# Dark energy from the fifth dimension

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(Received 5 July 2022; accepted 9 December 2022; published 3 January 2023)

After generalizing the Regge-Teitelboim formulation of gravity to include the case where the background embedding space is not flat, we examine the dynamics of the four-dimensional $k = 0$ Robertson-Walker (RW) manifold embedded in various five-dimensional backgrounds. We find that when the background is five-dimensional de Sitter space, the RW manifold undergoes a transition from a deaccelerating phase to an accelerating phase. This occurs before the inclusion of matter, radiation or cosmological constant sources, and thus does not require a balance of different components. We obtain a reasonable two-parameter fit of this model to the Hubble parameter data.

DOI: 10.1103/PhysRevD.107.024001

## I. INTRODUCTION

There has been much interest in applications of alternative theories of gravity to cosmology (for a review, see Ref. [1]), and in particular, toward using it to explain away dark matter [2–5]. Perhaps even more pressing is the need for an alternative gravity theory explanation of dark energy [6–12]. Here we present a new approach to the latter, whereby dark energy appears as an artifact of extra dimensions. The approach is based on a generalization of a formulation of gravity developed long ago by Regge and Teitelboim (RT) [13–19]. RT gravity is a fully diffeomorphism invariant theory, which has similarities to the original development of string theory, as well as current research on certain matrix models in the low energy continuum limit, [20–22] which has found application to cosmology [23–25]. The classical dynamics of RT gravity can be regarded as an extension (rather than a modification) of general relativity. This is since solutions of Einstein equations also satisfy the field equations of RT gravity. More generally, RT gravity may introduce new sources to the Einstein equations, which are not attributable to the energy-momentum tensor, but rather are a result of embedding our four-dimensional space-time in some fixed higher-dimensional background [26]. It is then natural to ask whether such new source terms could be responsible for phenomena such as cosmic acceleration. We explore this possibility in this article, and exhibit an example which gives a good fit to current observational data.

The dynamical degrees of freedom of RT gravity are associated with the embedding of our four-dimensional space-time manifold in some fixed higher dimensional background. The original formulation of Regge and Teitelboim makes the simplifying assumption that the higher dimensional background space is flat. This has severely restricted the dynamics of the embedded manifold in previous applications. For example, it did not lead to a realistic model of cosmic acceleration in a previous search [26]. Here we generalize the formalism to curved backgrounds, which allows for a wider range of application.

After generalizing RT gravity, we shall apply it to cosmology by embedding a four-dimensional Robertson-Walker (RW) manifold in three different five-dimensional background spaces. We specialize to the $k = 0$ RW metric since this case is currently favored (although the other cases can also be considered as well). The five-dimensional backgrounds we examine are: (i) $R^{4,1}$, (ii) $\mathrm{AdS}_5$, and (iii) $\mathrm{dS}_5$. Embeddings of the RW manifold into these spaces were obtained by Akbar [27], and shall be applied here. As a first approximation, we obtain the evolution of the scale factor on the RW manifold in the absence of matter, radiation or cosmological constant sources. We get that the acceleration of the scale factor is negative for all time for cases (i) and (ii). On the other hand, for case (iii) we find that a transition from the deaccelerating phase to an accelerating phase occurs at a finite time. The evolution in this case is determined by two free parameters, the curvature of the background de Sitter space and the strength of the RT source term. The two parameters allow for a fit to the Hubble parameter data. Unlike in the ΛCDM model, neither the matter density nor the cosmological constant play a role in the fit, meaning that their contributions should be significantly weaker than the RT source term, and furthermore, that they can have arbitrary strength relative to each other. So here we are able to avoid the coincidence puzzle of the ΛCDM model, where the matter contribution at the current time is coincidentally of the same order of magnitude as the cosmological constant contribution.

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The outline for the rest of the article is as follows: We generalize RT gravity to the case of curved backgrounds in Sec. II, and apply it to cosmology in Sec. III. Some concluding remarks are given in Sec. IV.

