Integrable teleparallel gravity in the presence of boundaries

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Symmetric teleparallel gravity is shown to be integrable in the presence of boundaries, given the consistent implementation of constraints in the covariant phase space formalism.

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I. INTRODUCTION

Teleparallelism is defined by $\mathbf{D}^2 = 0$, where we may consider $\mathbf{D}$ as the exterior covariant derivative of any general linear group $\mathrm{GL}(n)$ in any n dimensions. It has been argued that spacetime is *a priori* described by such a trivial algebra [1] and that the resulting theory of gravity improves general relativity (GR) by introducing a principle of relativity with the properly defined (equivalence classes of) inertial frames [2]. Besides the freedom to use arbitrary coordinates, the dynamics of the theory are equivalent in arbitrary *gravitational frames*.[1] In particular, in an inertial frame [3], the field equations read $\mathbf{D}\boldsymbol{h}_a \hat{=} \boldsymbol{t}^M_a$, where $\boldsymbol{h}_a$ is the excitation and $\boldsymbol{t}^M_a$ is the material energy current [4]. Recently, $\boldsymbol{h}_0$ was found to emerge as the Noether-Wald potential of the theory[2] [6–8]. However, the relation of $\oint \boldsymbol{h}_0$ to the Hamiltonian charge was not yet rigorously clarified.

A related issue is the integrability of the charges and, more generally, the consistency of the action principle in the presence of boundaries. This is the source of major technical complications in the conventional formulations of GR [9]. To wit, the Einstein field equations do not guarantee the stationarity of the Einstein-Hilbert action. Therefore, the action is amended, without clear physical interpretation, by a somewhat ambiguous piece which is confined to the boundary hypersurface. The issue is not avoided simply by turning to a first-order formulation, since the on-shell action should be stationary also when variations may not vanish at the boundary. If the density $\boldsymbol{L} = 0$ at the boundary, the variation $\delta \int \boldsymbol{L}$ is unproblematical. However, the conventional method is to cancel some of the unwanted terms by inserting a surface term to the action and to eliminate the remaining unwanted terms by restricting only to transformations adapted to the particular boundary geometry [10,11]. The possible new method we propose is based on the frame dependence of the density $\boldsymbol{L}$ in symmetric teleparallelism. We may adjust the frame according to quite arbitrary boundaries, such that the bounded action is manifestly differentiable with respect to arbitrary transformations.

In Sec. II, some basics of the covariant phase space formalism are introduced and extended to incorporate the frame transformations. In Sec. III, the formalism is adapted to symmetric teleparallelism and applied to both the gauge and the frame transformations. A simple example is worked out explicitly in Sec. IV, and Sec. V is the brief conclusion.

II. COVARIANT PHASE SPACE

Let us be given some generic boundary $\mathcal{B}$. (For concreteness, say we have a manifold $\mathcal{M}$ whose boundary $\partial\mathcal{M} = \mathcal{I}^- \cup \mathcal{B} \cup \mathcal{I}^+$ consists of a spatial part $\mathcal{B}$ that we are interested in and some past and future boundaries $\mathcal{I}^\pm$.)

Consider an action

$$I = \int_{\mathcal{M}} \boldsymbol{L} + \int_{\partial\mathcal{M}} \boldsymbol{\ell}, \tag{1}$$

where $\boldsymbol{L}$ is a $(n, 0)$-form and $\boldsymbol{\ell}$ is a $(n-1,\ 0)$-form. The variation of $\boldsymbol{L}$ can always be written as

$$\delta\boldsymbol{L} = \boldsymbol{E}_A \delta\phi^A + \mathbf{d}\boldsymbol{\Theta}, \tag{2}$$

where $\boldsymbol{E}_A = 0$ are the equations of motion for the fields $\phi^A(x)$ and the $(n-1,\ 1)$-form $\boldsymbol{\Theta}$ is the presymplectic potential. Stationarity of the action $\delta I = 0$ (up to the future and past boundary terms) requires

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[1]The dynamical or *frame symmetry* requires only that the action I in Eq. (1) is a scalar, whereas the redundancy or *gauge symmetry* implies an $(n, 1)$-form identity obtained from Eq. (2) and stating the covariance of the density $\boldsymbol{L}$.

