

Rotating (2 + 1)-dimensional black holes in Einstein-Maxwell-dilaton theoryThanasis Karakasis^{1,*}, Eleftherios Papantonopoulos,^{1,†} Zi-Yu Tang^{2,3,‡} and Bin Wang^{4,5,§}¹*Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece*²*School of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study, UCAS, Hangzhou 310024, China*³*School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*⁴*Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China*⁵*School of Aeronautics and Astronautics, Shanghai Jiao Tong University, Shanghai 200240, China*

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We consider Einstein-Maxwell-dilaton theory in (2 + 1) dimensions where the coupling between the scalar field and the Maxwell invariant is the dilatonic coupling $f(\phi) = \exp(-2\alpha\phi)$, and we obtain novel, exact, rotating black hole solutions. The dilatonic parameter α impacts the metric function, affecting the rotating properties of the black hole, its mass, and also its thermodynamics. Calculating the entropy we find that it is always positive, and the dilatonic black holes may have higher entropy than the BTZ black hole. Depending on the parameters, the dilatonic BTZ-like black hole may be thermodynamically preferred over the BTZ black hole which is recovered when $\alpha = 0$.

DOI: [10.1103/PhysRevD.107.024043](https://doi.org/10.1103/PhysRevD.107.024043)**I. INTRODUCTION**

In (2 + 1) dimensions one of the first exact black holes with a negative cosmological constant was discussed by Bañados, Teitelboim, and Zanelli (BTZ) [1,2]. This spacetime is a solution of a theory that consists of the Ricci scalar, a negative cosmological constant, and the electromagnetic invariant which can be linear [1–3] or nonlinearly coupled to gravity [4], which are regular solutions having a nonsingular center. Regular solutions were found recently in [5] involving a scalar field. In [6,7] (2 + 1)-dimensional black holes with a conformally coupled scalar field, being regular everywhere, were discussed. After these first results other hairy black holes in (2 + 1) dimensions were discussed [8–13]. In [14] (2 + 1)-dimensional gravity with a negative cosmological constant in the presence of a scalar field and an Abelian gauge field was considered. Both fields are conformally coupled to gravity, the scalar field through a nonminimal coupling to the curvature and the gauge field given by a power of the Maxwell field. A sixth-power self-interacting potential, which does not spoil conformal invariance, is also included in the action, introducing a relation between the scalar curvature and the cosmological constant. In [15,16] (2 + 1)-dimensional charged black holes with scalar hairs were derived, where

the scalar potential is not fixed *ad hoc* but derived from Einstein's equations. In [17] exact three-dimensional black holes with nonminimal coupling scalar fields were discussed, while (2 + 1)-dimensional $f(R)$ gravity black holes coupled to scalar fields were also obtained [18,19].

In (3 + 1) dimensions there is also a large amount of literature in black holes coupled to scalar fields. The first attempts were carried out by Bekenstein, Bocharova, Bronnikov, and Melnikov, called the BBMB black hole [20]. The resulting spacetime resembles the extremal Reissner-Nordström one, but the scalar field diverges at the horizon. It was also shown in [21] that this solution is unstable under scalar perturbations. It was shown that the resultant temperature is zero and the entropy diverges, allowing one to argue that the BBMB solution does not describe a black hole spacetime or that the thermodynamics of this solution is not well understood [22]. To make the scalar field regular at the horizon, a gravitational scale was introduced via a cosmological constant [23], resulting in an asymptotically de Sitter spacetime which is also found to be unstable [24]. The thermodynamics of this solution was also found to have an odd behavior [25,26]. Recently, another gravitational scale was introduced via the modification of the curvature part of the action by adding a nonlinear function of the Ricci scalar [$f(R)$ gravity] [27], making the scalar field regular at the horizon. The thermodynamics of both the general relativity and $f(R)$ gravity cases also suffer from the same problems (negative entropy), and in both cases an electric charge has to be introduced to cure this problem [25–27]. Black holes with

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scalar fields, minimally or nonminimally coupled to gravity and other matter fields, have been extensively discussed in the literature [28–41].

A dilatonic field appears naturally in the low energy limit of various string theories [42], in $(3 + 1)$ dimensions. As a result black hole solutions were obtained in these theories [43]. These pioneering results showed that the produced compact object is determined by the black hole mass, the electric charge, and the asymptotic value of the dilatonic scalar field. The scalar field dresses the black hole with a secondary scalar hair, and there is no independent scalar charge that can be detected asymptotically by an observer since the scalar charge is given in terms of the conserved quantities of the black hole, i.e., the mass and the electric charge. Black holes in $(3 + 1)$ dimensions with a dilatonic coupling have been a very active field of research, and different scenarios with self-interacting scalar potentials, different types of electrodynamics and Yang-Mills fields, and spacetimes with different types of topology have been studied through the years [44–51].

Dilatonic black hole solutions were also found in $(2 + 1)$ dimensions. Modification of the BTZ black hole by a dilaton scalar field has been considered in [52], where it was found that a second degree of freedom has to be added in the metric ansatz to obtain nontrivial solutions, which is expected since a simple self-interacting scalar field theory does not satisfy $\rho = -p_r$ [53]. This work was further extended in [54] in which a dilaton field $\phi = \ln(r\beta)$ coupled to an electromagnetic field was introduced and in the presence of a potential $V(\phi) = \exp(b\phi)\Lambda$ the causal structures were investigated, and the thermodynamical temperature and entropy were computed. In [55] using the same forms of the dilaton field and the potential term, spinning black hole solutions were obtained, where the coupling term does not account since the solutions are uncharged.

Black holes were also investigated in [56] where a self-interacting potential that is a linear combination of two Liouville-type potentials was considered. One can also consider different types of electrodynamics to find black hole solutions as in [57], where black holes with Born-Infeld electrodynamics were obtained. In [58] Born-Infeld-dilaton black holes were also found, and the conserved charges and the thermodynamics were also discussed, as were dilaton black holes with power-Maxwell electrodynamics [59].

