}\right)\right], \tag{75}$$

where still $\lambda=12m^2/\phi_0^2$. The logarithmically divergent quadratic term can be absorbed in a redefinition of μ^2 through the renormalized mass squared

$$\mu_{\text{R}}^2=\mu^2+\frac{\hbar\lambda}{8\pi}\ln\left(\frac{\Lambda^2}{m^2}\right), \tag{76}$$

and the remaining terms are either finite or φ-independent. Ignoring the $\mathcal{O}(\hbar^2)$ and divergent but φ-independent terms, the renormalized effective potential reads

$$U_{\text{R}}^{(1)}(\varphi)=\frac{1}{2}\mu_{\text{R}}^2\varphi^2-\frac{\lambda}{24}\varphi^4-\frac{\hbar}{8\pi}(\mu_{\text{R}}^2-\lambda\varphi^2/2)\times\ln\left(\frac{\mu_{\text{R}}^2-\lambda\varphi^2/2}{m^2}\right). \tag{77}$$

VI. CONCLUSION

$\mathcal{PT}$-symmetric theories may open a new window to phenomenological model building for new physics. However, compared to quantum mechanics, quantum field theory is much more complicated since in the latter there are an infinite number of degrees of freedom. One particular issue that is absent from quantum-mechanical models but appears in quantum field theory is renormalization. Therefore, when extending $\mathcal{PT}$-symmetric quantum-mechanical models to quantum field theory, renormalizability has to be taken into account. In this paper, we have studied the renormalization of a $\mathcal{PT}$-symmetric scalar field theory with the bare potential given by Eq. (2). This theory is a direct generalization of the well-studied quantum-mechanical models given by the Hamiltonian (1).

In contrast to Refs. [36,45,48] where the renormalization of the theory (2) is studied with the ε-expansion, our study is based on the Wilsonian approach and more specifically, the Wetterich equation for the running effective action. In this approach, the renormalization of the theory (2) can be studied without doing the ε-expansion. We first carry out our analysis for regions of ε, e.g., $\varepsilon\in[0,1/3]$ or $\varepsilon=1$, in which the scalar field can be kept real in the Euclidean path integral. We have solved the Wetterich equation in the local potential approximation either in the UV regime or at the one-loop order. We obtained the scale-dependent one-loop effective potentials for arbitrary ε, Eq. (56) ($d=4$) and (62) ($d=2$), and their IR limits, Eqs. (57) and (63). We found that for a generic noninteger ε, one-loop corrections generate new interactions with divergent coefficients that cannot be absorbed by the bare parameters. Therefore at the one-loop level, the theory is renormalizable only for integer values of ε. The aforementioned general formulas are then applied particularly for $\varepsilon = 1$. Although for $\varepsilon = 2$, a deformation for the integration contour of the Euclidean path integral is necessary to ensure the convergence of the latter, we argue that the general formulas can still be applied for $\varepsilon = 2$. One-loop beta functions for the coupling associated with the interaction $i\phi^3$ and $-\phi^4$ are then computed. It is confirmed that the $-\phi^4$ theory has asymptotic freedom in four-dimensional spacetime. We have also shown that a consequence of $\mathcal{PT}$ symmetry is that all the odd-point correlation functions are imaginary and all the even-point correlation functions, including the partition function itself, are real.

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