Causal diamonds in (2 + 1)-dimensional quantum gravity

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We develop the reduced phase space quantization of causal diamonds in pure $(2+1)$-dimensional gravity with a nonpositive cosmological constant. The system is defined as the domain of dependence of a topological disc with fixed boundary metric. By solving the initial value constraints in a constant-mean-curvature time gauge and removing all the spatial gauge redundancy, we find that the phase space is the cotangent bundle of $\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$. To quantize this phase space we apply Isham's group-theoretic quantization scheme, with respect to a BMS_3 group, and find that the quantum theory can be realized by wave functions on some coadjoint orbit of the Virasoro group, with labels in irreducible unitary representations of the corresponding little group. We find that the twist of the diamond boundary loop is quantized in integer or half-integer multiples of the ratio of the Planck length to the boundary length.

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I. INTRODUCTION

Among the many challenges to understanding nonperturbative quantum gravity are that standard canonical quantization is inapplicable due to the nonlinearity of the phase space, that local observables are not available, and that general relativity in four or more spacetime dimensions is (likely) not an ultraviolet-complete quantum field theory. On top of those is the obstacle of removing the diffeomorphism gauge redundancy (a.k.a. "coordinate freedom"), and the fact that spacetime diffeomorphisms include deformations in timelike directions, making time evolution a gauge transformation, which leads to the vexing "problem of time" [1–3]. To make progress it is worthwhile to consider simplified settings, and over the past several decades much work of that nature has been done. Here we consider a new such setting, in which all of the above-mentioned challenges can be met, namely, causal diamonds in $(2+1)$-dimensional general relativity with a nonpositive cosmological constant.

By a $(2+1)$-dimensional causal diamond we mean the domain of dependence of a spacelike topological disc with fixed boundary metric. To quantize the system we employ the reduced phase space approach, in which we first impose all the initial value constraints and remove the gauge ambiguities at the classical level, and then proceed with the quantization. Since there are no local degrees of freedom in $(2+1)$-dimensional gravity, and we choose the topology of the spatial slices to be that of a disc, the classical states (solutions to the Einstein equation, up to gauge transformations) can only correspond to all possible shapes of causal diamonds, with boundary length ℓ determined by the fixed boundary metric, embedded in anti-de Sitter space (AdS_3) if $\Lambda < 0$ or in Minkowski space (Mink_3) if $\Lambda = 0$ (see Fig. 1). We find that the corresponding phase space is the cotangent bundle $T^*\mathcal{Q}$ of a configuration space $\mathcal{Q} = \mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$ that is the quotient of the infinite dimensional group of orientation preserving smooth maps of the boundary loop into itself, by the projective special linear group in two real dimensions (which is the finite dimensional subgroup of $\mathrm{Diff}^+(S^1)$ induced by conformal isometries[1] of the unit flat disc). Similar $(2+1)$-dimensional gravity systems have been considered in the literature, such as spacetimes with closed spatial slices (where the reduced phase space is finite-dimensional) [4–12], spacetimes with finite timelike boundary [13–15], and asymptotically AdS_3 spacetimes [16–23]. The causal diamonds provide a novel, quasi-local system of quantum gravity in globally hyperbolic spacetimes that, while simple enough to be exactly solvable classically, has an infinite-dimensional reduced phase space of "boundary gravitons."

In this paper we describe the classical reduction process and explain how to quantize the resulting phase space using Isham's scheme [24,25] in which the quantization is

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[1] In this paper a *conformal transformation* acts on tensors as multiplication by a positive function followed with the push-forward by a diffeomorphism. Metrics related by such a transformation are said to be *conformally equivalent*; and a transformation that leaves the metric invariant is called a *conformal isometry.*

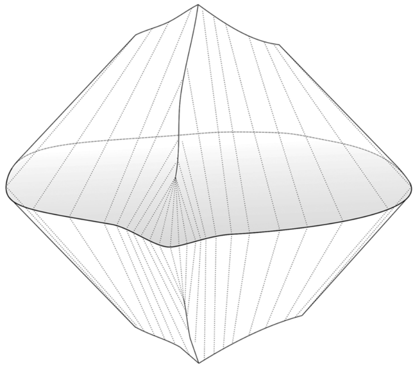

FIG. 1. A generic classical state corresponds to a causal diamond in AdS_3 (or in $Mink_3$ if $\Lambda = 0$) with boundary length ℓ. Note that in general the Cauchy horizon is not smooth since the null generators exit at caustics.

designed to preserve a group of symplectic (a.k.a. canonical) transformations of the phase space. This "quantization group" in our case is the three-dimensional Bondi-Metzner-Sachs group BMS_3. We discuss the representation theory of the algebra of quantum observables, and deduce that the twist of the diamond boundary loop—which is proportional to the spin of the diamond—is quantized in terms of the ratio of the Planck length to the boundary length. This paper is a brief summary of some aspects of our study, the full details of which will appear in [26].

II. CLASSICAL

In the Arnowitt-Deser-Misner (ADM) formulation of general relativity [27], the phase space before reduction is described by Riemannian metrics h_{ab} and conjugate momenta $\pi^{ab} = \sqrt{h}(K^{ab} - Kh^{ab})$, where K_{ab} is the extrinsic curvature on an initial value spatial surface (Cauchy slice), here assumed to have the topology of a disc D.[2] We shall restrict to metrics that induce a fixed metric on the boundary, $h|_{\partial D} = \gamma$. Note however that the total length ℓ of the boundary loop is the only gauge invariant attribute of the boundary geometry that is fixed by this condition. The maximal development of any data (h, π) that satisfy the initial value constraints of general relativity defines a *causal diamond*.