[2]And, furthermore, it was shown to be the local and covariant generalization (obtained by minimal coupling) of a variety of energy complexes, introduced by Bergmann-Thomson, von Freud, Landau-Lifshitz, Papapetrou, and Weinberg [5].

$$\boldsymbol{E}_A \delta\phi^A = 0, \tag{3a}$$

$$(\boldsymbol{\Theta} + \delta\boldsymbol{\ell})|_{\mathcal{B}} = \mathbf{d}\boldsymbol{C}, \tag{3b}$$

where $\boldsymbol{C}$ is a $(n-2,\,1)$-form (which we can arbitrarily choose due to the so-called $\boldsymbol{y}$ ambiguity) [12]. Then the presymplectic form Ω' is defined as an integral of the presymplectic $(n-1,\,2)$-current as

$$\Omega' = \int_{\mathcal{C}} \delta(\boldsymbol{\Theta} - \mathbf{d}\boldsymbol{C}), \tag{4}$$

over a Cauchy slice $\mathcal{C}$.

A symplectic manifold is defined as the pair of an abstract space $\mathcal{P}$ and a closed and nondegenerate symplectic form Ω. In a covariant Hamiltonian approach, one considers the phase space $\mathcal{P}$ of field configurations, such that the $T\mathcal{P}$ is spanned by the differentials $\delta\phi^A$. The idea is that the space of solutions (wherein one usually works in the noncovariant Dirac-Bergmann Hamiltonian formalism) should be isomorphic to the space of all valid initial data on a given $\mathcal{C}$ and can, therefore, be considered in a covariant fashion in terms of the $\mathcal{P}$. The space of stationary field configurations is called the prephase space $\mathcal{P}'$, and the phase space is obtained by quotienting by the gauge symmetry. It is for this reason the Ω' in Eq. (4) was called the presymplectic form. A nondegenerate Ω is obtained as its quotient.

However, here we intend only to suggest some first steps toward the covariant phase space analysis of teleparallel theories, and instead of Eq. (4) we focus on the stationarity conditions in Eq. (3).

A. Gauge transformation

Next, consider a $\mathrm{Diff}(n)$ generated by ξ. Define, for an arbitrary (i,j)-form $\boldsymbol{X}[\delta\phi,\ldots]$, the two contractions, the usual $\xi \lrcorner \boldsymbol{X}$, which results in an $(i-1,j)$-form, and $\xi\cdot\boldsymbol{X}[\delta\phi,\ldots] = \boldsymbol{X}[\pounds_\xi\phi,\ldots]$, which results in an $(i,j-1)$-form. The latter contraction can be understood in terms of the vector V_ξ defined in the configuration space such that

$$V_\xi = \int \mathbf{d}^n x \pounds_\xi \phi^A(x)\frac{\delta}{\delta\phi^A} \Rightarrow \mathbf{d}(\xi\cdot\boldsymbol{X}) = \xi\cdot\mathbf{d}\boldsymbol{X},$$
$$\delta(\xi\lrcorner\boldsymbol{X}) = \xi\lrcorner\delta\boldsymbol{X}, \quad \delta_\xi\delta\phi^A(x) = \delta(\delta_\xi\phi(x)). \tag{5}$$

In this notation, the familiar Noether current $(n-1,\,0)$-form can be expressed as

$$\boldsymbol{J}_\xi = \xi\cdot\boldsymbol{\Theta} - \xi\lrcorner\boldsymbol{L}. \tag{6}$$

A central result in the covariant phase space formalism is the formula for the Hamiltonian [12] (up to an irrelevant constant) that generates the family of Diffs:

$$H_\xi = \int_{\partial\mathcal{C}} (\boldsymbol{j}_\xi + \xi\lrcorner\boldsymbol{\ell} - \xi\cdot\boldsymbol{C}), \tag{7}$$

where the $(n-2,\,0)$-form $\boldsymbol{j}_\xi$ is the Noether-Wald potential which can always be locally found such that $\boldsymbol{J}_\xi = \mathbf{d}\boldsymbol{j}_\xi$ [6]. Using Eq. (3b), one can check that the Hamiltonian is independent of the choice of $\mathcal{C}$.

The $\mathrm{Diff}(n)$ gauge transformation could be called a total covariance,[3] since the passive operation in $\mathcal{M}$ is made tautologically active, and we may write, e.g., $\delta_\xi\phi^A(x) = \pounds_\xi\phi^A(x) = \{\mathbf{D},\xi\lrcorner\}\phi^A(x) = V_\xi\lrcorner\delta\phi^A(x) = \pounds_{V_\xi}\phi^A(x) = \{\delta,\xi\cdot\}\phi^A(x)$.