## II. GENERALIZED REGGE-TEITELBOIM GRAVITY

We begin with a very brief discussion of RT gravity, or more precisely, its generalization to the case where the $d$−dimensional background space $\mathbf{M}_d$, $d > 4$, is not necessarily flat. We denote a local set of coordinates on $\mathbf{M}_d$ by $Y^a$, $a, b, \dots = 0, \dots, d-1$, and its associated metric tensor $\mathbf{g}_{ab}(Y)$. Next embed a four-dimensional space-time manifold $\mathcal{M}_4$ in $\mathbf{M}_d$. This can be done by introducing the set of functions $Y^a = Y^a(x)$, where $x^\mu$, $\mu, \nu, \dots = 0, \dots, 3$, span $\mathcal{M}_4$. The metric tensor $g_{\mu\nu}(x)$ on $\mathcal{M}_4$ is defined to be induced from $\mathbf{g}_{ab}(Y)$. So

$$g_{\mu\nu}(x) = \mathbf{g}_{ab}(Y)\partial_\mu Y^a \partial_\nu Y^b, \tag{1}$$

$\partial_\mu$ denoting differentiation with respect to $x^\mu$. As is usual $g_{\nu\lambda}$ is required to be invertible, and metric compatible on $\mathcal{M}_4$, $\nabla_\mu g_{\nu\lambda} = 0$, and $\nabla_\mu$ being the covariant derivative on $\mathcal{M}_4$. The latter leads to the identity:

$$\mathbf{g}_{ab}\nabla_\lambda \partial_\mu Y^a \partial_\nu Y^b + \frac{1}{2}\frac{\partial \mathbf{g}_{ab}}{\partial Y^c}(\partial_\mu Y^a \partial_\nu Y^b \partial_\lambda Y^c + \partial_\nu Y^a \partial_\lambda Y^b \partial_\mu Y^c - \partial_\lambda Y^a \partial_\mu Y^b \partial_\nu Y^c) = 0 \tag{2}$$

To derive this compute $\nabla_\lambda g_{\mu\nu} + \nabla_\mu g_{\nu\lambda} - \nabla_\nu g_{\lambda\mu}$ using (1), and apply metric compatibility and the Leibniz rule.

RT gravity assumes the usual Einstein-Hilbert action $S_{\mathrm{EH}}$ for the gravitational field, however the dynamical degrees of freedom are the embedding functions, not $g_{\mu\nu}$. So upon including a source term $S_{\mathrm{source}}$, one has

$$S = S_{\mathrm{EH}} + S_{\mathrm{source}}, \qquad S_{\mathrm{EH}} = \frac{1}{16\pi G}\int_M d^4x\sqrt{|g|}R, \tag{3}$$

with the scalar curvature constructed from (1). Field dynamics is obtained from variations of $Y^a$. This gives

$$\partial_\mu(\sqrt{|g|}E^{\mu\nu}\mathbf{g}_{ab}\partial_\nu Y^b) - \frac{1}{2}\sqrt{|g|}E^{\mu\nu}\frac{\partial \mathbf{g}_{bc}}{\partial Y^a}\partial_\mu Y^b \partial_\nu Y^c = 0, \tag{4}$$

$$E^{\mu\nu} = G^{\mu\nu} - 8\pi G T^{\mu\nu}, \tag{5}$$

$G^{\mu\nu}$ and $T^{\mu\nu}$ being the Einstein tensor and stress-energy tensor, respectively. As in Einstein gravity, $T^{\mu\nu}$ must be covariantly conserved. To see this one can first rewrite the field equations as

$$\nabla_\mu(E^{\mu\nu}\mathbf{g}_{ab}\partial_\nu Y^b) - \frac{1}{2}E^{\mu\nu}\frac{\partial \mathbf{g}_{bc}}{\partial Y^a}\partial_\mu Y^b \partial_\nu Y^c = 0, \tag{6}$$

and then expand the first term using the Bianchi identity to obtain

$$-8\pi G\nabla_\mu T^{\mu\nu}\mathbf{g}_{ab}\partial_\nu Y^b + E^{\mu\nu}\left(\nabla_\mu(\mathbf{g}_{ab}\partial_\nu Y^b) - \frac{1}{2}\frac{\partial \mathbf{g}_{bc}}{\partial Y^a}\partial_\mu Y^b \partial_\nu Y^c\right) = 0. \tag{7}$$