A natural choice of intrinsic time function τ is given by (minus) the mean extrinsic curvature on the leaves of a foliation of the diamond by constant-mean-curvature (CMC) Cauchy surfaces, $\tau = -K^{ab}h_{ab}$. The nonpositive cosmological constant $\Lambda \leq 0$ ensures that, as τ ranges from $-\infty$ to $+\infty$, the CMC surfaces foliate the diamond [28–31]. This gauge-fixing of time also confers great simplification to the Lichnerowicz method [32] of solving the Einstein constraint equations [6,33], which consist of a scalar constraint and a vector constraint. In this method, we start with "seed data" (h_{ab}, π^{ab}) on a CMC slice with a given value of τ, satisfying the boundary condition on h_{ab} and the vector constraint $\nabla_a \sigma^{ab} = 0$, where $\sigma^{ab} \coloneqq K^{ab} + \frac{1}{2}\tau h^{ab}$ is the traceless part of K^{ab} and ∇_a is the covariant derivative determined by h_{ab}. Then, by means of a Weyl-transformation, we use this seed data to generate initial data $(\widetilde{h}_{ab}, \widetilde{\pi}^{ab})$ that satisfy both the vector and the scalar constraints. The new data, defined by $\widetilde{h}_{ab} = e^{\phi}h_{ab}$, $\widetilde{\sigma}^{ab} = e^{-2\phi}\sigma^{ab}$ and $\widetilde{\tau} = \tau$, continue to satisfy the vector constraint (for any ϕ), satisfy the boundary condition iff $\phi|_{\partial D} = 0$, and satisfy the scalar constraint iff ϕ satisfies the (two-dimensional) Lichnerowicz equation

$$\nabla^2\phi - R_{(h)} + e^{-\phi}\sigma^{ab}\sigma_{ab} - e^{\phi}\chi = 0, \tag{1}$$

where $R_{(h)}$ is the scalar curvature of the metric h_{ab} and $\chi = -2\Lambda + \tau^2/2$. The fact that $\chi \geq 0$ ensures that this equation always has a unique solution for ϕ given a boundary condition [26,34].

Since any element in the family of Weyl-deformed data, $(e^{\lambda}h_{ab}, e^{-2\lambda}\sigma^{ab}, \tau)$, leads to the same solution $(\widetilde{h}_{ab}, \widetilde{\pi}^{ab})$ of the initial value problem, the constraint surface on the phase space can be identified with the set of equivalence classes $[(h_{ab}, \sigma^{ab}) \sim (e^{\lambda}h_{ab}, e^{-2\lambda}\sigma^{ab})]$. Spatial diffeomorphisms that act trivially at the boundary, and only those, correspond to gauge transformations [26], hence the reduced phase space (i.e., the space of physically inequivalent solutions to the equations of motion) can be identified as the set of equivalence classes of seed data,

$$[(h_{ab}, \sigma^{ab}) \sim (\Psi_* e^{\lambda}h_{ab}, \Psi_* e^{-2\lambda}\sigma^{ab})], \tag{2}$$

where Ψ is a boundary-trivial diffeomorphism on D (and Ψ_* is the push-forward) and λ is a function on D vanishing at the boundary [26]. This happens to be the cotangent bundle $T^*\mathcal{Q}$ of the space $\mathcal{Q}$ of metrics on the disc with fixed induced boundary metric, modulo diffeomorphisms and Weyl transformations that are trivial on the boundary; and, as one might expect, the symplectic structure is the natural one on the cotangent bundle. In fact, $\mathcal{Q}$ is the homogeneous space $\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$,[3] and thus the reduced phase space is $\widetilde{\mathcal{P}} = T^*[\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})]$. This is the first of our main results.

There is another approach to the phase space reduction based on a suitable change of coordinates from ADM variables to "conformal coordinates," which exploits the fact that all metrics on a disc are conformally equivalent.

[2]We adopt units with $c = 16\pi G = 1$.

[3]In brief, $\mathrm{Diff}^+(S^1)$ (orientation preserving diffeomorphisms of the boundary loop, acting together with the corresponding Weyl transformation that preserves the boundary metric) acts transitively on $\mathcal{Q}$ (since all metrics on a disc are equivalent under conformal transformations that are allowed to act nontrivially at the boundary). The subgroup that leaves invariant each point of $\mathcal{Q}$, e.g., the (equivalence class of the) Euclidean round disc, is $\mathrm{PSL}(2,\mathbb{R})$. Therefore $\mathcal{Q} = \mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$.

This alternate approach provides an explicit projection map from the concrete geometrical ADM variables to abstract variables describing $\widetilde{\mathcal{P}}$ [26]. It is useful for several constructions, and relevant when physically interpreting the meaning of observables in the quantum theory, but we postpone its discussion to Sec. IV since it is not required for the quantization procedure.