B. Frame transformation

Previous works on Hamiltonian analysis of $[\nabla,\nabla]=0$ gravity have proven that the $\mathcal{P}$ for actions (1) can be well defined [15,16]. [Since, by fixing the $\mathrm{GL}(n)$ invariance of the density $\boldsymbol{L}_Q$ we shall introduce Eq. (10) to the coordinate frame $\mathbf{e}^a = \boldsymbol{\delta}^a$ in Eq. (20), it becomes the density of the coincident GR [17], and, by fixing further the gravitational frame imposing the coincident gauge, this density $\boldsymbol{L}_Q$ becomes the same $\boldsymbol{L}_{ADM}$ used in the standard Arnowitt-Deser-Misner (ADM) Hamiltonian treatment, it is clear that the Cauchy problem can be well posed and that we recover the $\mathcal{P}$ of GR.]

Here, our approach is completely different, since the aim is to take into account the frame dependence of the symplectic structure, the Hamiltonian, and other charges, etc. In the case that an I involves background fields, some of the nice identities at the end of Sec. II A are violated. We have demanded the covariance of each ϕ^A under a gauge-$\mathrm{Diff}(n)$, but this will not be the case under a frame-$\widetilde{\mathrm{Diff}}(n)$. The latter can indeed be interpreted as transformations which leave some of the fields frozen into the role of background fields. It is convenient to realize a frame transformation as the reconfiguration of the affine structure on $\mathcal{M}$.

The suggested construction of $\mathcal{P}$ proceeds as follows.

(i) Assume an invariant functional I of torsor connection.[4] The symmetry is then $\mathrm{GL}(n)\times\widetilde{\mathrm{Diff}}(n)$.

[3]General coordinate invariance has been regarded an "improper" symmetry (Noether) and coordinatizations, in general, "a formal scaffolding" (Weyl) to be discarded at a later stage. We will arrive at the result that the Diff (n) is a *trivial symmetry* according to Freidel, Geiller, and Pranzetti's [13] definition $H_\xi = 0$; i.e., not even a surface Hamiltonian survives on shell. [In this particular case, the Diff (n) may be technically regarded a "fake symmetry" achieved via Stückelbergization or Kretchmannization [14].]

[4]This means that we can freely translate the connection. The translation invariance can be straightforwardly generalized to full connection independence, extending the symmetry to $\mathrm{GL}(n)\times\tilde{\mathrm{GL}}(n)$ [18].

TABLE I. The generic set of gravitational fields ϕ^A.

The metric	(0,0)-form	g_{ab}	$\Rightarrow$ nonmetricity	$\boldsymbol{Q}_{ab} = \mathbf{D}g_{ab}$	$= \mathbf{d}g_{ab} + 2\boldsymbol{A}_{ab}$
The n-bein	(1,0)-form	$\mathbf{e}^a$	$\Rightarrow$ torsion	$\boldsymbol{T}^a = \mathbf{D}\mathbf{e}^a$	$= \mathbf{d}\mathbf{e}^a - \boldsymbol{A}_b{}^a \wedge \mathbf{e}^b$
The connection	(1,0)-form	$\boldsymbol{A}_a{}^b$	$\Rightarrow$ curvature	$\boldsymbol{R}_a{}^b = (\mathbf{D}\boldsymbol{A})_a{}^b$	$= \mathbf{d}\boldsymbol{A}_a{}^b + \boldsymbol{A}_a{}^c \wedge \boldsymbol{A}_c{}^b$

(ii) Imposing $\boldsymbol{E}^A = 0$ and, in particular, symmetric teleparallelism,[5] the symmetry is reduced to $\mathrm{Diff}(n) \times \tilde{\mathrm{Diff}}(n)$. We are then looking at some metaphase space $\mathcal{P}''$, accommodating physically distinct classes of theories and their gauge-degenerate field configurations. The same configuration $\{\phi^A(x)\}$ can represent many different theories, since the mappings from a configuration to the observables are frame dependent.

(iii) Imposing $\boldsymbol{L}|_{\mathcal{B}} = 0$, the frame is fixed such that the action principle is well defined in the presence of the given boundary $\mathcal{B}$. If these boundary conditions determine the frame completely, the remaining symmetry is $\mathrm{Diff}(n)$. This is the prephase space $\mathcal{P}'$.

(iv) The nondegenerate phase space $\mathcal{P} = \mathcal{P}'/\mathrm{Diff}(n)$ is well defined, though, in general, inequivalent to the $\mathcal{P}$ of GR.

The application of this schema will be illustrated, after setting up the general formalism in Sec. III, with a simple example in Sec. IV.

