Magnetized matter effects on dilaton photon mixing

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Dilatons $[\phi(x)]$ are a class of bosonic scalar particles associated with scaling symmetry and its compensation (under the violations of the same). They are capable of interacting gravitationally with other massive bodies. As they have coupling to two photons (γ), they are (also) capable of decaying to the two photons. However, the decay time is long and that makes them a good candidate for dark matter. Furthermore due to two photon coupling, they can produce optical signatures in a magnetic field. In a vacuum or plain matter they couple to one of the transversely polarized states of the photon. But in magnetized matter, they couple to both the transversely polarized state of photons (due to the emergence of a parity violating part of the photon self-energy contribution from magnetized matter). Being spin zero scalar, they could mix with spin zero longitudinal part of photons but they do not. A part of this work is directed towards understanding this issue of mixing the scalar with various polarization states of photons in a medium (magnetized or unmagnetized) due to the constraints from different discrete symmetries, e.g., charge conjugation (**C**), parity (**P**) and time reversal (**T**) associated with the interaction. Based on these symmetry aided arguments, the structure of the mixing matrix is found to be 3×3, as in the case of neutrino flavor mixing matrix. Thus there exists nonzero finite probabilities of oscillation between different polarization states of photon to dilaton. Our analytical and numerical analysis show no existence of periodic oscillation length either in temporal or spatial direction for the most general values of the parameters in the theory. Possible astrophysical consequences of these results can be detected through the discussed observations.

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I. INTRODUCTION

The issue of unification of four fundamental forces of nature by the introduction of an additional scalar (field) $(\phi(x))$ is probably one issue that may hold a key to the solution of the dark energy/matter puzzle. These postulated fields are found to make their appearance in two kinds of theories, one is in quantum theories of unification and other is in higher dimensional theories of cosmology/gravity. In theories beyond the standard model (of particle physics), these additional scalar fields often appear in theories of unification [1–15], like in five-dimensional Kaluza-Klien theory, in super-string theory, and also in theories of extended super gravity [1] etc. In higher dimensional theories, like the string theory, the scalar fields—termed moduli (fields) [2–12]—are necessary to produce a four-dimensional effective theory from the original higher dimensional theory by compactifing the extra dimensions. On the other hand, scalar fields (both interacting and massive) had also been postulated independently in models of cosmology to shed light on the aspects of dark matter. Those whose presence remains imprinted in the cosmological observations of cosmic microwave background radiations are called "chameleon" [16–28]. Also, the compelling observational aspects of dark matter physics has motivated the particle-astrophysics community to construct a particle physics models with similar fields [29–34] and verify their suitability in explaining the experimental data. A detailed description of the possible candidates of dark matter can be found in [34]. Other possible applications can be found in [7–35]. It is not so often that the solution of an unresolved issue in one area of physics holds the key to another unresolved issue in other area of the same. The issues of unification of four forces of nature and the missing mass and energy (dark matter and dark energy) problem of cosmology might turn out to represent such an event of rarity. The notable feature that this field $\phi(x)$ exhibits in these theories lie in the structure of their interaction with photons. This is given by an interaction Lagrangian of the form $\frac{1}{M}\phi F^{\mu\nu}F_{\mu\nu}$, where M is the energy scale of the physics. In this place we may write $g_{\phi\gamma\gamma} = \frac{1}{M}$, where $g_{\phi\gamma\gamma}$ is the dimension full coupling constant. This interaction term has most of the desirable properties of an ideal Lagrangian except one. Although it is local, it remains invariant under Lorentz, gauge, and **CPT** symmetry transformations; however, it is nonrenormalizable and compromised for having an interaction term of mass dimension five. Incidentally, similar interaction terms are also possible for a loop induced Higgs photon