Finally, contract with $\partial_\lambda Y^a$ and apply (2) to get $\nabla_\mu T^\mu{}_\lambda = 0$.

The field equations (4) are obviously satisfied for solutions to Einstein equations, $E^{\mu\nu} = 0$ (in which case the above derivation of $\nabla_\mu T^\mu{}_\lambda = 0$ is no longer necessary). More generally, $E^{\mu\nu}$ need not vanish. Thus, RT gravity is less constrained than general relativity [14]. Alternatively, we can argue that the Einstein equations effectively pick up additional source terms, which we denote by $T^{\mu\nu}_{\mathrm{RT}}$, which are not associated with the standard stress-energy tensor but rather are due to the embedding in the background space,

$$G^{\mu\nu} = 8\pi G(T^{\mu\nu} + T^{\mu\nu}_{\mathrm{RT}}), \tag{8}$$

Obviously, $T^{\mu\nu}_{\mathrm{RT}}$ is covariantly conserved since $T^{\mu\nu}$ is. (8) can be regarded as the definition of the source terms $T^{\mu\nu}_{\mathrm{RT}} = \frac{1}{8\pi G}E^{\mu\nu}$. Note that the field equations (4) only involve $T^{\mu\nu}_{\mathrm{RT}}$ and the embedding functions. $T^{\mu\nu}$ does not separately contribute to (4). Therefore any nontrivial solution one obtains for $T^{\mu\nu}_{\mathrm{RT}}$ will not depend on the choice of the stress-energy tensor $T^{\mu\nu}$.

## III. APPLICATION TO COSMOLOGY

Next we want to apply this dynamical system to the case where the embedded manifold $\mathcal{M}_4$ is that of standard cosmology, i.e., it is given by the RW metric tensor. Here we will specialize to the currently favored case of $k = 0$

$$ds^2 = -dt^2 + a(t)^2 dx^i dx^i, \tag{9}$$

where $t = x^0$ and $a(t)$ is the scale factor. As a first approximation let us consider source free RT gravity, i.e., $T^{\mu\nu} = 0$. From (8) we know that the Einstein tensor need not vanish. $T^{\mu\nu}_{\mathrm{RT}}$ in (8) needs to be computed from the particular choice of embedding, however from consistency with homogeneity and isotropy, we anticipate that its form should be analogous to that of a perfect fluid in the comoving frame

$$T^{00}_{\mathrm{RT}} = \rho_{\mathrm{RT}} \qquad T^{11}_{\mathrm{RT}} = T^{22}_{\mathrm{RT}} = T^{33}_{\mathrm{RT}} = a(t)^2 p_{\mathrm{RT}}, \tag{10}$$

with $\rho_{\mathrm{RT}}$ and $p_{\mathrm{RT}}$ being functions of $t$. Since it is covariantly conserved we have

$$\dot{\rho}_{\mathrm{RT}} + 3\frac{\dot{a}}{a}(\rho_{\mathrm{RT}} + p_{\mathrm{RT}}) = 0, \tag{11}$$

the dot denoting a $t$–derivative.

