Radiated momentum and radiation reaction in gravitational two-body scattering including time-asymmetric effects

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We compute to high post-Newtonian accuracy the 4-momentum (linear momentum and energy), radiated as gravitational waves in a two-body system undergoing gravitational scattering. We include, for the first time, all the relevant time-asymmetric effects that arise when consistently going three post-Newtonian orders beyond the leading post-Newtonian order. We find that the inclusion of time-asymmetric radiative effects (both in tails and in the radiation-reacted hyperbolic motion) is crucial to ensure the mass polynomiality of the post-Minkowskian expansion (G expansion) of the radiated 4-momentum. Imposing the mass polynomiality of the corresponding individual impulses determines the conservativelike radiative contributions at the fourth post-Minkowskian order and strongly constrains them at the fifth post-Minkowskian order.

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I. INTRODUCTION

Gravitational scattering has attracted a renewed interest in recent years, both for conservative and dissipative (i.e., gravitational-radiation-related) effects. Various approximation methods (post-Newtonian, post-Minkowskian, quantum perturbation theory, effective field theory, string theory) have been applied to this problem. For a sample of results on (classical or quantum) post-Minkowskian (PM) gravitational scattering, see, e.g., Refs. [1–19]. For recent PM results on radiative losses during gravitational scattering and related results, see, e.g., Refs. [20–25].

The state of the art for the PM scattering of spinless bodies is $O(G^3)$ for radiation-reacted scattering [23,26–28], and $O(G^4)$ for the *conservative* case [10,14]. The state of the art for radiative losses during gravitational scattering is $O(G^3)$ for radiated angular momentum [24] and $O(G^3)$ for radiated 4-momentum [9,20,21,23]. While finalizing this work, a $O(G^4)$-accurate computation of the (radiation-reacted) individual 4-momentum changes (or "impulses") $\Delta p_{a\mu}$ and of the loss of 4-momentum of the system appeared on arXiv [25].

The relation between radiative losses of energy, linear momentum, and angular momentum and the radiation-reaction contribution to scattering has been worked out, to linear order in radiation reaction, in Refs. [29,30]. One of the aims of the present work is to go beyond the purely linear-in-radiation-reaction treatment of Refs. [29,30]. This will be done by focusing on the various "time-asymmetric" effects arising in the radiative losses of energy and linear momentum during hyperbolic encounters.

The post-Newtonian (PN) approximation method has also recently played a useful role in tackling gravitational scattering. The state of the art for PN scattering (of spinless bodies) in the conservative case is the fourth post-Newtonian (4PN) accuracy [31]. This was generalized in Refs. [32–34] to the 5PN and 6PN accuracies (modulo the knowledge of a few, yet undetermined, Hamiltonian coefficients). The state of the art for the PN-expanded computation of the radiative losses (to gravitational waves) of energy, angular momentum, and linear momentum[1] is as follows: the radiated energy and angular momentum (for spinless bodies) have been computed at the absolute 4.5PN order (corresponding to a 2PN fractional accuracy) in Refs. [30,35,36]. Higher-order terms (corresponding to, at least, the 3PN fractional accuracy) have been computed in Refs. [37–39]. The radiated linear momentum is currently known to the (absolute) 5.5PN order [30,40] (corresponding to a 2PN fractional accuracy). The state of the art for the PN-expanded computation of the scattering of spinless bodies is the 5PN level, at which an inconsistency with the mass polynomiality of the conservative G^4 (4PM) contribution was highlighted in [30], and remains puzzling despite recent work on the additional radiative contributions [41].

The aims of the present paper are:

(1) to complete the PN knowledge of the radiated energy by including both the fractional 2.5PN contribution (linked to the 2.5PN radiation-reaction

[1]We recall that the leading PN orders of radiative losses is the 2.5PN order for energy and angular momentum, while it is the 3.5PN order for linear momentum.

modification of the hyperbolic motion), which was incorrectly argued to vanish in Ref. [37], and the "instantaneous" 3PN-level contribution first derived in Ref. [37] and rederived here;

(2) to improve the knowledge of the radiated angular momentum by including both the fractional 2.5PN contribution (computed here for the first time) and the 3PN-level contribution (obtained here by adding instantaneous 3PN terms [37] and higher-order tails [36]);

(3) to raise the knowledge of the radiated linear momentum to the fractional 3PN accuracy (corresponding to the absolute 6.5PN order);

(4) to bring new light on the mass-polynomiality structure of the scattering at the 4PM and 5PM orders.

The accuracy increase (from 2PN to 3PN fractional accuracy) in the radiated linear momentum requires that many new physical effects be taken into account; indeed, we will need to take into account: (i) 2.5PN radiation-reaction effects in the hyperbolic motion, (ii) 2.5PN instantaneous contributions to the radiative multipole moments [42,43], (iii) the 1PN fractional correction to the leading-order tail contribution[2] to the radiated linear momentum (which was first computed in Ref. [30]), (iv) 3PN accuracy in several multipoles and in the hyperboliclike motion, and (v) higher-order tails in the momentum loss [40].

To complete the so-obtained increased PN-expanded knowledge of the radiated 4-momentum $P^\mu_{\rm rad} = (E_{\rm rad}, P^i_{\rm rad})$ we will reexpress it in terms of Lorentz-invariant form factors by decomposing it on the basis u^μ_{1-}, u^μ_{2-}, $\hat b^\mu_{12}$, defined by the initial four velocities of the bodies and the direction of the vectorial impact parameter $\hat b^\mu_{12} = b^\mu_{12}/b$, with $b^\mu_{12} = b^\mu_1 - b^\mu_2$. More precisely, it will be useful to decompose it as

$$P^\mu_{\rm rad} = P^{\rm rad}_{1+2}(u^\mu_{1-} + u^\mu_{2-}) + P^{\rm rad}_{1-2}(u^\mu_{1-} - u^\mu_{2-}) + P^{\rm rad}_{b_{12}} \hat b^\mu_{12}. \tag{1.1}$$

We will show below (generalizing considerations introduced in Refs. [30,45]) that, at each order in G, the PM expansion of the form factors $P^{\rm rad}_{1+2}$, $P^{\rm rad}_{1-2}$, $P^{\rm rad}_{b_{12}}$ (expressed as functions of b and of the relative Lorentz factor $\gamma \equiv -u^\mu_{1-} u_{2-\mu}$),[3] have a polynomial structure in the two masses m_1, m_2, e.g.,

$$P^{\rm rad}_{1+2} = \frac{G^3}{b^3} m_1^2 m_2^2 \hat P^{\rm rad}_{1+2}, \tag{1.2}$$

with

$$\hat P^{\rm rad}_{1+2} = \sum_{n\geq 3} \frac{G^{n-3}}{b^{n-3}} SP^{1+2}_{n-3}(m_1, m_2; \gamma). \tag{1.3}$$

Here and in the following, the notation $SP^{1+2}_N(m_1, m_2; \gamma)$ denotes a homogeneous "symmetric polynomial" of order N in the two masses, with coefficients depending on the Lorentz factor γ.

At the 3PM level [$O(G^3)$], only one form factor of $P^\mu_{\rm rad}$ is nonvanishing, namely, $P^{{\rm rad}\,G^3}_{1+2}$, with

$$P^{{\rm rad}\,G^3}_{1+2} = \frac{G^3}{b^3} m_1^2 m_2^2 \frac{\mathcal{E}(\gamma)}{\gamma + 1}. \tag{1.4}$$

The exact value of the function $\mathcal{E}(\gamma)$ has been computed in Refs. [9,23,24,46,47], while its PN expansion was computed to order v^{15} included in [30], see Eq. (5.19) there. For illustration, let us display the beginning of the PN expansion of $\mathcal{E}(\gamma)$, when expressed in terms of $p_\infty \equiv \sqrt{\gamma^2 - 1}$,

$$\mathcal{E}(\gamma) = \pi\left(\frac{37}{15} p_\infty + \frac{1357}{840} p_\infty^3 + \frac{27953}{10080} p_\infty^5 - \frac{676273}{354816} p_\infty^7 + O(p_\infty^9)\right). \tag{1.5}$$

Using our newly acquired PN-expanded knowledge on the values of $E_{\rm rad}$ and $P^i_{\rm rad}$ (computed in the c.m. frame), we will be able both to check the mass-polynomiality structure of the form factors $P^{\rm rad}_{1+2}$, $P^{\rm rad}_{1-2}$, $P^{\rm rad}_{b_{12}}$ entering the decomposition (1.1) and to compute their expansions in powers of p_∞ at the fractional 3PN accuracy.

Finally, we will use the so-acquired improved knowledge of $P^\mu_{\rm rad}$ to constrain the radiation-reaction-induced contributions to the individual changes Δp^μ_a (also called impulses) of the 4-momenta of the two bodies. As we will recall in more detail below, Refs. [29,30] have derived the effect of radiation reaction on the individual momentum changes Δp^μ_a only to linear order in radiation reaction and within a restricted set of assumptions. Namely, writing the equations of motion of each particle as a perturbed "conservative" (Hamiltonian) system involving an additional "radiation-reaction force" $\mathcal{F}^\mu_{\rm rr}$, Refs. [29,30] worked only to *linear order* in $\mathcal{F}^\mu_{\rm rr}$, and, furthermore, often assumed that the latter radiation-reaction force was time antisymmetric.[4] Under these assumptions, Refs. [29,30] derived an expression for Δp^μ_a of the form

$$\Delta p_{a\mu} = \Delta p^{\rm cons}_{a\mu} + \Delta p^{\rm rr\,lin}_{a\mu} + \Delta p^{\rm rr\,nonlin}_{a\mu}. \tag{1.6}$$

Here the term $\Delta p^{\rm rr\,lin}_{a\mu}$ denotes the contribution linear in the radiation reaction derived in [30], while the term $\Delta p^{\rm rr\,nonlin}_{a\mu}$ denotes the missing remainder, due to nonlinear effects in $\mathcal{F}^\mu_{\rm rr}$. Reference [30] had illustrated the existence of

[2]We recall that tail contributions to gravitational radiation start at the fractional 1.5PN order [42,44].

[3]We use a mostly plus signature.

[4]The time-reversal operation is taken around the moment of closest approach of the time-symmetric unperturbed conservative dynamics, considered in the center-of-mass frame.

nonlinear effects in $\mathcal{F}^{\mu}_{\rm rr}$ by computing (within the standard PN approach) a contribution to $\Delta p_{a\mu}$ quadratic in $\mathcal{F}^{\mu}_{\rm rr}$. It has been known for a long time [48,49] that there are *hereditary*, tail-related contributions to the equations of motion. These contributions are time asymmetric, i.e., neither time even, nor time odd. At the 4PN level, they can be uniquely decomposed in a time-even conservative piece (contributing to the Hamiltonian) and a time-odd piece, giving a nonlocal-in-time contribution to $\mathcal{F}^{\mu}_{\rm rr}$ (see Sec. VI of [49]). However, this simple decomposition becomes more tricky at the 5PN level. This is indeed the PN level where quadratic effects in $\mathcal{F}^{\mu}_{\rm rr}$ enter and where past-related tail effects contribute to the linear-response results of [30] [via the presence of a "conservativelike," 5PN-level, past-tail contribution to $P^{\mu}_{\rm rad}$; see Eq. (H3) there]. [These 5PN-level subtleties arise at the 4PM level ($O(G^4)$).] The new results presented here will complete the results of [30] by fully taking into account time-asymmetric effects in various observables. First, in $P^{\mu}_{\rm rad}$ (which we compute here with higher PN accuracy than before, including all needed hereditary tail effects), and second, in the radiative contributions to the impulses $\Delta p^{\rm rr}_{a\mu}$. We will improve below the results of [30] by completing the linear-response term $\Delta p^{\rm rr\,lin}_{a\mu}$ with the effect of the time-even part of $\mathcal{F}^{\mu}_{\rm rr}$ on the relative scattering angle. In addition, our strategy to scrutinize the mass polynomiality of the impulses will allow us to obtain valuable information on the remainder term $\Delta p^{\rm rr\,nonlin}_{a\mu}$ in Eq. (1.6). This information is enough to uniquely determine $\Delta p^{\rm rr\,nonlin}_{a\mu}$ at order G^4 to strongly constrain its value at order G^5.

II. FRAMEWORK

To set the stage for our computations below, let us recall that the general expressions for the radiative fluxes (at infinity) of energy, linear momentum, and angular momentum in terms of the *radiative* multipole moments U_L and V_L (defined at future null infinity) read [48,50–54]

$$\frac{dE^{\rm rad}}{dt_{\rm ret}} \equiv \mathcal{F}_E = \sum_{l=2}^{\infty} \frac{G}{c^{2l+1}}\left[\frac{(l+1)(l+2)}{(l-1)ll!(2l+1)!!}U_L^{(1)}U_L^{(1)} + \frac{4l(l+2)}{c^2(l-1)(l+1)!(2l+1)!!}V_L^{(1)}V_L^{(1)}\right], \tag{2.1}$$

$$\frac{dP_i^{\rm rad}}{dt_{\rm ret}} \equiv \mathcal{F}_{Pi} = \sum_{l=2}^{\infty}\left[\frac{G}{c^{2l+3}}\frac{2(l+2)(l+3)}{l(l+1)!(2l+3)!!}U_{iL}^{(2)}U_L^{(1)} + \frac{G}{c^{2l+5}}\frac{8(l+3)}{(l+1)!(2l+3)!!}V_{iL}^{(2)}V_L^{(1)} + \frac{G}{c^{2l+3}}\frac{8(l+2)}{(l-1)(l+1)!(2l+1)!!}\epsilon_{iab}U_{aL-1}^{(1)}V_{bL-1}^{(1)}\right], \tag{2.2}$$

and

$$\frac{dJ_i^{\rm rad}}{dt_{\rm ret}} \equiv \mathcal{F}_{J_i} = \epsilon_{iab}\frac{G}{c^{2l+1}}\sum_{l=2}^{\infty}\left[\frac{(l+1)(l+2)}{(l-1)l!(2l+1)!!}U_{aL-1}U_{bL-1}^{(1)} + \frac{4l^2(l+2)}{c^2(l-1)(l+1)!(2l+1)!!}V_{aL-1}V_{bL-1}^{(1)}\right]. \tag{2.3}$$

Here, $t_{\rm ret} = t - \frac{r}{c} - \frac{2G\mathcal{M}}{c^3}\ln(\frac{r}{r_0}) + O(G^2)$ is the retarded time [with $\mathcal{M}$ denoting the total Arnowitt-Deser-Misner (ADM) mass of the spacetime, and r_0 a constant length scale], while U_L and V_L are the mass- and current-type radiative multipole moments, respectively (with $L = i_1 i_2 \cdots i_l$ being a multi-index consisting of l spatial indices). They are related to the *source* multipole moments I_L and J_L by relations having the structure[5] [42]

$$U_L(t) = I_L^{(l)}(t) + \frac{G}{c^3}(\text{tail} + \text{semihered} + \text{instantaneous}) + \frac{G}{c^5}(\text{semihered} + \text{instantaneous}) + \text{higher-order tails},$$
$$V_L(t) = J_L^{(l)}(t) + \frac{G}{c^3}(\text{tail} + \text{instantaneous}) + \frac{G}{c^5}(\text{instantaneous}) + \text{higher-order tails}. \tag{2.4}$$

Here the "tail terms" are given by integrals extending over the full past history of the source of the type

$$+\frac{2G\mathcal{M}}{c^3}\int_0^{\infty} d\tau I_L^{(l+2)}(t-\tau)\left(\ln\frac{\tau}{\tau_0} + \text{const}\right), \tag{2.5}$$

with $\tau_0 = r_0/c$. The semihereditary (semihered.) terms (also known as memory terms) are time antiderivatives of products of multipole moments, whereas the instantaneous terms are polynomials in (time derivatives of) the source multipole moments. Notice that there are no semihereditary contributions to the radiative current moments. In the case of radiative mass moments, instead, the $O(\frac{G}{c^3})$ semihereditary terms first appear for $l = 4$, while at the next order $O(\frac{G}{c^5})$ they are already present for $l = 2$. Furthermore, both the energy and the linear-momentum fluxes (2.2) only

[5] Henceforth, we replace the argument $t_{\rm ret}$ of the radiative multipole moments simply by the dynamical time variable $t = t_{\rm ret} + \text{cst}$ describing the binary motion (in the center-of-mass system).

contain time derivatives of the radiative moments (2.4), so that all the semihereditary terms give instantaneous contributions to both $\mathcal{F}_E(t)$ and $\mathcal{F}_{Pi}(t)$.

The higher-order tail contributions (tail-squared, tails-of-tails, etc.) start at fractional order $(\frac{2G\mathcal{M}}{c^3})^2$, i.e., 3PN. We will take into account these fractional 3PN contributions in all radiated quantities: energy, angular momentum, and linear momentum. To reach the 3PN accuracy, we also need to take into account all semihereditary and instantaneous terms that contribute at the fractional 2.5PN level. Among the 2.5PN effects, an important, and subtle, one comes from the 2.5PN-level correction to the hyperbolic motion induced by the leading-order radiation-reaction force. It is the subject of the next section.

III. 2.5PN CORRECTION TO THE QUASI-KEPLERIAN PARAMETRIZATION FOR HYPERBOLICLIKE ORBITS

In order to explicitly compute the 2.5PN correction to hyperbolic motion[6] caused by the leading-order radiation-reaction force (considered as a first-order perturbation of the 2PN equations of motion), it is convenient to follow Ref. [55] in using Lagrange's method of variation of constants. This is done by rewriting the *hyperbolic* version[7] [56] of the solution of the 2PN-level equations of motion [57–59] (which depends on four integration constants, say, c_1, c_2, c_l, c_ϕ) in terms of four time-dependent versions of the integration constants, say, $c_1(t), c_2(t), c_l(t), c_\phi(t)$. Namely, one writes

$$
\begin{aligned}
r &= S(l, c_1(t), c_2(t)),\\
\dot{r} &= \bar{n}(c_1(t), c_2(t))\frac{\partial S(l, c_1(t), c_2(t))}{\partial l},\\
\phi &= c_\phi(t) + \bar{W}(l, c_1(t), c_2(t)),\\
\dot{\phi} &= \bar{n}(c_1(t), c_2(t))\frac{\partial \bar{W}(l, c_1(t), c_2(t))}{\partial l}.
\end{aligned}
\tag{3.1}
$$

Here the functions $S(l, c_1, c_2)$ and $\bar{W}(l, c_1, c_2)$ are defined by eliminating the auxiliary variables v and V (by expressing them as functions of l, c_1 and c_2) from the four equations

$$
\begin{aligned}
S &= \bar{a}_r(e_r\cosh v - 1),\\
\bar{W} &= K[V + f_\phi \sin 2V + g_\phi \sin 3V],\\
l &= e_t \sinh v - v + f_t V + g_t \sin V,\\
V &= 2\arctan\left[\sqrt{\frac{e_\phi + 1}{e_\phi - 1}}\tanh\frac{v}{2}\right].
\end{aligned}
\tag{3.2}
$$

In these equations, the quasi-Keplerian orbital parameters $a_r, e_r, e_t, e_\phi, K \equiv 1 + k, f_\phi, g_\phi, f_t, g_t$ are functions of the two (2PN) integrals of motion c_1, c_2. Similar to S and $\bar{W}$, the auxiliary variable v can be considered as a function of l, c_1, c_2: $v = v(l, c_1, c_2)$. One could choose as basic 2PN constants, c_1, c_2, the energy E of the system [or the specific binding energy $\bar{E} \equiv (E - Mc^2)/(\mu c^2)$] and the angular momentum J of the system [or the dimensionless angular momentum $j = cJ/(GM\mu)$; see, e.g., Table VIII of Ref. [33] for the harmonic-coordinates-case expressions of the orbital parameters]. In the following, we find more convenient to use $c_1 = \bar{a}_r$ and $c_2 = e_r$. The harmonic-coordinates expressions of the quasi-Keplerian orbital parameters, as functions of $\bar{a}_r$ and e_r, will be presented below when discussing the generalization of this representation at the 3PN level. The auxiliary variable[8] v is then considered as a function of the form $v = v(l, \bar{a}_r, e_r)$, with the dependence on $\bar{a}_r$ entering only beyond the leading order (LO).

The perturbed motion is then expressed, besides allowing c_1, c_2, c_ϕ to be functions of time, by describing the time dependence of the basic angle l of the hyperboliclike planar motion in the following way:

$$
l(t) = \int_{t_0}^{t} \bar{n}(c_1(t), c_2(t))dt + c_l(t). \tag{3.3}
$$

Here, t_0 is an arbitrary reference time, and the four former "constants" $c_1(t), c_2(t), c_l(t)$, and $c_\phi(t)$ are now time varying. Inserting Eqs. (3.1) and (3.3) in the perturbed equations of motion determines the system of four first-order evolution equations that must be satisfied by the four quantities $c_1(t), c_2(t), c_l(t), c_\phi(t)$, say,

$$
\frac{dc_\alpha}{dt} = F_\alpha(l, c_\beta), \qquad \alpha, \beta = 1, 2, l, \phi, \tag{3.4}
$$

where the functions F_α are linear in the perturbing (relative) acceleration. They generally read

[6] For our present purpose, it is enough to study the relative two-body planar motion, considered in the center-of-mass system, and in harmonic coordinates.

[7] A straightforward analytic continuation to positive binding energies of the *ellipticlike* 2PN quasi-Keplerian parametrization would involve complex parameters.

[8] The variable v is the hyperbolic analog of the usual Kepler eccentric anomaly u [solution of Kepler's equation $l = u - e_t \sin u + O(\frac{1}{c^2})$] used in the description of elliptic motions. See Appendix B for a discussion of the complex analytic continuation relating elliptic and hyperbolic motions.

$$\frac{dc_1}{dt} = \frac{\partial c_1(\mathbf{x},\mathbf{v})}{\partial \mathbf{v}} \cdot \mathbf{A}_{\rm rr},$$
$$\frac{dc_2}{dt} = \frac{\partial c_2(\mathbf{x},\mathbf{v})}{\partial \mathbf{v}} \cdot \mathbf{A}_{\rm rr},$$
$$\frac{dc_l}{dt} = -\left(\frac{\partial S}{\partial l}\right)^{-1}\left[\frac{\partial S}{\partial c_1}\frac{dc_1}{dt} + \frac{\partial S}{\partial c_2}\frac{dc_2}{dt}\right],$$
$$\frac{dc_\phi}{dt} = -\frac{\partial \bar{W}}{\partial l}\frac{dc_l}{dt} - \frac{\partial \bar{W}}{\partial c_1}\frac{dc_1}{dt} - \frac{\partial \bar{W}}{\partial c_2}\frac{dc_2}{dt}, \qquad (3.5)$$

where $\mathbf{A}^{\rm rr}$ denotes the (relative) radiation-reaction acceleration (which starts at 2.5 PN). When choosing $c_1 = E$ and $c_2 = J$, and when working in the Hamiltonian formalism, the first two varying-constant equations read

$$\frac{dE}{dt} = \frac{\partial H}{\partial p_i}\mathcal{F}_i^{\rm rr}, \qquad \frac{dJ}{dt} = \mathcal{F}_\phi^{\rm rr}, \qquad (3.6)$$

where $\mathcal{F}_i^{\rm rr}$ denotes the relative radiation-reaction force.

When choosing $c_1 = \bar{a}_r$ and $c_2 = e_r$, and when working at the leading PN order, these two equations read

$$\frac{d\bar{a}_r}{dt} = -2\bar{a}_r^2 \mathbf{v}\cdot\mathbf{A}_{\rm rr},$$
$$\frac{de_r}{dt} = \frac{e_r^2-1}{e_r}\bar{a}_r \mathbf{v}\cdot\mathbf{A}_{\rm rr} + \frac{\sqrt{e_r^2-1}}{e_r\sqrt{\bar{a}_r}}[\mathbf{x}\times\mathbf{A}_{\rm rr}]_z. \qquad (3.7)$$

As we need to compute the time dependence of the source multipole moments expressed in *harmonic* coordinates, we shall use here the (leading order) value of $\mathbf{A}^{\rm rr}$ in harmonic coordinates, namely (denoting $\nu \equiv \mu/M$),

$$\mathbf{A}^{\rm rr} = -\frac{8}{5}\nu\frac{G^2}{c^5}\frac{M^2}{r^3}\left[-\left(3v^2 + \frac{17}{3}\frac{GM}{r}\right)\dot{r}\mathbf{n} + \left(v^2 + 3\frac{GM}{r}\right)\mathbf{v}\right]. \qquad (3.8)$$

Working at the leading 2.5PN order, denoting

$$\mathcal{X} \equiv e_r\cosh v - 1, \qquad (3.9)$$

[where the auxiliary variable v is the same as in Eq. (3.2)] and decomposing the four varying constants $c_\alpha(t)$ as

$$c_\alpha(t) = c_\alpha^0 + \delta^{\rm rr}c_\alpha(t), \qquad (3.10)$$

with constants c_α^0,[9] one finds the following explicit (2.5 PN-accurate) evolution system[10] for the four $\delta^{\rm rr}c_\alpha(t)$'s:

$$\frac{d\delta^{\rm rr}\bar{a}_r}{dt} = \frac{\nu}{\bar{a}_r^3}\left[-\frac{32}{5\mathcal{X}^3} - \frac{512}{15\mathcal{X}^4} + \frac{16(-49+9e_r^2)}{15\mathcal{X}^5} + \frac{112(e_r^2-1)}{3\mathcal{X}^6}\right],$$
$$\frac{d\delta^{\rm rr}e_r}{dt} = \frac{\nu(e_r^2-1)}{\bar{a}_r^4 e_r}\left[-\frac{56(e_r^2-1)}{3\mathcal{X}^6} - \frac{8(9e_r^2-49)}{15\mathcal{X}^5} + \frac{136}{15\mathcal{X}^4} + \frac{8}{5\mathcal{X}^3}\right],$$
$$\frac{d\delta^{\rm rr}c_l}{dt} = \frac{\nu\sinh v}{\bar{a}_r^4 e_r}\left[-\frac{56(e_r^2-1)^2}{3\mathcal{X}^6} - \frac{8(e_r^2-1)(9e_r^2-14)}{15\mathcal{X}^5} + \frac{8(43e_r^2-3)}{15\mathcal{X}^4} + \frac{32}{5}\frac{e_r^2}{\mathcal{X}^3}\right],$$
$$\frac{d\delta^{\rm rr}c_\phi}{dt} = \frac{\nu\sinh v}{\bar{a}_r^4}\frac{\sqrt{e_r^2-1}}{e_r}\left[\frac{8}{5\mathcal{X}^4} - \frac{8}{15}\frac{9e_r^2-14}{\mathcal{X}^5} - \frac{56}{3}\frac{e_r^2-1}{\mathcal{X}^6}\right]. \qquad (3.11)$$

Let us note in passing that we have checked these results on the 2.5PN-level variation of the 2PN quasi-Keplerian parameters of *hyperboliclike* motions by relating them to the results of Ref. [55] on the 2.5PN-level, radiation-reaction correction to the quasi-Keplerian parametrization of ellipticlike motions. In order to relate the two types of results, we used the fact that the latter 2.5PN-level, radiation-reaction correction only depends on the Newtonian-level Keplerian parametrization (which admits a smooth analytic continuation when changing the sign of the binding energy). We then had to go through two different steps: (i) to relate the elliptic and hyperbolic quasi-Keplerian parametrizations by a simple analytic continuation (as used, e.g., at the 1PN level in Ref. [60]) and (ii) to take into account the fact that Ref. [55] worked in a different coordinate system (namely, ADM coordinates), corresponding to a different explicit expression for the radiation-reaction force. Some partial results on the comparison to the results of Ref. [55] are given in Appendix B.

It is convenient to integrate perturbed quantities with respect to the auxiliary variable v by using the unperturbed relation $\frac{dt}{dv} = \bar{a}_r^{3/2}\mathcal{X}$. The explicit solution of the above-evolution system then reads

[9]Below, we ease the notation by denoting c_α^0 simply as c_α, while the full quantity $c_\alpha(t)$ is always indicated with its time dependence.

[10]Following usual practice, we often use scaled variables (factoring out some appropriate powers of M or μ) when studying the relative motion.

$$\begin{aligned}
\delta^{\rm rr}\bar{a}_r(t) &= \frac{\nu}{\bar{a}_r^{3/2}}\Bigg\{\frac{4(37e_r^4+292e_r^2+96)}{15(e_r^2-1)^{7/2}}At(v)+\sinh v\left[\frac{28e_r}{3\mathcal{X}^4}+\frac{4e_r(36e_r^2+49)}{45(e_r^2-1)\mathcal{X}^3}+\frac{2e_r(111e_r^2+314)}{45(e_r^2-1)^2\mathcal{X}^2}\right.\\
&\quad\left.+\frac{2e_r(673e_r^2+602)}{45(e_r^2-1)^3\mathcal{X}}\right]+\frac{2(673e_r^2+602)}{45(e_r^2-1)^3}\Bigg\},\\
\delta^{\rm rr}e_r(t) &= \frac{\nu}{\bar{a}_r^{5/2}}\left[-\frac{2e_r(121e_r^2+304)}{15(e_r^2-1)^{5/2}}At(v)+\sinh v\left(-\frac{14(e_r^2-1)}{3\mathcal{X}^4}-\frac{2(36e_r^2+49)}{45\mathcal{X}^3}-\frac{(291e_r^2+134)}{45\mathcal{X}^2(e_r^2-1)}\right.\right.\\
&\quad\left.\left.-\frac{(72e_r^4+1069e_r^2+134)}{45(e_r^2-1)^2\mathcal{X}}\right)-\frac{(72e_r^4+1069e_r^2+134)}{45(e_r^2-1)^2e_r}\right],\\
\delta^{\rm rr}c_l(t) &= \frac{\nu}{\bar{a}_r^{5/2}}\left[\frac{14(e_r^2-1)^2}{3e_r^2\mathcal{X}^4}+\frac{8(e_r^2-1)(9e_r^2-14)}{45e_r^2\mathcal{X}^3}-\frac{4(-3+43e_r^2)}{15e_r^2\mathcal{X}^2}-\frac{32}{5\mathcal{X}}\right],\\
\delta^{\rm rr}c_\phi(t) &= \frac{\nu\sqrt{e_r^2-1}}{\bar{a}_r^{5/2}e_r^2}\left[\frac{14(e_r^2-1)}{3\mathcal{X}^4}+\frac{8}{45}\frac{9e_r^2-14}{\mathcal{X}^3}-\frac{4}{5\mathcal{X}^2}\right],
\end{aligned}\tag{3.12}$$

where

$$At(v)\equiv\arctan\left[\alpha\tanh\left(\frac{v}{2}\right)\right]+\arctan\alpha,\tag{3.13}$$

with

$$\alpha\equiv\sqrt{\frac{e_r+1}{e_r-1}},\tag{3.14}$$

and where the dependence of v on t is the unperturbed one. Here we have assumed the boundary conditions $\lim_{t\to-\infty}\delta c_\alpha(t)=0$.

By looking at this solution, one sees that $\delta^{\rm rr}c_l(t)$ and $\delta^{\rm rr}c_\phi(t)$ are *even* functions of t, so that they tend to the same value (here chosen to be zero both at $t=-\infty$ and at $t=+\infty$). By contrast, the two other quantities $\delta^{\rm rr}\bar{a}_r(t)$ and $\delta^{\rm rr}e_r(t)$ vary between $t=-\infty$ and $t=+\infty$. More precisely, one gets total variations $[f]\equiv f(+\infty)-f(-\infty)$ given by

$$\begin{aligned}
[\delta^{\rm rr}\bar{a}_r] &= \frac{4}{15}\frac{\nu}{\bar{a}_r^{3/2}(e_r^2-1)^3}\left[\frac{673e_r^2+602}{3}+\frac{37e_r^4+292e_r^2+96}{(e_r^2-1)^{1/2}}\arccos\left(-\frac{1}{e_r}\right)\right],\\
[\delta^{\rm rr}e_r] &= -\frac{2}{15}\frac{\nu}{\bar{a}_r^{5/2}(e_r^2-1)^2e_r}\left[\frac{72e_r^4+1069e_r^2+134}{3}+\frac{e_r^2(121e_r^2+304)}{(e_r^2-1)^{1/2}}\arccos\left(-\frac{1}{e_r}\right)\right],
\end{aligned}\tag{3.15}$$

with $[\delta^{\rm rr}\bar{a}_r]=(2\bar{a}_r^2/\nu)\delta^{\rm rr}E^{\rm N}$, as from Eqs. (C7)–(C9) of Ref. [30]. These total variations agree with the total scattering changes in $\bar{a}_r$ and e_r obtained in Eqs. (6.1) and (6.2) of Ref. [39] by assuming (to leading PN order) balance equations for energy and angular momentum, between the system and radiation.