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interaction, given by $g' H F^{\mu\nu} F_{\mu\nu}$, where g' is the effective coupling constant, obtained after integrating out the heavy degrees of freedom (d.o.f.), H is the Higgs field, and the rest of the pieces have their usual meaning. Many of these fields can have a coupling (very weak though, since $\frac{m}{M} \ll 1$) to standard model particles, thereby, they may be detected indirectly through collider or astrophysical observations. Some of their possible signatures are (i) the existence of a fifth force distinguishable from gravity, (ii) the violation of a Lorentz invariance [36], and (iii) the spectropolarimetric signals discussed in the phenomenal papers like [37–40] among many others of similar quality. A comprehensive list of other possible signatures can be found in [41–49]. A notable and important issue related to this kind of interaction that is capable of distinguishing similar looking scalar fields from each other (owing their origin to various symmetries) is related to the magnitude of their mass (m). The same is not fixed by any symmetry argument. However the lower limit to the same (m) is fixed by the torsion-balance fifth force experiment, which is estimated to be greater than 10^{-2} eV [50–54]. And the upper limit on the same (i.e., m) can go up to a few TeV [55] depending on the type of model (moduli) one is interested in. In cosmology for example, a limit to the mass m and energy scale M is set from the estimate of their lifetime to two photon decay, given by $\tau_\phi = \frac{16\pi M^2}{m^3}$ [56–59], so that the produced photons do not interfere with the big bang nucleosynthesis constraints [60]. Although lately there are proposals of a new kind of dark matter termed as "fuzzy" dark matter [61] available in the literature, for which torsion balance bounds are not applicable. The bounds on their masses are obtained from the Sachs-Wolfe effect [62,63]. The laboratory based experimental search for these particles were initially suggested in [39,40] and some of the variants of the same were in [64–68]. Many of these lab based experiments have provided some bounds on the axion coupling constant with other fundamental particles like electron, photon, etc. These two experiments [67,68] happen to be one of those. However the launch of the satellite missions—EUVE [69], ROSAT [70], BeppoSAX [71], XMM-Newton [72], Chandra [71,73,74], and Suzaku [75]—sensitive to soft x-ray emissions around 5–10 KeV from the galaxy clusters have opened up another possibility of their astrophysical confirmation. Interestingly enough, ever since their launch, evidence of 0.5–1.0 MeV lines [76], 3.5 KeV lines [77], and 511 keV lines [78–80]) have been reported in the literature. Some of these signals are believed to be due to dark matter. Though many of the laboratory and astrophysical electromagnetic (EM) signals are complementary to each other however, for few of the astrophysical ones are better than the laboratory ones. The presence of strong coherent magnetic field over a large length scale, an ambient plasma and abundance of highly energetic photons make it convenient to look for EM signals from astrophysical sources to test many types of interactions. For the same reason, the EM signatures from astrophysical sources for the $\frac{1}{M}\phi F^{\mu\nu} F_{\mu\nu}$ interaction are arguably better than the ones from the laboratory. Although a large volume of literature [10,37,38,81–83] is already available on many aspects of the relevant issues.

One of the notable aspects of these studies had been the incorporation of magnetized vacuum effects by considering the Euler-Heisenberg Lagrangian. This incorporation lifts the degeneracy between the two transversely polarized photons and makes only one of the polarized states of photon mix with Axion like particles (ALP)-like particles leaving the other one free. This turns the vacuum dichroic and birefringent that in principle can be detected if ALPs exist in nature.

However, in this work we would like to point out another aspect that has rarely been considered important in such investigations: that is, the background dependence of the dynamics of scalar or pseudoscalar photon interaction.

Photons propagating in a magnetized vacuum with a $\phi F^{\mu\nu} F_{\mu\nu}$ interaction have two transverse polarization states. One of them ($|\gamma_\perp\rangle$) is orthogonal to B and the other one ($|\gamma_\parallel\rangle$) lies on the $k - \mathrm{B}$ plane. Following the reasoning of [38] (performed originally for axion photon system), the two polarization states of the photon, for this case too, would transform differently under parity **P** and charge conjugation **C** symmetry transformations. Since, under **CP** transformation, the scalar and the **CP** even polarization state of photon would remain even, only these two would couple during propagation; the **CP** odd polarized state of the photon would propagate freely.