Substituting (9) into (4) gives

$$\partial_t(F_1(t)\mathbf{g}_{ab}\partial_t Y^b) - \frac{1}{2}F_1(t)\partial_t Y^b \partial_t Y^c \frac{\partial \mathbf{g}_{bc}}{\partial Y^a} = F_2(t)\left(\partial_i(\mathbf{g}_{ab}\partial_i Y^b) - \frac{1}{2}\partial_i Y^b \partial_i Y^c \frac{\partial \mathbf{g}_{bc}}{\partial Y^a}\right), \qquad (12)$$

where

$$F_1(t) = 3a\dot{a}^2 \qquad F_2(t) = 2\ddot{a} + \frac{\dot{a}^2}{a} \qquad (13)$$

(12) will produce equations for $\dot{a}$ and $\ddot{a}$ which can be written in the form of $k = 0$ Friedmann equations

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_{\mathrm{RT}} \qquad (14)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_{\mathrm{RT}} + 3p_{\mathrm{RT}}), \qquad (15)$$

allowing us to identify $\rho_{\mathrm{RT}}$ and $p_{\mathrm{RT}}$ in (10). The resulting expressions for $\rho_{\mathrm{RT}}$ and $p_{\mathrm{RT}}$ will in general depend on the background space and the choice of embedding, as we illustrate in the examples that follow.

As stated previously, the background spaces we consider are $R^{4,1}$, $\mathrm{AdS}_5$, and $\mathrm{dS}_5$. Following [27], we use the same expression for the embedding in all three cases:

$$\begin{pmatrix} Y^0 \\ Y^1 \\ Y^2 \\ Y^3 \\ Y^4 \end{pmatrix} = \begin{pmatrix} b(t) \\ x^1 \\ x^2 \\ x^3 \\ h(t) \end{pmatrix}, \qquad (16)$$

where the functions $b(t)$ and $h(t)$ need to satisfy certain constraints in order to recover the $k = 0$ Robertson-Walker metric on the embedded four-dimensional manifold.

We next deduce $\rho_{\mathrm{RT}}$ and $p_{\mathrm{RT}}$ for the three different cases.

(1) Flat five-dimensional background $R^{4,1}$

A trivial system results if one chooses Cartesian coordinates for $R^{4,1}$ and maps to $\mathcal{M}_4$ using (16), as this restricts the scale factor in (9) to be one. Alternatively, a nontrivial function $a(t)$ can result from a different coordinatization on $R^{4,1}$, such as is in [27,28] where

$$(ds^2)_{R^{4,1}} = -(dY^0)^2 + (Y^0 + Y^4)^2((dY^1)^2 + (dY^2)^2 + (dY^3)^2) + (dY^4)^2. \qquad (17)$$

It can be checked that the five-dimensional curvature resulting from this metric is zero. Now using (16) to map to (9) one gets that $b(t)$ and $h(t)$ should satisfy

$$b(t) + h(t) = a(t) \qquad \dot{b}^2 - \dot{h}^2 = 1. \qquad (18)$$

Substituting (16) in (12) gives

$$\partial_t(\dot{b}F_1) = 3F_2 a \qquad (19)$$

$$\partial_t(\dot{h}F_1) = -3F_2 a. \qquad (20)$$

The sum of these two equations leads to a constant of motion $\partial_t(\dot{a}F_1) = 0$, from which we get the following expression for $\rho_{\mathrm{RT}}$

$$\rho_{\mathrm{RT}} = \frac{c_0}{a^3\dot{a}}, \qquad (21)$$

$c_0$ being a constant. The Friedmann equation (14) then gives $\dot{a}^3 \propto \frac{1}{a}$, and so there is no acceleration as $a$ increases. One gets a simple solution for the scale factor in this case: $a(t) \propto t^{3/4}$ for $a(0) = 0$. This coincides with the time evolution of the scale factor in the presence of a perfect fluid with equation of state $p = -\frac{1}{9}\rho$. The same result was observed in [29] for a different choice of embedding.

