Extremal black holes have external light rings

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It is proved that spherically symmetric extremal black holes possess at least one external light ring. Our remarkably compact proof is based on the dominant energy condition which characterizes the external matter fields in the nonvacuum asymptotically flat extremal black-hole spacetimes.

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I. INTRODUCTION

The nonlinearly coupled Einstein-matter field equations predict, under plausible physical assumptions [1–5], that curved spacetimes of highly compact objects are characterized by the presence of closed null circular trajectories (external light rings). The fact that massless particles can perform orbital motions along closed circular geodesics plays a key role in understanding many of the fundamental physical properties of the corresponding central compact objects [1–19].

In particular, the intriguing physical phenomenon of strong gravitational lensing, which can be used as an important observational tool to identify the existence of cosmological black holes, is a direct outcome of the presence of closed null geodesics in the highly curved near-horizon regions of the corresponding black-hole spacetimes [6–8]. In addition, it is well established (see [9–12] and references therein) that the relaxation rates of perturbed black-hole spacetimes are closely related to the instability timescales that characterize the geodesic motions of massless particles along the null circular geodesics of the corresponding curved spacetimes.

Interestingly, it has been proved [13,14] that, as judged by far away asymptotic observers, the innermost null circular geodesic of a black-hole spacetime provides the fastest way to travel around the central black hole. In addition, it has been revealed [4,11,15,16] that, in spherically symmetric hairy (nonvacuum) black-hole spacetimes, the effective lengths of the external matter fields are bounded from below by the radii of the innermost null circular geodesics that characterize the corresponding curved spacetimes.

The fact that null circular geodesics have a significant role in determining many of the fundamental physical properties of black-hole spacetimes naturally raises the following important question: Do the Einstein-matter field equations guarantee the existence of external null circular trajectories (light rings) in all black-hole spacetimes? Intriguingly, the existence of closed null circular geodesics in the external regions of asymptotically flat *nonextremal* spherically symmetric black-hole spacetimes has been proved in [4]. A general (and mathematically elegant) proof for the existence of null circular geodesics in *nonextremal* stationary axisymmetric black-hole spacetimes has recently been presented in the physically interesting work [5].

It is important to point out that the theorems presented in [4,5] for the existence of external null circular geodesics in generic black-hole spacetimes seem to fail for *extremal* hairy (nonvacumm) black-hole configurations. Motivated by this fact, it has recently been proved [20] that extremal black-hole spacetimes with positive tangential pressures on their horizons [$p_{\mathrm{T}}(r=r_{\mathrm{H}})>0$, see Eq. (2) below] possess external light rings. However, as emphasized in [20], the existence theorem presented in [20] is not valid for extremal black holes with nonpositive horizon tangential pressures. This fact leaves open the possibility of finding extremal black-hole spacetimes with nonpositive horizon tangential pressures that do not have external light rings.

The main goal of the present compact paper is to complete our knowledge about the (in)existence of external null circular geodesics in extremal black-hole spacetimes. In particular, using analytical techniques, we shall explicitly prove below that the nonlinearly coupled Einstein-matter field equations guarantee that asymptotically flat spherically symmetric hairy (nonvacuum) extremal black-hole spacetimes whose external matter fields respect the dominant energy condition are always characterized by the presence of external null circular geodesics (closed light rings).

II. DESCRIPTION OF THE SYSTEM

We consider spherically symmetric extremal black-hole spacetimes which, using the familiar Schwarzschild spacetime coordinates $\{t,r,\theta,\phi\}$, are characterized by the curved line element [13,19,21]

$$ds^2=-e^{-2\delta}\mu dt^2+\mu^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2), \qquad (1)$$

where the radially-dependent functions $\mu = \mu(r)$ and $\delta = \delta(r)$ are determined by the matter content of the nonvacuum spacetime.