The Hamiltonian generating evolution in τ on the reduced phase space can be obtained by starting with the Einstein-Hilbert action in the ADM form and then reexpressing it in terms of variables on the reduced phase space. The action $S[\mathcal{C}]$ along a curve $\mathcal{C}$ in the (constrained) ADM phase space is

$$S[\mathcal{C}] = \int_{\mathcal{C}} dt \int_D d^2x \pi^{ab} \dot{h}_{ab} = \int_{\tilde{\mathcal{C}}} \left(\tilde{\theta} - d\tau \int_D d^2x \sqrt{h} \right), \qquad (3)$$

where $\widetilde{\mathcal{C}}$ is the projection of $\mathcal{C}$ to $\widetilde{\mathcal{P}}$, and $\tilde{\theta}$ is the symplectic potential on $\widetilde{\mathcal{P}}$ (which is locally equal to a sum $\sum_i p_i dq^i$ over a complete set of canonically conjugate coordinates). Thus the reduced (time-dependent) Hamiltonian is identified as $\widetilde{H}(\tau) = \int_D d^2x \sqrt{h}$, that is, the area of the CMC surface with $K = -\tau$ [33].

III. QUANTUM

As the reduced phase space does not seem to admit a natural global coordinate chart, the traditional Dirac canonical quantization rule $\{q, p\} = 1 \mapsto \frac{1}{i\hbar}[\widehat{q}, \widehat{p}] = \hat{1}$ cannot be straightforwardly implemented. Isham developed a generalization of Dirac's canonical quantization rule that, rather than being based on a preferred coordinate system, is designed to preserve the structure of a group of symplectic (canonical) transformations acting transitively on the phase space [24,25]. In the simple case of a particle on a line $\mathbb{R}$, the functions x and p on phase space, acting as Hamiltonian "charges," generate the group of phase space translations, which is represented projectively, unitarily and irreducibly in the quantum theory. More generally, given a group G of symplectic symmetries acting on the phase space, we can generate a set of observables whose Poisson algebra closes. These observables are the Hamiltonian charges Q_i associated with the algebra $\mathfrak{g}$ of G, and their Poisson algebra is homomorphic to $\mathfrak{g}$, up to possible central extensions. If there are central extensions, we extend G to include them as generators, so that the Poisson algebra is then homomorphic to $\mathfrak{g}$. If the group action is transitive then the set $\{Q_i\}$ is complete in the sense that any function on the phase space can be locally written in terms of them. Quantization then proceeds by replacing the Poisson algebra by a commutator algebra, $\{Q_i, Q_j\} = c^k_{ij} Q_k \mapsto \frac{1}{i\hbar}[\widehat{Q}_i, \widehat{Q}_j] = c^k_{ij} \widehat{Q}_k$, and finding unitary irreducible representations of this algebra.

Isham quantization is particularly natural when the phase space is the cotangent bundle of a homogeneous space, $\widetilde{\mathcal{P}} = T^*(K/H)$, where H is a subgroup of a group K. The configuration space K/H carries a natural action of K that lifts to the cotangent bundle, and this provides "half" of the quantization group. There is a simple way to extend this group by "momentum translations" generated by charges defined globally on the phase space: given any function f on K/H, the 1-form df at every point can be subtracted from the momentum 1-forms at that point. This defines a symplectic map of the phase space that is generated by the function f. To define a transitive action on the phase space together with the K action one must choose a sufficiently large collection of such functions; and, to minimize the inclusion of algebra representations that fail to produce the desired classical limit, this collection of functions should presumably be as small as possible. Isham identified a construction that does exactly this, provided K can be linearly represented on a vector space V in such a way that at least one of the K orbits in V is homeomorphic to K/H: linear functions on V, i.e., elements of the dual V^*, induce on the orbit, and therefore on K/H, a suitable collection of functions. Together with K the corresponding momentum translations define a transitive group $G = V^* \rtimes K$ of symmetries on $\widetilde{\mathcal{P}}$.[4]

In our case, $\widetilde{\mathcal{P}} = T^*\mathcal{Q}$, where $\mathcal{Q} = \mathrm{Diff}^+(S^1)/\mathrm{PSL}(2, \mathbb{R})$, the group $K = \mathrm{Diff}^+(S^1)$ naturally acts from the left on $\mathcal{Q}$, but we have not found a representation of $\mathrm{Diff}^+(S^1)$ containing an orbit homeomorphic to $\mathcal{Q}$. Fortunately, however, for the purpose of identifying a suitable set of functions on $\mathcal{Q}$ we can take K to be the Virasoro group *Vira*, which is a central extension of $\mathrm{Diff}^+(S^1)$ and thus can also act on $\mathcal{Q}$ (where the central element just acts trivially). The coadjoint representation of *Vira*, which acts on $V = \mathfrak{vira}^*$ (where $\mathfrak{vira}$ is the Lie algebra of *Vira*), does contain an orbit isomorphic to $\mathcal{Q}$ [36–39], hence we can take $G = (\mathfrak{vira}^*)^* \rtimes \mathrm{Vira}$ as the group to be quantized. This group is a central extension of BMS_3 [40,41].[5]

In this way, the quantum theory is based on irreducible unitary (projective) representations of $(\mathfrak{vira}^*)^* \rtimes \mathrm{Vira}$. Since this group has the form of a semi-direct product with an abelian factor [namely $(\mathfrak{vira}^*)^*$ with its vector space group structure], we could hope to use Mackey's theory of induced representations to classify the representations [42]. (Mackey's classification has not been rigorously established for infinite dimensions, however [43].)