(2) $\mathrm{AdS}_5$ background

Here we cover a patch of $\mathrm{AdS}_5$ using Poincaré coordinates. The background metric is

$$(ds^2)_{\mathrm{AdS}_5} = -\frac{(Y^4)^2}{L^2}(dY^0)^2 + \frac{(Y^4)^2}{L^2}((dY^1)^2 + (dY^2)^2 + (dY^3)^2) + \frac{L^2(dY^4)^2}{(Y^4)^2}, \qquad (22)$$

the constant $L$ denoting the $\mathrm{AdS}_5$ radius of curvature. Utilizing the embedding (16), the $k = 0$ RW metric (9) is recovered provided that

$$h = La \qquad a^2\dot{b}^2 - L^2\frac{\dot{a}^2}{a^2} = 1. \qquad (23)$$

Substituting (16) in (12) gives

$$\partial_t(a^2\dot{b}F_1) = 0 \qquad (24)$$

$$L^2\partial_t\left(\frac{\dot{a}}{a^2}F_1\right) + \left(a\dot{b}^2 + L^2\frac{\dot{a}^2}{a^3}\right)F_1 = -3aF_2. \qquad (25)$$

From (14) and (24) we then get

$$\rho_{\mathrm{RT}} = \frac{c_0}{a^3\sqrt{L^2\dot{a}^2 + a^2}}. \qquad (26)$$

Note that the form (21) resulting from the flat background is recovered in the limit $L \to \infty$, or more precisely when $|\frac{\dot{a}}{a}| \gg \frac{1}{L}$.

(3) $dS_5$ background

Using the so-called flat slicing the metric for $dS_5$ is

$$(ds^2)_{dS_5} = -(dY^0)^2 + e^{2Y^0/L}((dY^1)^2 + (dY^2)^2 + (dY^3)^2) + e^{2Y^0/L}(dY^4)^2, \tag{27}$$

$L$ again being the radius of curvature. Now (9) is recovered from the embedding (16) for

$$e^{b/L} = a \qquad L^2\frac{\dot{a}^2}{a^2} - a^2\dot{h}^2 = 1. \tag{28}$$

After substituting (16) in (12)

$$L^2\partial_t\left(\frac{\dot{a}}{a}F_1\right) + a^2\dot{h}^2F_1 - 3a^2F_2 = 0 \tag{29}$$

$$\partial_t(a^2\dot{h}F_1) = 0. \tag{30}$$

From (14) and (30) we then get

$$\rho_{\rm RT} = \frac{c_0}{a^3\sqrt{L^2\dot{a}^2 - a^2}} \tag{31}$$

$c_0$ is real which means we need that $|\frac{\dot{a}}{a}| > \frac{1}{L}$. The expression (21) is once again recovered for $|\frac{\dot{a}}{a}| \gg \frac{1}{L}$.

To summarize, the source term $\rho_{\rm RT}$ for the three different backgrounds has the form[1]

$$\rho_{\rm RT} = \frac{c_0}{a^3\sqrt{L^2\dot{a}^2 - k_5a^2}}, \tag{32}$$

where $k_5$ defines the curvature of the five-dimensional background space: $k_5 = 0, -1, 1$ for $R^{4,1}$, $AdS_5$, and $dS_5$, respectively. Moreover, from (14) one has that

$$\dot{a}^2a\sqrt{L^2\dot{a}^2 - k_5a^2} = \text{constant} \tag{33}$$

$p_{\rm RT}$ can be determined from the conservation law (11) leading to a time-dependent[2] equation of state

$$p_{\rm RT} = -\frac{a}{3\dot{a}}\dot{\rho}_{\rm RT} - \rho_{\rm RT} = \frac{a(L^2\ddot{a} - k_5a)}{3(L^2\dot{a}^2 - k_5a^2)}\rho_{\rm RT}. \tag{34}$$