III. SYMMETRIC TELEPARALLELISM

The fields ϕ^A listed in Table I are conventional in metric-affine gravity. If we set symmetric $\boldsymbol{T}^a = 0$ teleparallelism $\boldsymbol{R}_a{}^b = 0$ with $(n-2,\ 0)$-form multipliers, those formally count as yet additional fields. The action would be

$$I = \int \boldsymbol{L} = \int (\boldsymbol{L}_Q + \boldsymbol{\lambda}^a{}_b \wedge \boldsymbol{R}_a{}^b + \boldsymbol{\lambda}_a \wedge \boldsymbol{T}^a), \tag{8}$$

wherein the $\boldsymbol{L}_Q$ is responsible for the dynamics of nonmetricity. It is useful to also define

$$\text{the nonmetricity conjugate:} \quad \boldsymbol{q}_{ab} = \frac{\partial \boldsymbol{L}_Q}{\partial \boldsymbol{Q}^{ab}}, \tag{9a}$$

$$\text{the metric energy current:} \quad \boldsymbol{G}_{ab} = -\frac{\partial \boldsymbol{L}}{\partial g^{ab}} - \boldsymbol{Q}_a{}^c \wedge \boldsymbol{q}_{cb}, \tag{9b}$$

$$\text{the } n\text{-bein energy current:} \quad \boldsymbol{t}_a = -\frac{\partial \boldsymbol{L}}{\partial \mathbf{e}^a}. \tag{9c}$$

[5] $\mathbf{D}^2 = 0$ is the dynamical consequence of m_P being the mass of the connection [4], as the density $\boldsymbol{L}_Q$ clearly suggests. However, restricting here to sub-Planckian scales, we need not consider an explicit kinetic term for the connection but will set $\mathbf{D}^2 = 0$ using multipliers.

Thus, we have the density

$$\boldsymbol{L}_Q = \frac{1}{2}\boldsymbol{Q}^{ab} \wedge \boldsymbol{q}_{ab}, \tag{10}$$

with some generic $\boldsymbol{q}_{ab}$. Variations then yield us

$$\begin{aligned} \boldsymbol{E}_A \delta\phi^A = {} & \delta g^{ab}(g_{ac}\mathbf{D}\boldsymbol{q}^c{}_b - \boldsymbol{G}_{ab}) + \delta\mathbf{e}^a \wedge (\mathbf{D}\boldsymbol{\lambda}_a - \boldsymbol{t}_a) \\ & + \delta\boldsymbol{A}_a{}^b \wedge (2\boldsymbol{q}^a{}_b - \mathbf{e}^a \wedge \boldsymbol{\lambda}_b + \mathbf{D}\boldsymbol{\lambda}^a{}_b) \\ & + \delta\boldsymbol{\lambda}^a{}_b \wedge \boldsymbol{R}_a{}^b + \delta\boldsymbol{\lambda}_a \wedge \boldsymbol{T}^a \end{aligned} \tag{11a}$$

and the symplectic current

$$\boldsymbol{\Theta} = -\delta g^{ab}\boldsymbol{q}_{ab} + \delta\mathbf{e}^a \wedge \boldsymbol{\lambda}_a + \delta\boldsymbol{A}_a{}^b \wedge \boldsymbol{\lambda}^a{}_b. \tag{11b}$$

The equations of motion $\boldsymbol{E}_A \delta\phi^A = 0$ imply the three equations

$$2\boldsymbol{G}^a{}_b = -\mathbf{e}^a \wedge \boldsymbol{t}_b \Rightarrow \boldsymbol{t}_a = -2\mathfrak{e}_b \lrcorner \boldsymbol{G}^b{}_a, \tag{12a}$$

$$\mathbf{D}(\mathfrak{e}_a \lrcorner \mathbf{D}\boldsymbol{q}^a{}_b) = 0 \Rightarrow \overset{\circ}{\mathbf{D}} * \boldsymbol{t}^M_a = 0, \tag{12b}$$

$$2\mathbf{D}\boldsymbol{q}^a{}_b = -\mathbf{e}^a \wedge \mathbf{D}\boldsymbol{\lambda}_b \Rightarrow \boldsymbol{\lambda}_a = \boldsymbol{h}_a + \mathbf{D}\boldsymbol{z}_a. \tag{12c}$$

The first equation shows that the metric and the n-bein inertiality criterions, $\boldsymbol{G}_{ab} \hat{=} 0$ and $\boldsymbol{t}_a \hat{=} 0$, respectively, are equivalent. The second equation is the Bianchi identity of the frame invariance.[6] The third equation can be used to determine the excitation $\boldsymbol{h}_a$, and we have parametrized the arbitrariness of the solution with a $(n-3,\ 0)$-form $\boldsymbol{z}_a$.