When medium induced corrections are considered, the contribution from the in-medium polarization tensor $\Pi_{\mu\nu}(k)$ need to be taken into account. The same, in absence of any parity violating interaction or ambient external magnetic field B, would be **CP** symmetric. Hence even in an unmagnetized medium, the propagating modes of the scalar (ϕ) photon (γ), system with five-dimensional scalar photon interaction, would remain same as in magnetized vacuum.[1]

This picture, however, changes with the introduction of a new parity violating interaction term to the effective Lagrangian (L_{eff}) that originates from the magnetized medium induced corrections to the photon self energy tensor (PSET), $\Pi_{\mu\nu}(k, T, \mu, e\mathrm{B})$ (sometimes called the Faraday term) in the system. This tensor (PSET) in a magnetized medium has a part that is even in eB and a part that is odd in the same. It is the part that is odd in eB that is also odd in μ (chemical potential) and is parity violating. This part was originally evaluated in [84].

An effective Lagrangian of the form $A^\mu(-k) \times \Pi_{\mu\nu}(k, T, \mu, e\mathrm{B}) A^\nu(k)$ constructed with the same would change the mixing dynamics. The leading order magnetic

[1]However, that consideration of the ambient plasma effects brings changes in the size of the contribution to the oscillation probability $P_{\gamma_\parallel \to \phi}$.

field effects in a magnetized plasma for $m^2 > e\mathrm{B}$ can be obtained by retaining the $\mathrm{O}(e\mathrm{B})$ piece from $A^{\mu}(-k) \times \Pi_{\mu\nu}(k, T, \mu, e\mathrm{B})\mathrm{A}^{\nu}(k)$, in the interaction Lagrangian (L_{eff}). We perform the same in this study. This formalism was developed earlier in [38,81,85] to study the axion-photon interaction dynamics. We have extended this formalism to the case of scalar photon interaction dynamics by incorporating the correction mentioned above. This causes further mixing between the two transverse polarization states of the photon; therefore, the scalar and the two transversely polarized states of the photon mix with each other. As a result, the $|\gamma_{\perp}\rangle$ part of a photon beam, unlike a vacuum, would evolve with propagation due to the presence of PSET, and longitudinal d.o.f. would propagate freely. The effect of the same on polarimetric signatures of scalar photon mixing case has been discussed in [86]. Though a similar approach for polarimetric studies had been considered in [85] for an axion photon system also, but the conversion probabilities of parallel or perpendicular polarized photons to pseudoscalar axions remained unexplored.

We complement the same in this work by calculating these conversion probabilities for dilaton-photon system. With the introduction of PSET, the probability of conversion of perpendicularly polarized photon to dilaton and vice versa turns out to be finite. Till so far this aspect remained unreported in the literature. We explore the same here and its consequences.

We demonstrate here in this note that the inclusion of PSET causes the $\gamma - \phi$ mixing matrix to be 3×3 instead of 2×2, usually encountered for a similar process taking place in magnetized vacuum, or unmagnetized plasma.[2] Now, as a result of scalar photon mixing, the two transverse degrees of freedom of photon, the $\parallel$ and the $\perp$ can now oscillate into and out of scalar (dilaton) mode in addition to the oscillations among themselves, i.e., $\parallel \leftrightarrow \perp$. Under favorable circumstances signals of the same may be within the future detector sensitivity.

The organization of the document is as follows: in the next section, we have elaborated on the form of the action and propagators in flat and curved space-time. We have provided the logic behind sticking to the description in flat space-time because of pathological problems encountered in curved space-time results. Followed by that the tensorial structure of the polarization tensor for photons in an unmagnetized medium is discussed. This is followed by the description of the fermion propagator in coordinate space. The parity violating part of the photon polarization tensor is discussed and its tensorial structure is discussed next.