The evolution of the scale factor for the three cases $k_5 = 1, 0, -1$ is obtained from (33). As stated previously, for $k_5 = 0$ one gets $a(t) \propto t^{3/4}$. We resort to numerical integration to obtain solutions for the other two cases, $k_5 = \pm 1$. The results for all three cases are plotted in Fig. 1, using the initial condition $a(0) = 0$. All three cases agree for small $t$, i.e., $a(t) \propto t^{3/4}$ as $L\frac{\dot{a}}{a} \to \infty$, and so $\ddot{a} < 0$. For cases $k_5 = 0$ and $-1$, we find that $\ddot{a} < 0$, *for all t*. The situation is more interesting for $k_5 = 1$, corresponding to the de Sitter background. In this case, $\ddot{a}$ vanishes at finite $t$, when $L\frac{\dot{a}}{a} = \sqrt{2}$, thus signaling a transition from the de-accelerating phase to an accelerating phase. We get that $L\frac{\dot{a}}{a}$ goes asymptotically to one in the $t \to \infty$ limit, where the scale factor undergoes an exponential expansion at leading order,

$$a(t) \to a_1e^{t/L}(1 - a_2e^{-8t/L} + \cdots), \quad \text{as } t \to \infty, \tag{35}$$

$a_1$ and $a_2$ being positive constants. From (34) we can obtain the equation of state for the RT source as a function of time. The ratio $p_{\rm RT}/\rho_{\rm RT}$, standardly denoted by $w$, goes from $-\frac{1}{9}$, near $t = 0$, to $-\frac{1}{3}$, at the transition, to $-1$, in the limit $t \to \infty$. Note that unlike in the ΛCDM model, here we get a transition from the de-accelerating phase to an accelerating phase even without the inclusion of a matter component or cosmological constant component to the Friedmann equations.

Finally, we proceed with a fit of the $k_5 = 1$ case to observational data. (33) gives an algebraic relation between the Hubble parameter $H = \dot{a}/a$ and the redshift parameter $z = a_0/a - 1$, where $a_0$ is the scale parameter at the current time. It is

$$L^2H^2\sqrt{L^2H^2 - 1} = \tilde{c}_0(1 + z)^4, \tag{36}$$

where $\tilde{c}_0 = \frac{8\pi G}{3}L^2a_0^{-4}c_0$. In Fig. 2(a) we fit the real solution to Eq. (36) to observed results for $H$ versus $z$ using the

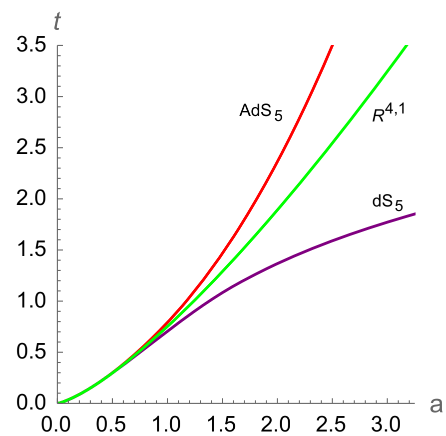
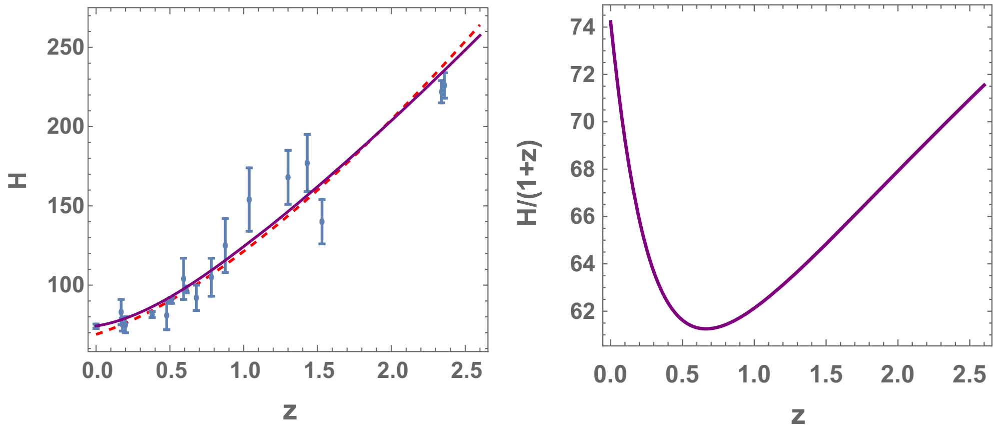



FIG. 1. Plot of $t$ vs $a$ for three different five-dimensional background spaces: $R^{4,1}$, $AdS_5$, and $dS_5$. (Here we set $L = 1$).