In the end, the full dynamics, taking into account the possible material current $\boldsymbol{t}^M_a$, are described by the gauge-covariant and frame-invariant field equation $\mathbf{D}\boldsymbol{h}_a = \boldsymbol{t}^M_a + \boldsymbol{t}_a$.

[6] This would generically be broken by modifications of GR introducing new degrees of freedom. Such could render Eq. (12b) only an on-shell identity, potentially spoiling the isomorphism of the space of solutions and the space $\mathcal{P}$. The question whether there exists a "properly parallelized" frame [19] (wherein the rank of Ω would be a constant) is outside our scope here, since the starting point (1) as stated in Sec. II B excludes the modifications of GR.

A. Gauge transformation

The presymplectic potential of a Diff ξ is

$$\xi\cdot\boldsymbol{\Theta} = (\xi\lrcorner\boldsymbol{Q}^{ab})\boldsymbol{q}_{ab} + (\xi\lrcorner\boldsymbol{T}^a + \mathbf{D}\xi^a)\wedge\boldsymbol{\lambda}_a + \xi\lrcorner\boldsymbol{R}_a{}^b\wedge\boldsymbol{\lambda}^a{}_b$$
$$= \mathbf{d}\boldsymbol{j}_\xi + \xi\lrcorner\boldsymbol{L} - \xi^a\boldsymbol{E}_a + \boldsymbol{T}^a\wedge\xi\lrcorner\boldsymbol{\lambda}_a + \boldsymbol{R}_a{}^b\wedge\xi\lrcorner\boldsymbol{\lambda}^a{}_b, \quad (13)$$

where

$$\xi\lrcorner\boldsymbol{L} = -\xi^a\boldsymbol{t}_a + (\xi\lrcorner\boldsymbol{Q}^{ab})\boldsymbol{q}_{ab} + \xi\lrcorner(\boldsymbol{T}^a\wedge\boldsymbol{\lambda}_a + \boldsymbol{R}_a{}^b\wedge\boldsymbol{\lambda}^a{}_b), \quad (14a)$$

$$\boldsymbol{j}_\xi = \xi^a\boldsymbol{h}_a. \quad (14b)$$

In the above, the $\boldsymbol{z}$ ambiguity is eliminated from Eq. (12c), since the integral of an exact form over a closed surface $= 0$ and, therefore, the $\boldsymbol{z}$ does not contribute to the observable charges.[7] We see that the stationarity condition (3b) holds (neglecting an irrelevant $\boldsymbol{y}$-type and $\boldsymbol{z}$-type ambiguity in $\boldsymbol{C}$) if

$$\xi\lrcorner\boldsymbol{L}|_{\mathcal{B}} = 0, \quad (15a)$$

$$\xi\cdot\boldsymbol{C}|_{\mathcal{B}} = \xi^a\boldsymbol{h}_a. \quad (15b)$$

We can satisfy Eq. (15a) with an arbitrary ξ by adjusting the frame of a configuration such that the density $\boldsymbol{L}$ vanishes at the given $\mathcal{B}$. This condition boils down to setting one scalar function $L(x\in\mathcal{B}) = 0$ to vanish, and, thus, we can safely assume a solution to exist. At least locally, the density-free frame can also be an inertial frame (pointwise, one can simply find a freely falling coordinate system in the coincident gauge). The density-free boundary condition agrees with the intuition extrapolated from the case which is best understood in GR, a flat $\mathcal{B}$ at spatial infinity. It is clear that, asymptotically far from the material sources, the density $\boldsymbol{L}$ may grow instead of decay only with respect to some kind of noninertial reference frame, and this has indeed been considered in the context of teleparallel gravity as a criterion for regularized energy expressions and actions [20,21]. The no-density consistency condition (15a) applies to a generic boundary and, thus, provides the local and covariant generalization of the physically acceptable boundary conditions for a flat $\mathcal{B}$ at infinity. Finally, we note that the identification (15b) vanishes the Hamiltonian (7), $\mathcal{H}_\xi = 0$.