In this work we present novel rotating BTZ-like solutions in the Einstein-Maxwell-dilaton theory. This work can be considered as a generalization of [52,54,55]. In our work we do not fix the form of the scalar field, but it is determined by the field equations. The scalar potential is also determined by the field equations. We follow this approach because the choice of different reasonable potentials leads to a system of equations which turn out to be untractable. As a result we follow a bottom-up approach and find a theory in which the field equations can be solved

analytically, determining the scalar potential also. Solutions for given potentials may still be found, but a numerical approach should be followed.

The spacetime is parametrized by six constants, M, Q, l, α, b, J , which are the black hole mass, electric charge, AdS scale, coupling constant between the Maxwell field and the dilaton, scalar length scale, and angular momentum, respectively. The solutions have the interesting feature of reducing to some versions of the BTZ black hole [1,2] spacetime by setting the parameters appropriately. For $J = 0$ the solutions reduce to the ones obtained in [60]. To derive these types of solutions we made two assumptions:

- (i) The coupling between the dilatonic scalar field and the Maxwell invariant is the dilatonic coupling $f(\phi) = \exp(-2\alpha\phi)$.
- (ii) The coupling function $f(\phi)$ modifies the circumference of the black hole according to $f(\phi)R(\phi) = r$ where R is the circumference metric function.

Under these two assumptions, we solve the field equations and derive spinning solutions. We compare both geometric and thermodynamical aspects of the new solutions with the rotating BTZ case, and we discuss the effects of the dilaton scalar field with the new ingredient, the angular momentum added in this work with respect to the existing literature.

The work is organized as follows: In Sec. II we set up the theory and derive the field equations. In Sec. III we solve the field equations to find new rotating dilatonic solutions, while in Sec. IV we discuss thermodynamics and compare the thermodynamical properties of the new solutions with the rotating BTZ case [1,2,60]. In Sec. V we discuss the particular scenario of $\alpha = 1$. Finally, in Sec. VI we conclude and point out possible extensions of this work that can be investigated in future works.

II. BASIC FIELD EQUATIONS

We consider Einstein-Maxwell-scalar (dilaton) (EMD) theory in $(2 + 1)$ dimensions,

$$S = \frac{1}{2} \int d^3x \sqrt{-g} [\kappa^{-1} R - 2\nabla_\mu \phi \nabla^\mu \phi - V(\phi) - f(\phi) F_{\mu\nu} F^{\mu\nu}], \quad (1)$$

where $V(\phi)$ is the self-interacting potential of the scalar field while $f(\phi) = \exp(-2\alpha\phi)$ is the dilatonic coupling function of the Maxwell field, and we set $\kappa = 8\pi G = 1$. The scalar potential is left arbitrary, and we determine its form from the field equations. Varying the action, we can obtain the Einstein equation, the Klein-Gordon equation, and the electromagnetic field equation,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}^\phi + T_{\mu\nu}^A, \quad (2)$$

$$4\Box\phi = V'(\phi) + f'(\phi)F, \quad (3)$$

$$\partial_\mu(\sqrt{-g}f(\phi)F^{\mu\nu}) = 0, \quad (4)$$

where $F \equiv F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A_\mu = \delta_\mu^0 q(r)$, and

$$T_{\mu\nu}^\phi = 2\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi - \frac{1}{2}g_{\mu\nu}V(\phi), \quad (5)$$

$$T_{\mu\nu}^A = 2f(\phi)\left(F_{\mu\sigma}F_\nu^\sigma - \frac{1}{4}g_{\mu\nu}F\right) \quad (6)$$

are stress-energy tensors for the scalar field and the electromagnetic field, respectively.

III. BLACK HOLE SOLUTIONS

Assuming the rotationally symmetric metric ansatz

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + R(r)^2(d\theta + u(r)dt)^2, \quad (7)$$

and inserting (7) into the field equations, we obtain five independent equations with six unknown functions $B(r)$, $q(r)$, $u(r)$, $R(r)$, $\phi(r)$, $V(r)$; therefore, one of the unknowns has to be fixed by hand. By looking at the electromagnetic equation, we can see that it yields a first integral

$$f(r) = \frac{Q_0}{R(r)q'(r)}, \quad (8)$$

where Q_0 is the charge parameter. It is clear that the above equation relates the dilatonic coupling with the electric potential and the area of the black hole. To solve the field equations, as discussed in the Introduction, we assume that the functions $R(r)$ and $f(r)$ satisfy $f(r)R(r)/r = 1$. This particular assumption is also needed in order to make our results comparable with the nonrotating case [60]. Moreover, the condition $f(r)R(r)/r = 1$ can be solved as a result by considering the electric potential $q(r) = Q_0 \ln r$, which is reasonable for a (2 + 1)-dimensional theory. Then, from (8) we can obtain the electric potential and the $R(r)$ function,

$$q(r) = Q_0 \ln\left(\frac{r}{l}\right), \quad (9)$$

$$R(r) = r \exp(2\alpha\phi(r)), \quad (10)$$

where l is the cosmological scale. We find that it is more convenient to express the solution in terms of a parameter x which is related to the dilatonic parameter α via

$$\alpha = \sqrt{\frac{x}{2(1-x)}}, \quad (11)$$

where x is restricted to take values in the range $0 \leq x < 1$. Given the above configurations, we can obtain the angular shift function $u(r)$ and the scalar field $\phi(r)$,

$$u(r) = \frac{J_0 r^{3x-2}}{3x-2} + c_u, \quad (12)$$

$$\phi(r) = (1-x)\sqrt{\frac{x}{2-2x}} \ln\left(\frac{b}{r}\right), \quad (13)$$

where J_0 is the parameter for the angular momentum of the black hole, c_u is an integration constant (which in order to remove global rotations of the coordinate system, we set to zero), and b is the scalar charge. In addition, finiteness of the angular velocity at large distances constrains the coupling constant x as $0 \leq x < 2/3$. We can calculate the lapse function $B(r)$ as