To complete the solution of the radiation-reacted motion, one needs to inject the results, Eqs. (3.12), in the definitions of $l(t)$ and $\phi(t)$. In other words, one must now evaluate the functions $l(t)=l^0(t)+\delta^{\rm rr}l(t)$ and $\phi(t)=\phi^0(t)+\delta^{\rm rr}\phi(t)$, where $l^0(t)=\bar{n}^0(t-t_0)$, $\phi^0(t)=c_\phi^0+\bar{W}(l^0(t),c_1^0,c_2^0)$, and where

$$\begin{aligned}
\delta^{\rm rr}l(t) &= \int_{t_0}^{t}\delta^{\rm rr}\bar{n}(t)dt+\delta^{\rm rr}c_l(t),\\
\delta^{\rm rr}\phi(t) &= \delta^{\rm rr}c_\phi(t)+\frac{\partial\bar{W}}{\partial l}\delta^{\rm rr}l(t)+\frac{\partial\bar{W}}{\partial c_1}\delta^{\rm rr}c_1(t)+\frac{\partial\bar{W}}{\partial c_2}\delta^{\rm rr}c_2(t).
\end{aligned}\tag{3.16}$$

If we work only to the leading PN order (i.e., the 2.5PN order) we can (in the radiation-reacted contributions) use the Newtonian-level approximation [notably $\bar{n}\approx(\bar{a}_r)^{-3/2}$, $K\approx1$] so as to get

$$\delta^{\rm rr} l(t) = \frac{\nu}{15\bar{a}_r^{5/2}(e_r^2-1)^3}\left[-(673e_r^2+602)[\mathcal{X}+1+e_r\sinh v - v] - (111e_r^2+314)(e_r^2-1)\ln\mathcal{X} + \frac{2(36e_r^2+49)(e_r^2-1)^2}{\mathcal{X}} + \frac{105(e_r^2-1)^3}{\mathcal{X}^2} - \frac{6(37e_r^4+292e_r^2+96)}{\sqrt{e_r^2-1}}[\mathcal{L}+(e_r\sinh v - v)\arctan\alpha]\right] + \text{const}, \tag{3.17}$$

where

$$\mathcal{L} \equiv \int^v dv \mathcal{X}(v)\arctan\left(\alpha\tanh\left(\frac{v}{2}\right)\right), \tag{3.18}$$

and where the integration constant can be chosen at will [e.g., to make $\delta^{\rm rr} l(t)$ vanish when $v=0$, i.e., at the moment of closest approach in the hyperbolic motion]. Changing the integration variable as $v = 2\text{arctanh}T$ in the above integral yields

$$\mathcal{L} = \int dT\left[-\frac{1}{1+T} - \frac{1}{1-T} + e_r\left(\frac{1}{(1+T)^2} + \frac{1}{(1-T)^2}\right)\right]\arctan(\alpha T), \tag{3.19}$$

which can be solved in terms of dilogarithms. Explicitly,

$$\int dT\frac{\arctan(\alpha T)}{(1\pm T)^2} = \mp\frac{\arctan(\alpha T)}{1\pm T} + \frac{\alpha}{2(1+\alpha^2)} \times [2\ln(1\pm T) - (1\pm i\alpha) \times \ln(1+i\alpha T) - (1\mp i\alpha)\ln(1-i\alpha T)], \tag{3.20}$$

and

$$\int dT\frac{\arctan(\alpha T)}{1\pm T} = \pm\frac{i}{2}\left[\ln\left(\frac{\alpha(1\pm T)}{\mp i+\alpha}\right)\ln(1-i\alpha T) - \ln\left(\frac{\alpha(1\pm T)}{\pm i+\alpha}\right)\ln(1+i\alpha T) - \text{Li}_2\left(\frac{i-\alpha T}{i\pm\alpha}\right) + \text{Li}_2\left(\frac{i+\alpha T}{i\mp\alpha}\right)\right]. \tag{3.21}$$

Finally, the solution for the orbit $x^i(t) = x^i_{\leq 2\text{PN}}(t) + \delta^{\rm rr} x^i(t)$, obtained by varying l, c_1, c_2, c_ϕ in the functions $r(l, c_1, c_2)$ and $\phi(l, c_1, c_2, c_\phi)$ defined by Eqs. (3.1) and (3.2), reads

$$\delta^{\rm rr} r(t) = \frac{1}{\mathcal{X}}\bar{a}_r e_r \sinh v\,\delta^{\rm rr} l(t) + \mathcal{X}\delta^{\rm rr}\bar{a}_r(t) + \frac{\bar{a}_r}{e_r}\left(-1+\frac{e_r^2-1}{\mathcal{X}}\right)\delta^{\rm rr} e_r(t),$$

$$\delta^{\rm rr}\phi(t) = \frac{\sqrt{e_r^2-1}}{\mathcal{X}^2}\delta^{\rm rr} l(t) + \delta^{\rm rr} c_\phi(t) - \frac{\sinh v}{\mathcal{X}}\left(\frac{\sqrt{e_r^2-1}}{\mathcal{X}} + \frac{1}{\sqrt{e_r^2-1}}\right)\delta^{\rm rr} e_r(t). \tag{3.22}$$

Taking into account the time-even character of $\delta^{\rm rr} c_l(t)$ and $\delta^{\rm rr} c_\phi(t)$, the total change $[\delta^{\rm rr}\phi]$ between $-\infty$ and $+\infty$ of the value of $\delta^{\rm rr}\phi(t)$ is then easily seen[11] to be

$$[\delta^{\rm rr}\phi] = -\frac{[\delta^{\rm rr} e_r]}{e_r\sqrt{e_r^2-1}}. \tag{3.23}$$

This agrees with the leading-PN-order result obtained in Ref. [29] for the radiation-reaction contribution to the (relative) scattering angle: $\chi_{\rm rr}^{2.5\text{PN}} = [\delta^{\rm rr}\phi]$. As already mentioned in Ref. [29], the general linear-response formula, Eq. (5.99) there, for $\chi_{\rm rr}(E, j)$ is generally valid (to linear order in radiation reaction) beyond the leading PN order, under the two conditions that the unperturbed conservative motion be time symmetric and that the radiation-reaction force be time antisymmetric. (These two conditions generally ensure that $\frac{d\delta^{\rm rr}\bar{a}_r}{dt}$ and $\frac{d\delta^{\rm rr} e_r}{dt}$ will be time symmetric, while $\frac{d\delta^{\rm rr} c_l}{dt}$ and $\frac{d\delta^{\rm rr} c_\phi}{dt}$ will be time antisymmetric, so that $[c_l] = 0$ and $[c_\phi] = 0$.) For completeness, we present in Appendix A the explicit expressions of the 2.5PN, 3.5PN, and 4.5PN contributions to the function $\chi_{\rm rr}(E, j)$.

IV. CONTRIBUTION TO THE RADIATED LINEAR MOMENTUM COMING FROM THE RADIATION-REACTION CORRECTION TO HYPERBOLIC MOTION

Having in hand the radiation-reaction correction to hyperbolic motion, we can now come back to the analytical determination of the linear-momentum loss at the fractional 3PN accuracy.

[11]The term proportional to $\delta^{\rm rr} l(t)$ in $\delta^{\rm rr}\phi(t)$ vanishes at infinity. Furthermore, $[\delta^{\rm rr} c_\phi] = 0$ as from Eq. (3.12).

Inserting in Eq. (2.2) the expressions (2.4) for the radiative moments in terms of the source moments and taking into account all instantaneous, semihereditary, and hereditary terms contributing at the 3PN level, we get a radiative linear-momentum flux of the form

$$\mathcal{F}_{P\,i}=\mathcal{F}_{P\,i}{}^{\mathrm{inst}\,I,J\leq 3\mathrm{PN}}+\Delta\mathcal{F}_{P_i^{\mathrm{inst}\,I,J}}+\mathcal{F}_{P_i^{\mathrm{tail}}}+\mathcal{F}_{P\,i}{}^{\text{higher-order tails}}. \tag{4.1}$$

Here, the "leading-order instantaneous" term $\mathcal{F}_{P\,i}{}^{\mathrm{inst}\,I,J\leq 3\mathrm{PN}}$ is defined by replacing in Eq. (2.2) the radiative moments U_L and V_L by the source ones, I_L and J_L; the "supplementary instantaneous" contribution $\Delta\mathcal{F}_{P_i^{\mathrm{inst}\,I,J}}$ combines contributions bilinear in (the derivatives of) I_L and J_L coming both from the instantaneous terms and the semihereditary ones in Eq. (2.4); finally the tail terms (both linear tails and higher-order tails) $\mathcal{F}_{P_i^{\mathrm{tail}}}+\mathcal{F}_{P\,i}{}^{\text{higher-order tails}}$ denote the contribution bilinear in I_L and J_L and in the various hereditary contributions to U_L and V_L.

The complete expression for the linear-momentum flux at the 2.5PN fractional accuracy level is given in Eqs. (2.3)–(2.5) of Ref. [43]. The notation used there is

$$\begin{aligned}(\mathcal{F}^i_P)_{\mathrm{inst}}&=\mathcal{F}_{P\,i}{}^{\mathrm{inst}\,I,J}+\Delta\mathcal{F}_{P_i^{\mathrm{inst}\,I,J}},\\ (\mathcal{F}^i_P)_{\mathrm{hered}}&=\mathcal{F}_{P_i^{\mathrm{tail}}}.\end{aligned} \tag{4.2}$$

In order to reach the 3PN accuracy, we need (i) to insert in these expressions the 3PN-accurate expressions of the source moments $I_L(t), J_L(t)$ considered as functions of dynamical time t and (ii) to add the higher-order tail contribution to the hereditary term $(\mathcal{F}^i_P)_{\mathrm{hered}}$. When evaluating 3PN-accurate values of the relevant kth time derivatives, $I_L^{(k)\leq 3\mathrm{PN}}(t), J_L^{(k)\leq 3\mathrm{PN}}(t)$, of the source moments, one needs to use the 3PN-level equations of motion (including the 2.5PN radiation-reaction contribution), and then to express these time-differentiated moments along radiation-reacted hyperboliclike solutions of the equations of motion. The latter are obtained by adding the 2.5PN-level, radiation-reaction effects discussed in the previous section to the conservative 3PN hyperboliclike solutions (which will be discussed below).

Let us symbolically write the motions as

$$\begin{aligned}x^{\leq 3\mathrm{PN}}(t)&=x^{3\mathrm{PN,cons}}(t)+\delta^{\mathrm{rr}}x(t),\\ v^{\leq 3\mathrm{PN}}(t)&=v^{3\mathrm{PN,cons}}(t)+\delta^{\mathrm{rr}}v(t).\end{aligned} \tag{4.3}$$

As a consequence, the first contribution, $\mathcal{F}_{P\,i}{}^{\mathrm{inst}\,I,J\leq 3\mathrm{PN}}(t)$, to the linear-momentum flux is naturally decomposed as a sum of two terms,

$$\mathcal{F}_{P\,i}{}^{\mathrm{inst}\,I,J\leq 3\mathrm{PN}}(t)=\mathcal{F}_{P\,i}{}^{\mathrm{inst}\,I,J\,3\mathrm{PN,cons}}(t)+\delta^{\mathrm{rr}}\mathcal{F}_{P\,i}{}^{\mathrm{inst}\,I,J}(t). \tag{4.4}$$

In these expressions, and below, the symbol δ^{rr} will be used to denote the 2.5PN-level radiation-reaction-generated contribution to some physical quantity, $Q(t)=Q(x(t),v(t))$, considered as a function of dynamical time t. In the previous section, we obtained (at leading order) the various needed radiation-reaction contributions $\delta^{\mathrm{rr}}x(t), \delta^{\mathrm{rr}}v(t), \delta^{\mathrm{rr}}Q(t)$ by using Lagrange's method of variation of constants.

Finally, integrating $\mathcal{F}_{P\,i}$ over t (from $-\infty$ to $+\infty$) we get the total linear momentum radiated in gravitational waves during a full hyperbolic encounter,

$$P_i^{\mathrm{rad}}=\int_{-\infty}^{+\infty}dt\mathcal{F}_{P\,i}(t). \tag{4.5}$$

The 6.5PN-accurate value of P_i^{rad} is then obtained as a sum of various contributions, say,

$$\begin{aligned}P_i^{\mathrm{rad}}&=P_i^{\mathrm{rad\,inst}\,I,J3\mathrm{PN,cons}}+\delta^{\mathrm{rr}}P_i^{\mathrm{rad\,inst}\,I,J}\\ &\quad+\Delta P_i^{\mathrm{rad\,inst}\,I,J}+P_i^{\mathrm{rad\,tail}}+P_i^{\text{rad higher-order tails}}.\end{aligned} \tag{4.6}$$

The resulting vectorial contributions will be projected on an orthonormal basis $\mathbf{e}_x$, $\mathbf{e}_y$ defined in terms of the vectorial impact parameter $b_{12}^\mu=b\hat{b}_{12}^\mu$, the initial four velocities u_{1-}^μ and u_{2-}^μ of the two bodies, and the conservative[12] part of the scattering angle, χ_{cons} [see, e.g., Table X of Ref. [30], also recalled at 2PN in Eqs. (C5) and (C6) below for convenience]. The basis $\mathbf{e}_x$, $\mathbf{e}_y$ was already used in Ref. [30] [see Eq. (3.49) there]. Its definition is recalled in Appendix A. Let us only mention here that $\mathbf{e}_x$ is in the direction of the major axis of the hyperboliclike relative orbit (direction of closest approach).

The 2PN-accurate value of the instantaneous contribution to linear-momentum loss has been evaluated in Ref. [30], see Eqs. (G6)–(G9) there. We have extended this result by including both the higher-order tail effects (which were computed in Ref. [40]) and the 3PN-level conservative effects. The technology (including a 3PN-accurate quasi-Keplerian representation of hyperboliclike motions) needed for computing 3PN-level conservative instantaneous contribution will be discussed below.

Let us discuss here the evaluation of the radiation-reaction-related contribution $\delta^{\mathrm{rr}}\mathcal{F}_{P\,i}{}^{\mathrm{inst},I,J}$. To obtain it, it is enough to evaluate the Newtonian flux

$$\mathcal{F}_{P\,i}{}^{\mathrm{inst},I,J\mathrm{N}}=-\frac{64}{105}\frac{G^3M^4}{r^4c^7}\frac{m_2-m_1}{M}\nu^2(A^{\mathrm{N}}n_i+B^{\mathrm{N}}v_i), \tag{4.7}$$

with

[12]In our treatment below, the coordinate basis $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ enters via the 2PN-accurate quasi-Keplerian representation of the scattering motion.

$$A^{\rm N} = \dot{r}\left(\frac{55}{8}v^2 - \frac{45}{8}\dot{r}^2 + \frac{3}{2}\frac{GM}{r}\right),$$
$$B^{\rm N} = -\left(\frac{25}{4}v^2 - \frac{19}{4}\dot{r}^2 + \frac{GM}{r}\right), \tag{4.8}$$

along the radiation-reaction-perturbed orbit, i.e., by substituting in it

$$r(t) = r_{\rm N}(t) + \delta^{\rm rr}r(t), \quad \dot{r}(t) = \dot{r}_{\rm N}(t) + \delta^{\rm rr}\dot{r}(t),$$
$$\phi(t) = \phi_{\rm N}(t) + \delta^{\rm rr}\phi(t), \quad \dot{\phi}(t) = \dot{\phi}_{\rm N}(t) + \delta^{\rm rr}\dot{\phi}(t), \tag{4.9}$$

taking then the time integral, retaining only linear corrections. [The magnitude of the relative velocity in Eq. (4.8) above should not be confused with the auxiliary variable used to parametrize the orbit, denoted by the same letter v.]

The variations $\delta^{\rm rr}r(t)$ and $\delta^{\rm rr}\phi(t)$ are given by Eq. (3.22). The related variations $\delta^{\rm rr}\dot{r}(t)$ and $\delta^{\rm rr}\dot{\phi}(t)$ are obtained either by taking the time derivatives of $\delta^{\rm rr}r(t)$ and $\delta^{\rm rr}\phi(t)$ or by varying the *functions* $\dot{r}(l, c_1, c_2)$ and $\dot{\phi}(l, c_1, c_2, c_\phi)$ in Eqs. (3.1) and (3.2). This yields

$$\delta^{\rm rr}\dot{r}(t) = \frac{e_r}{\bar{a}_r^{1/2}\mathcal{X}^3}(e_r - \cosh v)\delta^{\rm rr}l(t) - \frac{1}{2}\frac{e_r\sinh v}{\bar{a}_r^{3/2}\mathcal{X}}\delta^{\rm rr}\bar{a}_r(t) - \frac{(e_r^2-1)\sinh v}{\bar{a}_r^{1/2}\mathcal{X}^3}\delta^{\rm rr}e_r(t),$$
$$\delta^{\rm rr}\dot{\phi}(t) = -2\frac{e_r\sqrt{e_r^2-1}}{\bar{a}_r^{3/2}\mathcal{X}^4}\sinh v\,\delta^{\rm rr}l(t) - \frac{3}{2}\frac{\sqrt{e_r^2-1}}{\bar{a}_r^{5/2}\mathcal{X}^2}\delta^{\rm rr}\bar{a}_r(t) + \frac{e_r}{\bar{a}_r^{3/2}\mathcal{X}^2\sqrt{e_r^2-1}}\left[1 + \frac{2(e_r^2-1)}{e_r^2\mathcal{X}}\left(1 - \frac{e_r^2-1}{\mathcal{X}}\right)\right]\delta^{\rm rr}e_r(t). \tag{4.10}$$

We have checked that they satisfy $\frac{d\delta^{\rm rr}r(t)}{dt} = \delta^{\rm rr}\dot{r}(t)$ and $\frac{d\delta^{\rm rr}\phi(t)}{dt} = \delta^{\rm rr}\dot{\phi}(t)$.

We finally get the 2.5PN correction to the Newtonian flux

$$\mathcal{F}_{P\,i}{}^{{\rm inst},I,J{\rm N}}|_{x^\mu = x^\mu_{\rm N}} + \delta^{\rm rr}\mathcal{F}_{P\,i}{}^{{\rm inst},I,J}, \tag{4.11}$$

which has to be integrated along the orbit to yield

$$\delta^{\rm rr}P_i^{{\rm rad\,inst},I,J} = \int dt\delta^{\rm rr}\mathcal{F}_{P\,i}{}^{{\rm inst},I,J}. \tag{4.12}$$

The final exact results are given by the following functions of $\bar{a}_r$ and e_r (here and below, η is a place holder to indicate a half PN order $\frac{1}{c}$):

$$\begin{aligned}\delta^{\rm rr}P_x^{{\rm rad\,inst},I,J} &= -(Mc)\frac{m_2-m_1}{M}\nu^3\eta^5\frac{1}{e_r[\bar{a}_r(e_r^2-1)]^{13/2}}\\
&\times\left[\arccos\left(-\frac{1}{e_r}\right)\left(\frac{110416}{675} + \frac{132304}{135}e_r^2 + \frac{5134544}{4725}e_r^4 + \frac{1365802}{4725}e_r^6 + \frac{30331}{1800}e_r^8\right)\right.\\
&\left.+\frac{(e_r^2-1)^{1/2}}{e_r^2}\left(\frac{8576}{2025} + \frac{11644714}{33075}e_r^2 + \frac{22762729}{18375}e_r^4 + \frac{1623094259}{1984500}e_r^6 + \frac{159585499}{1323000}e_r^8 + \frac{15872}{6125}e_r^{10}\right)\right],\\
\delta^{\rm rr}P_y^{{\rm rad\,inst},I,J} &= (Mc)\frac{m_2-m_1}{M}\nu^3\eta^5\frac{1}{[\bar{a}_r(e_r^2-1)]^{13/2}}\left[e_r\arccos^2\left(-\frac{1}{e_r}\right)\left(\frac{2479}{225}e_r^6 + \frac{22616}{45}e_r^4 + \frac{35416}{75}e_r^2 + \frac{48256}{75}\right)\right.\\
&+\frac{(e_r^2-1)^{1/2}}{e_r}\arccos\left(-\frac{1}{e_r}\right)\left(\frac{296}{25}e_r^8 + \frac{277966}{675}e_r^6 + \frac{752812}{675}e_r^4 + \frac{75592}{45}e_r^2 + \frac{1072}{27}\right)\\
&\left.+\frac{e_r^2-1}{e_r^3}\left(\frac{9352}{75}e_r^8 + \frac{8027}{45}e_r^6 + \frac{2686964}{2025}e_r^4 - \frac{10084}{2025}e_r^2 + \frac{8576}{2025}\right)\right].\end{aligned} \tag{4.13}$$

The first terms of their expansions in inverse powers of j (equivalent, remembering $j \propto G^{-1}$ to a PM expansion) read

$$\delta^{\rm rr}P_x^{{\rm rad\,inst},I,J} = -(Mc)\frac{m_2-m_1}{M}\nu^3\eta^5\left[\frac{15872}{6125}\frac{p_\infty^8}{j^5} + \frac{30331}{360}\pi\frac{p_\infty^7}{j^6} + \frac{24234752}{165375}\frac{p_\infty^6}{j^7} + O\left(\frac{1}{j^8}\right)\right],$$
$$\delta^{\rm rr}P_y^{{\rm rad\,inst},I,J} = (Mc)\frac{m_2-m_1}{M}\nu^3\eta^5\left[\frac{148}{25}\pi\frac{p_\infty^8}{j^5} + \left(\frac{2048}{15} + \frac{2479}{900}\pi^2\right)\frac{p_\infty^7}{j^6} + \frac{160406}{675}\pi\frac{p_\infty^6}{j^7} + O\left(\frac{1}{j^8}\right)\right]. \tag{4.14}$$

V. 2.5PN INSTANTANEOUS CONTRIBUTIONS TO THE RADIATED LINEAR MOMENTUM

Let us now evaluate the third contribution to $P_i^{\rm rad}$ denoted $\Delta P_i^{\rm rad\,inst}$ in Eq. (4.6). This contribution is obtained by integrating over time the (2.5PN level) instantaneous part of the linear-momentum flux in terms of the source multipole moments. Using the results of Ref. [61], Ref. [43] has explicated this instantaneous part as the $O(\frac{1}{c^5})$ term in Eq. (2.3) there. Recently, Ref. [62] has provided an explicit expression for this 2.5PN instantaneous part of the linear-momentum flux as a function of the (relative) position and velocity along the orbit, see Eq. (4.1) there. For clarity, we reproduce here this explicit expression,

$$\Delta\mathcal{F}_{P_i^{{\rm inst}\,I,J}}=-\frac{64}{105}\frac{G^3M^4}{r^4c^7}\frac{m_2-m_1}{M}\nu^2(A^{\rm 2.5PN}n_i+B^{\rm 2.5PN}v_i), \tag{5.1}$$

where

$$\begin{aligned}A^{\rm 2.5PN}&=\frac{GM}{rc^5}\nu\left[\frac{701}{90}v^6-\frac{51137}{96}v^4\dot r^2+\frac{41611}{40}v^2\dot r^4-\frac{49219}{96}\dot r^6-\frac{4}{15}\frac{G^3M^3}{r^3}+\frac{G^2M^2}{r^2}\left(\frac{1237}{90}v^2-\frac{6607}{180}\dot r^2\right)\right.\\&\left.\quad-\frac{GM}{r}\left(\frac{4261}{120}v^4+\frac{8397}{40}v^2\dot r^2-\frac{3778}{15}\dot r^4\right)\right],\\B^{\rm 2.5PN}&=\frac{GM}{rc^5}\nu\dot r\left[\frac{157787}{480}v^4-\frac{39869}{60}v^2\dot r^2+\frac{31913}{96}\dot r^4+\frac{GM}{r}\left(\frac{10773}{40}v^2-\frac{99277}{360}\dot r^2\right)+\frac{737}{36}\frac{G^2M^2}{r^2}\right].\end{aligned} \tag{5.2}$$

The integral along a hyperboliclike orbit of $\Delta\mathcal{F}_{P_i}{}^{{\rm inst}\,I,J}$ can be explicitly evaluated. After projection on the x and y axes defined in Eq. (A3), one finds

$$\begin{aligned}\Delta P_x^{{\rm rad\,inst}\,I,J}&=(Mc)\frac{m_2-m_1}{M}\nu^3\eta^5\frac{e_r}{[\bar a_r(e_r^2-1)]^{13/2}}\\&\quad\times\left[\arccos\left(-\frac{1}{e_r}\right)\left(\frac{491447}{33600}e_r^8+\frac{123798}{175}e_r^6+\frac{30800977}{9450}e_r^4+\frac{13714844}{4725}e_r^2+\frac{2125082}{4725}\right)\right.\\&\quad\left.+\frac{(e_r^2-1)^{1/2}}{e_r^2}\left(\frac{797859313}{3528000}e_r^8+\frac{7556008631}{3175200}e_r^6+\frac{4935155857}{1323000}e_r^4+\frac{652923197}{661500}e_r^2+\frac{5266216}{496125}\right)\right]\\&=(Mc)\frac{m_2-m_1}{M}\nu^3\eta^5\left[\frac{491447}{67200}\frac{\pi p_\infty^9}{j^4}+\frac{13272832}{55125}\frac{p_\infty^8}{j^5}+\frac{494871}{1280}\frac{\pi p_\infty^7}{j^6}+\frac{1954525568}{496125}\frac{p_\infty^6}{j^7}+O\left(\frac{1}{j^8}\right)\right],\\\Delta P_y^{{\rm rad\,inst}\,I,J}&=0.\end{aligned} \tag{5.3}$$

In the last line of the first equation, we have also given the first few terms of its large-j expansion. Let us note that this contribution is (contrary to the other 2.5PN contribution discussed in the previous section) purely oriented along the x axis, i.e., along the vectorial distance of closest approach.

VI. NEW CONTRIBUTIONS TO THE RADIATED ENERGY

Let us repeat for the radiated energy the above treatment for the radiated linear momentum, namely,

$$\begin{aligned}E^{\rm rad}&=E^{{\rm rad\,inst}\,I,J\le 3{\rm PN,cons}}+\delta^{\rm rr}E^{{\rm rad\,inst}\,I,J}\\&\quad+\Delta E^{{\rm rad\,inst}\,I,J}+E^{\rm rad\,tail}+E^{\rm rad\,higher\text{-}order\,tails}.\end{aligned} \tag{6.1}$$

Here, we have indicated the (fractional) 3PN level of accuracy for the instantaneous term $E^{{\rm rad\,inst}\,I,J\,3{\rm PN,cons}}$. The 2PN-accurate instantaneous energy loss $E^{{\rm rad\,inst}\,I,J\,2{\rm PN,cons}}$ was first obtained in [30] [see Eqs. (C7)–(C13)]; its extension at the 3PN level was obtained in [37]. We have redone an independent 3PN-accurate computation of the energy loss and found agreement with the final results of Ref. [37] (after correcting several typos in the 3PN quasi-Keplerian expressions of Ref. [56], see Appendix D). The leading-PN-order contribution to the linear-tail $E^{\rm rad\,tail}$ has been obtained in [30] [see Eq. (D26)], while its 1PN correction is given in Eq. (5.20) of Ref. [39]; see also Ref. [35] for a Fourier space analysis. The higher-order tail contribution $E^{\rm rad\,higher\text{-}order\,tails}$ has been derived in Refs. [36,38]. As discussed in the text below Eq. (3.1) of Ref. [39], the last contribution $\Delta E^{{\rm rad\,inst}\,I,J}$ vanishes (because of the time-odd character of its integrand),

$$\Delta E^{\mathrm{rad\,inst}\,I,J} = 0. \tag{6.2}$$

Reference [37] claimed [see below Eq. (42) there] that, because of the time-odd character of radiation reaction, the term $\delta^{\mathrm{rr}} E^{\mathrm{rad\,inst}\,I,J}$ was similarly vanishing. We found that this was not correct because of the time-asymmetric character of the motion perturbation $\delta^{\mathrm{rr}} x(t), \delta^{\mathrm{rr}} v(t)$. We got a nonzero result for $\delta^{\mathrm{rr}} E^{\mathrm{rad\,inst}\,I,J}$. We further found that this nonvanishing contribution plays a crucial role in obtaining a correct mass-polynomiality behavior for the radiated (four) momentum.

The exact expression of $\delta^{\mathrm{rr}} E^{\mathrm{rad\,inst}\,I,J}$ in terms of $\bar{a}_r$ and e_r reads

$$\begin{aligned}
\delta^{\mathrm{rr}} E^{\mathrm{rad\,inst}\,I,J} = {}& (Mc^2)\nu^3\eta^5 \frac{1}{[\bar{a}_r(e_r^2-1)]^6}\left[\arccos^2\left(-\frac{1}{e_r}\right)\left(\frac{7696}{225}e_r^6+\frac{53936}{75}e_r^4+\frac{150272}{225}e_r^2+\frac{14336}{25}\right)\right.\\
&+(e_r^2-1)^{1/2}\arccos\left(-\frac{1}{e_r}\right)\left(\frac{592}{25}e_r^6+\frac{465952}{675}e_r^4+\frac{44176}{27}e_r^2+\frac{1106464}{675}\right)\\
&\left.+\frac{e_r^2-1}{e_r^2}\left(\frac{44848}{225}e_r^6+\frac{11056}{25}e_r^4+\frac{313024}{225}e_r^2-\frac{8576}{225}\right)\right].
\end{aligned} \tag{6.3}$$

The beginning of its expansion in $\frac{1}{j}$ (i.e., of its PM expansion in powers of G) reads

$$\delta^{\mathrm{rr}} E^{\mathrm{rad\,inst}\,I,J} = (Mc^2)\nu^3\eta^5\left[\frac{296}{25}\pi\frac{p_\infty^7}{j^5}+\left(\frac{50176}{225}+\frac{1924}{225}\pi^2\right)\frac{p_\infty^6}{j^6}+\frac{56008}{135}\pi\frac{p_\infty^5}{j^7}+O\left(\frac{p_\infty^4}{j^8}\right)\right]. \tag{6.4}$$

Adding this term to the 1PN corrections to the LO tails [36,38,39] then gives the following complete expression for the 2.5PN radiated energy:

$$\begin{aligned}
E^{\mathrm{rad}}_{2.5\mathrm{PN}} = {}& (Mc^2)\nu^2\eta^5\left[\left(\frac{1216}{105}-\frac{2848\nu}{15}\right)\frac{p_\infty^8}{j^4}+\left(\left(\frac{296}{25}-\frac{15291\pi^2}{280}\right)\nu-\frac{24993\pi^2}{1120}+\frac{9216}{35}\right)\pi\frac{p_\infty^7}{j^5}\right.\\
&+\left(\left(-\frac{71488}{75}-\frac{2974508\pi^2}{4725}\right)\nu+\frac{2898\zeta(3)}{5}+\frac{1024\pi^2}{135}+\frac{56708}{105}\right)\frac{p_\infty^6}{j^6}\\
&\left.+\left(\left(\frac{56008}{135}-\frac{23514\pi^2}{7}+\frac{30285\pi^4}{112}\right)\nu+\frac{689985\pi^4}{3584}-\frac{13138915\pi^2}{7392}+\frac{210176}{225}\right)\pi\frac{p_\infty^5}{j^7}+O\left(\frac{p_\infty^4}{j^8}\right)\right].
\end{aligned} \tag{6.5}$$

Let us also exhibit the $\frac{1}{j}$ expansion of the full 3PN-level contribution to the energy loss, which combines terms from several sources: the (exact) instantaneous contribution linked to 3PN-level multipole moments [37] and the higher-order tails (tails-of-tails and tail-squared) [36,38,39]

$$\begin{aligned}
E^{\mathrm{rad}}_{3\mathrm{PN}} = {}& (Mc^2)\nu^2\eta^6\left[\left(-\frac{148\nu^3}{15}+\frac{321\nu^2}{280}-\frac{2699\nu}{504}-\frac{676273}{354816}\right)\pi\frac{p_\infty^{10}}{j^3}+\left(-\frac{2366\nu^3}{9}+\frac{164\nu^2}{3}-\frac{1223594\nu}{33075}-\frac{151854}{13475}\right)\frac{p_\infty^9}{j^4}\right.\\
&+\left(-\frac{1823\nu^3}{5}+\frac{12269\nu^2}{80}+\left(\frac{76897}{480}-\frac{4059\pi^2}{640}\right)\nu-\frac{10593}{350}\ln\left(\frac{p_\infty}{2}\right)+\frac{99\pi^2}{10}+\frac{29573617463}{310464000}\right)\pi\frac{p_\infty^8}{j^5}\\
&+\left(-\frac{150892\nu^3}{45}+\frac{4201976\nu^2}{1575}+\left(\frac{875976284}{297675}-\frac{212216\pi^2}{1575}\right)\nu-\frac{18955264}{23625}\ln(2p_\infty)\right.\\
&\left.+\frac{177152\pi^2}{675}+\frac{36589282372}{11694375}\right)\frac{p_\infty^7}{j^6}\\
&+\left(-\frac{13955\nu^3}{6}+\frac{1419153\nu^2}{448}+\left(\frac{68898691}{36288}-\frac{51947\pi^2}{384}\right)\nu-\frac{337906}{315}\ln\left(\frac{p_\infty}{2}\right)\right.\\
&\left.\left.-\frac{58957\zeta(3)}{32}+\frac{3158\pi^2}{9}+\frac{37546579757}{8467200}\right)\pi\frac{p_\infty^6}{j^7}+O\left(\frac{p_\infty^5}{j^8}\right)\right].
\end{aligned} \tag{6.6}$$

An equivalent expression (and extended up to $1/j^{15}$), can be found in Ref. [38]. [Note that Eq. (B3) of the published version (and of the arXiv version 1) uses a different parametrization, $p \neq p_\infty$, while Eq. (C3) of the arXiv version 2 has been updated with the notation $p = p_\infty$.]