[4]For the example $K/H = SO(3)/SO(2)(= S^2)$, the $SO(3)$ charges are the components of angular momentum, the momentum translations are the Cartesian coordinates of the $\mathbb{R}^3$ in which the configuration space S^2 is realized as an orbit of $SO(3)$, and the quantizing group is $\mathbb{R}^{3*} \rtimes SO(3)$, the Euclidean group in three dimensions [35].

[5]BMS_3 is familiar as the symmetry of asymptotically Minkowskian spacetime acting on the null cone at future null infinity. Here it appears as a natural group of symplectic transformations acting on the phase space of the diamond. Perhaps there is a different way to view the reduction of the phase space of the diamond and the action of this group, in terms of the null surfaces that bound the diamond.

Basically, for any K-orbit $\mathcal{O}$ in $(\mathfrak{vira}^*)^{**}$, with corresponding little group $H_{\mathcal{O}}$, one can construct a unitary irreducible representation (irrep) consisting of wave functions on $\mathcal{O}$ taking values in unitary irreps of $H_{\mathcal{O}}$. Note that, modulo issues of infinite-dimensionality, $(\mathfrak{vira}^*)^{**} \sim \mathfrak{vira}^*$, and one of the orbits in $\mathfrak{vira}^*$ is just $\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$, so there exist representations given by wave functions on $\mathcal{Q}$, taking values in unitary irreps of the corresponding little group $\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}$ (where $\mathbb{R}$ is the central element of *Vira*). In particular, taking the trivial irrep of $\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}$ gives the usual Hilbert space of $\mathbb{C}$-valued wave functions on $\mathcal{Q}$, but it is worth noting that this is only one among a plethora of possibilities. Much as the quantization of a relativistic particle revealed the possibility of intrinsic spin, which is in fact realized in nature, perhaps the nontrivial representations of the little group $\mathrm{PSL}(2,\mathbb{R}) \times \mathbb{R}$ have physical significance for quantum gravity.

We can also think in terms of the representations of the algebra of G, $\mathfrak{g} = \mathfrak{vira}^c \Subset \mathfrak{vira}$, where $\mathfrak{vira}^c$ is the commutative algebra of momentum translations [which is isomorphic to $(\mathfrak{vira}^*)^* \sim \mathfrak{vira}$ as a vector space] and $\Subset$ denotes a semidirect sum, indicating that there is a nontrivial commutator between the two algebras. Note that $\mathfrak{vira}$ is a central extension of $\mathfrak{diff}(S^1)$ by $\mathbb{R}$, so its elements can be characterized by a vector field on S^1 plus a real number corresponding to the central direction. A convenient basis is defined by Fourier modes of the vector field, that is, $L_n = e^{in\theta}\partial_\theta$, with the central element denoted by R. Similarly, $\mathfrak{vira}^c$ is spanned by elements $A_n = e^{in\theta}\partial_\theta$ and the central element denoted by T. The algebra reads

$$\begin{aligned}
[L_n, L_m] &= i(n-m)L_{n+m} - 4\pi i n^3 \delta_{n+m,0} R,\\
[A_n, L_m] &= i(n-m)A_{n+m} - 4\pi i n^3 \delta_{n+m,0} T,\\
[A_n, A_m] &= 0,\\
[R, \cdot\,] &= 0,\\
[T, \cdot\,] &= 0,
\end{aligned} \tag{4}$$

where $n, m \in \mathbb{Z}$.[6] We reiterate that the L's and R are associated with the "configuration translations" (i.e., the K action), and the A's and T with the "momentum translations" (i.e., the V^* action), but note that R and T act trivially on the phase space. We find that this algebra can be realized by Poisson brackets on the phase space with a suitable choice of the charges P_n and Q_n corresponding to L_n and A_n, respectively, provided that the central charges R and T are realized by the constant functions 0 and 1, respectively. (This choice of charges is discussed in Sec. IV.) The resulting Poisson algebra is

$$\begin{aligned}
\{P_n, P_m\} &= i(n-m)P_{n+m},\\
\{Q_n, P_m\} &= i(n-m)Q_{n+m} - 4\pi i n^3 \delta_{n+m,0,}\\
\{Q_n, Q_m\} &= 0.
\end{aligned} \tag{5}$$

This is a centrally extended $\mathfrak{bms}_3$ algebra [40]. Finally, quantization amounts to associating operators $\widehat{P}_n$ and $\widehat{Q}_n$ to P_n and Q_n, respectively, and replacing $\{\,,\}$ by $\frac{1}{i\hbar}[\,,]$,

$$\begin{aligned}
[\widehat{P}_n, \widehat{P}_m] &= \hbar(m-n)\widehat{P}_{n+m},\\
[\widehat{Q}_n, \widehat{P}_m] &= \hbar(m-n)\widehat{Q}_{n+m} + 4\pi\hbar n^3 \delta_{n+m,0},\\
[\widehat{Q}_n, \widehat{Q}_m] &= 0.
\end{aligned} \tag{6}$$

The classical charges are not real and instead satisfy $(P_n)^* = P_{-n}$ and $(Q_n)^* = Q_{-n}$, so their associated operators must satisfy analogous adjoint relations, $(\widehat{P}_n)^\dagger = \widehat{P}_{-n}$ and $(\widehat{Q}_n)^\dagger = \widehat{Q}_{-n}$. Some aspects of the representation theory of this algebra have been studied recently [41,44–47].