$$B(r) = J_0^2 \frac{b^{2x} r^{4x-2}}{(2-3x)^2} + \frac{2r^2}{l^2(3x^2-5x+2)} \left(\frac{b}{r}\right)^{2x} + \frac{4Q_0^2}{3x-2} \left(\frac{b}{r}\right)^{-x} \ln\left(\frac{r}{l}\right) - M_0 r^x, \quad (14)$$

which breaks down for $x = 2/3$ [corresponding to $\alpha = 1$ (low energy strings)], M_0 is the mass parameter of the black hole (corresponding to the BTZ black hole mass), and the potential supporting this solution is found to be

$$V(r) = -\frac{2b^{2x} r^{-2x}}{l^2} + J_0^2 \frac{xb^{2x} r^{4x-4}}{4-6x} + Q_0^2 \frac{2xb^{-x} r^{x-2}}{2-3x}, \quad (15)$$

which vanishes at spatial infinity and is independent of the mass M_0 , while as a function of ϕ it reads

$$V(\phi) = -\frac{2}{l^2} \exp\left(\frac{2\sqrt{2}x\phi}{\sqrt{(1-x)x}}\right) - J_0^2 \frac{xb^{6x-4}}{2(3x-2)} \times \exp\left(\frac{4\sqrt{2-2x}\phi}{\sqrt{x}}\right) + Q_0^2 \frac{2x}{b^2(2-3x)} \times \exp\left(\frac{\sqrt{2-2x}(x-2)\phi}{(x-1)\sqrt{x}}\right). \quad (16)$$

We can see that the resultant potential is $V(\phi) \sim \sum_{i=1}^3 a_i e^{\lambda_i \phi}$, i.e., a linear combination of Liouville-type potentials, where λ_i are the coefficients of the exponents and a_i are the coupling constants. For $\alpha \rightarrow 0$ (i.e., $x \rightarrow 0$), the solution yields the BTZ black hole [1]

$$B(r) = \frac{J^2}{4r^2} + \frac{r^2}{l^2} - 2Q^2 \ln\left(\frac{r}{l}\right) - M_0, \quad (17)$$

$$u(r) = -\frac{J}{2r^2}, \quad (18)$$

$$V(r) = -\frac{2}{l^2}, \quad (19)$$

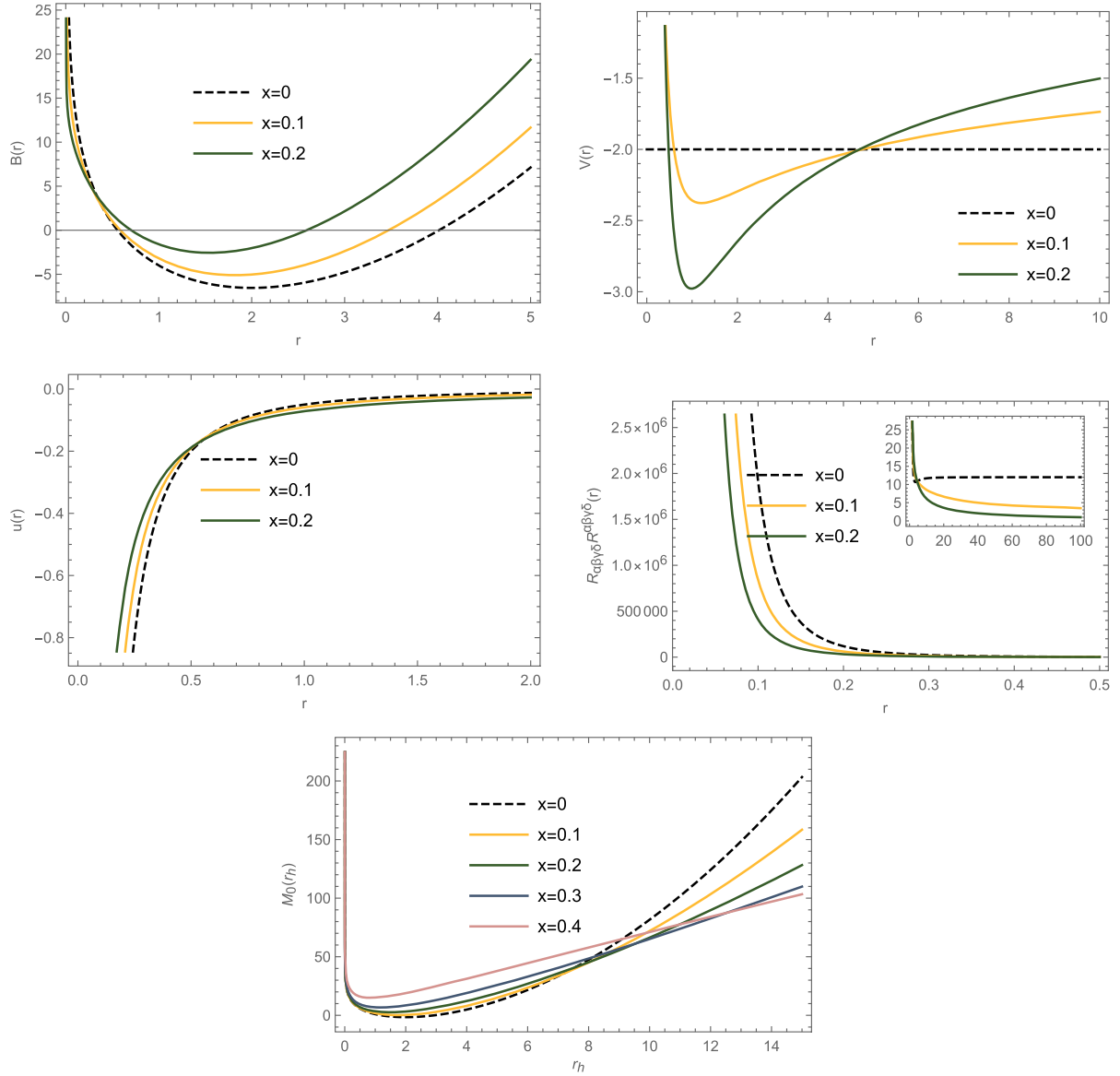


FIG. 1. Functions $B(r)$, $V(r)$, $u(r)$, and the Kretschmann scalar versus r , with the model parameter α changing through x , having set $l = 1, b = M_0 = 5, J_0 = 0.1, Q_0 = 2$. We also plot $M_0(r_h)$ as a function of r_h .