VII. NEW CONTRIBUTIONS TO THE RADIATED ANGULAR MOMENTUM

Similarly, for the angular momentum, we have

$$J^{\rm rad} = J^{\rm rad\,inst\,\mathit{I},\mathit{J}\leq 3PN,cons} + \delta^{\rm rr} J^{\rm rad\,inst\,\mathit{I},\mathit{J}} + J^{\rm rad\,mem\mathit{I},\mathit{J}} + \Delta J^{\rm rad\,inst\,\mathit{I},\mathit{J}} + J^{\rm rad\,tail} + J^{\rm rad\,higher\text{-}order\,tails}. \tag{7.1}$$

The 2.5PN instantaneous term is also vanishing in this case [39]. Therefore, the only contributions at that order come from the 1PN corrections to the LO tails, a memory term [36,39], and the radiation-reaction correction to hyperbolic motion. The latter turns out to be

$$\begin{aligned}\delta^{\rm rr} J^{\rm rad\,inst\,\mathit{I},\mathit{J}} = \frac{GM^2}{c}\nu^3\eta^5 \frac{1}{[\bar{a}_r(e_r^2-1)]^{9/2}}\bigg[&\arccos^2\left(-\frac{1}{e_r}\right)\left(\frac{5264}{75}e_r^4+\frac{1792}{15}e_r^2+\frac{8192}{25}\right)\\ &+(e_r^2-1)^{1/2}\arccos\left(-\frac{1}{e_r}\right)\left(\frac{752}{25}e_r^4+\frac{59792}{225}e_r^2+\frac{33248}{45}\right)\\ &+\frac{e_r^2-1}{e_r^2}\left(\frac{128}{25}e_r^6+\frac{1328}{75}e_r^4+\frac{23968}{45}e_r^2-\frac{8576}{225}\right)\bigg],\end{aligned} \tag{7.2}$$

as an exact expression in terms of $\bar{a}_r$ and e_r. The beginning of its $\frac{1}{j}$ expansion is

$$\delta^{\rm rr} J^{\rm rad\,inst\,\mathit{I},\mathit{J}} = \frac{GM^2}{c}\nu^3\eta^5\left[\frac{128}{25}\frac{p_\infty^6}{j^3}+\frac{376}{25}\pi\frac{p_\infty^5}{j^4}+\left(\frac{4352}{75}+\frac{1316}{75}\pi^2\right)\frac{p_\infty^4}{j^5}+\frac{52456}{225}\pi\frac{p_\infty^3}{j^6}+O\left(\frac{p_\infty^2}{j^7}\right)\right]. \tag{7.3}$$

Adding all terms leads to the final result,

$$\begin{aligned}J^{\rm rad}_{\rm 2.5PN} = \frac{GM^2}{c}\nu^2\eta^5\bigg[&\left(\frac{1184}{21}-\frac{431936\nu}{1575}\right)\frac{p_\infty^6}{j^3}+\left(\left(\frac{7816}{525}-\frac{2232\pi^2}{35}\right)\nu-\frac{1305\pi^2}{112}+\frac{7488}{25}\right)\pi\frac{p_\infty^5}{j^4}\\ &+\left(\left(-\frac{225536}{525}-\frac{201724\pi^2}{33075}\right)\nu+\frac{4116\zeta(3)}{5}-\frac{130688\pi^2}{6615}+\frac{147064}{315}\right)\frac{p_\infty^4}{j^5}\\ &+\left(\left(\frac{365392}{1575}-\frac{57037\pi^2}{21}+\frac{102619\pi^4}{448}\right)\nu+\frac{163083\pi^4}{1792}-\frac{18227\pi^2}{28}+\frac{32}{15}\right)\pi\frac{p_\infty^3}{j^6}+O\left(\frac{p_\infty^2}{j^7}\right)\bigg].\end{aligned} \tag{7.4}$$

New with this work is also the computation of the full 3PN-level contribution to the angular momentum loss. It is obtained by combining the (exact) instantaneous contribution of Ref. [37] (which we independently recomputed) and higher-order tails [36]. We got

$$\begin{aligned}J^{\rm rad}_{\rm 3PN} = \frac{GM^2}{c}\nu^2\eta^6\bigg[&\left(-\frac{16\nu^3}{5}+\frac{24\nu^2}{7}+\frac{878\nu}{315}+\frac{3712}{3465}\right)\frac{p_\infty^9}{j}+\left(-\frac{553\nu^3}{24}+\frac{9235\nu^2}{672}+\frac{1469\nu}{504}+\frac{115769}{126720}\right)\pi\frac{p_\infty^8}{j^2}\\ &+\left(-\frac{6224\nu^3}{15}+\frac{67432\nu^2}{315}+\frac{1459694\nu}{11025}-\frac{4955072}{121275}\right)\frac{p_\infty^7}{j^3}\\ &+\left(-\frac{861\nu^3}{2}+\frac{74693\nu^2}{280}+\left(\frac{2048629}{7560}-\frac{123\pi^2}{32}\right)\nu-\frac{4922}{175}\ln\left(\frac{p_\infty}{2}\right)+\frac{46\pi^2}{5}-\frac{561803611}{10584000}\right)\pi\frac{p_\infty^6}{j^4}\\ &+\left(-\frac{136976\nu^3}{45}+\frac{13320808\nu^2}{4725}+\left(\frac{85939786}{42525}-\frac{3362\pi^2}{75}\right)\nu-\frac{931328}{1575}\ln(2p_\infty)+\frac{8704\pi^2}{45}+\frac{7781823776}{16372125}\right)\frac{p_\infty^5}{j^5}\\ &+\left(-\frac{6517\nu^3}{4}+\frac{794749\nu^2}{336}+\left(\frac{46277}{432}-\frac{861\pi^2}{64}\right)\nu-\frac{21614}{35}\ln\left(\frac{p_\infty}{2}\right)-\frac{45261\zeta(3)}{40}+202\pi^2+\frac{5288341351}{4233600}\right)\pi\frac{p_\infty^4}{j^6}\\ &+O\left(\frac{p_\infty^3}{j^7}\right)\bigg].\end{aligned} \tag{7.5}$$

VIII. 1PN-ACCURATE TAIL CONTRIBUTION TO THE RADIATED LINEAR MOMENTUM

Let us now tackle the technically challenging (fractionally 1PN) tail contribution to the radiated linear momentum, namely, the term $P_i^{\rm rad\,tail}$ in Eq. (4.6). It is the time integral of the following linear-momentum-flux integrand (see Eq. (2.5) of Ref. [43]):

$$\mathcal{F}_P^{i\,\mathrm{tail}} = \frac{G^2\mathcal{M}}{c^{10}}\left\{\frac{4}{63}\left(F^i_{I_3^{(4)}I_2^{(5)}}+F^i_{I_2^{(3)}I_3^{(6)}}\right)+\frac{32}{45}\left({}^*F^i_{I_2^{(3)}J_2^{(5)}}+{}^*F^i_{J_2^{(3)}I_2^{(5)}}\right)+\frac{1}{c^2}\left[\frac{1}{567}\left(F^i_{I_4^{(5)}I_3^{(6)}}+F^i_{I_3^{(4)}I_4^{(7)}}\right)\right.\right.$$
$$\left.\left.+\frac{1}{63}\left({}^*F^i_{I_3^{(4)}J_3^{(6)}}+{}^*F^i_{J_3^{(4)}I_3^{(6)}}\right)+\frac{8}{63}\left(F^i_{J_3^{(4)}J_2^{(5)}}+F^i_{J_2^{(3)}J_3^{(6)}}\right)\right]\right\}, \tag{8.1}$$

where $\mathcal{M}=M(1+\nu\bar{E})$ is the total ADM mass of the system, and the definitions of the quantities $F^i_{X_L^{(n)}Y_M^{(m)}}$ in terms of the source multipole moments are given in Table I.

Introducing the shorthand notation

$$\langle f\rangle = \int_{-\infty}^{\infty} dt f(t), \tag{8.2}$$

for the total time integral of an arbitrary function $f(t)$ over the full scattering process, we need to evaluate

$$P_i^{\mathrm{rad\,tail}} \equiv \langle \mathcal{F}_P^{i\,\mathrm{tail}}\rangle. \tag{8.3}$$

We found useful to evaluate this integral in the frequency domain by using a quasi-Keplerian parametrization of the motion in harmonic coordinates. We refer to previous works for a review of all necessary tools (see, e.g., Ref. [35]).

TABLE I. Definition of the various terms $F^i_{X_L^{(n)}Y_M^{(m)}}$ entering the tail part of the linear-momentum flux. Here we have introduced the following set of multipolar tail timescales: $C_{I_2}=2\tau_0 e^{-11/12}$, $C_{I_3}=2\tau_0 e^{-97/60}$, $C_{J_2}=2\tau_0 e^{-7/6}$, $C_{J_3}=2\tau_0 e^{-5/3}$, and $C_{I_4}=2\tau_0 e^{-59/30}$.

$F^i_{I_3^{(4)}I_2^{(5)}}$	$I^{(4)}_{ijk}(t)\int_0^\infty d\tau I^{(5)}_{jk}(t-\tau)\ln(\frac{\tau}{C_{I_2}})$
$F^i_{I_2^{(3)}I_3^{(6)}}$	$I^{(3)}_{jk}(t)\int_0^\infty d\tau I^{(6)}_{ijk}(t-\tau)\ln(\frac{\tau}{C_{I_3}})$
${}^*F^i_{I_2^{(3)}J_2^{(5)}}$	$\epsilon_{ijk}I^{(3)}_{ja}(t)\int_0^\infty d\tau J^{(5)}_{ka}(t-\tau)\ln(\frac{\tau}{C_{J_2}})$
${}^*F^i_{J_2^{(3)}I_2^{(5)}}$	$\epsilon_{ijk}J^{(3)}_{ka}(t)\int_0^\infty d\tau I^{(5)}_{ja}(t-\tau)\ln(\frac{\tau}{C_{I_2}})$
$F^i_{I_4^{(5)}I_3^{(6)}}$	$I^{(5)}_{ijkl}(t)\int_0^\infty d\tau I^{(6)}_{jkl}(t-\tau)\ln(\frac{\tau}{C_{I_3}})$
$F^i_{I_3^{(4)}I_4^{(7)}}$	$I^{(4)}_{jkl}(t)\int_0^\infty d\tau I^{(7)}_{ijkl}(t-\tau)\ln(\frac{\tau}{C_{I_4}})$
${}^*F^i_{I_3^{(4)}J_3^{(6)}}$	$\epsilon_{ijk}I^{(4)}_{jab}(t)\int_0^\infty d\tau J^{(6)}_{kab}(t-\tau)\ln(\frac{\tau}{C_{J_3}})$
${}^*F^i_{J_3^{(4)}I_3^{(6)}}$	$\epsilon_{ijk}J^{(4)}_{kab}(t)\int_0^\infty d\tau I^{(6)}_{jab}(t-\tau)\ln(\frac{\tau}{C_{I_3}})$
$F^i_{J_3^{(4)}J_2^{(5)}}$	$J^{(4)}_{ijk}(t)\int_0^\infty d\tau J^{(5)}_{jk}(t-\tau)\ln(\frac{\tau}{C_{J_2}})$
$F^i_{J_2^{(3)}J_3^{(6)}}$	$J^{(3)}_{jk}(t)\int_0^\infty d\tau J^{(6)}_{ijk}(t-\tau)\ln(\frac{\tau}{C_{J_3}})$

Expanding the various multipole moments as Fourier integrals

$$X_L(t)\equiv\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\hat{X}_L(\omega),$$
$$\hat{X}_L(\omega)\equiv\int_{-\infty}^{\infty}dt e^{i\omega t}X_L(t), \tag{8.4}$$

leads to (denoting $\int_\omega \equiv \int_0^\infty \frac{d\omega}{2\pi}$)

$$\left\langle F^i_{I_3^{(4)}I_2^{(5)}}+F^i_{I_2^{(3)}I_3^{(6)}}\right\rangle = \int_\omega \omega^8\left(i\pi\mathcal{S}_i^- - \frac{7}{10}\mathcal{S}_i^+\right),$$
$$\left\langle {}^*F^i_{I_2^{(3)}J_2^{(5)}}+{}^*F^i_{J_2^{(3)}I_2^{(5)}}\right\rangle = \int_\omega \omega^7\left(\pi\mathcal{R}_i^+ - \frac{i}{4}\mathcal{R}_i^-\right),$$
$$\left\langle F^i_{I_4^{(5)}I_3^{(6)}}+F^i_{I_3^{(4)}I_4^{(7)}}\right\rangle = \int_\omega \omega^{10}\left(i\pi\mathcal{U}_i^- - \frac{7}{20}\mathcal{U}_i^+\right),$$
$$\left\langle {}^*F^i_{I_3^{(4)}J_3^{(6)}}+{}^*F^i_{J_3^{(4)}I_3^{(6)}}\right\rangle = \int_\omega \omega^9\left(\pi\mathcal{V}_i^+ - \frac{i}{20}\mathcal{V}_i^-\right),$$
$$\left\langle F^i_{J_3^{(4)}J_2^{(5)}}+F^i_{J_2^{(3)}J_3^{(6)}}\right\rangle = \int_\omega \omega^8\left(i\pi\mathcal{Z}_i^- - \frac{1}{2}\mathcal{Z}_i^+\right), \tag{8.5}$$

where

$$\mathcal{S}_i^\pm(\omega) = \hat{I}_{ijk}(-\omega)\hat{I}_{jk}(\omega)\pm\hat{I}_{ijk}(\omega)\hat{I}_{jk}(-\omega),$$
$$\mathcal{R}_i^\pm(\omega) = \epsilon_{ijk}[\hat{I}_{ja}(\omega)\hat{J}_{ka}(-\omega)\pm\hat{I}_{ja}(-\omega)\hat{J}_{ka}(\omega)],$$
$$\mathcal{U}_i^\pm(\omega) = \hat{I}_{ijkl}(-\omega)\hat{I}_{jkl}(\omega)\pm\hat{I}_{ijkl}(\omega)\hat{I}_{jkl}(-\omega),$$
$$\mathcal{V}_i^\pm(\omega) = \epsilon_{ijk}[\hat{I}_{jab}(\omega)\hat{J}_{kab}(-\omega)\pm\hat{I}_{jab}(-\omega)\hat{J}_{kab}(\omega)],$$
$$\mathcal{Z}_i^\pm(\omega) = \hat{J}_{ijk}(-\omega)\hat{J}_{jk}(\omega)\pm\hat{J}_{ijk}(\omega)\hat{J}_{jk}(-\omega). \tag{8.6}$$

The leading-PN-order tail contribution (8.3) [i.e., the first two lines in Eq. (8.5)] has been already computed in Ref. [30]; see also Ref. [40]]. We focus here on the next-to-leading order (fractionally 1PN) tail contribution. We need to take into account the fractional 1PN corrections to the first two lines in Eqs. (8.5), whereas the leading PN order is enough for the remaining three lines in Eqs. (8.5). The final results for the large-j expansions of the (nonvanishing) components $P_x^{\mathrm{rad\,tail}}$ and $P_y^{\mathrm{rad\,tail}}$ are

$$P_x^{\rm rad\,tail}=-(Mc)\frac{m_2-m_1}{M}\nu^2\eta^3\left[\pi\frac{\frac{1491}{400}p_\infty^7+\eta^2\left(-\frac{9529}{67200}\nu-\frac{26757}{5600}\right)p_\infty^9}{j^4}+\frac{\frac{20608}{225}p_\infty^6+\eta^2\left(\frac{72512}{7875}\nu-\frac{1143232}{7875}\right)p_\infty^8}{j^5}\right.$$
$$\left.+\pi\frac{\frac{267583}{2400}p_\infty^5+\eta^2\left(\frac{1711123}{57600}\nu-\frac{12566143}{67200}\right)p_\infty^7}{j^6}+\frac{\frac{64576}{75}p_\infty^4+\eta^2\left(\frac{29244128}{70875}\nu-\frac{9920672}{7875}\right)p_\infty^6}{j^7}+O\left(\frac{1}{j^8}\right)\right],$$
$$P_y^{\rm rad\,tail}=-(Mc)\frac{m_2-m_1}{M}\nu^2\eta^3\left[\frac{-\frac{128}{3}p_\infty^7+\eta^2\left(\frac{320}{3}\nu-\frac{192}{175}\right)p_\infty^9}{j^4}+\pi\frac{-\frac{1509\pi^2}{140}p_\infty^6+\eta^2\left(\frac{2721}{80}\pi^2\nu-\frac{2432}{15}+\frac{75661}{4480}\pi^2\right)p_\infty^8}{j^5}\right.$$
$$+\frac{\left(-\frac{8768}{45}-\frac{521216\pi^2}{4725}\right)p_\infty^5+\eta^2\left[\left(\frac{43457024}{99225}\pi^2+\frac{37792}{45}\right)\nu-\frac{77389}{1050}+\frac{42827264}{1091475}\pi^2-\frac{9489}{20}\zeta(3)\right]p_\infty^7}{j^6}$$
$$+\pi\frac{\left(\frac{36885}{896}\pi^4-\frac{142391}{280}\pi^2\right)p_\infty^4+\eta^2\left[\left(-\frac{208525}{1024}\pi^4+\frac{8537719}{3360}\pi^2\right)\nu-\frac{44900896}{55125}-\frac{328765}{1792}\pi^4+\frac{989879573}{549120}\pi^2\right]p_\infty^6}{j^7}$$
$$\left.+O\left(\frac{1}{j^8}\right)\right]. \tag{8.7}$$

These tail contributions take into account the physical retarded-tail interaction between the bodies, so that they are asymmetric under time reversal (they were called "past tails" in Refs. [30,36]). Let us note in passing that replacing the retarded kernel in the time-domain tail integral by its *time-symmetric* projection would lead to the following integral:

$$P_i^{\rm rad\,sym\,tail}=\frac{G^2\mathcal{M}}{c^{10}}\left\{\frac{32}{45}\pi\int_0^\infty\frac{d\omega}{2\pi}\omega^7\mathcal{R}_i^+(\omega)+\frac{4}{63}i\pi\int_0^\infty\frac{d\omega}{2\pi}\omega^8\mathcal{S}_i^-(\omega)+\frac{1}{c^2}\left[\frac{1}{567}i\pi\int_0^\infty\frac{d\omega}{2\pi}\omega^{10}\mathcal{U}_i^-(\omega)\right.\right.$$
$$\left.\left.+\frac{1}{63}\pi\int_0^\infty\frac{d\omega}{2\pi}\omega^9\mathcal{V}_i^+(\omega)+\frac{8}{63}i\pi\int_0^\infty\frac{d\omega}{2\pi}\omega^8\mathcal{Z}_i^-(\omega)\right]\right\}, \tag{8.8}$$

implying

$$P_x^{\rm rad\,sym\,tail}=0,\qquad P_y^{\rm rad\,sym\,tail}=P_y^{\rm rad\,tail}. \tag{8.9}$$

The complete 2.5PN radiated linear momentum is then obtained by summing up all contributions, Eqs. (4.14), (5.3), and (8.7). The final result is listed in Tables II and III as a double PM-PN expansion [see Eq. (10.1) below].

IX. 3PN-LEVEL CONTRIBUTION TO THE RADIATED LINEAR MOMENTUM

The radiated instantaneous linear momentum at the fractional 3PN accuracy can be obtained by integrating the 3PN instantaneous linear-momentum flux,

$${\mathcal{F}_P}_i^{{\rm inst}\,I,J\le 3{\rm PN}}=\frac{G}{c^7}\left(f_i^0+\frac{1}{c^2}f_i^1+\frac{1}{c^4}f_i^2+\frac{1}{c^5}f_i^{2.5}+\frac{1}{c^6}f_i^3\right), \tag{9.1}$$

where

$$\begin{aligned}
f_i^0&=\frac{2}{63}I_{ijk}^{(4)}I_{jk}^{(3)}+\frac{16}{45}\epsilon_{ijk}I_{jc}^{(3)}J_{kc}^{(3)},\\
f_i^1&=\frac{4}{63}J_{ijk}^{(4)}J_{jk}^{(3)}+\frac{1}{1134}I_{ijkl}^{(5)}I_{jkl}^{(4)}+\frac{1}{126}\epsilon_{ijk}I_{jab}^{(4)}J_{kab}^{(4)},\\
f_i^2&=\frac{2}{945}J_{ijkl}^{(5)}J_{jkl}^{(4)}+\frac{1}{59400}I_{ijklm}^{(6)}I_{jklm}^{(5)}+\frac{2}{14175}\epsilon_{ijk}I_{jabc}^{(5)}J_{kabc}^{(5)},\\
f_i^3&=\frac{1}{22275}J_{ijklm}^{(6)}I_{jklm}^{(5)}+\frac{1}{4343625}I_{ijklmn}^{(7)}I_{jklmn}^{(6)}+\frac{1}{534600}\epsilon_{ijk}I_{jabcd}^{(6)}J_{kabcd}^{(6)},
\end{aligned} \tag{9.2}$$

TABLE II. New terms at the 2.5PN and 3PN level of fractional accuracy improving the PN expansion given in Table IX of Ref. [30] of the coefficients E_n, J_n, and P_{yn}, entering the PM expansion (10.1) of the radiated energy, angular momentum, and y component of the linear momentum, respectively.

$E_3^{>2\mathrm{PN}}$	$\pi\left[(-\frac{676273}{354816}-\frac{2699}{504}\nu+\frac{321}{280}\nu^2-\frac{148}{15}\nu^3)p_\infty^{10}+O(p_\infty^{11})\right]$
$E_4^{>2\mathrm{PN}}$	$(\frac{1216}{105}-\frac{2848}{15}\nu)p_\infty^8+(-\frac{151854}{13475}-\frac{1223594}{33075}\nu+\frac{164}{3}\nu^2-\frac{2366}{9}\nu^3)p_\infty^9+O(p_\infty^{10})$
$E_5^{>2\mathrm{PN}}$	$\pi\left\{\left[(\frac{296}{25}-\frac{15291\pi^2}{280})\nu-\frac{24993\pi^2}{1120}+\frac{9216}{35}\right]p_\infty^7+\left[\frac{29573617463}{310464000}+\frac{99}{10}\pi^2-\frac{10593}{350}\ln(\frac{p_\infty}{2})+(\frac{76897}{480}-\frac{4059}{640}\pi^2)\nu+\frac{12269}{80}\nu^2-\frac{1823}{5}\nu^3\right]p_\infty^8+O(p_\infty^9)\right\}$
$E_6^{>2\mathrm{PN}}$	$\left[(-\frac{2974508}{4725}\pi^2-\frac{71488}{75})\nu+\frac{56708}{105}+\frac{1024}{135}\pi^2+\frac{2898}{5}\zeta(3)\right]p_\infty^6$ $+\left[-\frac{18955264}{23625}\ln(2p_\infty)+\frac{36589282372}{11694375}+\frac{177152}{675}\pi^2+(-\frac{212216}{1575}\pi^2+\frac{875976284}{297675})\nu+\frac{4201976}{1575}\nu^2-\frac{150892}{45}\nu^3\right]p_\infty^7+O(p_\infty^8)$
$E_7^{>2\mathrm{PN}}$	$\pi\left\{\left[(\frac{56008}{135}-\frac{23514\pi^2}{7}+\frac{30285\pi^4}{112})\nu+\frac{689985\pi^4}{3584}-\frac{13138915\pi^2}{7392}+\frac{210176}{225}\right]p_\infty^5\right.$ $\left.+\left[\frac{3158}{9}\pi^2-\frac{337906}{315}\ln(\frac{p_\infty}{2})+\frac{37546579757}{8467200}-\frac{58957}{32}\zeta(3)+(-\frac{51947}{384}\pi^2+\frac{68898691}{36288})\nu+\frac{1419153}{448}\nu^2-\frac{13955}{6}\nu^3\right]p_\infty^6+O(p_\infty^7)\right\}$
$J_2^{>2\mathrm{PN}}$	$(\frac{3712}{3465}+\frac{878}{315}\nu+\frac{24}{7}\nu^2-\frac{16}{5}\nu^3)p_\infty^9+O(p_\infty^{10})$
$J_3^{>2\mathrm{PN}}$	$\pi\left[(\frac{115769}{126720}-\frac{553}{24}\nu^3+\frac{9235}{672}\nu^2+\frac{1469}{504}\nu)p_\infty^8+O(p_\infty^9)\right]$
$J_4^{>2\mathrm{PN}}$	$(-\frac{431936}{1575}\nu+\frac{1184}{21})p_\infty^6+(-\frac{4955072}{121275}+\frac{1459694}{11025}\nu+\frac{67432}{315}\nu^2-\frac{6224}{15}\nu^3)p_\infty^7+O(p_\infty^8)$
$J_5^{>2\mathrm{PN}}$	$\pi\left\{\left[(\frac{7816}{525}-\frac{2232\pi^2}{35})\nu-\frac{1305\pi^2}{112}+\frac{7488}{25}\right]p_\infty^5+\left[-\frac{4922}{175}\ln(\frac{p_\infty}{2})+\frac{46}{5}\pi^2-\frac{561803611}{10584000}+(-\frac{123}{32}\pi^2+\frac{2048629}{7560})\nu+\frac{74693}{280}\nu^2-\frac{861}{2}\nu^3\right]p_\infty^6+O(p_\infty^7)\right\}$
$J_6^{>2\mathrm{PN}}$	$\left[(-\frac{201724}{33075}\pi^2-\frac{225536}{525})\nu+\frac{4116\zeta(3)}{5}+\frac{147064}{315}-\frac{130688\pi^2}{6615}\right]p_\infty^4$ $+[\frac{8704}{45}\pi^2+\frac{7781823776}{16372125}-\frac{931328}{1575}\ln(2p_\infty)+(-\frac{3362}{75}\pi^2+\frac{85939786}{42525})\nu+\frac{13320808}{4725}\nu^2-\frac{136976}{45}\nu^3]p_\infty^5+O(p_\infty^6)$
$J_7^{>2\mathrm{PN}}$	$\pi\left\{\left[\left(\frac{365392}{1575}-\frac{57037\pi^2}{21}+\frac{102619\pi^4}{448}\right)\nu+\frac{163083\pi^4}{1792}-\frac{18227\pi^2}{28}+\frac{32}{15}\right]p_\infty^3\right.$ $\left.+\left[-\frac{21614}{35}\ln(\frac{p_\infty}{2})-\frac{45261}{40}\zeta(3)+\frac{5288341351}{4233600}+202\pi^2+(\frac{46277}{432}-\frac{861}{64}\pi^2)\nu+\frac{794749}{336}\nu^2-\frac{6517}{4}\nu^3\right]p_\infty^4+O(p_\infty^5)\right\}$
$P_{y3}^{>2\mathrm{PN}}$	$\pi(-\frac{1531643}{1182720}-\frac{27581}{10080}\nu-\frac{197}{560}\nu^2-\frac{74}{15}\nu^3)p_\infty^{11}+O(p_\infty^{12})$
$P_{y4}^{>2\mathrm{PN}}$	$(\frac{192}{175}-\frac{320}{3}\nu)p_\infty^9+(-140\nu^3-\frac{118676}{6615}\nu-\frac{76}{15}\nu^2-\frac{1218176}{72765})p_\infty^{10}+O(p_\infty^{11})$
$P_{y5}^{>2\mathrm{PN}}$	$\pi\left\{\left[\frac{2432}{15}-\frac{75661}{4480}\pi^2+(-\frac{2721}{80}\pi^2+\frac{148}{25})\nu\right]p_\infty^8+\left[-\frac{41053}{2450}\ln(\frac{p_\infty}{2})+\frac{503}{70}\pi^2+\frac{37806320227}{790272000}+(\frac{945563}{8064}-\frac{4059}{1280}\pi^2)\nu+\frac{17617}{840}\nu^2-\frac{6199}{30}\nu^3\right]p_\infty^9+O(p_\infty^{10})\right\}$
$P_{y6}^{>2\mathrm{PN}}$	$\left[\frac{9489}{20}\zeta(3)+\frac{77389}{1050}-\frac{42827264}{1091475}\pi^2+(-\frac{172734857}{396900}\pi^2-\frac{31648}{45})\nu\right]p_\infty^7$ $+(-\frac{85434368}{165375}\ln(2p_\infty)+\frac{1042432}{4725}\pi^2+\frac{393851925056}{191008125}+(-\frac{8528}{105}\pi^2+\frac{815056834}{297675})\nu+\frac{174074}{225}\nu^2-\frac{30422}{15}\nu^3)p_\infty^8+O(p_\infty^9)$
$P_{y7}^{>2\mathrm{PN}}$	$\pi\left\{\left[(\frac{208525}{1024}\pi^4-\frac{8537719}{3360}\pi^2+\frac{160406}{675})\nu+\frac{328765}{1792}\pi^4-\frac{989879573}{549120}\pi^2+\frac{44900896}{55125}\right]p_\infty^6\right.$ $\left.+(-\frac{35125513}{44100}\ln(\frac{p_\infty}{2})-\frac{303491}{224}\zeta(3)+\frac{907691}{2688}\pi^2+\frac{1006741665001549}{312947712000}+(-\frac{3017083}{30720}\pi^2+\frac{2124695071}{725760})\nu+\frac{1209467}{960}\nu^2-\frac{30181}{20}\nu^3)p_\infty^7+O(p_\infty^8)\right\}$

TABLE III. PN expansion of the coefficients P_{xn} of the x component of the radiated linear momentum through the 3PN fractional accuracy.