[1]Here we have done a rescaling of the constant $c_0$ for the case $k_5 = 0$.

[2]The case $k_5 = 0$ is an exception. After using (33) one gets the simple relation $p_{\rm RT} = -\frac{1}{9}\rho_{\rm RT}$.




(a) $H$ versus $z$ (b) $H/(1+z)$ versus $z$

FIG. 2. The solid purple curve in figure (a) represents a fit of Eq. (36) with the Hubble parameter data, while the dashed red curve is $\Lambda$CDM. $H$ is given in units of km s$^{-1}$ Mpc$^{-1}$. The best fit occurs for $\tilde{c}_0 \approx .26$ and $1/L \approx 72$ km s$^{-1}$ Mpc$^{-1}$. From figure (b) the minimum of $H/(1+z)$ for the best fit occurs at $z \approx .675$, corresponding to the transition from a de-accelerating phase to an acceleration phase.

data in Table I. The best fit occurs for $\tilde{c}_0 \approx .26$ and $1/L \approx 72$ km s$^{-1}$ Mpc$^{-1}$. For $H$ evaluated at $z = 0$ one gets $H(0) \approx 74$ km s$^{-1}$ Mpc$^{-1}$. Our fit in Fig. 2(a) is compared to that of $\Lambda$CDM, where the expression for the Hubble parameter is given by $H = H_0\sqrt{(1+z)^3\Omega_m + \Omega_\Lambda}$, with $\Omega_m = .3$, $\Omega_\Lambda = .7$, and $H_0 \approx 68.92$ km s$^{-1}$ Mpc$^{-1}$. $H/(1+z)$ (which is proportional to $\dot{a}$) versus $z$ is plotted in Fig. 2(b), using our fit for $H$ in Fig 2(a). It shows that the transition from a de-acceleration phase to acceleration phase occurs at $z \approx .675$, which is similar to the value predicted by $\Lambda$CDM.

TABLE I. Data used for fit in Fig. 2. Columns 1-4 are $z$, $H$, error in $H$ and citation respectively. Columns 2 and 3 are in units of km s$^{-1}$ Mpc$^{-1}$. Data was selected with $\sigma_H < .15H$.

| $z$ | $H$ | $\sigma_H$ | |
|---|---|---|---|
| 0 | 74.03 | 1.42 | [30] |
| 0.17 | 83 | 8 | [31] |
| 0.1791 | 75 | 4 | [32] |
| 0.1993 | 75 | 5 | [32] |
| 0.38 | 81.5 | 1.9 | [33] |
| 0.4783 | 80.9 | 9 | [34] |
| 0.51 | 90.4 | 1.9 | [33] |
| 0.5929 | 104 | 13 | [32] |
| 0.61 | 97.3 | 2.1 | [33] |
| 0.6797 | 92 | 8 | [32] |
| 0.7812 | 105 | 12 | [32] |
| 0.8754 | 125 | 17 | [32] |
| 1.037 | 154 | 20 | [32] |
| 1.3 | 168 | 17 | [31] |
| 1.43 | 177 | 18 | [31] |
| 1.53 | 140 | 14 | [31] |
| 2.34 | 222 | 7 | [35] |
| 2.36 | 226 | 8 | [36] |

## IV. CONCLUDING REMARKS

We now summarize some of the features of this model. After generalizing RT gravity to curved backgrounds, we found universal formulas for the effective density and pressure, (32) and (34), respectively, resulting from embedding the $k = 0$ RW manifold in three different five-dimensional background spaces. We suspect that the results found here could be dependent on the choice of embedding (in addition to the choice of background space), although we have not found any specific examples of this. In this regard, only a limited number of embeddings of the Robertson-Walker manifold are currently known (for example [27,37]). For other choices of embeddings of the $k = 0$ RW manifold we find that the Regge-Teitelboim field equations collapse to Einstein equations, i.e., $T^{\mu\nu}_{\mathrm{RT}} = 0$.