B. Frame transformation

A transformation which leaves the density $\boldsymbol{L}$ invariant only up to an exact form is sometimes called a pseudosymmetry. A remarkable property of coincident GR is the pseudosymmetry with respect to independent translations of the connection [17]:

$$\Delta\boldsymbol{L} = \mathbf{d}(2\Delta\boldsymbol{A}_a{}^b\wedge\boldsymbol{q}^a{}_b). \quad (16)$$

On the other hand, by adapting Eq. (11), we obtain, in the variation (2)

$$\xi\lrcorner\Delta\boldsymbol{L} = \Delta\boldsymbol{A}_a{}^b\wedge(2\boldsymbol{q}^a{}_b - \mathbf{e}^a\wedge\boldsymbol{\lambda}_b + \mathbf{D}\boldsymbol{\lambda}^a{}_b)$$
$$+ \mathbf{d}(\Delta\boldsymbol{A}_a{}^b\wedge\boldsymbol{\lambda}^a{}_b) \quad (17a)$$

and the presymplectic potential

$$\boldsymbol{\Theta} = \mathbf{D}(\mathbf{e}_a\lrcorner\mathbf{D}\xi^b)\wedge\boldsymbol{\lambda}^a{}_b$$
$$= \mathbf{e}_a\lrcorner\mathbf{D}\xi^b\wedge(\mathbf{e}^a\wedge\boldsymbol{h}_b - 2\boldsymbol{q}^a{}_b), \quad (17b)$$

where the second form follows by discarding a term that does not contribute to the variation (17a). We now find the Noether current

$$\boldsymbol{J}_\xi = \mathbf{D}\xi^a\wedge\boldsymbol{h}_a \hat{=} \mathbf{d}(\xi^a\boldsymbol{h}_a). \quad (18a)$$

The hatted equality assumed an inertial frame, $\boldsymbol{t}_a \hat{=} 0$. Thus, the Noether charges of the frame (pseudo)symmetry and the gauge symmetry are the same in an inertial frame but not otherwise. Furthermore, the quasilocal Noether charge is also the Hamiltonian generator of the frame transformation:

$$H_\xi = -\int_{\mathcal{C}}\mathbf{D}\xi^a\wedge\boldsymbol{t}_a + \int_{\partial\mathcal{C}}\xi^a\boldsymbol{h}_a \hat{=} \int_{\partial\mathcal{C}}\xi^a\boldsymbol{h}_a. \quad (18b)$$

Transition into a noninertial frame $\boldsymbol{t}^a \neq 0$ can generate a bulk Hamiltonian.

IV. COINCIDENT GENERAL RELATIVITY

We shall make explicit the relation between the symmetric $\mathbf{D}^2 = 0$ version of GR originally suggested by Nester and Yo [22,23] and the $[\nabla,\nabla] = 0$ version of GR introduced by Beltrán Jiménez *et al.* [17,24]. To make contact with the Palatini (tensor) formulation, we define the contravariant vectors

$$\boldsymbol{Q} = g^{ab}\boldsymbol{Q}_{ab}, \quad (19a)$$

$$\tilde{\boldsymbol{Q}} = (\mathbf{e}_a\lrcorner\boldsymbol{Q}^a{}_b)\mathbf{e}^b. \quad (19b)$$

The special case of a density (10) we consider in this section is

[7] This does not quite account for the difference of the Noether-Wald potentials obtained in the Palatini [8] and in the GL(n) [7] formulations, since it is not exact but instead $\Delta\boldsymbol{j}_\xi = m_P^2 * \boldsymbol{d}^\flat\xi$ [5].

TABLE II. Summary of our conclusions and the well-known results in GR.

Form	GR: gauge	CGR: gauge	CGR: frame
Density $\boldsymbol{L}$	Einstein-Hilbert	$\boldsymbol{L}_Q = \boldsymbol{Q}^{ab} \wedge \boldsymbol{q}_{ab}/2$, cf. Eq. (20)	
Surface density $\boldsymbol{\ell}$	Gibbons-Hawking-York	$\cdots$	
Noether charge $\oint \boldsymbol{j}$	Komar	$\oint \boldsymbol{h}$, cf. Eq. (21b)	$\hat{=} \oint \boldsymbol{h}$
Hamiltonian H	Brown-York	0	$\hat{=} \int_{\partial\mathcal{C}} \boldsymbol{h}$
Conditions @$\mathcal{B}$	ξ is tangential Killing	$\boldsymbol{L} = 0$	

$$
\begin{aligned}
m_P^{-2}\boldsymbol{L}_Q &= -\frac{1}{8}\boldsymbol{Q}_{ab} \wedge *\boldsymbol{Q}^{ab} + \frac{1}{4}\boldsymbol{Q}_{ac} \wedge \mathbf{e}^a \wedge *(\boldsymbol{Q}^{bc} \wedge \mathbf{e}_b) \\
&\quad -\frac{1}{8}\boldsymbol{Q} \wedge *\boldsymbol{Q} + \frac{1}{4}\boldsymbol{Q} \wedge *\tilde{\boldsymbol{Q}} \\
&= \frac{1}{2}\left(-\frac{1}{4}Q_{abc}Q^{abc} + \frac{1}{2}Q_{abc}Q^{bac} + \frac{1}{4}Q_aQ^a \right. \\
&\quad \left. -\frac{1}{2}Q_a\tilde{Q}^a\right)(*1). \qquad (20)
\end{aligned}
$$