Note that (6) corresponds to a representation of (4) in which the quantum Casimir operators $\widehat{T}$ and $\widehat{R}$ match the classical values of 1 and 0, respectively. In the Mackey construction of induced representations of $(\mathfrak{vira}^*)^* \rtimes Vira$ we must therefore select an orbit on which $\widehat{T}$ is represented as the identity and the central $\mathbb{R}$ factor in the little group is represented trivially. The natural $\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$ orbit is suitable for that purpose [26], in which case the wave functions transform under a representation of $\mathrm{PSL}(2,\mathbb{R})$.

IV. CONFORMAL COORDINATES AND THE CANONICAL CHARGES

In this section we briefly introduce the *conformal coordinates* which allow us to carry out the reduction process in an explicit fashion, providing the map between the geometrical variables (e.g., spatial metric and extrinsic curvature) and the abstract gauge-invariant variables describing the reduced phase space. Such a map is relevant in understanding the physical/geometrical meaning of observables like the Q and P charges. A treatment including all details is given in [26]. *This section is somewhat technical and can be skipped on a first read.*

By virtue of the uniformization theorem, any Riemannian metric h_{ab} on the disc D can be obtained from a reference metric $\bar{h}_{ab}$ via some conformal transformation. That is, there exists an (orientation-preserving) diffeomorphism $\Psi\colon D \to D$ and a positive scalar $\Omega\colon D \to \mathbb{R}^+$ such that $h_{ab} = \Psi_* \Omega \bar{h}_{ab}$. Because of the boundary condition on h, $h|_{\partial D} = \gamma$, the boundary value of Ω is determined from the boundary action $\psi \coloneqq \Psi|_{\partial D}$ via $\Omega\bar{h}|_{\partial D} = \psi_*^{-1}\gamma$. We shall choose the reference disc to be the unit Euclidean disc, so $\bar{h} = dr^2 + r^2 d\theta^2$ in the usual polar coordinates, and choose θ without loss of generality so as to satisfy $\gamma = (\ell/2\pi)^2 d\theta^2$. Note that, given h, Ψ is determined only up to a $\mathrm{PSL}(2,\mathbb{R})$

[6]Note that the Lie algebra bracket for the diffeomophism group is the negative of the Lie bracket of the corresponding vector fields on the manifold.

ambiguity since the transformation can be composed from the right with a conformal isometry of the reference disc, i.e., if $\Phi_* \Theta \bar{h} = \bar{h}$ then $(\Psi, \Omega) \circ (\Phi, \Theta) = (\Psi \circ \Phi, \Phi^* \Omega \Theta)$ also maps $\bar{h}$ to h. (We are introducing additional gauge in the description, which is fine since it will be all removed in the end.) We define the "pull-back" of σ^{ab} to the reference disc by $\bar{\sigma}^{ab} \coloneqq \Omega^2 \Psi_*^{-1} \sigma^{ab}$, which implies that $\bar{\sigma}^{ab}$ is symmetric, traceless and divergenceless with respect to $\bar{h}$ if and only if σ^{ab} has the same properties with respect to h. So far we have a "change of coordinates" from (h_{ab}, σ^{ab}) to $(\Psi, \Omega, \bar{\sigma}^{ab})$. Imposing the scalar constraint leads to a Lichnerowicz equation for Ω, and the boundary value of Ω is determined from ψ (and γ); since that equation has a unique solution for Ω, given ψ and $\bar{\sigma}^{ab}$, the constraint surface in phase space can be parametrized by $(\Psi, \bar{\sigma}^{ab})$, where $\bar{\sigma}^{ab}$ is symmetric, traceless and divergenceless with respect to $\bar{h}$. This space of $\bar{\sigma}$'s is isomorphic to a subspace of dual vector fields $\widehat{\sigma}$ on the boundary S^1; given the form of the symplectic structure, it is natural to realize the isomorphism as $\widehat{\sigma}(\xi) \coloneqq \int d\theta \widehat{\sigma}(\theta) \xi(\theta) \coloneqq -2 \int d\theta \bar{\sigma}^{ab} n_a \xi_b$, where $\xi = \xi(\theta) \partial_\theta$ is a vector field on the boundary and n is the unit outward-pointing normal vector field on the boundary. In this realization of the isomorphism, the space of $\widehat{\sigma}$'s is missing the Fourier modes 1, $\sin\theta$, $\cos\theta$, since they annihilate the vector fields $\xi(\theta) = 1, \sin\theta, \cos\theta$. Via this isomorphism, the constraint surface can be parametrized by $(\Psi, \widehat{\sigma})$. It is clear from the presymplectic form that any two Ψ's with the same boundary action ψ are gauge-equivalent, so we can quotient out the bulk diffeomorphisms and obtain a partially reduced phase space coordinatized by $(\psi, \widehat{\sigma})$. By further inspection of the symplectic form one discovers that there remains a $\mathrm{PSL}(2, \mathbb{R})$ group of gauge transformations, which acts on ψ from the right and on $\widehat{\sigma}$ via the coadjoint action. The quotient under this group finally leads to the reduced phase space $T^*[\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2, \mathbb{R})]$.