P_{x4}	$\pi\left[-\frac{1491}{400}p_\infty^7+(\frac{1491}{200}\nu+\frac{26757}{5600})p_\infty^9+O(p_\infty^{11})\right]$
P_{x5}	$-\frac{20608}{225}p_\infty^6+(\frac{10304}{45}\nu+\frac{1143232}{7875})p_\infty^8-\frac{196096}{945}p_\infty^9+O(p_\infty^{10})$
P_{x6}	$\pi\left[-\frac{267583}{2400}p_\infty^5+(\frac{2509097}{7200}\nu+\frac{12566143}{67200})p_\infty^7-\frac{20719}{320}\pi^2p_\infty^8+O(p_\infty^9)\right]$
P_{x7}	$-\frac{64576}{75}p_\infty^4+(\frac{3802976}{1125}\nu+\frac{9920672}{7875})p_\infty^6+(-\frac{42739712}{55125}\pi^2-\frac{9226496}{4725})p_\infty^7+O(p_\infty^8)$

with f_i^0 (namely, I_{ij}, I_{ijk}, and J_{ij}) to be evaluated at the 3PN level of accuracy, f_i^1 at 2PN, etc. The 2.5PN contribution $f_i^{2.5}$ has already been discussed in the previous sections.

Moreover, all multipoles are needed in modified harmonic coordinates and several of them already exist in the literature (mainly from Ref. [63]), while for the others only the expression in harmonic coordinates is known, and one has to transform their expression to modified harmonic coordinates, following Ref. [64], Sec. IV B. More precisely,

(1) I_{ij}, needed at 3PN, see Eqs. (3.1) and (3.2c) of Ref. [64]; see also Eqs. (3.19) and (3.20) of Ref. [63];
(2) I_{ijk}, needed at 3PN, see Eqs. (4.9) and (4.10) of Ref. [65] for the expression in standard harmonic coordinates;
(3) I_{ijkl}, needed at 2PN, see Eq. (3.23a) of Ref. [63];
(4) I_{ijklm}, needed at 1PN, see Eq. (3.23b) of Ref. [63];
(5) I_{ijklmn}, needed at N, see Eq. (3.23c) of Ref. [63];
(6) J_{ij}, needed at 3PN, see Eqs. (3.6) and (3.7) of Ref. [66] for the expression in standard harmonic coordinates;
(7) J_{ijk}, needed at 2PN, see Eq. (3.26a) of Ref. [63];
(8) J_{ijkl}, needed at 1PN, see Eq. (3.26b) of Ref. [63];
(9) J_{ijklm}, needed at N, see Eq. (3.26c) of Ref. [63].

The final 3PN instantaneous term for a generic orbit reads

$$\mathcal{F}_{P\,i}^{\mathrm{inst}\,I,J\,3\mathrm{PN}} = \frac{G^3M^3\nu^2}{r^4c^7}(m_2-m_1)\eta^6(A^{3\mathrm{PN}}\dot{r}n^i + B^{3\mathrm{PN}}v^i), \tag{9.3}$$

with

$$\begin{aligned}
A^{3\mathrm{PN}} = {} & \left(\frac{50647}{4095} - \frac{6891347}{45045}\nu + \frac{378098}{715}\nu^2 - \frac{17700712}{45045}\nu^3\right)v^8 + \left[\left(-\frac{1486192}{15015} + \frac{38072087}{45045}\nu - \frac{124611538}{45045}\nu^2 + \frac{7420632}{5005}\nu^3\right)\dot{r}^2\right.\\
& \left. + \left(-\frac{5742794}{35035} + \frac{2875777}{3465}\nu - \frac{10668793}{9009}\nu^2 + \frac{3851017}{6435}\nu^3\right)\frac{GM}{r}\right]v^6\\
& + \left[\left(\frac{2039066}{5005} - \frac{95290066}{45045}\nu + \frac{18801898}{4095}\nu^2 - \frac{9617408}{5005}\nu^3\right)\dot{r}^4\right.\\
& + \left(\frac{925151368}{945945} - \frac{682213787}{135135}\nu + \frac{865924949}{135135}\nu^2 - \frac{95805097}{45045}\nu^3\right)\frac{GM\dot{r}^2}{r}\\
& \left. + \left(\frac{173961024956}{165540375} - \frac{486464}{3675}\ln\left(\frac{r}{r_0}\right) + \left(-\frac{1845}{28}\pi^2 - \frac{312503921}{675675}\right)\nu + \frac{33465314}{27027}\nu^2 - \frac{16820996}{45045}\nu^3\right)\frac{G^2M^2}{r^2}\right]v^4\\
& + \left[\left(-\frac{683368}{1287} + \frac{20252888}{9009}\nu - \frac{27439754}{9009}\nu^2 + \frac{9326368}{9009}\nu^3\right)\dot{r}^6\right.\\
& + \left(-\frac{836106314}{525525} + \frac{88323799}{10010}\nu - \frac{21042751}{2310}\nu^2 + \frac{103758968}{45045}\nu^3\right)\frac{GM\dot{r}^4}{r}\\
& + \left(-\frac{742542259516}{165540375} + \frac{1647104}{3675}\ln\left(\frac{r}{r_0}\right) + \left(\frac{66462078}{25025} + \frac{3075}{14}\pi^2\right)\nu - \frac{50439274}{10395}\nu^2 + \frac{13610134}{12285}\nu^3\right)\frac{G^2M^2\dot{r}^2}{r^2}\\
& \left. + \left(-\frac{69490090246}{55180125} + \frac{60992}{3675}\ln\left(\frac{r}{r_0}\right) + \left(\frac{56867}{840}\pi^2 - \frac{1698895}{2457}\right)\nu - \frac{993904}{2457}\nu^2 + \frac{891622}{7371}\nu^3\right)\frac{G^3M^3}{r^3}\right]v^2\\
& + \left(\frac{58349}{273} - \frac{2486416}{3003}\nu + \frac{2067616}{3003}\nu^2 - \frac{195680}{1001}\nu^3\right)\dot{r}^8\\
& + \left(\frac{1228477436}{1576575} - \frac{408167713}{90090}\nu + \frac{69532495}{18018}\nu^2 - \frac{1022414}{1287}\nu^3\right)\frac{GM\dot{r}^6}{r}
\end{aligned}$$

$$+\left(\frac{111525644752}{33108075}-\frac{42496}{147}\ln\left(\frac{r}{r_0}\right)+\left(-\frac{615}{4}\pi^2-\frac{67591807}{27027}\right)\nu+\frac{153157904}{45045}\nu^2-\frac{4165558}{6435}\nu^3\right)\frac{G^2M^2\dot{r}^4}{r^2}$$
$$+\left(\frac{4786348054}{5016375}+\frac{174592}{3675}\ln\left(\frac{r}{r_0}\right)+\left(-\frac{20869}{280}\pi^2+\frac{1498465697}{4729725}\right)\nu+\frac{15571532}{27027}\nu^2-\frac{9925382}{135135}\nu^3\right)\frac{G^3M^3\dot{r}^2}{r^3}$$
$$+\left(-\frac{21844644124}{496621125}+\frac{11904}{1225}\ln\left(\frac{r}{r_0}\right)+\left(-\frac{41}{70}\pi^2-\frac{2315974202}{4729725}\right)\nu+\frac{179768}{3003}\nu^2-\frac{40396}{6237}\nu^3\right)\frac{G^4M^4}{r^4} \tag{9.4}$$

and

$$B^{3\mathrm{PN}}=\left(-\frac{438226}{15015}+\frac{40657}{231}\nu-\frac{1010414}{3003}\nu^2+\frac{3600536}{15015}\nu^3\right)v^8+\left[\left(\frac{9523744}{45045}-\frac{395467}{315}\nu+\frac{7045306}{3465}\nu^2-\frac{3058024}{3465}\nu^3\right)\dot{r}^2\right.$$
$$\left.+\left(\frac{143914678}{1576575}-\frac{3249853}{6435}\nu+\frac{1744231}{3003}\nu^2-\frac{2536901}{9009}\nu^3\right)\frac{GM}{r}\right]v^6$$
$$+\left[\left(-\frac{1756652}{3003}+\frac{5370304}{5005}\nu^3-\frac{159792518}{45045}\nu^2+\frac{19699546}{6435}\nu\right)\dot{r}^4\right.$$
$$+\left(-\frac{230962616}{315315}+\frac{2612939}{715}\nu-\frac{24270509}{6435}\nu^2+\frac{44195497}{45045}\nu^3\right)\frac{GM\dot{r}^2}{r}$$
$$\left.+\left(-\frac{429821166328}{496621125}+\frac{337504}{3675}\ln\left(\frac{r}{r_0}\right)+\left(\frac{6519}{280}\pi^2+\frac{80267816}{405405}\right)\nu-\frac{1917296}{4095}\nu^2+\frac{43333016}{405405}\nu^3\right)\frac{G^2M^2}{r^2}\right]v^4$$
$$+\left[\left(\frac{3974752}{6435}-\frac{27078616}{9009}\nu+\frac{1667210}{693}\nu^2-\frac{2625664}{5005}\nu^3\right)\dot{r}^6\right.$$
$$+\left(\frac{46881654}{35035}-\frac{124322753}{18018}\nu+\frac{115944133}{20790}\nu^2-\frac{6226289}{6435}\nu^3\right)\frac{GM\dot{r}^4}{r}$$
$$+\left(\frac{205470694976}{55180125}-\frac{1286752}{3675}\ln\left(\frac{r}{r_0}\right)+\left(-\frac{47299991}{45045}-\frac{3813}{35}\pi^2\right)\nu+\frac{8684332}{3861}\nu^2-\frac{7223842}{19305}\nu^3\right)\frac{G^2M^2\dot{r}^2}{r^2}$$
$$\left.+\left(\frac{25042228006}{70945875}+\frac{704}{175}\ln\left(\frac{r}{r_0}\right)+\left(-\frac{21607}{840}\pi^2+\frac{783374999}{1289925}\right)\nu+\frac{411716}{10395}\nu^2-\frac{8327414}{405405}\nu^3\right)\frac{G^3M^3}{r^3}\right]v^2$$
$$+\left(-\frac{1974958}{9009}+\frac{440666}{429}\nu-\frac{4954028}{9009}\nu^2+\frac{767248}{9009}\nu^3\right)\dot{r}^8$$
$$+\left(-\frac{219642736}{315315}+\frac{47166701}{12870}\nu-\frac{3852731}{1638}\nu^2+\frac{629897}{2145}\nu^3\right)\frac{GM\dot{r}^6}{r}$$
$$+\left(-\frac{875401568}{315315}+\frac{1600}{7}\ln\left(\frac{r}{r_0}\right)+\left(\frac{24149}{280}\pi^2+\frac{58379561}{45045}\right)\nu-\frac{31792976}{19305}\nu^2+\frac{23299642}{135135}\nu^3\right)\frac{G^2M^2\dot{r}^4}{r^2}$$
$$+\left(-\frac{1458606746}{165540375}-\frac{716608}{11025}\ln\left(\frac{r}{r_0}\right)+\left(\frac{9389}{280}\pi^2-\frac{5177354543}{14189175}\right)\nu-\frac{34654208}{135135}\nu^2+\frac{596782}{81081}\nu^3\right)\frac{G^3M^3\dot{r}^2}{r^3}$$
$$+\left(\frac{5316518908}{45147375}-\frac{44032}{11025}\ln\left(\frac{r}{r_0}\right)+\left(-\frac{41}{14}\pi^2+\frac{5234646748}{14189175}\right)\nu-\frac{140216}{12285}\nu^2+\frac{1524088}{405405}\nu^3\right)\frac{G^4M^4}{r^4}. \tag{9.5}$$

The integration along hyperboliclike orbits (see Appendix D) can be carried on exactly and the sought for 3PN contribution reads

$$P_x^{\mathrm{rad\,inst},I,J\,3\mathrm{PN}}=0,$$
$$P_y^{\mathrm{rad\,inst},I,J\,3\mathrm{PN}}=(Mc)\frac{m_2-m_1}{M}\nu^2\eta^6\frac{1}{e_r[\bar{a}_r(e_r^2-1)]^7}(Q_y^{A^0}+Q_y^{A^1}A+Q_y^{A^2}A^2+Q_y^{A^3}A^3), \tag{9.6}$$

with $A=\arccos(-1/e_r)$ and

$$
\begin{aligned}
Q_x^{A^3} &= e_r^8\left(-\frac{37\nu}{20}-\frac{375}{112}\right)+e_r^6\left(\frac{72427}{420}-\frac{7\nu}{2}\right)+e_r^4\left(\frac{1104\nu}{5}+482\right)+e_r^2\left(\frac{804\nu}{5}+\frac{8798}{105}\right)+\frac{32}{5},\\
Q_y^{A^2} &= \sqrt{e_r^2-1}\left[e_r^6\left(\frac{311517}{2800}-\frac{849\nu}{20}\right)+e_r^4\left(\frac{1831\nu}{5}+\frac{6287443}{4200}\right)+e_r^2\left(\frac{3861\nu}{5}+\frac{61543}{105}\right)+\frac{164\nu}{5}+\frac{15473}{525}\right],\\
Q_y^{A^1} &= e_r^2\left(-\frac{41053e_r^8}{1225}-\frac{3180022e_r^6}{2205}-\frac{4558096e_r^4}{735}-\frac{19581152e_r^2}{3675}-\frac{9002752}{11025}\right)\ln\left(\frac{2\bar{a}_r(e_r^2-1)}{e_r r_0}\right)\\
&\quad+e_r^{12}\left(-\frac{37\nu^3}{48}+\frac{2447\nu^2}{448}-\frac{349061\nu}{20160}+\frac{26726213}{591360}\right)\\
&\quad+e_r^{10}\left(-\frac{351\nu^3}{40}-\frac{4891\nu^2}{630}+\left(\frac{5171197}{22400}-\frac{9143\pi^2}{1920}\right)\nu-\frac{5601182987}{32928000}\right)\\
&\quad+e_r^8\left(\frac{46\nu^3}{15}-\frac{230705\nu^2}{252}+\left(\frac{691093489}{453600}-\frac{3284551\pi^2}{15360}\right)\nu+\frac{253778681389}{17385984}\right)\\
&\quad+e_r^6\left(\frac{47\nu^3}{5}-\frac{407258\nu^2}{105}+\left(-\frac{7006220329}{226800}-\frac{644561\pi^2}{960}\right)\nu+\frac{15212690520617}{244490400}\right)\\
&\quad+e_r^4\left(\frac{8\nu^3}{5}-\frac{1305494\nu^2}{315}+\left(-\frac{772592833}{14175}-\frac{54899\pi^2}{192}\right)\nu+\frac{326345642761}{7546000}\right)\\
&\quad+e_r^2\left(-\frac{7040\nu^2}{9}+\left(\frac{2419\pi^2}{60}-\frac{127239209}{9450}\right)\nu+\frac{253937658533}{50935500}\right)\\
&\quad-\frac{208\nu^2}{15}+\left(-\frac{9398}{175}-\frac{82\pi^2}{15}\right)\nu-\frac{183451039}{1984500},\\
Q_y^{A^0} &= e_r^2\left(-\frac{41053e_r^8}{1225}-\frac{3180022e_r^6}{2205}-\frac{4558096e_r^4}{735}-\frac{19581152e_r^2}{3675}-\frac{9002752}{11025}\right)\mathrm{Cl}_2\left(2A\right)\\
&\quad+\sqrt{e_r^2-1}\left[\left(-\frac{79892213e_r^8}{165375}-\frac{153055244e_r^6}{33075}-\frac{42259956e_r^4}{6125}-\frac{296405824e_r^2}{165375}-\frac{3215456}{165375}\right)\ln\left(\frac{\bar{a}_r e_r}{2r_0}\right)\right.\\
&\quad+e_r^{10}\left(-\frac{283\nu^3}{48}+\frac{292361\nu^2}{11200}-\frac{78936947\nu}{2116800}+\frac{679015961}{186278400}\right)\\
&\quad+e_r^8\left(-\frac{214\nu^3}{45}-\frac{2544931\nu^2}{7200}+\left(\frac{364880588983}{190512000}-\frac{1027583\pi^2}{13440}\right)\nu+\frac{4811241578461}{1629936000}\right)\\
&\quad+e^6\left(\frac{2143\nu^3}{180}-\frac{103970549\nu^2}{37800}+\left(-\frac{2926723837}{235200}-\frac{935846033\pi^2}{1612800}\right)\nu+\frac{189910151942603}{4656960000}\right)\\
&\quad+e^4\left(\frac{143\nu^3}{45}-\frac{18824609\nu^2}{3780}+\left(-\frac{478614974947}{7938000}-\frac{417611363\pi^2}{806400}\right)\nu+\frac{2221845272842369}{48898080000}\right)\\
&\quad+e^2\left(\frac{4\nu^3}{45}-\frac{7722154\nu^2}{4725}+\left(\frac{2537449\pi^2}{67200}-\frac{154305135683}{5953500}\right)\nu+\frac{6582692584319}{814968000}\right)\\
&\quad\left.-\frac{70376\nu^2}{1575}+\left(-\frac{395368697}{992250}-\frac{143459\pi^2}{33600}\right)\nu-\frac{369018184091}{6112260000}\right],
\end{aligned}
\tag{9.7}
$$

where

$$
\mathrm{Cl}_2(x)=\frac{i}{2}[\mathrm{Li}_2(e^{-ix})-\mathrm{Li}_2(e^{ix})],
\tag{9.8}
$$

is the Clausen function of order 2.

As expected, these terms involve the arbitrary length scale r_0 (entering the retarded time as well as the relation connecting harmonic to modified harmonic coordinates), which disappears in the complete expression when all 3PN hereditary terms are included, i.e.,

$$P_i^{\mathrm{rad\,3PN}} = P_i^{\mathrm{rad\,inst}\,I,J\,\mathrm{3PN}} + P_i^{\mathrm{rad\,higher\text{-}order\,tails}}. \tag{9.9}$$

Indeed, this is exactly the case when using the results of Ref. [40] for the higher-order tail contributions. We list below the final large-j expansion (including terms from $1/j^3$ up to terms $1/j^7$) of both P_x^{rad} and P_y^{rad},

$$\begin{aligned}
P_x^{\mathrm{rad\,3PN}} &= -(Mc)\frac{m_2-m_1}{M}\nu^2\eta^6\left[\frac{196096}{945}\frac{p_\infty^9}{j^5}+\frac{20719\pi^3}{320}\frac{p_\infty^8}{j^6}+\left(\frac{42739712\pi^2}{55125}+\frac{9226496}{4725}\right)\frac{p_\infty^7}{j^7}+O\left(\frac{p_\infty^6}{j^8}\right)\right],\\
P_y^{\mathrm{rad\,3PN}} &= (Mc)\frac{m_2-m_1}{M}\nu^2\eta^6\Bigg\{\left(-\frac{1531643}{1182720}-\frac{27581}{10080}\nu-\frac{197}{560}\nu^2-\frac{74}{15}\nu^3\right)\pi\frac{p_\infty^{11}}{j^3}\\
&\quad+\left(-\frac{1218176}{72765}-\frac{118676}{6615}\nu-\frac{76}{15}\nu^2-140\nu^3\right)\frac{p_\infty^{10}}{j^4}\\
&\quad+\left[\frac{37806320227}{790272000}+\frac{503\pi^2}{70}-\frac{41053}{2450}\ln\left(\frac{p_\infty}{2}\right)+\left(\frac{945563}{8064}-\frac{4059\pi^2}{1280}\right)\nu+\frac{17617\nu^2}{840}-\frac{6199\nu^3}{30}\right]\pi\frac{p_\infty^9}{j^5}\\
&\quad+\left[\frac{393851925056}{191008125}+\frac{1042432\pi^2}{4725}-\frac{85434368}{165375}\ln(2p_\infty)+\left(\frac{815056834}{297675}-\frac{8528\pi^2}{105}\right)\nu+\frac{174074}{225}\nu^2-\frac{30422\nu^3}{15}\right]\frac{p_\infty^8}{j^6}\\
&\quad+\left[\frac{1006741665001549}{312947712000}+\frac{907691\pi^2}{2688}-\frac{303491\zeta(3)}{224}-\frac{35125513}{44100}\ln\left(\frac{p_\infty}{2}\right)+\left(\frac{2124695071}{725760}-\frac{3017083\pi^2}{30720}\right)\nu\right.\\
&\quad\left.+\frac{1209467\nu^2}{960}-\frac{30181\nu^3}{20}\right]\pi\frac{p_\infty^7}{j^7}+O\left(\frac{p_\infty^6}{j^8}\right)\Bigg\}.
\end{aligned} \tag{9.10}$$

X. SUMMARY OF RESULTS FOR THE ENERGY, ANGULAR MOMENTUM AND LINEAR-MOMENTUM LOSSES IN THE C.M. FRAME

For the convenience of the reader, let us summarize here the new results derived in this work concerning the losses of energy, angular momentum, and linear momentum (radiated as gravitational waves), as recorded in the (initial) c.m. frame. In this section, we use the notation of our previous work [30] for parametrizing the PM expansions of the radiative losses by the coefficients of their power expansion in $\frac{1}{j}$, namely,

$$\begin{aligned}
\frac{E^{\mathrm{rad}}}{M} &= +\nu^2\sum_{n=3}^{\infty}\frac{E_n}{j^n},\\
\frac{J^{\mathrm{rad}}}{J_{\mathrm{c.m.}}} &= +\nu^1\sum_{n=2}^{\infty}\frac{J_n}{j^n},\\
\frac{P_x^{\mathrm{rad}}}{M} &= +\frac{m_2-m_1}{M}\nu^2\sum_{n=4}^{\infty}\frac{P_{xn}}{j^n},\\
\frac{P_y^{\mathrm{rad}}}{M} &= +\frac{m_2-m_1}{M}\nu^2\sum_{n=3}^{\infty}\frac{P_{yn}}{j^n}.
\end{aligned} \tag{10.1}$$

Here the left-hand sides have been adimensionalized, and we pulled out some powers of ν on the right-hand sides, to ensure that the expansion coefficients E_n, J_n, P_{xn}, P_{yn} are dimensionless and that their LO PN contribution is ν independent. (We recall that $J_{\mathrm{c.m.}} = b\mu p_\infty/h = GM^2\nu j$.) Note that in Ref. [30] we focused on the PM expansion of P_y^{rad}, because P_x^{rad} was subdominant and linked to time-asymmetric hereditary tail effects. See Eq. (H3) there, giving the LO contribution to P_x^{rad}.

A. Energy loss in the c.m. frame

The radiated c.m. energy E^{rad} has been evaluated at the 2PN fractional accuracy in our previous work Ref. [30]. The corresponding $\frac{1}{j}$-expansion PM coefficients were given (up to $\frac{1}{j^7}$) in the first five lines of Table IX there. In the present work, we have computed the heretofore unevaluated fractional 2.5PN instantaneous contribution due the radiation-reaction correction to hyperbolic motion (incorrectly argued to vanish in [37]), and we have used the results of [36–38] when computing the fractional 3PN contribution in the form of a $\frac{1}{j}$ expansion [see Eqs. (6.5) and (6.6)]. In order to confirm the value of the fractional 3PN contribution to the radiated energy, we have done an independent computation of the instantaneous, 3PN-level contribution. The technically most challenging part of the latter computation comes from inserting the 3PN-accurate hyperbolic motion in the 3PN-accurate quadrupole moment. Following Ref. [56], the computation uses a 3PN-level, hyperbolic version of the quasi-Keplerian representation of binary motion. In redoing the computation of the latter hyperbolic quasi-Keplerian representation, we found that there were several typos in the results displayed in Ref. [56]. For the convenience of the reader, we give the corresponding corrected results in Appendix D.

Our results are displayed in Table II. Many of the ν-dependent terms can be directly checked by using the polynomiality rule satisfied by the coefficients E_n, namely,

$$h^{n+1}E_n = P^{\gamma}_{[(n-2)/2]}(\nu), \tag{10.2}$$

where $P^{\gamma}_N(\nu)$ denotes a polynomial of order N in ν, having γ-dependent coefficients. This rule was pointed out in Ref. [33] [see also Eq. (7.7) in Ref. [30]]. We shall give below another simple proof of this polynomiality rule. Our results on the coefficients E_n satisfy this polynomiality rule after adding all separate contributions. For instance, at the 4PM order ($n = 4$), if one would consider separately the 3PN contribution [$\frac{1}{j^4}$ term on the second line of Eq. (6.6)], it would violate the polynomiality rule (10.2) because of the terms $(-\frac{2366\nu^3}{9} + \frac{164\nu^2}{3})$. In fact, these terms precisely cancel the rule-violating terms in $h^5 E_4$ coming from lower PN contributions in E_4.

While writing up our results, a PN-exact computation of the G^4 energy coefficient E_4 was made public [25]. Our (fractionally 3PN-accurate) PN-expanded result listed in Eq. (D27) of Ref. [30], and in Table II here, agrees (when expressed in terms of $\tilde{E}_4 \equiv h^5 E_4$) with the 3PN expansion of the curly bracket on the right-hand side of Eq. (8) in Ref. [25].

Let us also note that we have included in Table II the PN-acquired knowledge of the 3PM-level contribution E_3, though E_3 has been determined as an exact function of p_∞ [9,23]. It agrees with the corresponding term in Refs. [9,23] and thereby provides an additional check of our PN calculations.

B. Angular momentum loss in the c.m. frame

The fractionally 2PN-accurate expansion of the PM coefficients J_n of the radiated c.m. angular momentum $J^{\rm rad}$ can also be found in Table IX of Ref. [30], up to $n = 7$. In the present work, we have raised their accuracy to the 3PN order by computing the missing term in the instantaneous part of the radiated angular momentum at the 2.5PN level due the radiation-reaction correction to hyperbolic motion, thereby completing partial results available in the literature for the various contributions through 3PN order [36–39]. The final result is given by Eqs. (7.4) and (7.5) as an expansion in inverse angular momentum. The post-2PN coefficients are listed in Table II. The 2PM and 3PM coefficients J_2 and J_3 are known exactly (see Refs. [28,24], respectively), but are also shown in their PN-expanded form for completeness.

Concerning the ν structure of the coefficients J_n, they satisfy the polynomiality rule [30]

$$h^n J_n + h^{n-1}\nu E_n = P^{\gamma}_{[(n-2)/2]}(\nu), \tag{10.3}$$

with $n \geq 3$, whereas $h^2 J_2$ is independent of ν.

C. Linear-momentum loss in the c.m. frame

Table IX of Ref. [30] listed the PN expansion of the coefficients P_{yn} of the PM expansion of the y component of the radiated linear momentum $J^{\rm rad}$ in the c.m. frame, accurate to 2PN fractional order. The corresponding post-2PN contributions up to 3PN order are listed in Table II.

As pointed out in [30] (and as is further discussed below), the coefficients P_{yn} must satisfy the polynomiality property

$$h^{n+1}P_{yn} = P^{\gamma}_{[(n-3)/2]}(\nu). \tag{10.4}$$

Our results on the coefficients P_{yn} satisfy this polynomiality rule after adding all separate contributions. For instance, at order $n = 4$ the term proportional to $+\eta^2 \frac{320}{3}\frac{p_\infty^9}{j^4}$ in the fractionally 1PN tail term (8.7) would separately violate the rule (10.4), but is needed to cancel corresponding rule-violating terms in $h^5 P_{y4}$.

We recall that P_{y3} is exactly known in PM sense, being related to E_3 by

$$P_{y3} = \sqrt{\frac{\gamma-1}{\gamma+1}}E_3. \tag{10.5}$$

The PN expansion of the coefficients P_{xn} are instead listed in Table III. These expansions include the leading-order (past-tail) contribution computed in [30] and complete them by two further terms in the PN expansion (fractionally 2.5PN and 3PN).

The coefficients P_{xn} satisfy (see below) the polynomiality property

$$h^n P_{xn} = P^{\gamma}_{[(n-4)/2]}(\nu). \tag{10.6}$$

Our results on the coefficients P_{xn} were found to satisfy this polynomiality rule after adding all separate contributions and, notably, the one linked to radiation-reaction modifications of the orbital motion. For example, at order G^4 the term proportional to $-\eta^2 \frac{9529}{67200}\frac{p_\infty^9}{j^4}$ in the fractionally 1PN tail term (8.7) would separately violate the rule (10.6), but is needed to cancel corresponding rule-violating terms in $h^4 P_{x4}$, while, at order G^5, the term $-(Mc)\frac{m_2-m_1}{M}\nu^3\eta^5\frac{15872}{6125}\frac{p_\infty^8}{j^5}$ in $\delta^{\rm rr}P_x^{{\rm rad\, inst},I,J}$, Eq. (4.14), is nonpolynomial by itself, but corrects the nonpolynomiality of other contributions.

XI. LORENTZ-INVARIANT FORM FACTORS FOR $P^\mu_{\rm rad}$ AND MASS-POLYNOMIALITY RULES

In the sections above, we have discussed the values of the losses of energy, angular momentum, and linear momentum in the c.m. frame. This was motivated by the fact that the multipolar-post-Minkowskian approach [50–52] to gravitational radiation is conveniently applied within the c.m. frame of the binary system. Let us now reexpress these c.m.-based and PN-expanded results in a Lorentz-invariant way.

As was pointed out in previous works (e.g., [30,45]), if one expresses the individual momentum changes (or impulses) $\Delta p_1^\mu, \Delta p_2^\mu$ during gravitational scattering, and therefore also the radiated 4-momentum $P^\mu_{\rm rad} = -(\Delta p_1^\mu + \Delta p_2^\mu)$, in terms of the incoming 4-velocities u^μ_{1-}, u^μ_{2-} and of the vectorial impact parameter $b^\mu_{12} \equiv b^\mu_1 - b^\mu_2$, their expansion coefficients in powers of G must be polynomials in the two masses m_1 and m_2. Let us show here what information we can thereby get from such mass polynomiality.