and the electric potential is given by (9), while all other functions vanish due to their dependence on α . The obtained scalar field has the standard $(2 + 1)$ -dimensional dilaton form [60] and diverges at both the origin and spatial infinity. Despite the fact that the scalar field diverges at large distances, the action remains finite since

$$\lim_{r \rightarrow \infty} (\nabla_\sigma \phi \nabla^\sigma \phi) \rightarrow 0, \quad (20)$$

while all the other quantities of the action (1) also remain finite for large r .

In Fig. 1 we plot the metric function $B(r)$, the potential $V(r)$, the angular shift function $u(r)$, and the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ along with the $x = 0$ case, which

corresponds to the charged and rotating BTZ black hole and also the geometric mass M_0 with respect to the horizon. From these figures, we can see that the metric function develops a smaller radius for the event horizon of the black hole, and the potential $V(r)$ for the dilaton case $x \neq 0$ goes to zero at large distances, indicating that spacetime is not asymptotically AdS, as in the BTZ case. It also develops a potential well between the inner and event horizon of the black hole. The Kretschmann scalar diverges at the origin $r = 0$, indicating a physical singularity (the singularity in the BTZ case comes from the electric charge Q_0 ; if $Q_0 = 0$, the BTZ spacetime is completely regular everywhere), and asymptotically goes to zero for large r , while it is related to the AdS radius for the BTZ case. The black hole mass has a minima and grows with the increase of the

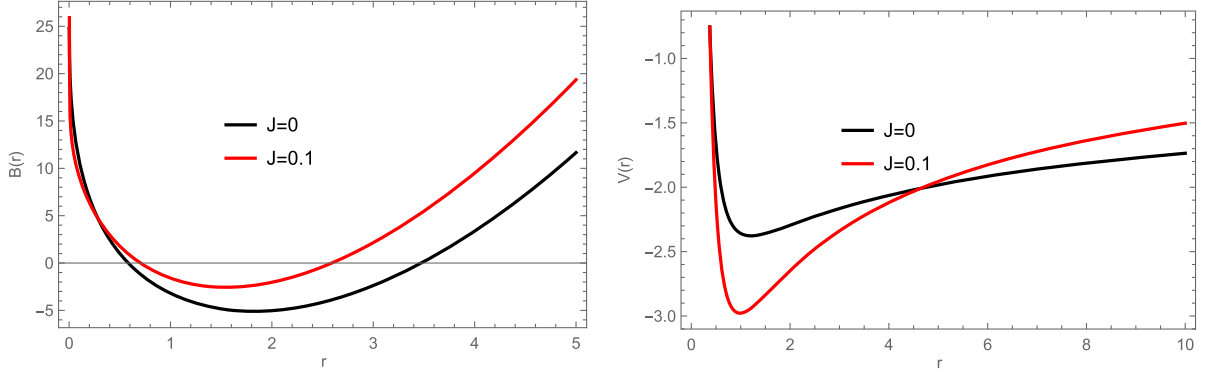


FIG. 2. Metric function $B(r)$ and potential $V(r)$ for $J_0 = 0$ and $J_0 = 0.1$, setting $x = 0.1$, $l = 1$, $Q_0 = 2$, $M_0 = b = 5$.

horizon; however, the BTZ case possesses a higher mass for large values of r_h .

In Fig. 2 we plot $B(r)$ and $V(r)$ for the static and rotating dilatonic black holes, respectively, in order to see the effect of the angular momentum on the resulting physics. We can see that the rotating solution possesses a smaller event horizon, while the potential develops a deeper potential well inside the event horizon of the black hole.

IV. THERMODYNAMICS AND CONSERVED CHARGES

In this section we discuss the thermodynamics of the obtained black hole solution. We first discuss the Hawking

temperature. To compute temperature we need to go to Euclidean space, which further specifies the periodicity of coordinates τ , θ as $\tau \rightarrow \tau + \beta$, $\theta \rightarrow \theta - \Omega\beta$, where Ω is the angular velocity of the horizon (see the discussion below) and β is related to the inverse temperature, which is given by

$$T(r_h) = \frac{1}{\beta} = \frac{B'(r_h)}{4\pi}, \quad (21)$$

where r_h represents the position of the outer horizon, while its explicit expression yields

$$T(r_h) = \frac{b^{-x} r_h^{-2x-3} (b^{3x} (J_0^2 l^2 (x-1) r_h^{6x} + (4-6x) r_h^4) + 4l^2 Q_0^2 (x-1) r_h^{3x+2})}{4\pi l^2 (x-1) (3x-2)}, \quad (22)$$

where we have already substituted M_0 from the $B(r_h) = 0$ condition of the black hole horizon. Its expansion for a large coupling constant $\alpha \gg 1$, $x \rightarrow 1$ yields

$$T(r_h) = -\frac{b^2}{2(x-1)(\pi l^2 r_h)} + \mathcal{O}((x-1)^0). \quad (23)$$

For $x \rightarrow 0$, we obtain

$$T(r_h) = -\frac{J_0^2}{8\pi r_h^3} + \frac{r_h}{2\pi l^2} - \frac{Q_0^2}{2\pi r_h}, \quad (24)$$

which is the BTZ black hole temperature. The zero point of the temperature corresponds to $B'(r_h) = 0$, i.e., extremal black holes, and can be obtained analytically as

$$r_{\text{extremal}} = (12x-8)^{\frac{1}{3x-2}} b^{\frac{3x}{3x-2}} \left(4l^2 Q_0^2 (x-1) - 2\sqrt{2} \sqrt{l^2 (x-1) (J_0^2 (3x-2) b^{6x} + 2l^2 Q_0^4 (x-1))} \right)^{\frac{1}{2-3x}}, \quad (25)$$