The canonical charges can be explicitly expressed in terms of the $(\psi, \widehat{\sigma})$ variables. (Only the results are presented here; the derivation can be found in [26].) As the canonical group acts on the phase space, each element ζ of the Lie algebra induces a vector field X_ζ on the phase space; the a corresponding Hamiltonian charge H_ζ is a solution of $\delta H_\zeta = -i_{X_\zeta} \omega$, where δ denotes the exterior derivative on phase space and $i_X \omega$ is the insertion of X into the first slot of the symplectic form ω. The "momentum" (P) charges are associated with the *Vira* part of the group, acting as configuration space "translations," therefore corresponding to algebra elements purely in the $\mathfrak{vira}$ factor of $\mathfrak{g} = \mathfrak{vira}^c \Subset \mathfrak{vira}$. If $\widehat{\xi} = (\xi(\theta)\partial_\theta, \xi_0) \in \mathfrak{vira}$, where ξ_0 is the central component, then

$$P_{\hat{\xi}}(\psi, \widehat{\sigma}) = \int d\theta \frac{\widehat{\sigma}(\theta)}{\psi'(\theta)} \xi(\psi(\theta)). \tag{7}$$

In the earlier notation, $P_n \coloneqq P_{\hat{\xi} = (e^{in\theta}\partial_\theta, 0)}$, and the central charge $R \coloneqq P_{\hat{\xi} = (0,1)} = 0$. The "position" ($Q$) charges are associated with the $(\mathfrak{vira}^*)^*$ part of the group, acting as "vertical translations" on phase space, thus corresponding to algebra elements purely in the $\mathfrak{vira}^c$ factor of $\mathfrak{g}$. If $\widehat{\eta} = (\eta(\theta)\partial_\theta, \eta_0) \in \mathfrak{vira}^c$, where η_0 is the central component, then

$$Q_{\hat{\eta}}(\psi, \widehat{\sigma}) = \int d\theta \frac{1 - 2S[\psi](\theta)}{\psi'(\theta)} \eta(\psi(\theta)) + \eta_0, \tag{8}$$

where $S[\psi](\theta) \coloneqq \psi'''(\theta)/\psi'(\theta) - \frac{3}{2}(\psi''(\theta)/\psi'(\theta))^2$ is the Schwarzian derivative of ψ. In the earlier notation, $Q_n \coloneqq Q_{\hat{\eta} = (e^{in\theta}\partial_\theta, 0)}$; and the central charge $T \coloneqq Q_{\hat{\eta} = (0,1)} = 1$.

It is straightforward to express the P_ξ charges in terms of the physical spatial metric and extrinsic curvature. This can be done by direct manipulation of expression (7), basically by reversing the map from the reference disc variables $(\bar{h}_{ab}, \bar{\sigma}^{ab})$ to the physical disc variables (h_{ab}, σ^{ab}) so as to express $(\psi, \widehat{\sigma})$ in terms of (h_{ab}, σ^{ab}). Instead of going through this formal derivation (which can be found in [26]), we can infer the answer by noticing that the charge must descend from a function on the unreduced phase space that generates a corresponding diffeomorphism on the spatial slice. We know that this charge must be related to $\int d^2x \pi^{ab} \pounds_\xi h_{ab}$, where ξ is now an arbitrary extension of the boundary vector field to the disc. However this function alone generates a pure diffeomorphism on the ADM phase space and thus does not generally respect the boundary conditions on the induced metric (unless ξ is an isometry of the boundary metric). That can be fixed by adding a constraint term which generates a compensating Weyl transformation. The appropriate constraint here comes from the gauge fixing of time $\tau = -K$, that is, $P_\xi = -\int d^2x \pi^{ab} \pounds_\xi h_{ab} + \int d^2x \sqrt{h} \zeta (K + \tau)$ for some scalar ζ. When this expression is evaluated imposing the CMC gauge condition and the vector constraint $\nabla_a \pi^{ab} = 0$, it reduces to $P_\xi = -2 \int d^2x \sqrt{h} \sigma^{ab} \nabla_a \xi_b$. This inferred form can be shown to agree with the pull-back to the (constrained, gauge-fixed) ADM phase space of the P_ξ's defined in (7). Using Stokes' theorem we get $P_\xi = -2 \int_\partial ds K_{ab} n^a \xi^b$, where n is the unit outward-pointing normal vector field at the boundary of the disc, and ds is the proper length along the boundary. Restoring the factor of $16\pi G$ that had previously been set to unity, this becomes $P_\xi = -\frac{1}{8\pi G} \int_\partial ds K_{ab} n^a \xi^b$. The vector field ξ that labels the charge P_0 is ∂_θ on the reference disk. In terms of the vector field t^a tangent to the boundary, with unit norm with respect to the physical metric γ, we have on the boundary $\partial_\theta = \frac{\ell}{2\pi} t^a$, hence $P_0 = -\frac{\ell}{16\pi^2 \ell_P} \int_\partial ds K_{ab} n^a t^b$. If u is the unit future-pointing vector field normal to the CMC slice, then $P_0 = -\frac{\ell}{16\pi^2 \ell_P} \int_\partial ds \boldsymbol{\nabla}_b u_a n^a t^b$. Integrating by parts we conclude that $P_0 = \frac{\ell}{16\pi^2 \ell_P} \int_\partial ds u_a t^b \boldsymbol{\nabla}_b n^a = \frac{\ell}{16\pi^2 \ell_P} \mathcal{T}$, where $\mathcal{T}$ is the twist of the boundary loop, as embedded in the spacetime, which is defined as the integral of the torsion $u_a t^b \nabla_b n^a$ with respect to proper length.