Section III contains a discussion of the effective Lagrangian. This is followed by a brief discussion on the discrete symmetry transformation properties of the gauge potentials and the terms of the equations of motion of the $\gamma\phi$ system obtained from the effective Lagrangian under consideration. In Sec. IV, we establish the unitary transformation matrix that diagonalizes the mixing matrix. In Sec. V, using the same unitary matrix obtained in Sec. IV, we diagonalize the equations of motion and identify its similarity with the Klein-Gordon equation in the diagonal form. We take this equation and, following the procedure of [38], we estimates the conversion probabilities of various modes into each other. In Sec. VI we discuss the physics of appropriate astrophysical environments where the mixing of the photons with the scalar can take place. We identify some possible signatures of this mixing from the EM signals coming out of these astrophysical environments. The implications of this modified system is discussed in Sec. VII. In Sec. VIII we provide possible implications of our work for some of the DM signatures existing in the literatures followed by Appendices where some technical details are elaborated.

II. THE POLARIZATOIN TENSOR

Considering the form of coordinate space dilaton-photon interaction term presented in [56] the effective action for scalar photon interaction including medium corrections [84–99] in configuration space can be expressed as

$$\begin{aligned} S = \int d^4x\sqrt{-g}\bigg(&-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \\ &-\frac{g_{\phi\gamma\gamma}}{4}\phi(x)F_{\mu\nu}(x)F^{\mu\nu}(x) \\ &-\frac{1}{2}\int d^4x' A_{\mu}(x)\Pi^{\mu\nu}(x,x')A_{\nu}(x') \\ &+\frac{1}{2}\partial_{\mu}\phi(x)\partial^{\mu}\phi(x) - \frac{1}{2}m_{\phi}^2\phi^2(x)\bigg). \end{aligned} \tag{2.1}$$

In Eq. (2.1) g is the determinant of the space-time metric that assumes the value -1 in Minkowski space. The same (factor $\sqrt{-g}$) is multiplied by d^4x to maintain a Lorentz invariance of the measure. In the same [i.e., Eq. (2.1)] $\Pi^{\mu\nu}(x,x')$ stands for the in-medium polarization tensor in configuration space. The polarization tensor can be expressed, following [90], as

$$\begin{aligned} \Pi^{\mu\nu}(x,x') = e^2 tr\int \sqrt{-g}d^4y'(\gamma^{\mu}S_F(x,y')\gamma^{\nu}S_F(y',x)) \\ \times \delta^4(x'-y'), \end{aligned} \tag{2.2}$$

[2] The situation is also different from axion photon system when PSET is considered, where there is mixing between all four degrees of freedom (three degrees of freedom of a photon in medium and a single degree of freedom of axion). Thus one has to deal with a 4×4 mixing matrix to study the evolution of the axion-photon system.

where the trace is defined over Dirac matrices. The in-medium propagator in coordinate space is defined, in terms of a vacuum propagator $S_F^v(x', x'') = \langle x'| \frac{1}{\not{\partial} - m + i\epsilon} |x''\rangle$, as

$$S_F(x', x'') = S_F^v(x', x'') - f(p.u)[S_F^v(x', x'') - S_F^{v*}(x', x'')], \tag{2.3}$$

where $f(p.u)$ happens to be the Fermi-distribution function in a frame moving with four velocity u^μ. The curved space-time Fermion propagator S_{Fc} can also be represented as [100]

$$(i\not{D} - m)S_{Fc}(x, x') = \frac{i}{\sqrt{-g}}\delta(x, x'). \tag{2.4}$$

The reason behind using this alternative representation is the inability of expressing the asymptotic vacuum states uniquely curved space-time. Here D stands for the covariant derivative in curved space-time. It can further be written in terms of the scalar green function

$$\left(D^2 + i\sigma^{\mu\nu}F_{\mu\nu} - \frac{R}{4} + m^2\right)G(x, x') = \frac{-i}{\sqrt{-g}}\delta(x, x'). \tag{2.5}$$