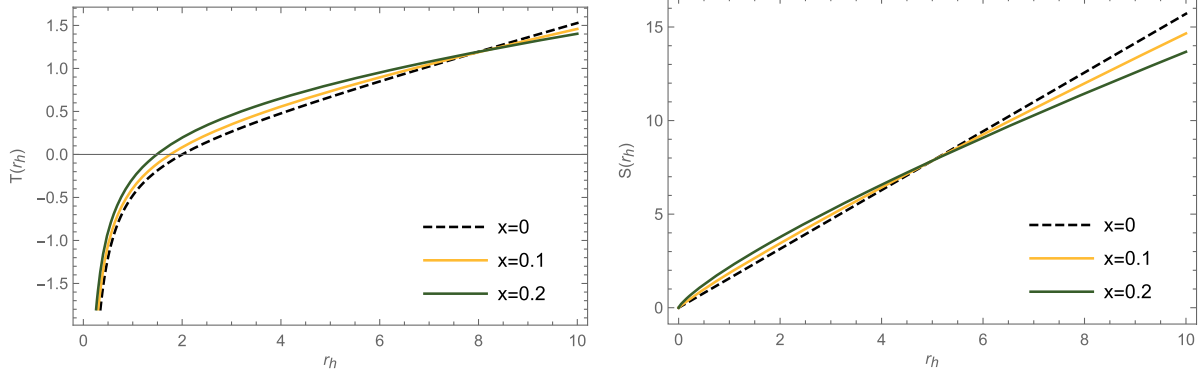
while the extremal horizon of the BTZ case is given by

$$r_{\text{extremal}} = \frac{1}{\sqrt{2}} \sqrt{l \left(\sqrt{J_0^2 + l^2 Q_0^4} + l Q_0^2 \right)}. \quad (26)$$

The entropy will be given by the Bekenstein-Hawking area law since the theory (1) is minimally coupled to gravity

theory and higher order invariants are absent; thus, the only term that will contribute to Wald entropy [61] is the pure Ricci scalar term, which yields the entropy as

$$S = \frac{\mathcal{A}}{4G} = 2\pi\mathcal{A} = 4\pi^2 b^x r_h^{1-x}, \quad (27)$$


 FIG. 3. Temperature and entropy versus r_h , for different values of x , setting $l = 1, J_0 = 0.1, Q_0 = 2, M_0 = b = 5$.

where $\mathcal{A} = 2\pi b^x r_h^{1-x}$ is the area of the black hole. It can be expanded near the no-coupling case $x = 0$ ($\alpha = 0$),

$$S = \frac{4\pi^2 r_h}{\kappa} + \frac{4\pi^2 r_h}{\kappa} \ln\left(\frac{b}{r_h}\right)x + \mathcal{O}(x^2), \quad (28)$$

while for a large coupling constant $\alpha \gg 1, x \rightarrow 1$, we have

$$T = \frac{4\pi^2 r_h^2 (r_h^2 - 4) - 1}{8\pi^3 r_h^3} + \frac{x(8\pi^2 (r_h^2 - 6)r_h^2 - 8(2\pi^2 (r_h^2 + 2)r_h^2 + 1) \ln(r_h) - 3)}{16\pi^3 r_h^3} + \mathcal{O}(x^2). \quad (30)$$

We can see that for a large coupling constant, the entropy takes a constant value and does not depend on the horizon radius at zeroth order as should be expected. This is another clue that large values of the coupling constant should be excluded, besides the one coming from the finiteness of the angular velocity function as we already discussed. For $x \rightarrow 0$ we recover the BTZ black hole entropy and temperature (24), while we can see that the scalar length scale b affects the entropy. We present plots of the temperature and the entropy for some values of x alongside $x = 0$, which corresponds to the BTZ case, in order to make comparisons in Fig. 3.

From Fig. 3 we can see that the temperature and the entropy of the rotating dilatonic black holes do not differ qualitatively from the corresponding quantities of the BTZ black hole spacetime. The entropy is always positive, and the dilatonic black holes may have higher entropy than the BTZ black hole, depending on the value of the scalar length scale b . The expression (28) and Fig. 3 both show that for $r_h < b$, the dilatonic black holes are thermodynamically preferred; otherwise the BTZ case is preferred.

To compute the physical electric charge we use [60]

$$\begin{aligned} Q &= \frac{1}{4\pi} \int \sqrt{-g} \frac{\partial \mathcal{L}_{\text{EM}}}{\partial F} F_{\mu\nu} n^\mu v^\nu d\Omega = \frac{1}{4\pi} \int dS^\mu f(\phi) \nabla_\mu q(r) \\ &= -\frac{Q_0}{2}, \end{aligned} \quad (31)$$

$$S = \frac{\pi b}{2} + \frac{1}{2} \pi b (x - 1) \ln\left(\frac{b}{r_h}\right) + \mathcal{O}((x - 1)^2). \quad (29)$$

The Hawking temperature of the black hole can also be expanded near the $x = 0$ case,

where \mathcal{L}_{EM} denotes the electromagnetic part of the Lagrangian of our theory (1) and n^μ, v^ν are the unit spacelike and timelike normal vectors to the hypersurface of radius r [60]. We can see that the dilatonic coupling does not affect the conserved electric charge, which is expected since we assume that $f(r)R(r) = r$, so the effect of the dilatonic coupling has the exact countereffect on the circumference of the black hole. It also coincides with the nonrotating case [60] as expected.