The nonmetricity conjugate $(n-1, 0)$-form (9) derived for Eq. (20) is

$$
\begin{aligned}
m_P^{-2}\boldsymbol{q}_{ab} &= -\frac{1}{4}*\boldsymbol{Q}_{ab} + \frac{1}{2}\mathbf{e}_{(a} \wedge *(\boldsymbol{Q}_{b)c} \wedge \mathbf{e}^c) \\
&\quad -\frac{1}{4}g_{ab}(*\boldsymbol{Q} - *\tilde{\boldsymbol{Q}}) + \frac{1}{4}\boldsymbol{Q}_{(a}(*\mathbf{e}_{b)}) \\
&= -\frac{1}{4}[-Q^c{}_{ab} + 2Q_{(ab)}{}^c \\
&\quad + g_{ab}(Q^c - \tilde{Q}^c) - Q_{(a}\delta^c_{b)}](*\mathbf{e}_c). \\
&= -m_P^{-2}*\boldsymbol{P}_{ab}. \qquad (21a)
\end{aligned}
$$

In the last step, we borrowed a notation for the 1-form $\boldsymbol{P}_{ab} = *\boldsymbol{q}_{ab}$ from the tensor formalism. Now we solve the excitation from Eq. (12c):

$$
\boldsymbol{h}_a = m_P^2 * [\boldsymbol{Q}_{ab} \wedge \mathbf{e}^b + \mathbf{e}_a \wedge (\boldsymbol{Q} - \tilde{\boldsymbol{Q}})]. \qquad (21b)
$$

The n-bein energy current (9c) is

$$
\begin{aligned}
\boldsymbol{t}_a &= -\partial_a \lrcorner \boldsymbol{L}_Q + \frac{m_P^4}{4}[\partial_a \lrcorner \boldsymbol{Q}_{bc}(2\mathbf{e}^b \wedge *(\boldsymbol{Q}^c{}_d \wedge \mathbf{e}^d) \\
&\quad + \partial^{(b} \lrcorner \boldsymbol{Q} * \mathbf{e}^{c)} - *\boldsymbol{Q}^{bc}) + \partial_a \lrcorner \boldsymbol{Q}(\partial_b \lrcorner \tilde{\boldsymbol{Q}} * \mathbf{e}^b - *\boldsymbol{Q})] \\
&= \left(-\frac{1}{2}\delta^b_a Q_{cde}P^{cde} + Q_a{}^{cd}P^b{}_{cd}\right)(*\mathbf{e}_b). \qquad (21c)
\end{aligned}
$$

We compute also the metric energy current (9b) and as a cross-check verify the identity (12a):

$$
\begin{aligned}
2\boldsymbol{G}^a{}_b &= \delta^a_b\boldsymbol{L}_Q - \partial_b \lrcorner \boldsymbol{Q}^{cd}\boldsymbol{P}_{cd} \wedge *\mathbf{e}^a \\
&= \left(\frac{1}{2}\delta^a_b Q_{cde}P^{cde} - Q_b{}^{cd}P^a{}_{cd}\right)(*1) \\
&= -\mathbf{e}^a \wedge \boldsymbol{t}_b. \qquad (21d)
\end{aligned}
$$

The results of the analysis are summarized in Table II.