We can decompose $P^\mu_{\rm rad}$ as follows:

$$\begin{aligned} P^\mu_{\rm rad} = &P^{\rm rad}_{1+2}(m_1,m_2,\gamma,b)(u^\mu_{1-}+u^\mu_{2-}) \\ &+ P^{\rm rad}_{1-2}(m_1,m_2,\gamma,b)(u^\mu_{1-}-u^\mu_{2-}) \\ &+ P^{\rm rad}_{b_{12}}(m_1,m_2,\gamma,b)\hat{b}^\mu_{12}. \end{aligned} \tag{11.1}$$

The basis $u^\mu_{1-}+u^\mu_{2-}$, $u^\mu_{1-}-u^\mu_{2-}$, $\hat{b}^\mu_{12}$ is *orthogonal*, though not orthonormal. While $(\hat{b}_{12})^2 = +1$ we have

$$(u^\mu_{1-}+u^\mu_{2-})^2 = -2(\gamma+1), \qquad (u^\mu_{1-}-u^\mu_{2-})^2 = +2(\gamma-1). \tag{11.2}$$

Taking into account the symmetry of $P^\mu_{\rm rad}$ under the $1 \leftrightarrow 2$ exchange, and the (anti)symmetry of $u^\mu_{1-}+u^\mu_{2-}$ ($u^\mu_{1-}-u^\mu_{2-}$, $\hat{b}^\mu_{12}$), we see that the first form factor $P^{\rm rad}_{1+2}(m_1,m_2,\gamma,b)$ must be $1\leftrightarrow 2$ symmetric, while $P^{\rm rad}_{1-2}(m_1,m_2,\gamma,b)$ and $P^{\rm rad}_{b_{12}}(m_1,m_2,\gamma,b)$ must be $1\leftrightarrow 2$ antisymmetric. We can then use the further facts that (i) radiative losses of energy and linear momentum being quadratic in the retarded-time derivative of the waveform must contain a factor $(m_1m_2)^2$; and (ii) $P^{\rm rad}_{1+2}(m_1,m_2,\gamma,b)$ starts at order G^3, while $P^{\rm rad}_{1-2}(m_1,m_2,\gamma,b)$ and $P^{\rm rad}_{b_{12}}(m_1,m_2,\gamma,b)$ start at order G^4. The mass polynomiality of the PM expansion coefficients of $P^\mu_{\rm rad}$ then allows us to write

$$\begin{aligned} P^{\rm rad}_{1+2}(m_1,m_2,\gamma,b) &= \frac{G^3}{b^3} m_1^2 m_2^2 \hat{P}^{\rm rad}_{1+2}, \\ P^{\rm rad}_{1-2}(m_1,m_2,\gamma,b) &= \frac{G^4}{b^4} m_1^2 m_2^2 (m_2-m_1) \hat{P}^{\rm rad}_{1-2}, \\ P^{\rm rad}_{b_{12}}(m_1,m_2,\gamma,b) &= \frac{G^4}{b^4} m_1^2 m_2^2 (m_2-m_1) \hat{P}^{\rm rad}_{b_{12}}, \end{aligned} \tag{11.3}$$

where the dimensionless factors $\hat{P}^{\rm rad}_{1+2}$, $\hat{P}^{\rm rad}_{1-2}$, $\hat{P}^{\rm rad}_{b_{12}}$ have PM expansions of the form

$$\begin{aligned} \hat{P}^{\rm rad}_{1+2} &= \sum_{n\geq 3} \frac{G^{n-3}}{b^{n-3}} SP^{1+2}_{n-3}(m_1,m_2) \\ &= \sum_{n\geq 3} \frac{G^{n-3}M^{n-3}}{b^{n-3}} p^{1+2,G^n}_{[\frac{n-3}{2}]}(\gamma,\nu), \\ \hat{P}^{\rm rad}_{1-2} &= \sum_{n\geq 4} \frac{G^{n-4}}{b^{n-4}} SP^{1-2}_{n-4}(m_1,m_2) \\ &= \sum_{n\geq 4} \frac{G^{n-4}M^{n-4}}{b^{n-4}} p^{1-2,G^n}_{[\frac{n-4}{2}]}(\gamma,\nu), \\ \hat{P}^{\rm rad}_{b_{12}} &= \sum_{n\geq 4} \frac{G^{n-4}}{b^{n-4}} SP^{b_{12}}_{n-4}(m_1,m_2) \\ &= \sum_{n\geq 4} \frac{G^{n-3}M^{n-3}}{b^{n-3}} p^{b_{12},G^n}_{[\frac{n-4}{2}]}(\gamma,\nu). \end{aligned} \tag{11.4}$$

Here, $SP^X_N(m_1,m_2)$ denotes a *symmetric* polynomial of order N in the two masses. By scaling out the total mass $M = m_1 + m_2$, each such polynomial can be rewritten as

$$SP^X_N(m_1,m_2) = M^N p^{X,G^n}_{[\frac{N}{2}]}(\gamma,\nu), \tag{11.5}$$

where $p^{X,G^n}_{[\frac{N}{2}]}(\gamma,\nu)$ is a polynomial in ν of order $[\frac{N}{2}]$ (the integer part of $\frac{N}{2}$), with γ-dependent coefficients. In order to keep track of the PM order n, we add a label G^n, and we also sometimes keep the notation $[\frac{N}{2}]$, with $N = n-3$ or $N = n-4$ (e.g., we write $[\frac{1}{2}]$ instead replacing it by its numerical value 0).

We thereby see that, while at order G^3 (3PM order), $P^\mu_{\rm rad}$ was described by only one function of γ, namely [see Eq. (1.4)],

$$SP^{1+2}_0(m_1,m_2) = p^{1+2,G^3}_{[\frac{0}{2}]}(\gamma) = \frac{\mathcal{E}(\gamma)}{\gamma+1}, \tag{11.6}$$

it will involve three functions of γ at order G^4, namely,

$$\begin{aligned} P^{{\rm rad},G^4}_{1+2} &= \frac{G^4}{b^4} m_1^2 m_2^2 SP^{1+2}_1(m_1,m_2) \\ &= \frac{G^4}{b^4} m_1^2 m_2^2 (m_1+m_2) p^{1+2,G^4}_{[\frac{1}{2}]}(\gamma), \\ P^{{\rm rad},G^4}_{1-2} &= \frac{G^4}{b^4} m_1^2 m_2^2 (m_2-m_1) SP^{1-2}_0(m_1,m_2) \\ &= \frac{G^4}{b^4} m_1^2 m_2^2 (m_2-m_1) p^{1-2,G^4}_{[\frac{0}{2}]}(\gamma), \\ P^{{\rm rad},G^4}_{b_{12}} &= \frac{G^4}{b^4} m_1^2 m_2^2 (m_2-m_1) SP^{b_{12}}_0(m_1,m_2) \\ &= \frac{G^4}{b^4} m_1^2 m_2^2 (m_2-m_1) p^{b_{12},G^4}_{[\frac{0}{2}]}(\gamma). \end{aligned} \tag{11.7}$$

At order G^5, we have four functions of γ,

$$P_{1+2}^{\mathrm{rad},G^5} = \frac{G^5}{b^5} m_1^2 m_2^2 SP_2^{1+2}(m_1,m_2)$$
$$= \frac{G^5}{b^5} m_1^2 m_2^2 (m_1+m_2)^2 p_{[\frac{2}{2}]}^{1+2,G^5}(\gamma,\nu),$$
$$P_{1-2}^{\mathrm{rad},G^5} = \frac{G^5}{b^5} m_1^2 m_2^2 (m_2-m_1) SP_1^{1-2}(m_1,m_2)$$
$$= \frac{G^5}{b^5} m_1^2 m_2^2 (m_2-m_1)(m_1+m_2) p_{[\frac{1}{2}]}^{1-2,G^5}(\gamma),$$
$$P_{b_{12}}^{\mathrm{rad},G^5} = \frac{G^5}{b^5} m_1^2 m_2^2 (m_2-m_1) SP_1^{b_{12}}(m_1,m_2)$$
$$= \frac{G^5}{b^5} m_1^2 m_2^2 (m_2-m_1)(m_1+m_2) p_{[\frac{1}{2}]}^{b_{12},G^5}(\gamma), \quad (11.8)$$

where $p_{[\frac{2}{2}]}^{1+2,G^5}(\gamma,\nu)$, being linear in ν, involves two independent functions of γ. At order G^n, P^μ_{rad} generally involves

$$N_{P_{\mathrm{rad}}}^{G^n} = \left[\frac{n-1}{2}\right] + 2\times\left[\frac{n-2}{2}\right] \quad (11.9)$$

functions of γ.

Let us now discuss how to relate the Lorentz-invariant building blocks $p_{[\frac{n-3}{2}]}^{1+2,G^n}(\gamma,\nu)$, $p_{[\frac{n-4}{2}]}^{1-2,G^n}(\gamma,\nu)$, $p_{[\frac{n-4}{2}]}^{b_{12},G^n}(\gamma,\nu)$ parametrizing the PM expansion of P^μ_{rad} to our previous c.m.-frame, PN-expanded results on $E^{\mathrm{rad}}, P_x^{\mathrm{rad}}, P_y^{\mathrm{rad}}$.

A first step in this direction consists in computing the projections of P^μ_{rad} on the three unit vectors U^μ, n_-^μ, and $\hat b_{12}$, where U^μ is the c.m. time axis, such that

$$MhU^\mu = m_1 u_{1-}^\mu + m_2 u_{2-}^\mu, \quad (11.10)$$

and where n_-^μ is the unit vector in the c.m.-frame direction of u_{1-}^μ, such that

$$Mhp_\infty n_-^\mu = (m_2+\gamma m_1)u_{1-}^\mu - (m_1+\gamma m_2)u_{2-}^\mu. \quad (11.11)$$

The definition of E^{rad}, namely, $E^{\mathrm{rad}} = -U^\mu P^{\mathrm{rad}}_\mu$ then yields

$$MhE^{\mathrm{rad}} = (m_1 u_{1-}^\mu + m_2 u_{2-}^\mu)P^{\mathrm{rad}}_\mu. \quad (11.12)$$

From the definition (A3) of $\mathbf{e}_x$ and $\mathbf{e}_y$, we deduce that

$$P_n^{\mathrm{rad}} \equiv n_-^\mu P_\mu^{\mathrm{rad}} = \sin\frac{\chi_{\mathrm{cons}}}{2} P_x^{\mathrm{rad}} + \cos\frac{\chi_{\mathrm{cons}}}{2} P_y^{\mathrm{rad}}, \quad (11.13)$$

while

$$P_b^{\mathrm{rad}} \equiv \hat b_{12}^\mu P_\mu^{\mathrm{rad}} = \cos\frac{\chi_{\mathrm{cons}}}{2} P_x^{\mathrm{rad}} - \sin\frac{\chi_{\mathrm{cons}}}{2} P_y^{\mathrm{rad}}. \quad (11.14)$$

Inserting the parametrization (12.24) into these results then yields the following links between $E^{\mathrm{rad}}, P_x^{\mathrm{rad}}, P_y^{\mathrm{rad}}$ [remembering the definitions (11.13) and (11.14)] and the form factors of P^{rad}_μ:

$$MhE^{\mathrm{rad}} = M(\gamma+1)P_{1+2}^{\mathrm{rad}} + (m_2-m_1)(\gamma-1)P_{1-2}^{\mathrm{rad}},$$
$$MhP_n^{\mathrm{rad}} = (m_2-m_1)p_\infty P_{1+2}^{\mathrm{rad}} + Mp_\infty P_{1-2}^{\mathrm{rad}},$$
$$P_b^{\mathrm{rad}} = P_{b_{12}}^{\mathrm{rad}}. \quad (11.15)$$

These simple links can be easily inverted to express P_{1+2}^{rad} and P_{1-2}^{rad} as linear combinations of hE^{rad} and hP_n^{rad}, and we have used them to extract the values of P_{1+2}^{rad} and P_{1-2}^{rad}. Before exhibiting our results, several remarks are in order.

Let us first note that, while the mass polynomiality of the form factor $P_{b_{12}}^{\mathrm{rad}}$ immediately implies the mass polynomiality of $P_b^{\mathrm{rad}} \equiv \hat b_{12}^\mu P_\mu^{\mathrm{rad}}$, the mass polynomiality of the two other form factors, P_{1+2}^{rad} and P_{1-2}^{rad}, implies the mass polynomiality of the combinations MhE^{rad} and MhP_n^{rad}. In these combinations, it is crucial to include the factor $Mh = M\sqrt{1+2\nu(\gamma-1)} = E_-^{\mathrm{c.m}}$. (including the extra mass factor M, which cannot be, generally, factored out on the right-hand sides).

In more detail, we have

$$MhE^{\mathrm{rad}} = \frac{G^3}{b^3} m_1^2 m_2^2 M(\gamma+1)\hat P_{1+2}^{\mathrm{rad}}$$
$$+ \frac{G^4}{b^4} m_1^2 m_2^2 (m_2-m_1)^2(\gamma-1)\hat P_{1-2}^{\mathrm{rad}},$$
$$MhP_n^{\mathrm{rad}} = \frac{G^3}{b^3}(m_2-m_1)p_\infty \hat P_{1+2}^{\mathrm{rad}}$$
$$+ \frac{G^4}{b^4} m_1^2 m_2^2 (m_2-m_1)Mp_\infty \hat P_{1-2}^{\mathrm{rad}}, \quad (11.16)$$

where we recall that the various dimensionless factors $\hat P_X^{\mathrm{rad}}$ have the more explicit structure

$$\hat P_X^{\mathrm{rad}} = \sum_{N\geq 0}\frac{G^N}{b^N} SP_N^X(m_1,m_2) = \sum_{N\geq 0}\left(\frac{GM}{b}\right)^N P_{[\frac{N}{2}]}^X(\nu). \quad (11.17)$$

These expressions give a direct proof of the ν structures pointed out in our previous works, notably,[13]

$$\left(\frac{hE^{\mathrm{rad}}}{M}\right)^{\frac{G^n}{b^n}} = \left(\frac{GM}{b}\right)^n \nu^2 P^\gamma_{[(n-2)/2]}(\nu), \quad (11.18)$$

and also

$$\left(\frac{hP_n^{\mathrm{rad}}}{M}\right)^{\frac{G^n}{b^n}} = \left(\frac{GM}{b}\right)^n \nu^2 \frac{m_2-m_1}{M} P^\gamma_{[(n-3)/2]}(\nu). \quad (11.19)$$

[13]Here we use the expansion in powers of $\frac{G}{b}$. When using the expansion in $\frac{1}{j} = \frac{GMh}{bp_\infty}$ one must add an extra factor h^n at order $\frac{1}{j^n}$, as used in Eq. (10.2).

Note also that, while in $MhE^{\rm rad}$ the dimensionless form factor $\hat{P}^{\rm rad}_{1-2}$ is multiplied by the small PN factor $\gamma-1=O(p_\infty^2)$, in $MhP^{\rm rad}_n$ the two form factors $\hat{P}^{\rm rad}_{1+2}$ and $\hat{P}^{\rm rad}_{1-2}$ contribute with the same PN weight (at any given order in G).

Inserting the mass-polynomiality structures of $P^{\rm rad}_b$ and $MhP^{\rm rad}_n$ in the expressions of $P^{\rm rad}_x$ and $P^{\rm rad}_y$ in terms of $P^{\rm rad}_b$ and $P^{\rm rad}_n$, and using the mass polynomiality of the magnitude of the conservative momentum transfer,

$$\frac{\mathsf{Q}}{2}=P_{\rm c.m.}\sin\frac{\chi_{\rm cons}}{2}=\frac{Gm_1m_2}{b}\left[\frac{2\gamma^2-1}{\gamma^2-1}+\frac{G}{b}SP_1(m_1,m_2)+\frac{G^2}{b^2}SP_2(m_1,m_2)+\cdots\right], \tag{11.20}$$

which yields

$$\sin\frac{\chi_{\rm cons}}{2}=\frac{GMh}{b}\left[\frac{2\gamma^2-1}{\gamma^2-1}+\frac{G}{b}SP_1(m_1,m_2)+\frac{G^2}{b^2}SP_2(m_1,m_2)+\cdots\right], \tag{11.21}$$

one can easily derive the following mass-polynomiality structures:

$$P^{\rm rad}_x=\frac{G^4}{b^4}m_1^2m_2^2(m_2-m_1)\left[SP_0(m_1,m_2)+\frac{G}{b}SP_1(m_1,m_2)+\frac{G^2}{b^2}SP_2(m_1,m_2)+\cdots\right], \tag{11.22}$$

and

$$MhP^{\rm rad}_y=\frac{G^3}{b^3}m_1^2m_2^2(m_2-m_1)\left[SP_0(m_1,m_2)+\frac{G}{b}SP_1(m_1,m_2)+\frac{G^2}{b^2}SP_2(m_1,m_2)+\cdots\right]. \tag{11.23}$$

As above, each such mass-polynomiality structure leads, after scaling out the appropriate power of $\frac{GM}{b}$, a polynomial structure in the symmetric mass ratio ν (with γ-dependent coefficients), namely,

$$\frac{G^N}{b^N}SP^X_N(m_1,m_2)=\left(\frac{GM}{b}\right)^N P^X_{[\frac{N}{2}]}(\nu). \tag{11.24}$$

One then easily checks that relations such as Eq. (7.27) in Ref. [30] and its G^5 generalization indicated in the caption of Table II there, follow from Eqs. (11.18) and (11.19) above.

We have already mentioned above that our c.m.-based and PN-based results on $E^{\rm rad}$, $P^{\rm rad}_x$, and $P^{\rm rad}_y$ were all in agreement (after adding all separate contributions and, notably, the one linked to radiation-reaction modifications of the orbital motion) with the ν-polynomiality rules rederived here. We can therefore encapsulate the full, current PN-expanded information on $P^\mu_{\rm rad}$ in the values of the γ-dependent ν polynomials $p^X_{[\frac{N}{2}]}(\gamma,\nu)$ parametrizing the form factors, see Eqs. (11.3)–(11.8).

At order G^3 our results yield

$$p^{1+2,G^3}_{[\frac{0}{2}]}(\gamma)=\pi\left(\frac{37}{30}p_\infty+\frac{839}{1680}p_\infty^3+\frac{2699}{2016}p_\infty^5-\frac{1531643}{1182720}p_\infty^7+O(p_\infty^9)\right), \tag{11.25}$$

which agrees with the fractionally 3PN-level expansion of the exact result

$$p^{1+2,G^3}_{[\frac{0}{2}]}(\gamma)=\pi\frac{\hat{\mathcal{E}}(\gamma)}{\gamma+1}. \tag{11.26}$$

At order G^4 we find

$$\begin{aligned}
p^{1+2,G^4}_{[\frac{1}{2}]}(\gamma)&=\frac{784}{45p_\infty}+\frac{2168}{175}p_\infty+\frac{1568}{45}p_\infty^2+\frac{98666}{11025}p_\infty^3-\frac{512}{105}p_\infty^4-\frac{2702747}{363825}p_\infty^5+O(p_\infty^6),\\
p^{1-2,G^4}_{[\frac{0}{2}]}(\gamma)&=\frac{176}{45p_\infty}-\frac{72}{25}p_\infty+\frac{352}{45}p_\infty^2-\frac{9746}{4725}p_\infty^3+\frac{448}{75}p_\infty^4-\frac{484019}{51975}p_\infty^5+O(p_\infty^6),\\
p^{b_{12},G^4}_{[\frac{0}{2}]}&=-\pi\left[\frac{37}{30}+\frac{1661}{560}p_\infty^2+\frac{1491}{400}p_\infty^3+\frac{23563}{10080}p_\infty^4-\frac{26757}{5600}p_\infty^5+\frac{700793}{506880}p_\infty^6+O(p_\infty^7)\right].
\end{aligned} \tag{11.27}$$

While writing up our results, a PN-exact computation of the 4PM contribution to $P^{\rm rad}_\mu$, and notably, its $\hat{b}^\mu_{12}$ projection, appeared on arXiv [25]. Our (fractionally 3PN-accurate) results, Eq. (11.27), are compatible with those given in Ref. [25].

Similarly, at $O(G^5)$ we have

$$
\begin{aligned}
p_{[\frac{2}{5}]}^{1+2,G^5}(\gamma,\nu) = \pi\Bigg[&\frac{61}{5p_\infty^3}+\frac{34073}{1680p_\infty}+\frac{297}{40}\pi^2-\frac{23923}{2880}p_\infty+\left(-\frac{31029}{2240}\pi^2+\frac{1484997}{11200}\right)p_\infty^2\\
&+\left(\frac{99}{20}\pi^2+\frac{34695068413}{620928000}-\frac{10593}{700}\ln\left(\frac{p_\infty}{2}\right)\right)p_\infty^3\Bigg]\\
&+\nu\pi\left[-\frac{55}{12p_\infty}+\frac{6427}{10080}p_\infty+\left(\frac{877}{400}-\frac{939}{560}\pi^2\right)p_\infty^2+\left(\frac{255491}{10080}-\frac{4059}{1280}\pi^2\right)p_\infty^3+O(p_\infty^4)\right],\\
p_{[\frac{1}{2}]}^{1-2,G^5}(\gamma) = \pi\Bigg[&\frac{82}{15p_\infty^3}-\frac{5207}{630p_\infty}-\frac{1491}{400}+\frac{939}{280}\pi^2-\frac{963239}{40320}p_\infty+\left(\frac{902743}{33600}-\frac{13603}{4480}\pi^2\right)p_\infty^2\\
&+\left(-\frac{4809573323}{434649600}-\frac{1591}{980}\ln\left(\frac{p_\infty}{2}\right)+\frac{313}{140}\pi^2\right)p_\infty^3+O(p_\infty^4)\Bigg],\\
p_{[\frac{1}{2}]}^{b_{12},G^5}(\gamma) = &-\frac{64}{3p_\infty^2}-\frac{37}{20}\pi^2-\frac{27392}{525}-\frac{30208}{225}p_\infty+\left(-\frac{856768}{33075}-\frac{3429}{1120}\pi^2\right)p_\infty^2+\frac{462592}{7875}p_\infty^3\\
&+\left(-\frac{74417152}{363825}-\frac{7915}{2688}\pi^2\right)p_\infty^4+O(p_\infty^5).
\end{aligned}
\tag{11.28}
$$

XII. INFORMATION ON THE INDIVIDUAL IMPULSES Δp_a^μ DERIVABLE FROM $P_{\rm rad}^\mu$

Let us now discuss what information on the individual momentum changes (or impulses), $\Delta p_1^\mu, \Delta p_2^\mu$, can be extracted from our results on $P_{\rm rad}^\mu$ by combining *six* different facts:

First, the coefficients of the PM expansion of $\Delta p_1^\mu, \Delta p_2^\mu$ in terms of the incoming 4-velocities u_{1-}^μ, u_{2-}^μ and of the vectorial impact parameter $b^\mu \equiv b_1^\mu - b_2^\mu$ must be polynomials in the two masses m_1 and m_2. More precisely, one has (for the first particle)

$$
\Delta p_{1\mu} = -2Gm_1m_2\frac{2\gamma^2-1}{\sqrt{\gamma^2-1}}\frac{b_{12\mu}}{b^2}+\sum_{n\geq 2}\Delta p_{1\mu}^{n\rm PM}, \tag{12.1}
$$

where each term $\Delta p_{1\mu}^{n\rm PM}$ is a combination of the three vectors b_{12}^μ/b, u_{1-}^μ, and u_{2-}^μ, with coefficients that are, at each order in G, homogeneous polynomials in m_1 and m_2, containing the product m_1m_2 as an overall factor. Symbolically,

$$
\begin{aligned}
\Delta p_{1\mu}^{n\rm PM} \sim \frac{Gm_1m_2}{b^n}&[(Gm_1)^{n-1}\\
&+(Gm_1)^{n-2}Gm_2+\cdots+(Gm_2)^{n-1}],
\end{aligned}
\tag{12.2}
$$

where each term is a combination of the three vectors b^μ/b, u_{1-}^μ, and u_{2-}^μ, with coefficients that are functions of γ. (Note that, contrary to the case of $P_{\rm rad}^\mu$, $\Delta p_{1\mu}^{n\rm PM}$ is not symmetric under particle exchange.)

Second, linear-momentum conservation implies that the radiated momentum is equal to

$$
\Delta p_1^\mu + \Delta p_2^\mu = -P_{\rm rad}^\mu. \tag{12.3}
$$

Third, we have the decomposition

$$
\Delta p_{a\mu} = \Delta p_{a\mu}^{\rm cons} + \Delta p_{a\mu}^{\rm rr\,lin} + \Delta p_{a\mu}^{\rm rr\,nonlin}. \tag{12.4}
$$

Here, (i) the conservative part $\Delta p_{a\mu}^{\rm cons}$ is known up to the sixth PN order (modulo six still unknown parameters [32–34,67]), while its G expansion is known exactly up to order G^4 included [10,25]; (ii) the linear-response contribution $\Delta p_{a\mu}^{\rm rr\,lin}$ is known (modulo some linear, time-even radiation-reaction effects discussed below) from our previous work [30]; while (iii) the remainder term $\Delta p_{a\mu}^{\rm rr\,nonlin}$ can be described as containing the contributions that are higher-order in radiation reaction [starting with the quadratic order $O(\mathcal{F}_{\rm rr}^{\mu\,2})$].

Fourth, as we are going to show, the linear-response contribution happens to satisfy, by itself, the momentum conservation law (12.3), namely,

$$
\Delta p_{1\mu}^{\rm rr\,lin} + \Delta p_{2\mu}^{\rm rr\,lin} = -P_{\rm rad}^\mu. \tag{12.5}
$$

Fifth, the linear-response contribution satisfies a linearized version of the mass-shell condition that must hold for the outgoing momenta, namely,

$$
p_{a\mu}^{+\rm cons}\Delta p_a^{\mu\,\rm rr\,lin} = 0. \tag{12.6}
$$

Sixth, the nonlinear contribution $\Delta p_{a\mu}^{\rm rr\,nonlin}$ to the impulse of the ath particle (as well as the additional contribution $\Delta p_{a\mu}^{\rm rr\,\Delta c_\phi}$ to $\Delta p_{a\mu}^{\rm rr\,lin}$ linked to the time-even part of $\mathcal{F}_{\rm rr}^\mu$ discussed below) must involve a factor m_a^3.

In the following, we explain the origin of these facts and then show how they determine the conservativelike radiative contributions $\Delta p_{a\mu}^{\mathrm{rr}\,\Delta c_\phi} + \Delta p_{a\mu}^{\mathrm{rr\,nonlin}}$ at the fourth post-Minkowskian order $[O(G^4)]$ and strongly constrain them at the fifth post-Minkowskian order $[O(G^5)]$.

A. Proof of the identity (12.5) and antisymmetry property of $\Delta p_{a\mu}^{\mathrm{rr\,nonlin}}$

The linear-response contribution $\Delta p_{a\mu}^{\mathrm{rr\,lin}}$ was obtained in [30] as the sum of two terms: a "relative motion" term $\Delta p_{a\mu}^{\mathrm{rr\,rel}}$ and a "recoil" term $\Delta p_{a\mu}^{\mathrm{rr\,rec}}$,

$$\Delta p_{a\mu}^{\mathrm{rr\,lin}} = \Delta p_{a\mu}^{\mathrm{rr\,rel}} + \Delta p_{a\mu}^{\mathrm{rr\,rec}}. \tag{12.7}$$

From Eqs. (3.32) and (3.33) in [30], we have

$$\Delta p_{a\mu}^{\mathrm{rr\,rel}} = \chi^{\mathrm{rr\,rel}} \frac{d}{d\chi_{\mathrm{cons}}} \Delta p_{a\mu}^{\mathrm{cons}} + \frac{\Delta P_{\mathrm{c.m.}}}{P_{\mathrm{c.m.}}} p_{a\mu}^{+} - \frac{m_a^2}{E_a} \frac{\Delta P_{\mathrm{c.m.}}}{P_{\mathrm{c.m.}}} U_\mu, \tag{12.8}$$

and

$$\Delta p_{a\mu}^{\mathrm{rr\,rec}} = -\frac{E_a}{E_{\mathrm{c.m.}}} P_\mu^{\mathrm{rad}} - \frac{(p_{a\nu}^{+} P_{\mathrm{rad}}^{\nu})}{E_{\mathrm{c.m.}}} U_\mu. \tag{12.9}$$

Here,

$$\Delta P_{\mathrm{c.m.}} = -\frac{E_1 E_2}{E_{\mathrm{c.m.}} P_{\mathrm{c.m.}}} E_{\mathrm{rad}}, \tag{12.10}$$

and the quantities E_a, $E_{\mathrm{c.m.}} = E_1 + E_2 = Mh$, $P_{\mathrm{c.m.}} = \frac{m_1 m_2 p_\infty}{E_{\mathrm{c.m}}}$., $p_{a\nu}^{+}$ (outgoing momenta), and $U_\mu \equiv (p_{1\nu}^{-} + p_{2\nu}^{-})/E_{\mathrm{c.m.}}$ are all taken along the unperturbed, conservative motion.

When summing over the particle label a, taking into account the fact that $\sum_a \Delta p_{a\mu}^{\mathrm{cons}} = 0$ and $\sum_a p_{a\nu}^{+} = \sum_a p_{a\nu}^{-} = E_{\mathrm{c.m.}} U_\mu$, one easily finds that Eq. (12.5) is (exactly) satisfied. This identity (together with the fact that $\sum_a \Delta p_{a\mu}^{\mathrm{cons}} = 0$) implies the somewhat remarkable identity that the remainder (nonlinear) term in the linear-response formula (12.4) must separately satisfy the identity

$$\Delta p_{1\mu}^{\mathrm{rr\,nonlin}} + \Delta p_{2\mu}^{\mathrm{rr\,nonlin}} = 0. \tag{12.11}$$

In other words, the nonlinear contribution $\Delta p_{a\mu}^{\mathrm{rr\,nonlin}}$ must be *antisymmetric* under particle exchange.

Another constraint on $\Delta p_{a\mu}^{\mathrm{rr\,nonlin}}$ is the mass-shell condition

$$p_{a\mu}^{+\mathrm{tot}} p_a^{\mu+\mathrm{tot}} = -m_a^2, \tag{12.12}$$

where the total outgoing momentum is

$$p_{a\mu}^{+\mathrm{tot}} = p_{a\mu}^{+\mathrm{cons}} + \Delta p_{a\mu}^{\mathrm{rr\,lin}} + \Delta p_{a\mu}^{\mathrm{rr\,nonlin}}. \tag{12.13}$$

Using the fact that $\Delta p_{a\mu}^{\mathrm{rr\,lin}}$ satisfies (independent of the value of χ^{rel}) Eq. (12.6), we get the following additional constraint on $\Delta p_{a\mu}^{\mathrm{rr\,nonlin}}$:

$$2 p_{a\mu}^{+\mathrm{cons}} \Delta p_a^{\mu\mathrm{rr\,nonlin}} + (\Delta p_{a\mu}^{\mathrm{rr\,lin}} + \Delta p_{a\mu}^{\mathrm{rr\,nonlin}})^2 = 0. \tag{12.14}$$

B. Completing the linear-response formula when $\mathcal{F}_a^{\mathrm{rr}}$ is time asymmetric, without being time antisymmetric

At this point, we need to complete one result derived in Ref. [30], namely, Eq. (3.25) there, giving the value of the radiation-reaction contribution $\chi^{\mathrm{rr\,rel}}$ to the relative scattering angle. Note first that the actual value of $\chi^{\mathrm{rr\,rel}}$ did not matter in the proof of the validity of Eq. (12.5) we have just given. Indeed, after summing over a, the coefficient of $\chi^{\mathrm{rr\,rel}}$ is

$$\frac{d}{d\chi_{\mathrm{cons}}} \left[\sum_a \Delta p_{a\mu}^{\mathrm{cons}} \right], \tag{12.15}$$

which vanishes because $\sum_a \Delta p_{a\mu}^{\mathrm{cons}}$ vanishes, independent of the value of χ_{cons}.

The only place were the assumption of time antisymmetry of the radiation reaction force was crucial in the derivation of the linear-response formula in Ref. [30] was in the derivation of the value of $\chi^{\mathrm{rr\,rel}}$ [leading to Eq. (3.25) there]. Going back to the previous derivation of $\chi^{\mathrm{rr\,rel}}$ in Ref. [29], it was explained, around Eq. (5.98) there, that one could (when using Lagrange's method of variation of constants) directly relate $\chi^{\mathrm{rr\,rel}}$ to the radiative losses of (c.m.) energy and angular momentum if the time derivatives of $\frac{dc_l(t)}{dt}$ and $\frac{dc_\phi(t)}{dt}$ were odd functions of time (around the moment of closest approach in the conservative motion). As $\frac{dc_l(t)}{dt}$ and $\frac{dc_\phi(t)}{dt}$ are linear expressions in the radiation-reaction force, their time-odd character is directly linked to the time-odd character of $\mathcal{F}_{\mathrm{rr}}$ (as was discussed at the end of Sec. III above, when working with the LO, 2.5PN radiation-reaction force). As we were aware of this limitation in Ref. [30], we limited our study of radiation-reaction effects to the 4.5PN level, because we had shown there [see Eq. (H3) there] that at the 5PN level there arose a nonzero value of P_x^{rad} (while a time-odd $\mathcal{F}_{\mathrm{rr}}$ implies a vanishing value for P_x^{rad}).