Regarding the appearance of the Schwarzian in the expression (8) for the $Q_{\hat{\eta}}$ charges we offer here a brief explanation. When the configuration space is embedded as a coadjoint orbit in $\mathfrak{vira}^*$, each point $x \in \mathcal{Q}$ corresponds to an element of $\mathfrak{vira}^*$. In this context, the charge $Q_{\hat{\eta}}$ evaluated at x is the value of the dual vector $x \in \mathfrak{vira}^*$ acting on the vector $\hat{\eta} \in \mathfrak{vira}$, i.e., $Q_{\hat{\eta}}(x) = x(\hat{\eta})$. The point x is labeled by a diffeomorphism ψ, relative to a reference point $x_0 \in \mathcal{Q}$, via the coadjoint action $x = \mathrm{coad}_\psi x_0$. (Of course this labeling system is not one-to-one because x_0 is invariant under a $\mathrm{PSL}(2,\mathbb{R})$ subgroup of $\mathrm{Diff}^+(S^1)$.) This yields the expression $Q_{\hat{\eta}}(x) = \mathrm{coad}_\psi x_0(\hat{\eta})$, which for a simple choice of x_0 corresponds to (8). The Schwarzian appears in this expression because it figures in the coadjoint action.

Note that the Q's do not depend on $\hat{\sigma}$ and, as can be shown from basic properties of the Schwarzian derivative, depend only on the right $\mathrm{PSL}(2,\mathbb{R})$ equivalence classes $[\psi] \in \mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$. A given spatial metric h uniquely determines one such equivalence classes $[\psi]$, and one class $[\psi]$ determines a spatial metric up to boundary-trivial conformal transformations, $[h] = [\Phi_* \Theta h]$, where $\Phi \in \mathrm{Diff}^+(D)$ acts as the identity on the boundary and the function Θ is 1 at the boundary. Therefore, the Q charges evidently depend only on the conformal class of the spatial metric.

It can be shown that Q_0 is bounded from above, attaining a maximum value of 2π when $[\psi] = [I]$ [26,41,48]. In that configuration, $Q_{\hat{\eta}} = \int d\theta \eta(\theta) + \eta_0$, hence all Q_n with $n \neq 0$ vanish. Classically it corresponds to a spatial geometry that is related to the round disc by a boundary-trivial conformal transformation.

V. SPIN/TWIST

An interesting observable to discuss in more detail is P_0. It is the "zero Fourier mode" of $\mathrm{Diff}^+(S^1) \subset Vira$, i.e., it generates the $SO(2)$ subgroup of rotations, suggesting that it corresponds to the *spin* of the diamond. This interpretation can be further strengthened by noticing that it is precisely (minus) the on-shell value of the ADM charge associated with a vanishing lapse and a shift that acts as an isometry of the boundary loop. The charge P_0 generates not only a symmetry of the symplectic form (as do all of the P's and Q's), but also a true dynamical symmetry. That is, it commutes with the CMC time evolution Hamiltonian [defined below (3)], $[P_0, \tilde{H}] = 0$, as will become clear presently. We have argued that the physical states correspond (classically) to shapes of diamonds embedded in AdS_3 (or Mink_3 if $\Lambda = 0$), with boundary length ℓ, so P_0 must correspond to some aspect of the shape. As shown in Sec. IV, it turns out that P_0 is proportional to the *twist* $\mathcal{T}$ of the diamond boundary loop, i.e., the loop integral (with respect to proper length) of the torsion of the curve (as embedded in the spacetime). The twist can also be interpreted as the holonomy of Fermi-Walker transport of an orthogonal frame around the loop, i.e., the (hyperbolic) angle of the boost relating the final frame to the initial one. The precise relation (which is obtained using the previously mentioned "conformal coordinates" characterization of the reduced phase space) is

$$P_0 = \frac{\ell}{16\pi^2 G}\mathcal{T}. \tag{9}$$

Note that the twist of the boundary is clearly independent of the CMC slice of the diamond, hence it is time independent and thus commutes with $\tilde{H}$ as stated above.[7]

At the quantum level, note that the Poisson brackets (5) imply $[\hat{P}_0, \hat{P}_n] = n\hbar\hat{P}_n$ and $[\hat{P}_0, \hat{Q}_n] = n\hbar\hat{Q}_n$, so the P's and Q's act as ladder operators for $\hat{P}_0$. That is, if $|s\rangle$ is an eigenvector of $\hat{P}_0$ with eigenvalue $s\hbar$, then $\hat{P}_n|s\rangle$ and $\hat{Q}_n|s\rangle$ have eigenvalue $(s+n)\hbar$. Since the P's and Q's are represented irreducibly in the Hilbert space, the spectrum of $\hat{P}_0$ is $\{(s+n)\hbar,\ \forall\ n \in \mathbb{Z}\}$, where without loss of generality we can take $s \in [0,1)$. Classically, τ-time reversal flips the sign of P_0; if this (antisymplectic) symmetry of the phase space is represented by an antiunitary transformation in the quantum theory—as one might expect given that the classical Hamiltonian is invariant under this symmetry—then in particular the spectrum of $\hat{P}_0$ will be symmetric under sign reversal. In this case, only $s = 0$ and $s = \frac{1}{2}$ are allowed. From formula (9) we conclude that the twist is quantized as

$$\mathcal{T} = \frac{16\pi^2 \ell_P}{\ell}(s+n), \quad n \in \mathbb{Z}, \tag{10}$$

where (in 3d) $\ell_P = \hbar G$ is the Planck length, in units with $c = 1$. In the classical limit $\ell \gg \ell_P$, the twist quantum is very small, so that a continuum of twist values is recovered.