Using the line element (1) and the notations [22]

$$\rho \equiv -T^t_t, \qquad p \equiv T^r_r, \qquad p_T \equiv T^\theta_\theta = T^\phi_\phi \tag{2}$$

for the radially-dependent energy density ρ, radial pressure p, and tangential pressure p_T of the external static matter configurations, one finds that the Einstein-matter field equations $G^\mu_\nu = 8\pi T^\mu_\nu$ yield the radial differential equations [13,19]

$$\frac{d\mu}{dr} = -8\pi r\rho + \frac{1-\mu}{r} \tag{3}$$

and

$$\frac{d\delta}{dr} = -\frac{4\pi r(\rho + p)}{\mu}, \tag{4}$$

which relate the metric functions to the external matter sources.

Extremal black-hole spacetimes are characterized by the horizon boundary conditions [23]:

$$\mu(r = r_{\rm H}) = 0, \tag{5}$$

$$\left[\frac{d\mu}{dr}\right]_{r=r_{\rm H}} = 0, \tag{6}$$

$$\left[\frac{d^2\mu}{dr^2}\right]_{r=r_{\rm H}} > 0, \tag{7}$$

$$\delta(r = r_{\rm H}) < \infty; \qquad \left[\frac{d\delta}{dr}\right]_{r=r_{\rm H}} < \infty, \tag{8}$$

and

$$p(r = r_{\rm H}) = -\rho(r = r_{\rm H}) = -(8\pi r^2_{\rm H})^{-1}. \tag{9}$$

In addition, the radially-dependent metric functions of asymptotically flat black-hole spacetimes are characterized by the functional relations

$$\mu(r \to \infty) \to 1 \tag{10}$$

and

$$\delta(r \to \infty) \to 0. \tag{11}$$

Taking cognizance of the Einstein equation (3), one finds the expression

$$\mu(r) = 1 - \frac{2m(r)}{r} \tag{12}$$

for the dimensionless metric function $\mu(r)$, where

$$m(r) = \frac{r_{\rm H}}{2} + \int_{r_{\rm H}}^{r} 4\pi r^2 \rho(r) dr \tag{13}$$

is the gravitational mass contained within a sphere of radius r [here $m(r = r_{\rm H}) = r_{\rm H}/2$ is the mass contained within the black-hole horizon]. Taking cognizance of Eqs. (10), (12), and (13), one deduces the characteristic functional relation

$$r^3\rho(r) \to 0 \quad \text{for } r \to \infty \tag{14}$$

for the external energy density in asymptotically flat black-hole spacetimes.

Our proof, to be presented below, for the existence of external null circular geodesics in extremal black-hole spacetimes is based on the well known dominant energy condition which, for a given density profile of the matter fields, bounds from above the (radial and tangential) pressure components of the corresponding matter distribution [23]:

$$|p|, |p_T| \leq \rho. \tag{15}$$

III. THE PROOF FOR THE EXISTENCE OF EXTERNAL NULL CIRCULAR GEODESICS IN EXTREMAL BLACK-HOLE SPACETIMES

In the present section we shall explicitly prove that extremal hairy (nonvacuum) black-hole spacetimes that respect the dominant energy condition (15) necessarily possess at least one external light ring whose radius satisfies the inequality $r_\gamma > r_{\rm H}$.

To this end, we shall analyze the radial functional behavior of the dimensionless function

$$\mathcal{N}(r) \equiv 3\mu - 1 - 8\pi r^2 p \tag{16}$$

in the extremal black-hole spacetime (1). It has been explicitly proved [15] that, in spherically symmetric black-hole spacetimes, the radii of null circular geodesics are determined by the mathematically compact relation

$$\mathcal{N}(r = r_\gamma) = 0. \tag{17}$$

Taking cognizance of Eqs. (5), (9), and (16), one deduces the boundary relation

$$\mathcal{N}(r = r_{\rm H}) = 0 \tag{18}$$

on the outer horizon of the extremal black hole. In addition, the functional relation (14), which characterizes asymptotically flat black-hole spacetimes, together with the assumed dominant energy condition (15), imply the asymptotic functional behavior

$$r^3 p(r) \to 0 \quad \text{for } r \to \infty \tag{19}$$

for the external radial pressure. From Eqs. (10) and (19) one finds the characteristic large-r behavior

$$\mathcal{N}(r \to \infty) \to 2 \tag{20}$$

of the dimensionless radial function (16).