To compute the mass of the black hole, we use the quasilocal method [62,63]. The metric (7) can be transformed to a more general form,

$$ds^2 = -N^2(R)dt^2 + \frac{dR^2}{H^2(R)} + R^2[d\theta + V^\theta(R)dt]^2, \quad (32)$$

with

$$N^2(R) = B(b^{\frac{x}{x-1}} R^{\frac{1}{1-x}}), \quad (33)$$

$$H^2(R) = B(b^{\frac{x}{x-1}} R^{\frac{1}{1-x}}) (1-x)^2 b^{\frac{2x}{1-x}} R^{-\frac{2x}{1-x}}, \quad (34)$$

$$V^\theta(R) = u(b^{\frac{x}{x-1}} R^{\frac{1}{1-x}}). \quad (35)$$

For a D -dimensional spacetime manifold \mathcal{M} , which is topologically the product of a spacelike hypersurface and a real line interval $\Sigma \times I$, the total quasilocal energy is defined as [62,63]

$$E = \int_B d^{D-2}x \sqrt{\sigma} \varepsilon, \quad (36)$$

where $B \equiv \partial\Sigma$ is the $(D - 2)$ -dimensional boundary, σ is the determinant of the induced metric σ_{ab} on B , and ε is the energy density.

The boundary $\partial\mathcal{M}$ consists of initial and final spacelike hypersurfaces t' and t'' , respectively, and a timelike hypersurface $\mathcal{T} = B \times I$ joining them. The $(D - 1)$ -metric γ_{ij} on \mathcal{T} can be written according to the Arnowitt, Deser, Misner (ADM) decomposition as

$$\gamma_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{ab} (dx^a + V^a dt)(dx^b + V^b dt). \quad (37)$$

The conserved charge associated with a Killing vector field ξ^i is defined as

$$Q_\xi = \int_B d^{D-2}x \sqrt{\sigma} (\varepsilon u^i + j^i) \xi_i, \quad (38)$$

where u^i is the unit normal to spacelike hypersurfaces t' or t'' , and j^i is the momentum density.

If the system contains a rotational symmetry given by a Killing vector field $\zeta^i = (\partial/\partial\theta)^i$ on \mathcal{T} , and the $(D - 2)$ -surface B is chosen to contain the orbits of ζ^i , then the angular momentum can be expressed as

$$J = Q_\zeta = \int_B d^{D-2}x \sqrt{\sigma} j_i \zeta^i. \quad (39)$$

In addition, a conserved mass can be given by

$$M = -Q_\xi = - \int_B d^{D-2}x \sqrt{\sigma} (\varepsilon u_i + j_i) \xi^i, \quad (40)$$

where $\xi^i = Nu^i + V^a (\partial/\partial x^a)^i$ is the associated timelike Killing vector field.

The energy density ε and quasilocal energy E can be calculated in metric form (32),

$$\varepsilon(R) = k(R) - \varepsilon_0(R), \quad (41)$$

$$E(R) = 2\pi R \varepsilon(R) = 2\pi R k(R) - 2\pi R \varepsilon_0(R), \quad (42)$$

where $k(R) = -H(R)/R$ is the trace of the extrinsic curvature of B . To make the total quasilocal energy finite, the condition $0 \leq x < 2/3$ is required, and ε_0 is chosen to be

$$\varepsilon_0(R) = -\frac{1}{l} \sqrt{\frac{2(1-x)}{2-3x}} \left(\frac{b}{R}\right)^{\frac{1}{1-x}}. \quad (43)$$

The only nonzero components of the momentum P^{ij} and the momentum density j_i are

$$P^{r\theta} = P^{\theta r} = -\frac{RH(R)}{4N(R)} \frac{dV^\theta(R)}{dR} = -\frac{b^{3x} J_0}{4R^2}, \quad (44)$$

$$j_\theta = \frac{b^{3x} J_0}{2R}. \quad (45)$$

Then we can calculate the total angular momentum associated with the Killing vector field $\zeta^i = (\partial/\partial\theta)^i$ as

$$J = 2\pi R j_\theta = \pi b^{3x} J_0. \quad (46)$$

Finally, the conserved mass can be obtained by taking the limit at spatial infinity,

$$M = \lim_{R \rightarrow \infty} (N(R)E(R) - 2\pi R j_\theta \xi^\theta) = \pi(1-x)b^x M_0, \quad (47)$$

which can reduce to the ADM mass M_0 of the BTZ case when $x = 0$. We can see that the scalar length scale b enters in both the conserved mass and the conserved angular momentum; therefore, the scalar hair of the solution is of the secondary type, which is not given by a unique Gauss law but is related to the black hole conserved quantities. Using the relations (46) and (47) we get

$$\frac{J}{M} = \frac{b^{2x} J_0}{(1-x)M_0}. \quad (48)$$

Substituting the conserved mass M , conserved angular momentum J , and conserved charge $Q = -Q_0/2$ into the metric and taking $B(r_h) = 0$, we obtain

$$M(r_h, Q, J) = \frac{\pi(x-1)b^x r_h^{-x}}{\kappa} \left(-\frac{\kappa^2 J^2 b^{-4x} r_h^{4x-2}}{\pi^2 (2-3x)^2} - \frac{2b^{2x} r_h^{2-2x}}{l^2 (3x^2 - 5x + 2)} + \frac{16Q^2 b^{-x} r_h^x}{2-3x} \ln\left(\frac{r_h}{l}\right) \right). \quad (49)$$

In Fig. 4 we plot the figure of conserved mass M as a function of the event horizon r_h with various values of x . We can see that the conserved mass M always grows with the increase of r_h , and its tendency decreases as the coupling constant x (or α) rises, which means that in 3D EMD theory, black holes of the same size contain less energy.