VI. DISCUSSION

We studied quantization of causal diamonds of fixed boundary length in pure $(2+1)$-dimensional general relativity gravity with a nonpositive cosmological constant, via the reduced phase space approach. The low dimensionality allowed us to solve the constraints exactly and remove

[7]This relationship between twist and spin seems to be related to a result in [49]. Working in an extended phase space including edge modes in $3+1$ spacetime dimensions, they find that the generator of volume-preserving diffeomorphisms of the "corner," S^2, is essentially the curvature of the natural connection on the normal bundle of S^2 (as embedded in the ambient spacetime). In our case the corner is the boundary loop, S^1; volume-preserving diffeomorphisms are just the isometries of the boundary metric; and, although the curvature of the normal bundle connection vanishes (because S^1 is 1-dimensional), there is a nontrivial holonomy (around the loop) which is equal to the twist.

all the gauge ambiguities, resulting in the phase space $T^*\mathcal{Q}$ with $\mathcal{Q} = \mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$. Further, this phase space could be quantized exactly, at the kinematical level, with all the rigor and generality of Isham's group quantization scheme. We ended up with a classification of all possible quantizations based on irreducible unitary representations of the $\mathfrak{bms}_3$ algebra. This differs from the canonical quantization of pure asymptotically AdS_3 (with trivial topology) based on the group $Vira \times Vira$ [20], whose algebra is $\mathfrak{vira} \oplus \mathfrak{vira}$. Note that although the quantization groups differ the phase spaces are the same, since $T^*\mathcal{Q} \sim \mathcal{Q} \times \mathcal{Q}$ [21,50].

The quantization was strictly kinematical because only the canonical charges Q_n and P_n ("coordinates" on phase space) have been quantized. This was sufficient to reveal that the spin of the diamond, or equivalently the twist of the diamond boundary loop, is quantized in integer or half-integer multiples of $16\pi^2\ell_P/\ell$. To fully characterize the quantum theory one must also represent the Hamiltonian $\widetilde{H}$ that generates evolution in CMC time. This Hamiltonian is, however, a very complicated function on the reduced phase space, for which we have not found any preferred operator ordering or even any explicit expression in terms of the canonical charges. It may be that progress could be made using a perturbative approach. There are certain regimes where the Hamiltonian simplifies, even as much as becoming "free" (quadratic in P_n) in the limit $\ell \gg |\Lambda|^{-1/2}$ when the maximal slice is nearly a hyperbolic disc. This includes the case where the boundary loop approaches the boundary of AdS, in which the diamond approaches a "Wheeler-DeWitt patch" of AdS. It would be interesting to explore such regimes, and in particular the possible connection to quantization of the diamond from the perspective of AdS/CFT duality (and its TTbar deformations).

Another important open question is the geometrical meaning of the charges Q_n. Unlike the P_n, which have a simple interpretation as Fourier components of the torsion of the boundary curve, the Q_n are related to the shape of the diamond in a complicated, implicit fashion. As explained in Sec. IV we know that the Q charges depend only on the configuration space variables $[\psi] \in \mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$, which implies that they depend only on the conformal class of the spacial metric, $[h_{ab}]$, where two metrics are identified if they can be related by a conformal transformation that is trivial on the boundary. But, despite some effort, we have not yet been able to express Q_n directly in terms of the spatial conformal metric. In the asymptotically flat case, whose group of symmetries at null infinity is also BMS_3, $-Q_0$ plays the role of energy (i.e., the generator of u-coordinate translations, up to a scaling factor), so by analogy this suggests that $-Q_0$ should be some sort of quasilocal mass. In fact, it is noteworthy that for many representations of $(\mathfrak{vira}^*)^* \rtimes \mathrm{Vira}$, including the one associated with the orbit $\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$ with $T = 1$, $-Q_0$ is bounded from below and unbounded from above [41,48]. In the case of the $\mathrm{Diff}^+(S^1)/\mathrm{PSL}(2,\mathbb{R})$ orbit the minimum value of $-Q_0$ is equal to -2π, and it is attained by a (non-normalizable) state corresponding to a wave function localized at $[\psi] = [I]$, i.e., at the spatial geometry conformal to a flat round disc.

One would also like to understand what is the nature of a "quantum causal diamond," given that the classical "spacetime shape" interpretation, which requires that Q_n and P_n are all simultaneously specified, fails to make sense in the quantum theory. (We note that there are certain observables that do commute among themselves, such as the set including P_0, Q_0 and any operators of the form $Q_{n_1}Q_{n_2}\cdots$ such that $n_1 + n_2 + \cdots = 0$; some of these operators are actually self-adjoint, like $Q_{-n}Q_n$ for all n.) A perhaps related question is whether the quantized theory depends upon the CMC time gauge choice used for the phase space reduction. Finally, it might be interesting to analyse the system using the formulation of this gravity theory as a pair of $SL(2,R)$ Chern-Simons theories [4,51]. The fixed metric boundary condition that we have imposed would be a complicated condition that couples those two theories.

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