When staying at the level of linear effects in $\mathcal{F}_{\mathrm{rr}}$, a reexamination of the proof of the linear-response formula in Ref. [30] shows that the only $O(\mathcal{F}_{\mathrm{rr}})$ modification to take into account is the presence of an extra contribution in Eq. (3.25) there. One gets an explicit expression for the latter extra contribution by using the varying-constant version of the quasi-Keplerian representation, Eq. (3.1). From the equation parametrizing $\phi(t)$, and the link

$\chi = [\phi]_{-\infty}^{+\infty} - \pi$ between the total scattering angle χ and the variation of ϕ, we get (using $c_1 = E$ and $c_2 = J$)

$$\chi = [\bar{W}(l, E(t), J(t))] - \pi + [c_\phi(t)]. \tag{12.16}$$

The first term yields (when separating out the conservative contribution and linearly expanding in the radiative losses of energy and angular momentum) our usual linear-response formula for the radiative contribution to the c.m. relative scattering angle. The second contribution is new [and exists only when $\mathcal{F}_{\rm rr}(t)$ is time asymmetric, rather than time odd]. This yields the result

$$\begin{aligned}\chi^{\rm rr\,rel} &= -\left(\frac{1}{2}\frac{\partial\chi^{\rm cons}}{\partial E}E_{\rm rad} + \frac{1}{2}\frac{\partial\chi^{\rm cons}}{\partial J}J_{\rm rad}\right) + \Delta c_\phi \\ &\equiv \bar{\chi}^{\rm rr\,rel} + \Delta c_\phi,\end{aligned} \tag{12.17}$$

where the first contribution $\bar{\chi}^{\rm rr\,rel}$ has been evaluated at the $O(\mathcal{F}_{\rm rr})$ accuracy and where a formal, but explicit, expression for the additional contribution $\Delta c_\phi = [c_\phi] = \int_{-\infty}^{+\infty} dt \frac{dc_\phi(t)}{dt}$ is obtained from the last equation in Eq. (3.5) and reads

$$\begin{aligned}\Delta c_\phi = \int_{-\infty}^{+\infty} dt\Bigg[&\frac{\partial\bar{W}}{\partial l}\left(\frac{\partial S}{\partial l}\right)^{-1}\left(\frac{\partial S}{\partial E}\frac{dE}{dt} + \frac{\partial S}{\partial J}\frac{dJ}{dt}\right) \\ &- \frac{\partial\bar{W}}{\partial E}\frac{dE}{dt} - \frac{\partial\bar{W}}{\partial J}\frac{dJ}{dt}\Bigg].\end{aligned} \tag{12.18}$$

Here $\frac{dE}{dt}$ and $\frac{dJ}{dt}$ are linear expressions in $\mathcal{F}_{\rm rr}(t)$, defined by the first two equations in Eq. (3.5) [or, explicitly, Eq. (3.6) in the Hamiltonian formalism].

We leave to future work the use of this result to directly estimate the additional term (starting at the 5PN level) Δc_ϕ, in $\chi^{\rm rr\,rel}$, linked to time-asymmetric radiation-reaction effects.

C. Proof that time-asymmetric radiation-reaction contributions to $\Delta p_{a\mu}$ involve m_a^3

One of the aims of the present paper is to go beyond the limitations of Ref. [30] and to discuss the physical effects present in $P^{\rm rad}_{a\mu}$ and in $\Delta p_{a\mu}$ that are related to time-asymmetric (rather than simply time-odd) radiative processes. Time-asymmetric effects in the equations of motion first enter at the 4PN (and 4PM) level via tail-transported hereditary processes [48]. However, at the 4PN level one can still uniquely decompose these contributions to the dynamics into a nonlocal-in-time conservative (time-symmetric) contribution and a nonlocal-in-time dissipative (time-antisymmetric) one [49]. This postpones the presence of genuinely time-asymmetric effects to the 5PN level (still being at the 4PM level).

Additional information on the structure of time-asymmetric contributions to, say, the impulse of particle 1, is obtained by considering the small mass-ratio limit (say, $m_1 \ll m_2$). This limit is usefully tackled by using the gravitational self-force approximation method (i.e., perturbations around the probe limit in which a test particle of infinitesimal mass m_1 moves around a Schwarzschild black hole of mass m_2). It was shown in Ref. [68] that, if one works at the *first-order self-force* approximation, i.e., if one keeps only terms of order m_1 in the acceleration of particle 1, i.e., terms of order m_1^2 in the force acting on particle 1, one can uniquely decompose the dynamics in a conservative (time-symmetric) contribution and a nonlocal-in-time dissipative (time-antisymmetric) one. This proves that the level where the separation time even versus time odd becomes ambiguous is the second-order self-force approximation, corresponding to terms of order m_1^3 in the force acting on particle 1. The corresponding contributions to $\Delta p_{1\mu}$ will therefore also involve a factor m_1^3. (When scaling out the total mass, such terms contain a factor ν^3.)

D. Contribution to the impulses proportional to $P_x^{\rm rad}$ and its nonpolynomiality in the masses

As recalled above, Ref. [30] generalized the linear-response formula of Ref. [29] by including recoil[14] effects. However, while the effects proportional to the $\mathbf{e}_y$ component $P_y^{\rm rad}$ of the recoil were kept (and analyzed) in all the formulas derived in Ref. [30], in some of the formulas there the contributions proportional to the $\mathbf{e}_x$ component $P_x^{\rm rad}$ were set to zero. Here we explicitly include (and analyze) the contribution to the impulses proportional to $P_x^{\rm rad}$.

Accordingly, it is henceforth useful to decompose the radiation-reaction contribution $\Delta p^{\rm rr}_{a\mu}$ to the impulses in the following new way:

$$\Delta p^{\rm rr}_{a\mu} = \Delta p^{\rm rr\,lin-odd}_{a\mu} + \Delta p^{\rm rr\,P_x^{rad}}_{a\mu} + \Delta p^{\rm rr\,remain}_{a\mu}. \tag{12.19}$$

Here, $\Delta p^{\rm rr\,lin-odd}_{a\mu}$ denotes the part of our linear-response formula obtained when assuming that $\mathcal{F}_{\rm rr}$ is time odd (keeping the full[15] $E^{\rm rad}$, $J^{\rm rad}$, and $P_y^{\rm rad}$ contributions, but setting $\Delta c_\phi = 0$, and $P_x^{\rm rad} = 0$),

$$\Delta p^{\rm rr\,P_x^{rad}}_{a\mu} \equiv -\frac{E_a}{E_{\rm c.m.}}P_x^{\rm rad}e_{x\mu} - \frac{(p^+_{ax}P_x^{\rm rad})}{E_{\rm c.m.}}U_\mu \tag{12.20}$$

is the contribution linked to a nonzero value of $P_x^{\rm rad}$ contained in Eq. (3.33) of Ref. [30], and finally,

$$\Delta p^{\rm rr\,remain}_{a\mu} \equiv \Delta p^{\rm rr\,\Delta c_\phi}_{a\mu} + \Delta p^{\rm rr\,nonlin}_{a\mu}, \tag{12.21}$$

where

[14]As $P^{\rm rad}_\mu = O(G^3)$, it is enough to work linearly in recoil to reach the $O(G^6)$ accuracy.

[15]The adjective "full" means here that we keep all the time-asymmetric (tail) contributions to the radiative losses.

$$\Delta p_{a\mu}^{\mathrm{rr}\,\Delta c_\phi} = \Delta c_\phi \frac{d}{d\chi_{\mathrm{cons}}} \Delta p_{a\mu}^{\mathrm{cons}} \qquad (12.22)$$

is the additional term linked to a nonzero Δc_ϕ and where $\Delta p_{a\mu}^{\mathrm{rr\,nonlin}}$ is the same remainder term as in our previous decomposition [nonlinear in radiation reaction and satisfying the antisymmetry constraint Eq. (12.11)].

An important fact for the following reasonings is that, as Δc_ϕ is symmetric under particle exchange, while $\sum_a \frac{d}{d\chi_{\mathrm{cons}}} \Delta p_{a\mu}^{\mathrm{cons}} = 0$, the contribution $\Delta p_{a\mu}^{\mathrm{rr}\,\Delta c_\phi}$ is antisymmetric under particle exchange. As the same was proven to be true for $\Delta p_{a\mu}^{\mathrm{rr\,nonlin}}$ [see Eq. (12.11)], we conclude that $\Delta p_{a\mu}^{\mathrm{rr\,remain}}$ also satisfies the antisymmetry constraint

$$\Delta p_{1\mu}^{\mathrm{rr\,remain}} + \Delta p_{2\mu}^{\mathrm{rr\,remain}} = 0. \qquad (12.23)$$

From our previous work, and from the considerations above, we know that both Δc_ϕ and P_x^{rad} start at order $\frac{G^4}{c^{10}}$, i.e., at 4PM and 5PN. Therefore, $\Delta p_{a\mu}^{\mathrm{rr\,remain}}$ starts also at order $\frac{G^4}{c^{10}}$.

One useful source of information on the various contributions to $\Delta p_{a\mu}^{\mathrm{rr}}$ in the decomposition (12.19) is that they should combine to ensure the mass polynomiality of $\Delta p_{a\mu}^{\mathrm{rr}}$. (We assume here, consistent with previous works, that $\Delta p_{a\mu}^{\mathrm{cons}}$ has been defined so as to be mass polynomial.)

It was shown in Ref. [30] that $\Delta p_{a\mu}^{\mathrm{rr\,lin-odd}}$ (in the precise sense defined above) is polynomial in the masses under some constraints on the mass structure of E_{rad}, J^{rad}, and P_y^{rad}. It is easily checked that the constraints discussed in Ref. [30] are all implied by the more general constraints on the mass structure of E_{rad}, J^{rad}, and P_y^{rad}, which have been deduced above from the mass polynomiality of P_μ^{rad}, considered as a function of b (see Sec. XI above). Therefore, the contribution $\Delta p_{a\mu}^{\mathrm{rr\,lin-odd}}$ to $\Delta p_{a\mu}^{\mathrm{rr}}$ in the decomposition (12.19) is separately polynomial in masses.

By contrast, we see that the presence of denominators $E_{\mathrm{c.m}}$. in $\Delta p_{a\mu}^{\mathrm{rr}\,P_x^{\mathrm{rad}}}$, Eq. (12.20), implies that the P_x^{rad} contribution to $\Delta p_{a\mu}^{\mathrm{rr}}$ is *nonpolynomial* in the masses. We are going to see that the need to cancel the nonpolynomiality of P_x^{rad} by the remaining contribution $\Delta p_{a\mu}^{\mathrm{rr\,remain}}$, together with the antisymmetric character, Eq. (12.23), and the second-self-force character ($\propto m_a^3$) of the remaining contribution, uniquely determines $\Delta p_{a\mu}^{\mathrm{rr\,remain}}$ (and therefore $\Delta p_{a\mu}^{\mathrm{rr}}$) at order G^4 and determines it nearly completely at order G^5.

E. Uniqueness of $\Delta p_{a\mu}^{\mathrm{rr\,remain}}$ and $\Delta p_{a\mu}^{\mathrm{rr}}$ at 4PM and strong constraints on them at 5PM

To discuss the uniqueness of $\Delta p_{a\mu}^{\mathrm{rr\,remain}}$, it is useful to consider its form factors on the same basis as the one used in Sec. XI, namely, $u_{1-}^\mu + u_{2-}^\mu$, $u_{1-}^\mu - u_{2-}^\mu$, and $\hat{b}_{12}^\mu$. Namely, for $a = 1$, and for any label $X = \mathrm{rr\,remain}$, $\mathrm{rr}\,P_x^{\mathrm{rad}}$, rr lin-odd, etc., we write

$$\begin{aligned}\Delta p_1^{\mu,X} &= c_{1+2}^{1X}(m_1, m_2, \gamma, b)(u_{1-}^\mu + u_{2-}^\mu) \\ &\quad + c_{1-2}^{1X}(m_1, m_2, \gamma, b)(u_{1-}^\mu - u_{2-}^\mu) \\ &\quad + c_b^{1X}(m_1, m_2, \gamma, b)\hat{b}_{12}^\mu. \end{aligned} \qquad (12.24)$$

For $a = 2$, one should exchange $1 \leftrightarrow 2$, including in the basis vectors.

Among the basis vectors, the first one is symmetric under particle exchange, while the other two are antisymmetric. The exchange antisymmetry of $\Delta p_{1\mu}^{\mathrm{rr\,remain}}$ then implies that its component $c_{1+2}^{1\,\mathrm{remain}}$ along $u_{1-}^\mu + u_{2-}^\mu$ will be antisymmetric, while its components, $c_{1-2}^{1\,\mathrm{remain}}, c_b^{1\,\mathrm{remain}}$ along $u_{1-}^\mu - u_{2-}^\mu$, and $\hat{b}_{12}^\mu$ will be symmetric. Let us assume that we can construct (as we will do next) one particular $\Delta p_{a\mu}^{\mathrm{rr\,remain}}$ that satisfies the needed conditions of canceling the nonpolynomiality of $\Delta p_{a\mu}^{\mathrm{rr}\,P_x^{\mathrm{rad}}}$ (so as to lead to a mass-polynomial $\Delta p_{a\mu}^{\mathrm{rr}}$) and of being $\propto m_a^3$. The most general $\Delta p_{a\mu}^{\mathrm{rr\,remain}}$ satisfying the latter condition will then be obtained by adding to this particular solution a general additional term, say, $\Delta p_{a\mu}^{\mathrm{rr\,remain\,add}}$ that must satisfy several conditions. Namely, (i) it must be antisymmetric; (ii) it must be mass polynomial; and (iii) it must contain a factor m_a^3 (in addition to containing the factor $m_1^2 m_2^2$ which is a common factor of all contributions to $\Delta p_{a\mu}^{\mathrm{rr}}$).

Let us prove that there cannot exist such a $\Delta p_{a\mu}^{\mathrm{rr\,remain\,add}}$ at order G^4. Indeed, at order G^4, mass polynomiality of an impulse means that it must be quintic in masses. After factoring the universal factor $m_1^2 m_2^2$, we find that the mass dependence of the (antisymmetric) component of $\Delta p_{a\mu}^{\mathrm{rr\,remain\,add}}$ along $u_{1-}^\mu + u_{1-}^\mu$ must be proportional to $m_1^2 m_2^2 (m_1 - m_2)$, while the (symmetric) components of $\Delta p_{a\mu}^{\mathrm{rr\,remain\,add}}$ along $u_{a-}^\mu - u_{a'-}^\mu$ and $\hat{b}_{aa'}^\mu$ (with $a' \neq a$) must be proportional to $m_1^2 m_2^2 (m_1 + m_2)$. Neither of these types of components can also satisfy the last condition of containing a factor m_a^3.

When going at order G^5, we must discuss antisymmetric, or symmetric, *sextic* polynomials in masses. In the antisymmetric case ($u_{1-}^\mu + u_{2-}^\mu$ component), such polynomials must be proportional to $m_1^2 m_2^2 (m_1 - m_2)(m_1 + m_2)$, and the m_a^3 condition does not allow such terms. By contrast, in the symmetric case ($u_{1-}^\mu - u_{2-}^\mu$ and b_{12}^μ components), such polynomials must be proportional to a combination $m_1^2 m_2^2 (c_{M^2}(m_1 + m_2)^2 + c_{m_1 m_2} m_1 m_2)$. The first combination (with coefficient c_{M^2}) is forbidden by the m_a^3 condition. However, the second combination, namely, $c_{m_1 m_2} m_1^3 m_2^3$ is compatible with the m_a^3 condition. The conclusion is that at order G^5 there are two different types of contributions that can be added to any specific solution of all the conditions, namely,

$$\Delta p_1^{\mu\,\mathrm{rr\,remain\,add}} = \frac{G^5 m_1^3 m_2^3}{b^5}(f^{G^5}_{1-2}(\gamma)(u_1^\mu - u_2^\mu) + f_b^{G^5}(\gamma)\hat{b}_{12}^\mu), \tag{12.25}$$

involving two *a priori* unconstrained functions of γ: $f^{G^5}_{1-2}(\gamma)$ and $f_b^{G^5}(\gamma)$.

We show below how to construct a particular solution of all the constraints. The general solution at order G^5 is then obtained by adding the specific ($\propto m_1^3 m_2^3$) additional terms displayed in Eq. (12.25).

F. Determining the unique transverse components $\Delta p_{ab}^{\mathrm{rr\,remain}}$ and $\Delta p_{ab}^{\mathrm{rr}}$ at 4PM

For definiteness, we henceforth consider the impulse of the first particle, $a = 1$. It is easily seen from its definition in Eq. (12.20) that, at order G^4, the only nonzero component of $\Delta p_{1\mu}^{\mathrm{rr}\,P_x^{\mathrm{rad}}}$ is the one along $\hat{b}_{12}^\mu$, say,

$$\Delta p_{1b}^{\mathrm{rr}\,P_x^{\mathrm{rad}}} \equiv \Delta p_{1\mu}^{\mathrm{rr}\,P_x^{\mathrm{rad}}}\hat{b}_{12}^\mu, \tag{12.26}$$

which is equal to

$$\Delta p_{1b}^{\mathrm{rr}\,P_x^{\mathrm{rad}}} = -\frac{E_1}{E_{\mathrm{c.m.}}}P^{\mathrm{rad}}_{x\,G^4}. \tag{12.27}$$

The problem to be solved is the following: given the nonpolynomial term in the $\hat{b}_{12}^\mu$ component of $\Delta p_{1\mu}^{\mathrm{rr}\,P_x^{\mathrm{rad}}}$,

$$\Delta p_{1b}^{\mathrm{rr}\,P_x^{\mathrm{rad}}} = -\frac{E_1}{E_{\mathrm{c.m.}}}P^{\mathrm{rad}}_{x\,G^4} = -\frac{m_1(m_1+\gamma m_2)}{M^2 h^2}P^{\mathrm{rad}}_{x\,G^4}, \tag{12.28}$$

where $P^{\mathrm{rad}}_{x\,G^4}$ is mass polynomial and of the type [see Eq. (11.22)]

$$P^{\mathrm{rad}}_{x\,G^4} = \frac{G^4}{b^4}m_1^2 m_2^2(m_2 - m_1)p_x^{G^4}(\gamma), \tag{12.29}$$

what type of extra contribution $\Delta p_{1b}^{\mathrm{rr\,remain}} \equiv \Delta p_{1\mu}^{\mathrm{rr\,remain}}\hat{b}_{12}^\mu$ (satisfying the constraints discussed above) can be added to it to guarantee that the sum becomes polynomial in the masses.

It is easily seen that

$$\begin{aligned}\Delta p_{1b\,G^4}^{\mathrm{rr\,remain}} &= \frac{G^4}{b^4}m_1^2 m_2^2\frac{m_1 E_2 + m_2 E_1}{E}p_x^{G^4}(\gamma)\\ &= \frac{G^4}{b^4}m_1^2 m_2^2\frac{m_1 m_2(\gamma+1)}{M h^2}p_x^{G^4}(\gamma)\end{aligned} \tag{12.30}$$

satisfies the needed constraints (symmetry, $\propto m_1^3$) and solves the problem at hand. Indeed,

$$-\frac{E_1}{E_{\mathrm{c.m.}}}P^{\mathrm{rad}}_{x\,G^4} + \Delta p_{1b\,G^4}^{\mathrm{rr\,remain}} = +\frac{G^4}{b^4}m_1^3 m_2^2 p_x^{G^4}(\gamma). \tag{12.31}$$

As proven above, this solution is unique.

Therefore, we have proven that the full radiation-reaction contribution to the impulse (including the time-even contribution $\Delta p_{1\ \mu G^4}^{\mathrm{rr}\,\Delta c_\phi}$ and the nonlinear one $\Delta p_{1\ \mu G^4}^{\mathrm{rr\,nonlin}}$) is given by

$$\Delta p_{1\ \mu G^4}^{\mathrm{rr}} = \Delta p_{1\ \mu G^4}^{\mathrm{rr\,lin-odd}} + \frac{G^4}{b^4}m_1^3 m_2^2 p_x^{G^4}(\gamma)\hat{b}_{12}^\mu, \tag{12.32}$$

or, equivalently [using the definition Eq. (11.22) of $p_x^{G^4}(\gamma)$],

$$\Delta p_{1\,\mu G^4}^{\mathrm{rr}} = \Delta p_{1\,\mu G^4}^{\mathrm{rr\,lin-odd}} + \frac{m_1}{m_2 - m_1}P^{\mathrm{rad}}_{x\,G^4}\hat{b}_{12}^\mu. \tag{12.33}$$

In other words, the full, 4PM-level, transverse impulse of the first particle reads

$$\begin{aligned}\Delta p_{1b\,G^4} &= \Delta p_{1b\,G^4}^{\mathrm{cons}} + \Delta p_{1b\,G^4}^{\mathrm{rr\,lin-odd}} + \frac{G^4}{b^4}m_1^3 m_2^2 p_x^{G^4}(\gamma)\\ &= \Delta p_{1b\,G^4}^{\mathrm{cons}} + \Delta p_{1b\,G^4}^{\mathrm{rr\,lin-odd}} + \frac{m_1}{m_2 - m_1}P^{\mathrm{rad}}_{x\,G^4}.\end{aligned} \tag{12.34}$$

The latter equation corresponds to Eq. (18) in Ref. [24], with the value $\frac{G^4}{b^4}m_1^3 m_2^2 p_x^{G^4}(\gamma)$ for the (undefined) term denoted $\frac{G^4}{b^4}\nu M^5 c_{b,4}^{\mathrm{rr,even}}$ there. Note that our reasoning has given a direct relation between this term and the value of $P^{\mathrm{rad}}_{x\,G^4}$, namely,

$$\frac{G^4}{b^4}m_1^3 m_2^2 p_x^{G^4}(\gamma) = \frac{m_1}{m_2 - m_1}P^{\mathrm{rad}}_{x\,G^4}. \tag{12.35}$$

Our results above yield only the beginning of the PN expansion of the function $p_x^{G^4}(\gamma)$, namely,

$$\begin{aligned}p_x^{G^4}(\gamma) &= \frac{h^4 P_{x4}}{(\gamma^2-1)^2}\\ &= \pi\left(-\frac{1491}{400}p_\infty^3 + \frac{26757}{5600}p_\infty^5 + O(p_\infty^7)\right).\end{aligned} \tag{12.36}$$

Concerning the first term, $\Delta p_{1b\,G^4}^{\mathrm{rr\,lin-odd}}$, its general expression as a function of $E^{\mathrm{rad}}_{G^3}$, $J^{\mathrm{rad}}_{G^2}$, and $J^{\mathrm{rad}}_{G^3}$ was derived in Eq. (7.16) of Ref. [30]. At the time, only $E^{\mathrm{rad}}_{G^3}$ [9,23] and $J^{\mathrm{rad}}_{G^2}$ [28] were known (in a PN-exact sense). Since then, the exact value of $J^{\mathrm{rad}}_{G^3}$ has been obtained in Ref. [24]. This leads to the following exact value of $\Delta p_{1b\,G^4}^{\mathrm{rr\,lin-odd}}$:

$$\Delta p_{1b\,G^4}^{\mathrm{rr\,lin-odd}} = \frac{G^4}{b^4}m_1^2 m_2^2[C_{bM}^{\mathrm{4PM}}(\gamma)M + C_{bm_1}^{\mathrm{4PM}}(\gamma)m_1], \tag{12.37}$$

with coefficients [see Eq. (7.31) of Ref. [30] and Eq. (19) of Ref. [24]]

$$C^{4\mathrm{PM}}_{bM}(\gamma)=\pi\hat{\mathcal{E}}\frac{\gamma(6\gamma^2-5)}{(\gamma^2-1)^{3/2}}-\pi\frac{3}{4}\hat{J}_2\frac{(5\gamma^2-1)}{(\gamma^2-1)^{3/2}}-\hat{J}_3\frac{(2\gamma^2-1)}{(\gamma^2-1)^2},$$
$$C^{4\mathrm{PM}}_{bm_1}(\gamma)=-\pi\hat{\mathcal{E}}\frac{2\gamma^2-1}{(\gamma+1)\sqrt{\gamma^2-1}}. \quad (12.38)$$

Here, $\hat{\mathcal{E}}=\mathcal{E}/\pi$, $\hat{J}_2=2(2\gamma^2-1)(\gamma^2-1)^{1/2}\mathcal{I}$ (with $\mathcal{I}$ defined in [28]), and $\hat{J}_3=(\gamma^2-1)(\mathcal{C}+2\mathcal{D})$ (with $\mathcal{C}$ and $\mathcal{D}$ defined in [24]).

When separating out the 4PM conservative contribution $\Delta p^{\mathrm{cons}}_{1b\,G^4}$ [10,25] from the $\hat{b}^{\mu}_{12}$-projected impulse in our Eq. (12.34), the term $\Delta p^{\mathrm{rr\,lin-odd}}_{1b\,G^4}$ coincides with the term $c^{(4)\mathrm{diss}}_{1\mathrm{b},1\,\mathrm{rad}}$ in Eq. (15) of [25], while the remaining term $\frac{G^4}{b^4}m_1^3m_2^2p_x^{G^4}(\gamma)$ has the same mass structure as the term $c^{(4)\mathrm{diss}}_{1\,\mathrm{b},2\,\mathrm{rad}}$ in Eq. (16) of [25]. Moreover, not only the first two terms in the PN expansion of $c^{(4)\mathrm{diss}}_{1\,\mathrm{b},2\,\mathrm{rad}}$ given in Eq. (16) of [25] agree with those given by inserting our PN-derived result Eq. (12.36) in the last term in Eq. (12.34), but the PN-exact value of $c^{(4)\mathrm{diss}}_{1\,\mathrm{b},2\,\mathrm{rad}}$ [25] satisfies the exact relation $c^{(4)\mathrm{diss}}_{1\,\mathrm{b},2\,\mathrm{rad}}=\frac{m_1}{m_2-m_1}P^{\mathrm{rad}}_{x\,G^4}$ derived here between this remaining term and the x component of the radiated momentum.

G. High-energy behavior of $\Delta p^{G^4}_{1b}$

Let us remark in passing that, if one considers the result Eq. (12.34), the mass scaling of the term $\frac{G^4}{b^4}m_1^3m_2^2p_x^{G^4}(\gamma)$ makes it impossible to tame the high-energy behavior of $\Delta p^{G^4}_{1b}$.

When considering the high-energy (HE) limit $\gamma\to\infty$ for a fixed value of the scattering angle $\chi_1\sim\frac{GE_{\mathrm{c.m.}}}{b}$, with $E_{\mathrm{c.m.}}=Mh\propto\gamma^{\frac{1}{2}}$, one would expect, in this limit (suitably scaled[16]) scattering observables to admit a finite limit. If the formal $G\to0$ limit commuted with the HE limit, this would imply, in particular, that each term in the PM expansion of the impulse would admit a finite HE limit (at fixed $\chi_1\sim\frac{GE_{\mathrm{c.m.}}}{b}$). This is the case at orders G^1 and G^2. At the G^3 level, the conservative contribution $\Delta p^{\mathrm{cons},G^3}_{1b}/P_{\mathrm{c.m.}}$ [5] is logarithmically larger than its expected contribution $\sim\chi_1^3$. However, it was found [27,28] that this logarithmic divergence is tamed when completing the conservative impulse by the radiative correction $\Delta p^{\mathrm{rr},G^3}_{1b}$. This raises the hope that a similar taming might occur at order G^4.

At order G^4 the ratio $\Delta p^{\mathrm{cons},G^4}_{1b}/(P_{\mathrm{c.m}}\chi_1^4)$ is power-law divergent, being proportional to $\gamma^{\frac{1}{2}}$. In terms of the unrescaled impulse, this divergence is $\Delta p^{\mathrm{cons},G^4}_{1b}\propto\gamma^3$. Parametrizing the various contributions to the HE limit of the impulse according to

$$\Delta p^{X,G^4}_{1b}\approx\frac{G^4m_1^2m_2^2}{b^4}\pi C^{X,G^4}\gamma^3, \quad (12.39)$$

the coefficient entering the conservative contribution $\Delta p^{\mathrm{cons},G^4}_{1b}$ is

$$C^{\mathrm{cons},G^4}=-\frac{105}{8}(4\ln(2)-1+4\ln(2)^2)(m_1+m_2). \quad (12.40)$$

As pointed out in [24], the linear-response radiative contribution $\Delta p^{\mathrm{rr\,lin-odd}}_{1b\,G^4}$ is similarly $\propto\gamma^3$. However, the corresponding coefficient is

$$C^{\mathrm{rr,lin},G^4}=\frac{35}{4}[m_1(1+8\ln(2))+2m_2(1+5\ln(2))], \quad (12.41)$$

which has the correct sign, but not the correct value to cancel the "bad" high-energy behavior of the conservative contribution. If we assume that the function $p_x^{G^4}(\gamma)$ entering our additional contribution has a HE behavior of the type $p_x^{G^4}(\gamma)\approx\pi c_x\gamma^3$, it will contribute another term of order γ^3, with a coefficient

$$C^{\mathrm{rr,p_x},G^4}=c_xm_1. \quad (12.42)$$

It is, however, easy to see that, whatever the value of c_x, such an additional term (proportional only to m_1) cannot tame the contribution proportional to m_2, i.e., cannot yield a vanishing total coefficient $C^{\mathrm{tot},G^4}=C^{\mathrm{cons},G^4}+C^{\mathrm{rr,lin},G^4}+C^{\mathrm{rr,p_x},G^4}$. Indeed, the latter turns out to be[17]

$$C^{\mathrm{tot},G^4}=\left(\frac{35}{2}\ln(2)+\frac{175}{8}-\frac{105}{2}\ln(2)^2+c_x\right)m_1+\left(35\ln(2)+\frac{245}{8}-\frac{105}{2}\ln(2)^2\right)m_2. \quad (12.43)$$

In order to tame the HE behavior of $\Delta p^{\mathrm{tot},G^4}_{1b}$, i.e., to reduce it from γ^3 to, say, γ^2 or $\gamma^2\ln\gamma$,[18] one would need to add a suitable extra contribution of the (disallowed) symmetric type $\frac{G^4}{b^4}m_1^2m_2^2(m_1+m_2)f_{\mathrm{sym}}(\gamma)$.

[16]For example, one should consider the ratio $\Delta p_{1b}/P_{\mathrm{c.m.}}$.

[17]The recent result of Ref. [25] happens to lead to a coefficient c_x which precisely annuls the coefficient of m_1 in C^{tot,G^4}.

[18]Such a reduction would ensure the HE vanishing of the ratio $\Delta p^{\mathrm{tot},G^4}_{1b}/(P_{\mathrm{c.m}}\chi_1^4)$, as expected from the structure of the massless scattering discussed in Ref. [19].

We do not view the inability of the additional term to tame the HE behavior of the *G-expanded* impulse as a blemish. It seems indeed probable that the $G \to 0$ limit does not commute with the HE limit $\gamma \to \infty$. This is notably indicated by the studies of the HE limit of the total gravitational-wave energy emitted during the collision of massless particles [69–71]. While the HE limit of the $O(G^3)$ leading-order radiative energy loss exceeds the energy $E_{\rm c.m.}$ available in the system by a factor $\propto \gamma^{\frac{1}{2}}$, the works [69–71] suggest that (due to coherence effects in the beamed radiation) the HE limit of radiative losses is finite and of order $\chi_1^2 \ln \frac{1}{\chi_1}$.

H. Longitudinal components of $\Delta p^{\rm rr}_{1\mu}$ at 4PM

To end our discussion of the radiative contributions to the impulse of the first particle $\Delta p^{\rm rr}_{1\mu}$, let us also consider its longitudinal components, i.e., the components along u_{1-} and u_{2-}. We have shown above that the only source of nonpolynomiality (namely, the P_x-related contribution $\Delta p_{1\mu}^{\rm rr\, P_x^{rad}}$) does not contribute to the longitudinal components. In addition, we have shown that there was, at the 4PM level, a unique value of $\Delta p^{\rm rr}_{1\mu}$ satisfying all the needed constraints. Namely, the one given by Eq. (12.32) or (12.33).