The partial derivative of the conserved mass M with respect to entropy S ,

$$\frac{\partial M}{\partial S} = \frac{\partial M / \partial r_h}{\partial S / \partial r_h} = \frac{B'(r_h)}{4\pi} = T, \quad (50)$$

is exactly the Hawking temperature at the event horizon. The electric potential Φ at the event horizon can be

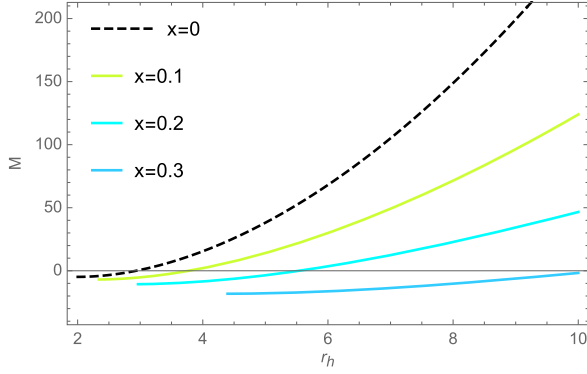


FIG. 4. Conserved mass M plotted as a function of the event horizon r_h with various values of x , where we have set $Q = J = b = l = 1$.

calculated using the null generator of the horizon $\chi = C_0 \partial^t + \Omega_h \partial^\theta$ as

$$\Phi = A_{\mu\chi^\mu}|_\infty - A_{\mu\chi^\mu}|_{r_h} = C_0 Q_0 \ln \frac{r_h}{l}, \quad (51)$$

while the derivative of the conserved mass M with respect to Q leads to

$$\frac{\partial M}{\partial Q} = \frac{32\pi Q(x-1)}{(2-3x)} \ln \left(\frac{r_h}{l} \right) = \Phi, \quad (52)$$

where we have set $C_0 = \frac{32\pi(x-1)}{(2-3x)}$. However, the derivative of the conserved mass M with respect to angular momentum J is proportional to the angular velocity $\Omega_h = -u(r_h)$ at the event horizon, with a coefficient depending on the coupling constant

$$\frac{\partial M}{\partial J} = -\frac{2J(x-1)b^{-3x}r_h^{3x-2}}{\pi(2-3x)^2} = \frac{2(1-x)}{2-3x} \Omega = \frac{1}{1-\alpha^2} \Omega, \quad (53)$$

which indicates that the first law is modified in EMD theory in $(2+1)$ dimensions,

$$dM = TdS + \Phi dQ + \frac{1}{1-\alpha^2} \Omega dJ. \quad (54)$$

We should note here that the conserved charge Q and conserved angular momentum J are parts of the theory under consideration since they appear in the potential $V(\phi)$. Consequently, these are not pure integration constants that are allowed to vary but constants of the theory, so varying Q, J is like varying the whole theory. The mass M does not appear in the potential; thus, the first law should (at least) satisfy $M'(r_h) = T(r_h)S'(r_h)$, and indeed this holds. In conclusion, the anomalous angular momentum term is a consequence of the fact that our theory can only yield solutions with a particular angular momentum.

Nevertheless, it is worth noting that in [55], the uncharged rotating solution in $(2+1)$ -dimensional EMD theory does not satisfy the dJ term of the first law either, i.e., $\partial M / \partial J \neq \Omega$, where J is not a parameter of the theory.

V. SOLUTION WITH $\alpha = 1$

In this section we discuss the case of $\alpha = 1$. The motivation for this consideration is that in the case of $\alpha = 1$ the action (1) is resulting from a string theory with a dilatonic field to couple to a gauge field. So it would be interesting to see if there are rotating black hole solutions coming from this string theory setup. We have seen that the general case breaks down for $\alpha = 1, x = 2/3$, and for this reason we set $\alpha = 1$ first and then solve the field equations. The solution is written as

$$\phi(r) = \frac{1}{3} \ln \left(\frac{b}{r} \right), \quad (55)$$

$$R(r) = f(r)^{-1} = b^{2/3} r^{1/3}, \quad (56)$$

$$u(r) = J_0 \ln \left(\frac{r}{b} \right), \quad (57)$$

$$B(r) = b^2 J_0^2 \ln^2 \left(\frac{r}{b} \right) \left(\frac{b}{r} \right)^{-2/3} - 6 \left(\frac{r}{b} \right)^{2/3} \ln \left(\frac{r}{l} \right) \times \left(Q^2 \left(\frac{1}{3} \ln \left(\frac{b^2}{lr} \right) + 1 \right) - \frac{b^2}{l^2} \right) - M_0 r^{2/3}, \quad (58)$$

$$V(r) = \frac{J_0^2 (4b^2 \ln(\frac{b}{r}) - 3b^2)}{6b^{2/3} r^{4/3}} - \frac{2b^{4/3}}{l^2 r^{4/3}} + \frac{4Q^2}{3b^{2/3} r^{4/3}} \ln \left(\frac{b}{r} \right), \quad (59)$$

$$V(\phi) = \frac{J_0^2 e^{4\phi} (4b^2 (\ln(\frac{l}{b}) + 3\phi) - 3b^2)}{6b^2} + \frac{4Q^2 e^{4\phi} \phi}{b^2} - \frac{2e^{4\phi}}{l^2}. \quad (60)$$

In the above equations M_0 and J_0 are constants of integration. In Fig. 5 we plot the metric function $B(r)$ and the potential $V(r)$ for different values of J_0 , including the $J_0 = 0$ case, in order to compare our solution with the nonrotating solution [60]. We can see that the nonrotating black hole possesses a larger event horizon radius in comparison to the rotating case, while in the rotating case the potential forms a deeper potential well between the inner and outer event horizons.