A reasonable fit to the Hubble parameter data was obtained in the case where the background was de Sitter space. This result holds even without considering the usual stress-energy contributions to the Einstein equations, which on the other hand, play an essential role for $\Lambda$CDM. Such components can easily be included in our model by adding appropriate terms to (13) and consequent equations. For the case of nonrelativistic matter, one ends up with the following modification to (36)

$$\frac{L^2H^2}{(1+z)^3} - \frac{\tilde{c}_0(1+z)}{\sqrt{L^2H^2-1}} = \tilde{c}_1, \tag{37}$$

which we derive in the Appendix. Here $\tilde{c}_1$ is an additional constant which quantifies the nonrelativistic matter component. The inclusion of the additional parameter $\tilde{c}_1$ does not appear to improve the previous fit in any significant manner.

The presence of the square root in (36) [and also in (37)] gives a lower bound on the Hubble parameter, $H(z) > 1/L$, which is in agreement with observation.

The fit we obtained to the Hubble parameter data holds for values of $z$ up to approximately 2.36. Concerning $z > 2.36$, the deviation of our fit in Fig. 2 with that of ΛCDM grows when extrapolating to higher $z$. However, our fit did not include contributions from the stress-energy tensor, which can play a more significant role at large $z$. For example, if one considers a nonrelativistic matter density $\rho_m$, which is proportional to $a^{-3}$, then its relative contribution is $\rho_m/\rho_{\mathrm{RT}} \propto \sqrt{L^2H^2-1}/(z+1)$, which grows like $LH/z$ for large $z$. Also, there is no reason to assume that the 5d de Sitter background is valid for all $z$.

A more challenging issue is that of stability. If one wishes to allow for all possible such four-manifolds in a stability analysis then one needs to consider embeddings in a much bigger space. It was shown in [38] that a 91-dimensional target space is necessary for a global embedding of a general 4d manifold (although there it was assumed that the background space is flat). On the other hand, if one restricts to say a five-dimensional background one can still recover many of the known physical four-manifolds, such as the Schwarzschild space-time. A stability analysis is further complicated by the fact that a nonperturbative treatment is required, as a weak field approximation in terms of embedding coordinates is out of reach, as was pointed out in [14]. These are among the issues which are open for further investigation/speculation.

## APPENDIX: DERIVATION OF (37)

For the inclusion of an energy-momentum source assume as usual that

$$T^{00} = \rho \qquad T^{11} = T^{22} = T^{33} = a(t)^2 p, \tag{A1}$$

in the comoving frame with $\rho$ and $p$ consistent with the conservation law

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{A2}$$

Substituting this along with (9) into (4) again gives (12), but with additional contributions to the functions $F_1$ and $F_2$

$$F_1(t) = 3a\dot{a}^2 - 8\pi G a^3 \rho \qquad F_2(t) = 2\ddot{a} + \frac{\dot{a}^2}{a} + 8\pi G a p \tag{A3}$$

Then upon repeating the analysis for cases 1–3 in Sec. III one gets the following Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left(\rho + \frac{c_0}{a^3\sqrt{L^2\dot{a}^2 - k_5 a^2}}\right) = \frac{8\pi G}{3}(\rho + \rho_{\mathrm{RT}}) \tag{A4}$$

Finally, for the case of nonrelativistic matter we set $p = 0$ in the conservation equation (A2) to get $\rho = c_1 a^{-3}$. Equation (37) then follows for $k_5 = 1$ and $\tilde{c}_1 = \frac{8\pi G L^2}{3a_0^3} c_1$.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