In view of Eq. (12.32), at order G^4, the longitudinal components of $\Delta p^{\rm rr}_{1\mu}$ are fully described by the time-odd-linear-response formula of Ref. [30], i.e., the term denoted $\Delta p_1^{\mu{\rm rr\, lin-odd}}$ above. Using the notation of Ref. [30], its longitudinal components are defined as follows:

$$\begin{aligned}\Delta p_1^{\mu{\rm rr\, longit}} &= \Delta p_1^{\mu{\rm rr\, lin\text{-}odd\, longit}}\\ &= c_{u_1}^{1\,{\rm rr}} u_{1-}^\mu + c_{u_2}^{1\,{\rm rr}} u_{2-}^\mu\\ &= c_{u_1}^{1\,{\rm rr,lin\text{-}odd}} u_{1-}^\mu + c_{u_2}^{1{\rm rr,lin\text{-}odd}} u_{2-}^\mu. \end{aligned} \qquad (12.44)$$

Using the expressions given in Table II of Ref. [30],[19] we find that the coefficients $c_{u_1}^{1\,{\rm rr}}$ and $c_{u_2}^{1\,{\rm rr}}$ are given by

$$\begin{aligned} c_{u_1}^{1\,{\rm rr,4\,PM}} &= -\frac{G^4 m_1^2 m_2^2}{b^4(\gamma^2-1)^3}\left[\gamma M \tilde{E}_4^0 + \frac{1}{2}\gamma m_1 \tilde{E}_4^1 + 2(2\gamma^2-1)^2(m_1\gamma + m_2)\hat{J}_2\right],\\ c_{u_2}^{1\,{\rm rr,4\,PM}} &= \frac{G^4 m_1^2 m_2^2}{b^4(\gamma^2-1)^3}\left[M \tilde{E}_4^0 + \frac{1}{2} m_1 \tilde{E}_4^1 + 2(2\gamma^2-1)^2(m_2\gamma + m_1)\hat{J}_2\right], \end{aligned} \qquad (12.45)$$

where $\tilde{E}_4^0$ and $\tilde{E}_4^1$ (defined by $h^5 E_4 = \tilde{E}_4^0 + \nu \tilde{E}_4^1$), as well as $\hat{J}_2 \equiv h^2 J_2$ are all functions only of γ. [See Eq. (8) of [25] for the exact value of $h^5 E_4$.]

The combination

$$b^4(c_{u_1}^{1\,{\rm rr,4\,PM}} + \gamma c_{u_2}^{1\,{\rm rr,4\,PM}}) = m_1^2 m_2^3 \frac{2\hat{J}_2(2\gamma^2-1)^2}{(\gamma^2-1)^2}, \qquad (12.46)$$

coincides with the impulse coefficient $c^{(4){\rm diss}}_{1\check{u}_1, 1\,{\rm rad}}$ given in Eq. (15) of Ref. [25]. The other combination

$$\begin{aligned} &b^4(c_{u_2}^{1\,{\rm rr,4\,PM}} + \gamma c_{u_1}^{1\,{\rm rr,4\,PM}})\\ &\quad = -\tfrac{m_1^2 m_2^2}{(\gamma^2-1)^2}[m_1(\tilde{E}_4^0 + \tfrac{1}{2}\tilde{E}_4^1 + 2(2\gamma^2-1)^2\hat{J}_2) + m_2 \tilde{E}_4^0], \end{aligned} \qquad (12.47)$$

coincides with the sum $c^{(4){\rm diss}}_{1\check{u}_1, 1\,{\rm rad}} + c^{(4){\rm diss}}_{1\check{u}_2, 2\,{\rm rad}}$ of the two $\check{u}_2$-type impulse coefficients given in Eqs. (15) and (16) in Ref. [25]. More precisely, the part called $c^{(4){\rm diss}}_{1\check{u}_1, 1\,{\rm rad}}$ corresponds to the part of the right-hand side of Eq. (12.47) featuring *odd* powers of p_∞ in its PN expansion, while the part called $c^{(4){\rm diss}}_{1\check{u}_2, 2\,{\rm rad}}$ corresponds to the part of the right-hand side of Eq. (12.47) featuring *even* powers of p_∞ in its PN expansion (the latter part is the one generated by the tail contribution to the radiated energy).

I. Radiative contribution to the impulse coefficients at 5PM: Transverse component

As in the above discussion of the impulse at 4PM, it is convenient to project the various radiative contributions (labeled by $X = $ rr lin-odd, rr $\rm P_x^{rad}$, rr remain) to the impulse,

$$\Delta p^{\rm rr}_{a\mu} = \Delta p^{\rm rr\, lin\text{-}odd}_{a\mu} + \Delta p^{\rm rr\, P_x^{rad}}_{a\mu} + \Delta p^{\rm rr\, remain}_{a\mu}, \qquad (12.48)$$

on the basis given in Eq. (12.24). For instance, for $a = 1$, the transverse ($\hat{b}^\mu_{12}$) component is the sum of the following contributions:

$$c_b^{1,{\rm rr}} = c_b^{1,{\rm rr\, lin\text{-}odd}} + c_b^{1,{\rm rr\, P_x^{rad}}} + c_b^{1,{\rm rr\, remain}}. \qquad (12.49)$$

Similar to what happened at 4PM, the nonpolynomial contribution generated by $c_b^{1,{\rm rr\, P_x^{rad}}}$ reads, at the 5PM level,

$$c_b^{1,{\rm rr\, P_x^{rad}} G^5} = -\frac{E_1}{E_{\rm c.m.}} P^{\rm rad}_{xG^5}. \qquad (12.50)$$

[19] We also use Eq. (7.26) there to replace the original expression in terms of the 4PM component P_4 of the y component $P_y^{\rm rad}$ of the recoil in terms of the rescaled 4PM component $\tilde{E}_4$ of the energy loss.

Again, the simplest solution (satisfying all the needed constraints) for the remaining contribution $c_b^{1,\mathrm{rr\,remain}\,G^5}$ to cancel the nonpolynomiality of $c_b^{1,\mathrm{rr}\,\mathrm{P}_x^{\mathrm{rad}}G^5}$ is

$$c_b^{1,\mathrm{rr\,remain\,simplest},G^5} = +\frac{m_1E_2+m_2E_1}{(m_2-m_1)E_{\mathrm{c.m.}}}P^{\mathrm{rad}}_{xG^5}. \tag{12.51}$$

Indeed, we have

$$c_b^{1,\mathrm{rr}\,\mathrm{P}_x^{\mathrm{rad}}G^5} + c_b^{1,\mathrm{rr\,remain\,simplest},G^5}, = +\frac{m_1}{(m_2-m_1)}P^{\mathrm{rad}}_{xG^5}, \tag{12.52}$$

which is polynomial in masses because $P^{\mathrm{rad}}_{xG^5}$ contains a factor (m_2-m_1).

As was discussed above, the most general solution for $c_b^{1,\mathrm{rr\,remain}\,G^5}$ is

$$c_b^{1,\mathrm{rr\,remain},G^5} = +\frac{m_1E_2+m_2E_1}{(m_2-m_1)E_{\mathrm{c.m.}}}P^{\mathrm{rad}}_{xG^5} + \frac{G^5}{b^5}m_1^3m_2^3f_b^{G^5}(\gamma). \tag{12.53}$$

Writing $P^{\mathrm{rad}}_{xG^5}$ as

$$P^{\mathrm{rad}}_{xG^5} = \frac{G^5}{b^5}m_1^2m_2^2(m_2-m_1)(m_1+m_2)p_{xG^5}(\gamma), \tag{12.54}$$

we finally get

$$c_b^{1,\mathrm{rr}\,\mathrm{P}_x^{\mathrm{rad}}G^5} + c_b^{1,\mathrm{rr\,remain},G^5} = +\frac{G^5}{b^5}m_1^3m_2^2(m_1+m_2)p_{xG^5}(\gamma) + \frac{G^5}{b^5}m_1^3m_2^3f_b^{G^5}(\gamma). \tag{12.55}$$

In other words, the most general 5PM transverse radiative impulse reads

$$c_b^{1,\mathrm{rr}\,G^5} = c_b^{1,\mathrm{rr\,lin-odd}\,G^5} + \frac{G^5}{b^5}m_1^3m_2^2(m_1+m_2)p_{xG^5}(\gamma) + \frac{G^5}{b^5}m_1^3m_2^3f_b^{G^5}(\gamma). \tag{12.56}$$

Tables I and II of Ref. [30] gave exact expressions for $c_b^{1,\mathrm{rr\,lin-odd}\,G^5}$ in terms of E_n and J_n with $n\le 4$. However, the PN-exact value of J_4 is unknown so that our 5.5PN-accurate determination of J_4 currently limits the knowledge of $c_b^{1,\mathrm{rr\,lin-odd}\,G^5}$ to the 5.5PN level. We so find

$$\begin{aligned}
c_b^{1,\mathrm{rr\,lin-odd}\,G^5} = \frac{G^5m_1^2m_2^2}{b^5}\Bigg\{&\frac{416}{45}(m_1+m_2)^2\frac{1}{p_\infty^4} + \left[\left(\frac{169664}{1575}-\frac{47}{5}\pi^2\right)m_1^2 + \left(\frac{409888}{1575}-\frac{94}{5}\pi^2\right)m_1m_2\right.\\
&\left.+\left(\frac{203264}{1575}-\frac{47}{5}\pi^2\right)m_2^2\right]\frac{1}{p_\infty^2} - \frac{896}{45}(m_1+m_2)^2\frac{1}{p_\infty} + \left(\frac{159232}{3675}-\frac{1243}{56}\pi^2\right)m_1^2\\
&+\left(\frac{2283544}{11025}-\frac{1489}{35}\pi^2\right)m_1m_2 + \left(\frac{116992}{1225}-\frac{5697}{280}\pi^2\right)m_2^2 + \left(\frac{8384}{45}m_1^2+\frac{694016}{1575}m_1m_2+\frac{10304}{45}m_2^2\right)p_\infty\\
&+\left[\left(\frac{241}{120}\pi^2-\frac{22294592}{363825}\right)m_1^2 + \left(\frac{67876972}{363825}+\frac{23783}{3360}\pi^2\right)m_1m_2 + \left(-\frac{9728}{275}+\frac{3407}{672}\pi^2\right)m_2^2\right]p_\infty^2\\
&+O(p_\infty^3)\Bigg\}.
\end{aligned} \tag{12.57}$$

The second contribution in $c_b^{1,\mathrm{rr}\,G^5}$ is known to 6.5PN absolute accuracy, because our results above give the following 6.5PN-accurate value of $p_{xG^5}(\gamma)$:

$$p_{xG^5}(\gamma) = -\frac{20608}{225}p_\infty + \frac{1143232}{7875}p_\infty^3 - \frac{196096}{945}p_\infty^4 + O(p_\infty^5). \tag{12.58}$$

By contrast, the only thing we know at this stage concerning the additional contribution $\propto f_b^{G^5}(\gamma)$ in Eq. (12.56) is that it could start at the 5PN level and be $f_b^{G^5}(\gamma) = O(p_\infty)$.

The latter result limits the PN accuracy of $c_b^{1,\mathrm{rr}\,G^5}$. However, more is known about the sum[20] $c_{b_{12}}^{1,\mathrm{rr}\,G^5} + c_{b_{12}}^{2,\mathrm{rr}\,G^5}$, in which the f_b term cancels. Indeed, the linear-odd contribution to this only depends on E_3 and E_4 (see Table II of Ref. [30]), which are exactly known [9,25]. The beginning of its PN expansion reads

[20] Note that the sum becomes a difference if one exchanges $\hat{b}_{12}$ into $\hat{b}_{21}$.

$$c_b^{1,\mathrm{rr\,lin-odd}\,G^5} + c_b^{2,\mathrm{rr\,lin-odd}\,G^5} = \frac{G^5}{b^5} m_1^2 m_2^2 (m_1+m_2)(m_1-m_2)\left[-\frac{64}{3p_\infty^2} - \left(\frac{37}{20}\pi^2 + \frac{27392}{525}\right) - \frac{128}{3}p_\infty - \left(\frac{856768}{33075} + \frac{3429}{1120}\pi^2\right)p_\infty^2 + O(p_\infty^3)\right]. \quad (12.59)$$

The second contribution is known to 6.5PN accuracy by using Eq. (12.58) and reads

$$\frac{G^5}{b^5} m_1^2 m_2^2 (m_1+m_2)(m_1-m_2) p_{xG^5}(\gamma), \quad (12.60)$$

where the 6.5PN value of $p_{xG^5}(\gamma)$ is given in Eq. (12.58) above.

J. Radiative contribution to the impulse coefficients at 5PM: Longitudinal components

Let us finally consider the nonpolynomial contributions to the $u_{1-} \pm u_{2-}$ components of $\Delta p_1^{\mathrm{rr\,P_x^{rad}}}$,

$$c_{1+2}^{1\,\mathrm{rr},\mathrm{P_x}G^5} = \frac{G^5}{2}\frac{m_1^3 m_2^2}{b^5}(m_1-m_2)\frac{(2\gamma^2-1)(\gamma-1)}{(\gamma^2-1)^{3/2}}\frac{-2m_2^2\gamma + m_1^2 - 2m_1m_2 - m_2^2}{(m_1^2+2\gamma m_1 m_2 + m_2^2)} p_x^{G^4}(\gamma),$$
$$c_{1-2}^{1\,\mathrm{rr},\mathrm{P_x}G^5} = \frac{G^5}{2}\frac{m_1^3 m_2^2}{b^5}(m_1-m_2)\frac{(2\gamma^2-1)(\gamma+1)}{(\gamma^2-1)^{3/2}}\frac{2m_2^2\gamma - m_2^2 + 2m_1m_2 + m_1^2}{(m_1^2+2\gamma m_1 m_2 + m_2^2)} p_x^{G^4}(\gamma), \quad (12.61)$$

where $p_x^{G^4}(\gamma)$ is the same function of γ as defined above, Eq. (12.36).

As before, we look for corresponding components of $\Delta p_{a\mu}^{\mathrm{rr\,remain}}$ that will cancel the nonpolynomiality of the above longitudinal components. As discussed above, there is a unique way to do so for the $u_{1-}+u_{2-}$ component, while the $u_{1-}-u_{2-}$ component is nonunique and can be augmented by a term of the form [see Eq. (12.25)]

$$\Delta c_{1-2}^{1\,\mathrm{rr,remain}} = \frac{G^5 m_1^3 m_2^3}{b^5} f_{1-2}^{G^5}(\gamma). \quad (12.62)$$

Let us start by considering the $u_{1-}+u_{2-}$ component $c_{1+2}^{1\,\mathrm{rr},\mathrm{P_x}G^5}$ and look for an additional mass-antisymmetric contribution $c_{1+2}^{1\,\mathrm{rr,remain}}$ able to cancel the nonpolynomiality of $c_{1+2}^{1\,\mathrm{rr},\mathrm{P_x}G^5}$. After scaling out

$$\frac{G^5 m_1^2 m_2^2}{2b^5}(2\gamma^2-1)(\gamma-1)(\gamma^2-1)^{-3/2} p_x^{G^4}(\gamma), \quad (12.63)$$

and multiplying by $m_1^2 + 2\gamma m_1 m_2 + m_2^2$, the problem to be solved involves quartic polynomials in the masses. Namely, we look for a rescaled

$$\hat{c}_{1+2}^{1\,\mathrm{rr,remain}} = c_+(m_1-m_2)(m_1+m_2)m_1m_2, \quad (12.64)$$

and two coefficients x, y, such that c_+, x, y satisfy the mass-polynomial equation

$$m_1(m_1-m_2)(m_1^2 - 2m_1m_2 - (2\gamma+1)m_2^2) + c_+(m_1-m_2)(m_1+m_2)m_1m_2 - (m_1^2+2\gamma m_1m_2+m_2^2)(xm_1^2 + ym_1m_2) = 0. \quad (12.65)$$

Here, we imposed the constraint that the resulting contribution to $\Delta c_{1+2}^{1\,\mathrm{rr}}$ be $\propto m_1^3$.

It is easily found that the mass-polynomiality equation (12.65) admits a unique solution, namely,

$$c_+ = 2(\gamma+1); \qquad x = 1; \qquad y = -1. \quad (12.66)$$

This proves that

$$c_{1+2}^{1\,\mathrm{rr,remain}} = \frac{G^5 m_1^2 m_2^2}{b^5}\frac{m_1m_2(m_1-m_2)(m_1+m_2)(2\gamma^2-1)}{(\gamma^2-1)^{1/2}(m_1^2+2m_1m_2\gamma+m_2^2)} \times p_x^{G^4}(\gamma), \quad (12.67)$$

and therefore that

$$c_{1+2}^{1\,\mathrm{rr}\,G^5} = c_{1+2}^{1\,\mathrm{rr,lin\text{-}odd}} + \frac{G^5 m_1^2 m_2^2}{b^5} m_1(m_1 - m_2)\frac{(\gamma-1)(2\gamma^2-1)}{2(\gamma^2-1)^{3/2}} p_x^{G^4}(\gamma). \tag{12.68}$$

Proceeding in a similar way for the particle-symmetric $u_{1-} - u_{2-}$ component, we find as a general solution for $c_{1-2}^{1\,\mathrm{rr,remain}}$,

$$c_{1-2}^{1\,\mathrm{rr,remain}} = -2\frac{G^5 m_1^2 m_2^2}{b^5}\frac{m_1^2 m_2^2(2\gamma^2-1)(\gamma+1)}{(m_1^2+2m_1m_2\gamma+m_2^2)\sqrt{\gamma^2-1}} p_x^{G^4}(\gamma) + \frac{G^5 m_1^3 m_2^3}{b^5} f_{1-2}^{G^5}(\gamma), \tag{12.69}$$

and therefore that

$$c_{1-2}^{1\,\mathrm{rr}\,G^5} = c_{1-2}^{1\,\mathrm{rr,lin\text{-}odd}} + \frac{G^5 m_1^2 m_2^2}{b^5}\frac{m_1(m_1+m_2-2m_2\gamma)(2\gamma^2-1)(\gamma+1)}{2(\gamma^2-1)^{3/2}} p_x^{G^4}(\gamma) + \frac{G^5 m_1^3 m_2^3}{b^5} f_{1-2}^{G^5}(\gamma). \tag{12.70}$$

At this stage, the constraints we used above leave undetermined the additional longitudinal term involving the function $f_{1-2}^{G^5}(\gamma)$ [in addition to the function $f_b^{G^5}(\gamma)$ entering the transverse component].

However, we still have one more constraint that we can use, namely, the mass-shell-related constraints, Eqs. (12.6) and (12.14). When using our new decomposition, the following analog of Eq. (12.6) holds (because the Δc_ϕ contribution vanishes separately):

$$p_{a\mu}^{+\mathrm{cons}}(\Delta p_a^{\mu\mathrm{rr\,lin\text{-}odd}} + \Delta p_a^{\mu\mathrm{rr}P_x^{\mathrm{rad}}}) = 0. \tag{12.71}$$

The analog of Eq. (12.14) then reads

$$p_a^{+\mathrm{cons}} \cdot \Delta p_a^{\mathrm{rr\,remain}} = -\frac{1}{2}(\Delta p_a^{\mathrm{rr\,tot}})^2, \tag{12.72}$$

where $\Delta p_a^{\mathrm{rr\,tot}}$ is the full (nonlinear) radiative impulse, as determined above at orders G^4 and G^5,

$$\Delta p_a^{\mathrm{rr\,tot}} = \Delta p_a^{\mathrm{rr\,lin\text{-}odd}} + \Delta p_a^{\mathrm{rr}\,P_x^{\mathrm{rad}}} + \Delta p_a^{\mathrm{rr\,remain}}. \tag{12.73}$$

Since $\Delta p_a^{\mathrm{rr\,tot}}$ starts at order G^3, the right-hand side of Eq. (12.73) starts at order G^6. Inserting the decomposition (for $a = 1$)

$$\Delta p_1^{\mathrm{rr\,remain}} = c_{b1}^{\mathrm{rr\,remain}}\hat{\mathbf{b}} + c_{1+2}^{1\ \mathrm{rr\,remain}}(u_{1-}+u_{2-}) + c_{1-2}^{1\ \mathrm{rr\,remain}}(u_{1-}-u_{2-}) \tag{12.74}$$

in Eq. (12.71), we find

$$p_1^{+\mathrm{cons}} \cdot \Delta p_1^{\mathrm{rr\,remain}} = -c_{b1}^{\mathrm{rr\,remain}} p_- \sin(\chi_{\mathrm{cons}}) + \frac{p_-^2}{m_1 m_2}\cos(\chi_{\mathrm{cons}})[c_{1+2}^{1\,\mathrm{rr,remain}}(m_2-m_1) + c_{1-2}^{1\,\mathrm{rr,remain}}(m_1+m_2)]. \tag{12.75}$$

Here we used

$$p_1^{+\mathrm{cons}} \cdot u_{1-} = \frac{p_-^2}{m_1}\cos(\chi_{\mathrm{cons}}),$$
$$p_1^{+\mathrm{cons}} \cdot u_{2-} = -\frac{p_-^2}{m_2}\cos(\chi_{\mathrm{cons}}),$$
$$p_1^{+\mathrm{cons}} \cdot \hat{\mathbf{b}} = -p_- \sin(\chi_{\mathrm{cons}}). \tag{12.76}$$

Working up to order G^5 we find

$$0 = -c_{b1}^{\mathrm{rr\,remain}\,G^4}\frac{2\chi_{1\mathrm{cons}}}{j} + \frac{p_-}{m_1 m_2} \times [c_{1+2}^{1\mathrm{rr,remain},G^5}(m_2-m_1) + c_{1-2}^{1\mathrm{rr,remain},G^5}(m_1+m_2)], \tag{12.77}$$

which determines the value of $f_{1-2}^{G^5}(\gamma)$, namely,

$$f_{1-2}^{G^5}(\gamma) = \frac{(2\gamma^2-1)(\gamma+1)}{(\gamma-1)\sqrt{\gamma^2-1}} p_{x\,G^4}(\gamma). \tag{12.78}$$

Consequently,

$$c_{1-2}^{1\,\mathrm{rr}\,G^5} = c_{1-2}^{1\,\mathrm{rr,lin\text{-}odd}} + \frac{G^5 m_1^3 m_2^2(m_1+3m_2)}{b^5}\frac{(2\gamma^2-1)(\gamma+1)}{2(\gamma^2-1)^{3/2}} p_{x\,G^4}(\gamma). \tag{12.79}$$

XIII. CONCLUDING REMARKS

In the present work, we improved the knowledge of radiative contributions to scattering observables in several directions.

We pushed the PN accuracy of the energy, angular momentum, and linear-momentum radiated during a scattering encounter to higher levels, namely, the *fractional* 3PN accuracy: for energy, we reached the absolute 5.5PN accuracy [see Eqs. (6.3) and (6.4)]; for angular momentum, we reached the absolute 5.5PN accuracy [see Eqs. (7.2) and (7.3)]; for linear momentum, we reached the absolute 6.5PN accuracy [see Eqs. (4.13), (5.3), (8.7), (9.6), and (9.10)]. See the summary of our results in Sec. X and, notably, in Tables II and III.

Our results have a limited PN accuracy, but are valid (at least) at order G^7.

We completed the linear-response computation of the radiative contribution to the individual impulses [30] by including two additional terms (see Sec. XII): (i) the additional contribution Δc_ϕ in the relative scattering angle linked to the time-asymmetric piece of the radiation-reaction force [see Eq. (12.18)] and (ii) the additional contribution $\Delta p_a^{\rm rr\,nonlin}$ linked to nonlinear radiation-reaction effects. We then wrote the total radiative contribution to the impulses in the following form:

$$\Delta p_a^{\rm rr\,tot} = \Delta p_a^{\rm rr\,lin\text{-}odd} + \Delta p_a^{\rm rr\,P_x^{rad}} + \Delta p_a^{\rm rr\,remain}, \qquad (13.1)$$

with

$$\Delta p_a^{\rm rr\,lin\text{-}odd} = \bar\chi^{\rm rr\,rel}\frac{d}{d\chi_{\rm cons}}\Delta p_{a\mu}^{\rm cons} + \frac{\Delta P_{\rm c.m.}}{P_{\rm c.m.}}p_{a\mu}^{+} - \frac{m_a^2}{E_a}\frac{\Delta P_{\rm c.m.}}{P_{\rm c.m.}}U_\mu - \frac{E_a}{E_{\rm c.m.}}\bar P_\mu^{\rm rad} - \frac{(p_{a\nu}^{+}\bar P_{\rm rad}^\nu)}{E_{\rm c.m.}}U_\mu, \qquad (13.2)$$

where

$$\Delta P_{\rm c.m.} = -\frac{E_1E_2}{E_{\rm c.m.}P_{\rm c.m.}}E_{\rm rad}. \qquad (13.3)$$

Here $\bar\chi^{\rm rr\,rel}$ is defined as

$$\bar\chi^{\rm rr\,rel} \equiv -\left(\frac{1}{2}\frac{\partial\chi^{\rm cons}}{\partial E}E_{\rm rad} + \frac{1}{2}\frac{\partial\chi^{\rm cons}}{\partial J}J_{\rm rad}\right), \qquad (13.4)$$

and $\bar P_{\rm rad}^\mu$ denotes the part of $P_{\rm rad}^\mu$ orthogonal to the x direction, namely,

$$\bar P_{\rm rad}^\mu \equiv P_{\rm rad}^\mu - P_{\rm rad}^x \mathbf{e}_x^\mu. \qquad (13.5)$$

All the radiative losses (in $E_{\rm rad}$, $J_{\rm rad}$, and $P_{\rm rad}^\mu$) entering here include time-asymmetric (hereditary) effects. The second term in Eq. (13.1), $\Delta p_a^{\rm rr\,P_x^{rad}}$, is the contribution linked to the x component of $P_{\rm rad}^\mu$, namely,

$$\Delta p_{a\mu}^{\rm rr\,P_x^{rad}} \equiv -\frac{E_a}{E_{\rm c.m.}}P_x^{\rm rad}e_{x\mu} - \frac{(p_{ax}^{+}P_x^{\rm rad})}{E_{\rm c.m.}}U_\mu. \qquad (13.6)$$

Finally, the remaining contribution in the decomposition (13.1) is

$$\Delta p_{a\mu}^{\rm rr\,remain} \equiv \Delta c_\phi \frac{d}{d\chi_{\rm cons}}\Delta p_{a\mu}^{\rm cons} + \Delta p_{a\mu}^{\rm rr\,nonlin}. \qquad (13.7)$$

Within our approach, the two contributions to $\Delta p_{a\mu}^{\rm rr\,remain}$, Eq. (13.7), have different physical origins: the Δc_ϕ-related one is linear in radiation reaction (but of time-asymmetric origin), while $\Delta p_{a\mu}^{\rm rr\,nonlin}$ is nonlinear in radiation reaction. However, they share common mathematical properties (antisymmetry under particle exchange, second-self-force character), and our mass-polynomiality constraints do not distinguish their origins.

We studied the consequences of the mass polynomiality of the Lorentz-invariant form factors as defined in Eqs. (11.3)–(11.8). The resulting structures were shown to imply the ν-polynomiality rules introduced in [30]. The latter ν rules ensure the mass polynomiality of the first contribution $\Delta p_a^{\rm rr\,lin\text{-}odd}$ to the impulses (see Table II of [30]). Then we showed how the nonpolynomiality of the P_x-related contribution $\Delta p_a^{\rm rr\,P_x^{rad}}$ could be cured by adding specific remaining contributions $\Delta p_a^{\rm rr\,remain}$. At order G^4, the various constraints to be satisfied by $\Delta p_a^{\rm rr\,remain}$ were shown to be sufficient to fully determine $\Delta p_a^{\rm rr\,remain}$ in terms of $P_x^{\rm rad}$, see Eq. (12.30). At order G^5, $\Delta p_a^{\rm rr\,remain}$ was determined up to the addition of one extra term, see Eq. (12.69).

All our 4PM-level results are compatible with those of Ref. [25] and provide an alternative way of understanding 4PM radiation reaction effects. Our 5PM-level results give benchmarks for future 5PM computations and hopefully will bring new light on the current puzzles concerning the 5PN dynamics of binary systems [41,72].

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APPENDIX A: NOTATION AND USEFUL FORMULAS

We list below some useful formulas that one often needs to have at hand. The incoming c.m. Lorentz factor $\gamma = -u_1^- \cdot u_2^-$ and its associated (dimensionless) momentumlike variable p_∞ are related by

$$p_\infty \equiv \sqrt{\gamma^2 - 1}. \qquad (A1)$$

The dimensionless angular momentum j is related to the original c.m. angular momentum J by

$$j \equiv \frac{cJ}{Gm_1 m_2}. \tag{A2}$$

The vectorial impact parameter (orthogonal to u_1^- and u_2^-) $\mathbf{b}_{12} = \mathbf{b}_1 - \mathbf{b}_2 = b\hat{\mathbf{b}}_{12}$ together with the conservative scattering angle $\chi_{\rm cons}$ enters the definition of the Cartesian-like basis vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ as follows [see Eq. (3.49) of Ref. [30]]:

$$\mathbf{e}_x = \cos\frac{\chi_{\rm cons}}{2}\hat{\mathbf{b}} + \sin\frac{\chi_{\rm cons}}{2}\mathbf{n}_-,$$
$$\mathbf{e}_y = -\sin\frac{\chi_{\rm cons}}{2}\hat{\mathbf{b}} + \cos\frac{\chi_{\rm cons}}{2}\mathbf{n}_-, \tag{A3}$$

where $\mathbf{n}_-$ is the direction of the incoming momenta,

$$\mathbf{n}_- = \frac{m_1 m_2}{P^-_{\rm c.m.}E^-_{\rm c.m.}}\left(\frac{E_2^-}{m_2}u_1^- - \frac{E_1^-}{m_1}u_2^-\right), \tag{A4}$$

and [see Eqs. (A4) and (A5) of Ref. [30]]

$$P^-_{\rm c.m.} = \frac{m_1 m_2}{E^-_{\rm c.m.}}\sqrt{\gamma^2 - 1},$$
$$E^-_{\rm c.m.} = Mc^2 h = Mc^2\sqrt{1 + 2\nu(\gamma - 1)}. \tag{A5}$$

An equivalent expression for $\mathbf{n}_-$ is the following:

$$\mathbf{n}_- = \frac{(u_{2-} \wedge u_{1-})\cdot U^-}{\sqrt{\gamma^2 - 1}}, \tag{A6}$$

where the wedge product of two vectors A and B is standardly defined as

$$A \wedge B = A \otimes B - B \otimes A, \tag{A7}$$

so that the contraction with a third vector C is given by $(A \wedge B)\cdot C = A(B\cdot C) - B(A\cdot C)$.

Boldface vectors denote spatial vectors in the c.m. frame with time axis U^-: $p_a^- = m_a u_a^- = E_a U^- + \mathbf{p}_a^-$ (where $\mathbf{p}_a^-$ is orthogonal to U^-, and $\mathbf{p}_1^- = -\mathbf{p}_2^- = \mathbf{p}^-$), with

$$U^- = \frac{p_1^- + p_2^-}{|p_1^- + p_2^-|} = \frac{1}{E^-_{\rm c.m.}}(m_1 u_1^- + m_2 u_2^-), \tag{A8}$$

and $E^-_{\rm c.m.} = E_1^- + E_2^-$.

To ease the notation, we often remove the “c.m.” label from both energy and linear momentum, e.g., $P^-_{\rm c.m.} \to p_-$. The label “−” (for incoming) is also frequently omitted: $E^-_{\rm c.m.} \to E$.

Let us also recall the following expressions [see Eqs. (A9) of Ref. [30]] for the incoming c.m. energy of each particle:

$$E_1^- = \frac{m_1(m_2\gamma + m_1)}{E}, \qquad E_2^- = \frac{m_2(m_1\gamma + m_2)}{E}, \tag{A9}$$

as well as the relation between the dimensionless angular momentum and the impact parameter,

$$\frac{1}{j} = \frac{GMh}{bp_\infty} = \frac{GE}{bp_\infty}. \tag{A10}$$

When describing the conservative scattering, it is useful to introduce the c.m. direction of the (conservative) outgoing momenta $\mathbf{n}_+^{\rm cons}$, as well as its associated orthogonal direction $\hat{\mathbf{B}}$, namely,

$$\hat{\mathbf{B}} = \cos(\chi_{\rm cons})\hat{\mathbf{b}} + \sin(\chi_{\rm cons})\mathbf{n}_-,$$
$$\mathbf{n}_+^{\rm cons} = -\sin(\chi_{\rm cons})\hat{\mathbf{b}} + \cos(\chi_{\rm cons})\mathbf{n}_-. \tag{A11}$$

In the text we used the relation

$$\hat{\mathbf{B}} = -\frac{d}{d\chi_{\rm cons}}\mathbf{n}_+^{\rm cons}. \tag{A12}$$

The dyad $(\hat{\mathbf{B}}, \mathbf{n}_+^{\rm cons},)$ differs from the incoming dyad $(\hat{\mathbf{b}}, \mathbf{n}_-)$ by a rotation of angle $\chi_{\rm cons}$. The dyad $(\mathbf{e}_x, \mathbf{e}_y)$ is midway between the latter two dyads, being obtained from the incoming dyad by a rotation of angle $\frac{1}{2}\chi_{\rm cons}$.