Defining the dimensionless function

$$\mathcal{F} \equiv r \frac{d\mu}{dr} \tag{21}$$

and using Eqs. (6) and (7), one finds the characteristic relation

$$\left[\frac{d\mathcal{F}}{dr}\right]_{r=r_{\rm H}} = r_{\rm H} \left[\frac{d^2\mu}{dr^2}\right]_{r=r_{\rm H}} > 0 \tag{22}$$

for extremal black holes. In addition, from the Einstein equation (3) and the boundary condition (6) one obtains the horizon relation

$$\left[\frac{d\mathcal{F}}{dr}\right]_{r=r_{\rm H}} = -\frac{d}{dr}[8\pi r^2 \rho]_{r=r_{\rm H}} \tag{23}$$

for the extremal black-hole spacetime (1). From Eqs. (22) and (23) one deduces that the dimensionless function $r^2\rho$ decreases in the vicinity of the extremal black-hole horizon:

$$\left[\frac{d(r^2\rho)}{dr}\right]_{r=r_{\rm H}} < 0. \tag{24}$$

Taking cognizance of the horizon boundary relation (9) and the assumed dominant energy condition (15), one deduces from (24) that the radial expression $r^2 p$ is a negative increasing function in the vicinity of the black-hole horizon:

$$[r^2 p]_{r=r_{\rm H}} < 0 \quad \text{and} \quad \left[\frac{d(r^2 p)}{dr}\right]_{r=r_{\rm H}} > 0. \tag{25}$$

From the analytically derived functional relation (25) and the horizon boundary condition (6) for extremal black holes, one finds the characteristic inequality [see Eq. (16)]

$$\left[\frac{d\mathcal{N}}{dr}\right]_{r=r_{\rm H}} = -8\pi \left[\frac{d(r^2 p)}{dr}\right]_{r=r_{\rm H}} < 0, \tag{26}$$

which, together with the relation (18), imply that the dimensionless radial function (16) is *nonpositive* in the vicinity of the black-hole horizon. In particular, the radial function $\mathcal{N}(r)$ is characterized by the near-horizon property:

$$\mathcal{N}(r/r_{\rm H} \to 1^+) \to 0^-. \tag{27}$$

Finally taking cognizance of the analytically derived near-horizon relation (27) and the characteristic asymptotic behavior (20) of the dimensionless radial function (16), one deduces that spherically symmetric extremal black-hole spacetimes whose external matter fields respect the dominant energy condition (15) possess at least one external null circular geodesic (closed light ring) which is characterized by the functional relation

$$\mathcal{N}(r = r_\gamma) = 0 \quad \text{with} \quad r_\gamma > r_{\rm H}. \tag{28}$$

IV. SUMMARY

Null circular geodesics play important roles in fundamental as well as observational studies of the physics of curved black-hole spacetimes (see [1–18] and references therein). Interestingly, the existence of closed light rings in asymptotically flat *nonextremal* black-hole spacetimes has been proved, using the Einstein-matter field equations, in [4] for spherically symmetric nonvacuum (hairy) black-hole configurations. A mathematically elegant proof for the existence of external null circular geodesics in *nonextremal* stationary axisymmetric black-hole spacetimes has been provided in the highly important work [5].

Intriguingly, the existence theorems presented in [4,5] seem to fail for *extremal* black-hole spacetimes. Motivated by this observation, we have raised the following physically important question [20]: Do extremal black-hole spacetimes always possess external light rings?

In the present paper we have presented a remarkably compact theorem, which is based on the nonlinearly coupled Einstein-matter field equations, that reveals the physically important fact that spherically symmetric asymptotically flat extremal hairy (nonvacuum) black holes whose external matter fields respect the dominant energy condition necessarily possess at least one external light ring (closed null circular geodesic).

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