A. Example

It can be useful to illustrate the role of the two types of transformations with a simple example. Set $n = 4$ and take the cosmological solution

$$
\mathbf{d}s^2 = -n^2(t)\mathbf{d}t^2 + a^2(t)\delta_{ij}\mathbf{d}x^i\mathbf{d}x^j. \qquad (22)
$$

One may want to consider this in an $\mathcal{M}$ bounded by the cosmological horizon or maybe to have an action bounded by a given event's past light cone. Such could be consistently described by a theory wherein the density $\boldsymbol{L}$ vanishes at the boundary. Given the line element (22), the choice of frame simply corresponds to the choice of connection. Hohmann has constructed the most general homogeneous and isotropic symmetric teleparallel geometry [25], and we adopt his first solution[8] characterized by one free function, $K(t)$. The $\boldsymbol{L}_Q$ depends upon this function as

$$
\boldsymbol{L}_Q = \frac{3m_P^2}{2}(2H^2 + 3HK + \dot{K})(*1), \quad \text{where } H = \frac{\dot{a}}{a}, \quad \frac{df}{dt} = n\dot{f}. \qquad (23)
$$

Given the dynamics encoded in $a(t)$, from $\boldsymbol{L}_Q = 0$ we obtain an inhomogeneous, first-order ordinary differential equation to determine the function $K(t)$. To find an explicit solution, let us add a matter source, in the simplest case a Λ term:

$$
\begin{aligned}
\boldsymbol{L} = \boldsymbol{L}_Q + \boldsymbol{L}_\Lambda &= \frac{m_P^2}{2}(6H^2 + 9HK + 3\dot{K} + 2\Lambda)a^3n\mathbf{d}^4x \\
&\overset{n=1}{=} \frac{m_P^2}{2}(4\Lambda + 3\sqrt{3\Lambda}K + 3\dot{K})e^{\sqrt{3\Lambda}t}a_0^3\mathbf{d}^4x. \qquad (24)
\end{aligned}
$$

Both the choice of the cosmological connection and the choice of the lapse are irrelevant to the dynamics, even though the former will recalibrate the energy units and the latter will change the interpretation of the coordinate time t. More generally, time-space intervals are varied in a gauge-

[8]Hohmann reported three branches of solutions for the connection [25], but our conclusions would be similar in the two other branches. The most general case of spherically symmetric geometry has also been nicely explored [26–28].

Diff, while the differences in the gauge-invariant energy-momentum charges $\oint \boldsymbol{h}_a$ are varied in a frame-$\widetilde{\mathrm{Diff}}$.

Let us now walk through the steps we recall from Sec. II B. (i) We have an invariant $\boldsymbol{L}$ such that it changes by an exact form when translating the connection. (ii) The metaphase space $\mathcal{P}''$ corresponds to all the available solutions, including now two arbitrary functions $n(t)$ and $K(t)$. (iii) By imposing the density-free boundary condition, we are given from the space $\mathcal{P}''$ the slice $\mathcal{P}'$ wherein K is fixed such that on the $n(t) = 1$ hyperslice of $\mathcal{P}'$ it is the constant $K = -4/3\sqrt{\Lambda/3}$ (a one-parameter family of solutions is found iff for some $\mathcal{C} \supset \mathcal{B}$). (iv) In the phase space $\mathcal{P} = \mathcal{P}'/\mathrm{Diff}$ also the degeneracy due to $n(t)$ is eliminated, since we *mod* out the time-reparametrization gauge invariance.

V. CONCLUSION

Symmetric teleparallel gravity features the so-called frame pseudosymmetry. Recovering the standard ADM formulation of GR is one way to the fix the frame, but in a manifold with a boundary $\mathcal{B}$, the well posedness of the action principle can provide the more appropriate criterion. By the consistent choice of frame one may incorporate arbitrary gauge transformations in arbitrary geometry.

(i) We see from Eq. (15) that the $\xi_\perp$'s normal to $\mathcal{B}$ are automatically integrable, avoiding artificial restrictions to diffeomorph the total $\mathcal{M}$ with boundaries.

(ii) The $\xi_\parallel$'s tangential to $\mathcal{B}$ require the no-density boundary condition. We emphasize the viewpoint that Eq. (15) is imposed in $\mathcal{P}''$; i.e., the boundary condition is a restriction upon the resulting covariant phase space $\mathcal{P}$.

An example in Sec. IV A demonstrated that the no-density boundary condition determines the gravitational frame at cosmological scales. It remains to be explored whether this could shed light on the initial conditions required for a viable inflation. At very high energies, we can no longer justify the approximation $\mathbf{D}^2 = 0$, and it seems possible that the frame is settled in a dynamical fashion.

Intuitively, the density-free boundary condition expresses the continuity of the action principle. It is a natural requirement also outside the context of teleparallelism and could, in principle, be used to reduce the number of free parameters in a consistent theory. In particular, in the context of a pregeometric gauge theory [29], the boundaries of spacetime are by construction described by $\boldsymbol{L} = 0$, since spacetime emerges via the spontaneous breaking of the symmetric phase characterized by $\mathbf{e}^a = 0$.

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