The conservative scattering of the particle 1 corresponds to the change $p_1^- \to p_1^{+\rm cons}$ of its linear momentum

$$p_1^- = E_1 U + p_-\mathbf{n}_-, \qquad p_1^{+\rm cons} = E_1 U + p_-\mathbf{n}_+^{\rm cons}, \tag{A13}$$

such that

$$\Delta p_1^{\rm cons} = p_1^{+\rm cons} - p_1^- = p_-(\mathbf{n}_+^{\rm cons} - \mathbf{n}_-). \tag{A14}$$

The following representation

$$\Delta p_1^{\rm cons} = c_b^{1\rm cons}\hat{\mathbf{b}} + c_{u_1}^{1\rm cons}u_1 + c_{u_2}^{1\rm cons}u_2, \tag{A15}$$

with

$$c_b^{1\rm cons} = -p_-\sin\chi_{\rm cons},$$
$$c_{u_1}^{1\rm cons} = \frac{m_1 E_2}{E}(\cos\chi_{\rm cons} - 1),$$
$$c_{u_2}^{1\rm cons} = -\frac{m_2 E_1}{E}(\cos\chi_{\rm cons} - 1), \tag{A16}$$

is also used.

For particle 2 we have instead

$$p_2^- = E_1 U - p_-\mathbf{n}_-, \qquad p_2^{+\rm cons} = E_1 U - p_-\mathbf{n}_+^{\rm cons}, \tag{A17}$$

with

$$\Delta p_2^{\rm cons} = p_2^{+\rm cons} - p_2^- = -p_-(\mathbf{n}_+^{\rm cons} - \mathbf{n}_-). \tag{A18}$$

Therefore, $\Delta p_1^{\rm cons} + \Delta p_2^{\rm cons} = 0$, and then

$$\frac{d}{d\chi^{\rm cons}}\Delta p_1^{\rm cons} = -p_-\hat{\mathbf{B}} = -\frac{d}{d\chi^{\rm cons}}\Delta p_2^{\rm cons}. \tag{A19}$$

APPENDIX B: RELATING HYPERBOLIC-MOTION RESULTS TO ELLIPTIC-MOTION ONES BY ANALYTIC CONTINUATION

As a check on our computation, in Sec. III, of the 2.5PN radiation-reaction correction to the quasi-Keplerian parametrization of hyperboliclike motions, we have (successfully) related it to the corresponding 2.5PN radiation-reaction correction to the quasi-Keplerian parametrization of ellipticlike motions derived in Ref. [55] (by using the elliptic version of Lagrange's method of variation of constants). As already mentioned in the text, this comparison used two different ingredients: (i) analytic continuation between elliptic and hyperbolic quasi-Keplerian parametrizations (at the Newtonian order) and (ii) the use of a different expression for the radiation-reaction force, because of a difference in coordinates (ADM versus harmonic).

Let us only mention a few technical steps of this comparison. The analytic continuation relating the elliptic eccentric anomaly u to the hyperbolic one v is simply $u \to iv$. This has to be taken together with the replacement $a_r \to -\bar{a}_r$. Concerning the gauge dependence of the radiation-reaction force, let us recall that, in a general coordinate system, the 2.5PN-level radiation-reaction acceleration depends on two gauge parameters, α and β, and reads [73,74]

$$\mathbf{A}^{\rm rr} = -\frac{8}{5}\nu\frac{G^2}{c^5}\frac{M^2}{r^3}[-A_{2.5\rm PN}\dot{r}\mathbf{n} + B_{2.5\rm PN}\mathbf{v}], \tag{B1}$$

where

$$A_{2.5\rm PN} = 3(1+\beta)v^2 + \frac{1}{3}(23+6\alpha-9\beta)\frac{GM}{r} - 5\beta\dot{r}^2,$$
$$B_{2.5\rm PN} = (2+\alpha)v^2 + (2-\alpha)\frac{GM}{r} - 3(1+\alpha)\dot{r}^2. \tag{B2}$$

For example, in harmonic coordinates $\alpha = -1$ and $\beta = 0$,

$$A_{2.5\rm PN,h} = 3v^2 + \frac{17}{3}\frac{GM}{r},$$
$$B_{2.5\rm PN,h} = v^2 + 3\frac{GM}{r}. \tag{B3}$$

Other useful gauge choices correspond to the Burke-Thorne reactive potential ($\alpha = 4$, $\beta = 5$) and to ADM coordinates ($\alpha = \frac{5}{3}$, $\beta = 3$).

One can then easily derive the variation of constants in a general gauge. For example the (α, β)-dependent equation for $\delta^{\rm rr} e_t$ reads

$$\frac{d\delta^{\rm rr}e_t}{dt} = \frac{8\nu(1-e_t^2)}{15a_r^4 e_t}\left\{\frac{12\alpha-6\beta+15}{\chi^3} + \frac{-48\alpha+33\beta-65}{\chi^4} + \frac{21(e_t^2-3)\beta - 9(2\alpha+3)e_t^2 + 60\alpha+109}{\chi^5} + \frac{24(e_t^2-1)(\alpha-\frac{17}{8}\beta+\frac{59}{24})}{\chi^6} - \frac{15\beta(e_t^2-1)^2}{\chi^7}\right\}. \tag{B4}$$

In the ADM case, this equation becomes

$$\frac{d\delta^{\rm rr}e_t}{dt} = \frac{8\nu(1-e_t^2)}{15a_r^4 e_t}\left\{\frac{17}{\chi^3} - \frac{46}{\chi^4} + \frac{6e_t^2+20}{\chi^5} - 54\frac{e_t^2-1}{\chi^6} - \frac{45(e_t^2-1)^2}{\chi^7}\right\}, \tag{B5}$$

as in Eq. (56.b) of Ref. [55], while in the harmonic case we find

$$\frac{d\delta^{\rm rr}e_t}{dt} = \frac{8\nu(1-e_t^2)}{15a_r^4 e_t}\left[\frac{3}{\chi^3} - \frac{17}{\chi^4} - \frac{9e_t^2-49}{\chi^5} + \frac{35(e_t^2-1)}{\chi^6}\right]. \tag{B6}$$

APPENDIX C: RADIATION-REACTION CONTRIBUTION TO THE RELATIVE SCATTERING ANGLE UP TO 4.5 PN ACCURACY

Reference [29] [see Eq. (5.99) there] has shown that, to linear order in radiation reaction and under the assumption of a time-odd radiation-reaction force, the radiation-reaction contribution to the relative scattering angle (in the c.m. frame) $\chi_{\rm rr,rel}$ can be computed through a linear-response formula involving the radiative losses of energy and angular momentum. We have generalized this linear-response formula above, see Eq. (12.17), by including the term Δc_ϕ that is nonzero when the radiation-reaction force contains a time-even piece. As discussed above, such a correction in $\chi_{\rm rr,rel}$ starts to contribute only at the 5PN (and 4PM) level. In other words, the first two terms on the right-hand side of Eq. (12.17) suffice to evaluate $\chi_{\rm rr,rel}$ up to the 4.5PN level, by using the known radiative losses at the 4.5PN accuracy (as the radiative losses start at the 2.5PN level, this corresponds to a fractional 2PN accuracy).

At the leading-order 2.5PN level, we have given in the text a direct rederivation of the value of $\chi_{\rm rr,rel}$, see Eq. (3.23). The explicit expression of $\chi^{\rm 2.5PN}_{\rm rr,rel}=[\delta^{\rm rr}\phi]^{\rm 2.5PN}$ in terms of a_r and e_r reads

$$\chi^{\rm 2.5PN}_{\rm rr,rel}(a_r,e_r)=\frac{2\nu}{15\bar{a}_r^{5/2}(e_r^2-1)^{5/2}}\left[\frac{72e_r^4+1069e_r^2+134}{3e_r^2}+\frac{121e_r^2+304}{\sqrt{e_r^2-1}}\arccos\left(-\frac{1}{e_r}\right)\right], \tag{C1}$$

which, when expressed in terms of the conserved energy and angular momentum, becomes

$$\chi^{\rm 2.5PN}_{\rm rr}(p_\infty,j)=\frac{2\nu}{15j^5}\left[\frac{72p_\infty^4j^4+1213p_\infty^2j^2+1275}{3(1+p_\infty^2j^2)}+\frac{121p_\infty^2j^2+425}{p_\infty j}\mathcal{A}(p_\infty,j)\right], \tag{C2}$$

where

$$\mathcal{A}(p_\infty,j)\equiv\arccos\left(-\frac{1}{\sqrt{1+p_\infty^2j^2}}\right). \tag{C3}$$

The large-j expansion of the latter expression reproduces the leading PN order of the PM expansion of $\chi_{\rm rr}$, the first terms of which [up to $O(G^7)$] are listed in Table XI of Ref. [30].

When going to higher PN levels in the radiative losses (still keeping below the absolute 5PN level), we must take into account that the radiative losses contain fractional corrections at the following levels: 1PN, 1.5PN, and 2PN. The 1.5PN correction to the losses is the leading-order tail effect (which is still described by a time-odd radiation reaction). Let us first discuss the 1PN and 2PN fractional corrections, leading to contributions to $\chi_{\rm rr,rel}$ at the 3.5PN and 4.5PN levels.

The expressions of $\chi_{\rm rr,rel}$ at the $(n+\frac{1}{2})$PN levels (for $n=3$, 4) have the general structure

$$\chi_{\rm rr}(p_\infty,j)^{\rm n.5PN}=A_2^{\rm n.5PN}(p_\infty,j;\nu)\mathcal{A}^2(p_\infty,j)+A_1^{\rm n.5PN}(p_\infty,j;\nu)\mathcal{A}(p_\infty,j)+A_0^{\rm n.5PN}(p_\infty,j;\nu). \tag{C4}$$

Using the 2PN conservative scattering angle, Eq. (45) of Ref. [31],

$$\frac{\chi_{\rm cons}}{2}=\frac{\chi^{\rm N}_{\rm cons}}{2}+\frac{\chi^{\rm 1PN}_{\rm cons}}{2}\eta^2+\frac{\chi^{\rm 2PN}_{\rm cons}}{2}\eta^4+O(\eta^6), \tag{C5}$$

where

$$\begin{aligned}\frac{\chi^{\rm N}_{\rm cons}}{2}&=\mathcal{A}(p_\infty,j)-\frac{\pi}{2},\\ \frac{\chi^{\rm 1PN}_{\rm cons}}{2}&=\frac{3}{j^2}\mathcal{A}(p_\infty,j)+\frac{p_\infty(3+2j^2p_\infty^2)}{j(1+j^2p_\infty^2)},\\ \frac{\chi^{\rm 2PN}_{\rm cons}}{2}&=-\frac{3[j^2p_\infty^2(2\nu-5)-35+10\nu]}{4j^4}\mathcal{A}(p_\infty,j)\\ &\quad-\frac{p_\infty}{4j^3(1+j^2p_\infty^2)^2}[j^4p_\infty^4(-81+26\nu)\\ &\quad+2j^2p_\infty^2(-95+28\nu)+30\nu-105],\end{aligned} \tag{C6}$$

and the fractionally 2PN-accurate expressions (when excluding tails) for the radiated energy and angular momentum given in Ref. [30], Eqs. (C10)–(C13) and (E4)–(E10), we get the following explicit results:

$$\begin{aligned}\chi^{\rm 3.5PN}_{\rm rr}(p_\infty,j)=\frac{2\nu}{j^7}\Bigg\{&\left(\frac{168(p_\infty j)^2}{5}+72\right)\mathcal{A}^2(p_\infty,j)\\ &+\left[(p_\infty j)^3\left(\frac{23111}{840}-\frac{437\nu}{30}\right)+(p_\infty j)\left(\frac{11647}{60}-\frac{424\nu}{3}\right)+\frac{\frac{13447}{40}-\frac{1127\nu}{6}}{(p_\infty j)}\right]\mathcal{A}(p_\infty,j)\\ &+\frac{1}{((p_\infty j)^2+1)^2}\left[(p_\infty j)^8\left(\frac{40}{7}-\frac{8\nu}{5}\right)+(p_\infty j)^6\left(\frac{92639}{1400}-\frac{7681\nu}{90}\right)+(p_\infty j)^4\left(\frac{5049251}{12600}-\frac{3503\nu}{10}\right)\right.\\ &\left.+(p_\infty j)^2\left(\frac{81889}{120}-\frac{8179\nu}{18}\right)-\frac{1127\nu}{6}+\frac{13447}{40}\right]\Bigg\},\end{aligned}$$

$$
\begin{aligned}
\chi_{\rm rr}^{4.5\rm PN}(p_\infty,j) = \frac{2\nu}{j^9}\Bigg\{&\left[(p_\infty j)^4\left(\frac{534}{7}-\frac{373\nu}{5}\right)+(p_\infty j)^2\left(816-\frac{2898\nu}{5}\right)-745\nu+1586\right]\mathcal{A}^2(p_\infty,j)\\
&+\frac{1}{(p_\infty j)((p_\infty j)^2+1)}\left[(p_\infty j)^8\left(\frac{511\nu^2}{24}-\frac{75253\nu}{1680}+\frac{44759}{1120}\right)+(p_\infty j)^6\left(366\nu^2-\frac{136789\nu}{168}+\frac{1020745}{1512}\right)\right.\\
&+(p_\infty j)^4\left(\frac{5237\nu^2}{4}-\frac{1579549\nu}{420}+\frac{16375901}{5040}\right)+(p_\infty j)^2\left(\frac{4949\nu^2}{3}-\frac{149209\nu}{24}+\frac{6034507}{1080}\right)\\
&\left.+\frac{5481\nu^2}{8}-\frac{258051\nu}{80}+\frac{5839651}{2016}\right]\mathcal{A}(p_\infty,j)\\
&+\frac{1}{((p_\infty j)^2+1)^3}\left[(p_\infty j)^{12}\left(\frac{8\nu^2}{5}-\frac{186\nu}{35}+\frac{256}{63}\right)+(p_\infty j)^{10}\left(\frac{19781\nu^2}{120}-\frac{105913\nu}{560}+\frac{45934963}{352800}\right)\right.\\
&+(p_\infty j)^8\left(\frac{439657\nu^2}{360}-\frac{47396053\nu}{25200}+\frac{3027711913}{3175200}\right)+(p_\infty j)^6\left(\frac{607627\nu^2}{180}-\frac{95753533\nu}{12600}+\frac{7101025663}{1587600}\right)\\
&+(p_\infty j)^4\left(\frac{796337\nu^2}{180}-\frac{170414669\nu}{12600}+\frac{434998411}{45360}\right)+(p_\infty j)^2\left(\frac{66997\nu^2}{24}-\frac{520709\nu}{48}+\frac{266996831}{30240}\right)\\
&\left.+\frac{5481\nu^2}{8}-\frac{258051\nu}{80}+\frac{5839651}{2016}\right]\Bigg\}. \qquad \text{(C7)}
\end{aligned}
$$

For completeness, the corresponding PN-expansion coefficients when considering $\chi_{\rm rr}$ as a function of $\bar{a}_r$ and e_r are Eq. (C1) (at the 2.5PN accuracy) together with

$$
\begin{aligned}
\chi_{\rm rr}^{3.5\rm PN}(a_r,e_r) &= \frac{\nu}{\bar{a}_r^{7/2}(e_r^2-1)^{7/2}}\left[C_2^{3.5\rm PN}\arccos^2\left(-\frac{1}{e_r}\right)+\frac{C_1^{3.5\rm PN}}{\sqrt{e_r^2-1}}\arccos\left(-\frac{1}{e_r}\right)+C_0^{3.5\rm PN}\right],\\
\chi_{\rm rr}^{4.5\rm PN}(a_r,e_r) &= \frac{\nu}{\bar{a}_r^{9/2}(e_r^2-1)^{9/2}}\left[C_2^{4.5\rm PN}\arccos^2\left(-\frac{1}{e_r}\right)+\frac{C_1^{4.5\rm PN}}{\sqrt{e_r^2-1}}\arccos\left(-\frac{1}{e_r}\right)+C_0^{4.5\rm PN}\right], \qquad \text{(C8)}
\end{aligned}
$$

where

$$
\begin{aligned}
C_2^{3.5\rm PN} &= \frac{336}{5}e_r^2+\frac{384}{5},\\
C_1^{3.5\rm PN} &= \left(\frac{2783}{420}+\frac{47}{15}\nu\right)e_r^4+\left(-\frac{260}{3}\nu-\frac{1507}{7}\right)e_r^2-\frac{1832}{15}\nu-\frac{14594}{105},\\
C_0^{3.5\rm PN} &= \left(\frac{8}{5}\nu+\frac{288}{35}\right)e_r^4+\left(-\frac{1253}{45}\nu-\frac{1396049}{6300}\right)e_r^2-\frac{7498}{45}\nu-\frac{71683}{450}+\left(-\frac{64}{5}\nu+\frac{39394}{1575}\right)\frac{1}{e_r^2}, \qquad \text{(C9)}
\end{aligned}
$$

and

$$
\begin{aligned}
C_2^{4.5\rm PN} &= \left(-\frac{1716}{35}+\frac{94}{5}\nu\right)e_r^4+\left(-\frac{10008}{35}-\frac{2624}{5}\nu\right)e_r^2-480\nu+\frac{16904}{35},\\
C_1^{4.5\rm PN} &= \left(\frac{9}{20}\nu^2+\frac{7783}{840}\nu+\frac{82489}{1680}\right)e_r^6+\left(\frac{49}{3}\nu^2+\frac{48821}{84}\nu-\frac{417001}{3780}\right)e_r^4+\left(\frac{514}{5}\nu^2+\frac{427622}{105}\nu-\frac{1607}{63}\right)e_r^2\\
&\quad+88\nu^2+\frac{19066}{15}\nu-\frac{19882}{27},\\
C_0^{4.5\rm PN} &= \left(-\frac{2}{5}\nu^2+\frac{242}{35}\nu+\frac{808}{45}\right)e_r^6+\left(\frac{1367}{180}\nu^2+\frac{72587}{2520}\nu+\frac{28987039}{176400}\right)e_r^4+\left(\frac{365}{6}\nu^2+\frac{72257}{18}\nu-\frac{147017953}{793800}\right)e_r^2\\
&\quad+\frac{5956}{45}\nu^2-\frac{98228321}{99225}+\frac{1299217}{630}\nu+\left(\frac{36}{5}\nu^2-\frac{56108}{315}\nu+\frac{16847071}{99225}\right)\frac{1}{e_r^2}. \qquad \text{(C10)}
\end{aligned}
$$

Let us finally discuss the tail-related contribution to $\chi_{\mathrm{rr,rel}}$. The leading-order 4PN tail contribution is obtained by inserting in the linear-response formula the (j-expanded) Eqs. (D26) and (F2) of Ref. [30]. The result is the following:

$$\chi^{\mathrm{4PN}}_{\mathrm{rr,rel}}(p_\infty, j) = \nu\left[\frac{7168}{45}\frac{p_\infty^3}{j^5} + \frac{573}{20}\pi^3\frac{p_\infty^2}{j^6} + \left(\frac{512}{9} + \frac{153856}{675}\pi^2\right)\frac{p_\infty}{j^7} + O\left(\frac{1}{j^8}\right)\right]. \tag{C11}$$

If we formally insert also the fractional 1PN correction to the linear tail, we get [by using the 2.5PN-accurate expressions for E^{rad} and J^{rad} derived above in Eqs. (6.5) and (7.4), respectively] the following 5PN-level contribution to $\chi_{\mathrm{rr,rel}}$:

$$\begin{aligned}\chi^{\mathrm{5PN\,from\,tail\,in\,losses}}_{\mathrm{rr,rel}}(p_\infty, j) = \nu\Bigg[&\left(\frac{4992}{35} - \frac{676096}{1575}\nu\right)\frac{p_\infty^5}{j^5} + \left(-\frac{32079}{1120}\pi^2 + \frac{145536}{175} - \frac{7767}{70}\nu\pi^2 + \frac{14032}{525}\nu\right)\pi\frac{p_\infty^4}{j^6}\\ &+ \left(\frac{7014}{5}\zeta(3) - \frac{515456}{33075}\pi^2 + \frac{206188}{105} + \frac{207}{5}\pi^4 - \frac{89216}{105}\nu - \frac{18853168}{33075}\nu\pi^2\right)\frac{p_\infty^3}{j^7} + O\left(\frac{p_\infty^2}{j^8}\right)\Bigg].\end{aligned} \tag{C12}$$

Note, however, that at this level there are several other contributions that should be added to this result.

APPENDIX D: 3PN-ACCURATE QUASI-KEPLERIAN PARAMETRIZATION OF THE HYPERBOLIC MOTION

The 3PN-accurate quasi-Keplerian parametrization of the hyperboliclike motion is

$$\begin{aligned} r &= \bar{a}_r(e_r\cosh v - 1),\\ \bar{n}t &= e_t\sinh v - v + f_t V + g_t\sin V + h_t\sin 2V + i_t\sin 3V,\\ \phi &= K[V + f_\phi\sin 2V + g_\phi\sin 3V + h_\phi\sin 4V + i_\phi\sin 5V],\end{aligned} \tag{D1}$$

with

$$V(v) = 2\arctan\left[\sqrt{\frac{e_\phi+1}{e_\phi-1}}\tanh\frac{v}{2}\right]. \tag{D2}$$

The 3PN orbital parameters in modified harmonic coordinates along hyperboliclike orbits were obtained in Ref. [56]. However, their expressions are affected by typos, which we discovered when rederiving the 3PN-accurate quasi-Keplerian parametrization of hyperboliclike motions. We list below these typos.

(1) Eq. (2.36b), third line: the term $4\eta^3$, should be replaced by

$$2\bar{E}j^2\frac{4\eta^3 - 195\eta^2 + 1120\eta - 1488}{430080}.$$

(2) Eq. (2.36c): The third term in parentheses should have an overall factor of 15 in front, and one should replace the $+3\eta^3$ by $-3\eta^3$.
(3) Eq. (2.36j): The prefactor η^3 should instead be η.
(4) Eq. (2.36k): The $+-$ sign of the third term in parentheses is a $-$.
(5) Eq. (2.36m), second line: the term $-30135\eta^2$ is $-30135\pi^2$.
(6) Eq. (2.36o): There is a missing overall $3/35$ in front.

It is convenient to express the orbital parameters in terms of $\bar{a}_r$ and e_r through the relations

$$\begin{aligned}\bar{E} = \frac{1}{2\bar{a}_r} &+ \left(\frac{7}{8} - \frac{1}{8}\nu\right)\frac{\eta^2}{\bar{a}_r^2} + \left[\frac{25}{16} - \frac{7}{16}\nu + \frac{1}{16}\nu^2 + \frac{(2-\frac{7}{2}\nu)}{e_r^2-1}\right]\frac{\eta^4}{\bar{a}_r^3}\\ &+ \left[\frac{363}{128} - \frac{149}{128}\nu + \frac{21}{64}\nu^2 - \frac{5}{128}\nu^3 + \frac{(5 + (\frac{41}{128}\pi^2 - \frac{17033}{840})\nu + \frac{7}{4}\nu^2)}{(e_r^2-1)} + \frac{(4 + (-\frac{12343}{420} + \frac{41}{32}\pi^2)\nu + \nu^2)}{(e_r^2-1)^2}\right]\frac{\eta^6}{\bar{a}_r^4},\end{aligned}$$

$$
\begin{aligned}
j = {}& \sqrt{\bar{a}_r}\sqrt{e_r^2-1} + \left[\left(1-\frac{1}{2}\nu\right)\sqrt{e_r^2-1} + \frac{(3-\frac{1}{2}\nu)}{\sqrt{e_r^2-1}}\right]\frac{\eta^2}{\sqrt{\bar{a}_r}} \\
&+ \left[\left(\frac{3}{2}-\frac{11}{8}\nu+\frac{3}{8}\nu^2\right)\sqrt{e_r^2-1} + \frac{(\frac{5}{2}-\frac{75}{8}\nu+\frac{1}{4}\nu^2)}{\sqrt{e_r^2-1}} + \frac{(\frac{7}{2}-\frac{25}{2}\nu-\frac{1}{8}\nu^2)}{(e_r^2-1)^{3/2}}\right]\frac{\eta^4}{\bar{a}_r^{3/2}} \\
&+ \left[\left(\frac{5}{2}-\frac{51}{16}\nu+\frac{27}{16}\nu^2-\frac{5}{16}\nu^3\right)\sqrt{e_r^2-1} + \frac{(6+(\frac{41}{128}\pi^2-\frac{19697}{560})\nu+9\nu^2-\frac{3}{16}\nu^3)}{\sqrt{e_r^2-1}}\right. \\
&\left.+ \frac{(5+(\frac{123}{32}\pi^2-\frac{22193}{280})\nu+\frac{71}{16}\nu^2+\frac{1}{16}\nu^3)}{(e_r^2-1)^{3/2}} + \frac{(\frac{11}{2}+(\frac{41}{8}\pi^2-\frac{32887}{420})\nu-\frac{15}{8}\nu^2-\frac{1}{16}\nu^3)}{(e_r^2-1)^{5/2}}\right]\frac{\eta^6}{\bar{a}_r^{5/2}}.
\end{aligned}
\tag{D3}
$$

We find

$$
\begin{aligned}
\bar{n} = {}& \frac{1}{\bar{a}_r^{3/2}} - \frac{1}{2}(-9+\nu)\frac{\eta^2}{\bar{a}_r^{5/2}} + \left[\frac{3}{8}\nu^2-\frac{25}{8}\nu+\frac{147}{8}+\frac{(-\frac{21}{2}\nu+6)}{\sqrt{e_r^2-1}}\right]\frac{\eta^4}{\bar{a}_r^{7/2}} \\
&+ \left[\frac{1181}{16}-\frac{235}{16}\nu+3\nu^2-\frac{5}{16}\nu^3+\frac{(39+(\frac{123}{128}\pi^2-\frac{29353}{280})\nu+\frac{35}{4}\nu^2)}{\sqrt{e_r^2-1}}+\frac{(12+(-\frac{12343}{140}+\frac{123}{32}\pi^2)\nu+3\nu^2)}{(e_r^2-1)^2}\right]\frac{\eta^6}{\bar{a}_r^{9/2}}, \\
K = {}& 1+\frac{3}{(e_r^2-1)}\frac{\eta^2}{\bar{a}_r}+\left[\frac{(\frac{3}{2}\nu-\frac{9}{4})}{e_r^2-1}+\frac{(-\frac{9}{2}\nu+\frac{33}{4})}{(e_r^2-1)^2}\right]\frac{\eta^4}{\bar{a}_r^2} \\
&+ \left[\frac{(\frac{3}{2}\nu-\frac{3}{8}\nu^2)}{e_r^2-1}+\frac{(-\frac{39}{4}+(\frac{13}{2}+\frac{123}{128}\pi^2)\nu-\frac{9}{4}\nu^2)}{(e_r^2-1)^2}+\frac{(\frac{135}{4}+(-122+\frac{615}{128}\pi^2)\nu+\frac{9}{8}\nu^2)}{(e_r^2-1)^3}\right]\frac{\eta^6}{\bar{a}_r^3}, \\
\frac{e_t}{e_r} = {}& 1+\left(4-\frac{3}{2}\nu\right)\frac{\eta^2}{\bar{a}_r}+\left(16-\frac{67}{8}\nu+\frac{15}{8}\nu^2+\frac{4-7\nu}{e_r^2-1}\right)\frac{\eta^4}{\bar{a}_r^2} \\
&+ \left(64-\frac{599}{16}\nu+\frac{219}{16}\nu^2-\frac{35}{16}\nu^3+\frac{(28+(\frac{41}{64}\pi^2-\frac{34463}{420})\nu+21\nu^2)}{e_r^2-1}+\frac{(8+(-\frac{12343}{210}+\frac{41}{16}\pi^2)\nu+2\nu^2)}{(e_r^2-1)^2}\right)\frac{\eta^6}{\bar{a}_r^3}, \\
\frac{e_\phi}{e_r} = {}& 1-\frac{\nu}{2}\frac{\eta^2}{\bar{a}_r}+\left(-\frac{29}{32}\nu+\frac{15}{32}\nu^2+\frac{(-5-\frac{357}{32}\nu+\frac{15}{32}\nu^2)}{e_r^2-1}\right)\frac{\eta^4}{\bar{a}_r^2}+\left(-\frac{213}{128}\nu+\frac{213}{128}\nu^2-\frac{61}{128}\nu^3\right. \\
&\left.+\frac{(4+(\frac{205}{256}\pi^2-\frac{10463}{448})\nu+\frac{799}{64}\nu^2-\frac{15}{64}\nu^3)}{e_r^2-1}+\frac{(-16+(\frac{533}{256}\pi^2+\frac{276553}{4480})\nu+\frac{585}{128}\nu^2+\frac{95}{128}\nu^3)}{(e_r^2-1)^2}\right)\frac{\eta^6}{\bar{a}_r^3}.
\end{aligned}
\tag{D4}
$$

The remaining 3PN orbital parameters still expressed as functions of $\bar{a}_r$ and e_r are listed in Table IV.

TABLE IV. The orbital parameters of the 3PN quasi-Keplerian hyperbolic representation in modified harmonic coordinates, expressed as functions of $\bar{a}_r$ and e_r. The corresponding (equivalent) expressions in terms of $\bar{E}$ and j have been given in Ref. [56].

f_t	$\frac{3(5-2\nu)}{2\bar{a}_r^2\sqrt{e_r^2-1}}\eta^4+\frac{144(4\nu^2-19\nu+40)e_r^2+\nu(-8768+576\nu+123\pi^2)}{192(e_r^2-1)^{3/2}\bar{a}_r^3}\eta^6$
f_ϕ	$\frac{e_r^2(1+19\nu-3\nu^2)}{8\bar{a}_r^2(e_r^2-1)^2}\eta^4-\frac{e_r^2[-280\nu(9\nu^2-177\nu+458)e_r^2-26880+26880\nu^3+36960\nu^2+(-107104+30135\pi^2)\nu]}{26880\bar{a}_r^3(e_r^2-1)^3}\eta^6$
g_t	$\frac{e_r\nu(15-\nu)}{8\bar{a}_r^2\sqrt{e_r^2-1}}\eta^4-\frac{e_r[-35\nu(27\nu^2-263\nu+717)e_r^2-22400+700\nu^3+8820\nu^2+(-5956+1435\pi^2)\nu]}{2240(e_r^2-1)^{3/2}\bar{a}_r^3}\eta^6$
g_ϕ	$\frac{(1-3\nu)\nu e_r^3}{32\bar{a}_r^2(e_r^2-1)^2}\eta^4-\frac{\nu e_r^3[-35(23\nu^2-87\nu+27)e_r^2+1960\nu^2+14840\nu+1435\pi^2-31856]}{8960\bar{a}_r^3(e_r^2-1)^3}\eta^6$
h_t	$\frac{(3\nu^2-49\nu+116)\nu e_r^2}{16(e_r^2-1)^{3/2}\bar{a}_r^3}\eta^6$
h_ϕ	$\frac{(15\nu^2-57\nu+82)e_r^4\nu}{192\bar{a}_r^3(e_r^2-1)^3}\eta^6$
i_t	$\frac{(13\nu^2-73\nu+23)\nu e_r^3}{192(e_r^2-1)^{3/2}\bar{a}_r^3}\eta^6$
i_ϕ	$\frac{(5\nu^2-5\nu+1)\nu e_r^5}{256\bar{a}_r^3(e_r^2-1)^3}\eta^6$

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