We now discuss the thermodynamics of this black hole. The Hawking temperature is found to be

$$T(r_h) = \frac{B'(r_h)}{4\pi} = \frac{3(b-lQ)(b+lQ) - l^2(b^2 J_0^2 + 2Q^2)(\ln(b/r_h))}{2\pi b^{2/3} l^2 \sqrt{3} r_h}, \quad (61)$$

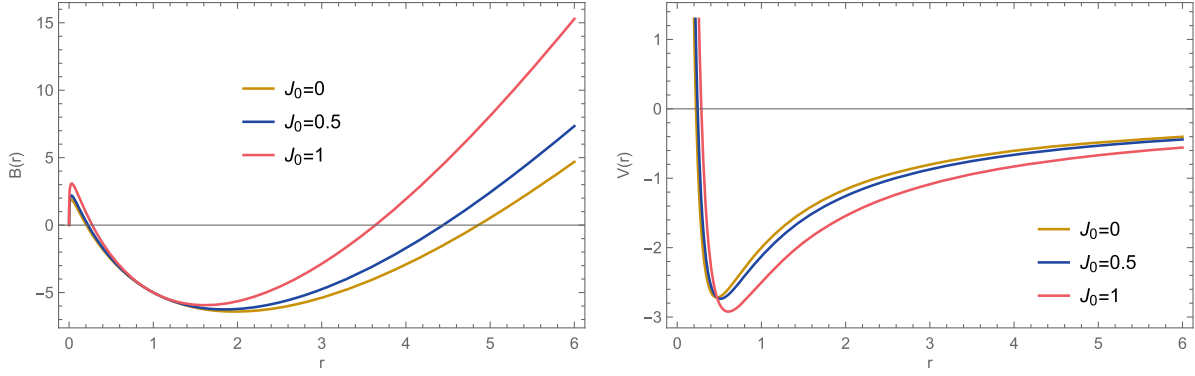


FIG. 5. Metric function $B(r)$ and potential $V(r)$ versus r , for $\alpha = 1$, setting $l = b = Q = 1, M_0 = 5$, while changing J_0 .

while the extremal black hole with zero temperature has an extremal horizon at

$$r_{\text{extremal}} = b \exp\left(\frac{3(lQ - b)(b + lQ)}{l^2(b^2 J_0^2 + 2Q^2)}\right). \quad (62)$$

In the case with $b = Ql$, the radius will be $r_{\text{extremal}} = b$, purely determined by the scalar length scale b , while the entropy and the conserved angular momentum will become

$$S(r_h) = 4\pi^2 b^{2/3} r_h^{1/3}, \quad (63)$$

$$J = \pi b^2 J_0. \quad (64)$$

Using the same method as before, we find that the conserved mass of the black hole cannot be finite. In particular, we can have a finite energy density; however, there is a contribution in the conserved mass being related to the angular velocity,

$$2\pi R_{j\phi}^{\xi\theta} \rightarrow 3\pi b^2 J_0^2 \ln \frac{R}{b}, \quad (65)$$

which is divergent for the observer at large distances ($R \rightarrow \infty$). This strange behavior is caused by the logarithmic term of the angular velocity, which is also divergent at large distances, indicating that the black hole rotates faster for larger r . Nevertheless, this pathology can be cured by a more general solution with a scalar field $\phi(r) = \frac{1}{3} \ln\left(\frac{b}{r} + 1\right)$ that is regular at spatial infinity, and the corresponding angular velocity $u(r) = u_0 + u_1\left(\ln\left(\frac{b}{r} + 1\right) - \frac{b}{b+r}\right)$ also becomes finite at large distances. In this case, the metric function and the potential can be solved explicitly but are very complicated; therefore, we do not show them here. Similar behaviors are also present in higher-dimensional, slowly rotating black holes [64] where the solution is ill defined for $\alpha = 1$. Additionally, in [65] it was found that for $\alpha = 1$, the resultant spacetime is asymptotically flat in the absence of potential, while for the particular Liouville

potential the authors used, the solution breaks down for $\alpha = \sqrt{3}$, which corresponds to Kaluza-Klein theory.

VI. CONCLUSIONS

In this work we considered Einstein-Maxwell-dilaton theory in (2 + 1) dimensions, in which the dilatonic matter is coupled to the electromagnetic field, and obtained exact rotating BTZ-like solutions. There are two important parameters that control the properties of the found black hole solutions. The dilatonic parameter α , which appears in the dilatonic function coupled to the electromagnetic field, has an important impact on the metric function, also affecting the rotating properties of the black hole and its thermodynamics. As expected, as α increases, the horizon of the black hole increases, as does its mass. We found that the dilatonic parameter α does not affect the conserved electric charge, while, with the increase of α , the angular momentum J becomes larger.

The other crucial parameter that influences the properties of the black hole solution is the scalar length scale b of the dilatonic matter ϕ . Since the scalar length scale b is of the secondary type, it enters in both the conserved mass and conserved angular momentum. With increasing α the angular momentum can be enlarged for $b > 1$, or it can be shrunk for $b < 1$, indicating that the dilatonic matter makes the black hole rotate more slowly. Calculating the entropy we found that it is always positive, and the dilatonic black holes may have higher entropy than the BTZ black hole, depending on the value of the scalar length scale b . For $b < r_h$, where r_h is the black hole horizon, the dilatonic BTZ-like black hole is thermodynamically preferred; for $b > r_h$ the BTZ black hole is preferred.

We also discussed the particular case that arises as the low energy limit of string theory, the case of $\alpha = 1$. In this case the solution of the field equations is ill defined because the angular velocity is divergent at large distances, and this results in the nonfiniteness of the conserved mass. However, the black hole horizon still exists, and this led the authors in [3], who studied the BTZ black hole and

found a logarithmic divergence of the electric potential, to compute the mass of the black hole in a finite radius r_0 that encloses the black hole. In our rotating case, the low energy limit of string theory cannot be realized unless we relax some of the assumptions we considered in order to derive the field equations. For example, we can introduce

another degree of freedom in the metric function, so that $g_{tt}g_{rr} \neq 1$, or consider another form for the potential V and the circumference function R . We can also introduce the three-form $H_{\mu\nu\rho}$ term in the action, which arises from the string theory. We leave these possibilities for a future work.

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