Finally, the Fermion propagator in curved space-time in a magnetic field is as follows from Eq. (2.5):

$$S_{Fc}(x, x') = (i\not{D} + m)G(x, x'), \tag{2.6}$$

where R happens to be the Ricci scalar. One can express the above expressions in momentum space by taking the Fourier transform using two point transforms [101,102]:

$$f(x) = \int \frac{dk_{\mu'}}{(2\pi)^d} g^{(-1/2)}(x') \exp(-ik_{\mu'}\sigma^{\mu'}(x, x'))\tilde{f}(k; x'). \tag{2.7}$$

A strong gravitational background is known to introduce some pathological problem in the external electromagnetic field. For example the photon velocity estimated for the Euler-Heisenberg Lagrangian system predicts a superluminal velocity of photon [103]. Therefore the search for particles like dilatonlike particles should be restricted to space where space-time curvature is negligible or flat. Usually the space-time curvature at a distance r from a body of mass M is given by the Kretschmann scalar [104],

$$R = \frac{48M^2}{r^6}. \tag{2.8}$$

For a body with mass close to solar mass or less, the curvature is negligible for most of the regions close to its surface. Moreover our region of interest happens to be close to the light cylinder of the star, hence we have not considered the curvature of the space-time for our estimates.

Coming back to flat space-time picture, $\Pi_{\mu\nu}(k, \mu, T, e\mathrm{B})$ has a contribution coming from (a) a magnetized vacuum, (b) an unmagnetized medium, and (c) a magnetized medium. The contribution from (c) can further be divided in two pieces: those having an algebraic structure that can be written as a polynomial, the first (c_1) that is even eB even μ and the second (c_2) that is a polynomial odd in eB and odd in μ. The even eB odd μ and odd eB even μ parts make vanishing contribution to $\Pi_{\mu\nu}$. In a parameter region where $e\mathrm{B} \ll m^2$ and momentum $k \ll m$ contributions from (a) and (c_1) are suppressed compared to (c_2) [105,106].

Though the contribution from (a) and (c_1) have their special significance in the mixing dynamics for charge symmetric medium ($\mu = 0$), the inclusion of (c_2) makes a paradigm shift in structure of mixing. So to initiate a discussion on effective Lagrangian we start with a discussion on the structure of polarization tensor in an unmagnetized media in momentum space in the following subsection.

A. Polarization tensor: Structure

The linear response to EM excitations of a medium at finite density, temperature, and an external field can studied by evaluating the in-medium photon polarization tensor $\Pi_{\mu\nu}(k)$ by the techniques of quantum statistical field theory [87–90]. This tensor is supposed to have few essential properties [87], e.g., it possesses a symmetry, i.e.,

$$\Pi_{\mu\nu}(k) = \Pi_{\nu\mu}(-k), \tag{2.9}$$

called Bose symmetry. It should obey the Ward identity,

$$k^\mu \Pi_{\mu\nu}(k) = 0, \tag{2.10}$$

ensuring gauge invariance (also called charge conservation law). The requirement of unitarity demands that the polarization tensor should have the property $\Pi_{\mu\nu}(k) = \Pi^*_{\nu\mu}(k)$. This, when combined with Bose symmetry [Eq. (2.9)], yields

$$\Pi_{\mu\nu}(k) = \Pi^*_{\mu\nu}(-k). \tag{2.11}$$

Therefore, in four dimensions, the tensor $\Pi_{\mu\nu}(k)$ now needs to be constructed from the available four-vectors and tensors with the system; namely the medium center-of-mass velocity four-vector u^μ, the photon four-momentum vector $k^\mu = (\omega, \vec{k})$, the metric tensor $g^{\mu\nu}$, Levi-Civita tensor $\epsilon_{\mu\nu\rho\lambda}$, and the Lorentz scalar form factors such that Eqs. (2.9)–(2.10) are satisfied.

The tensorial structure of the in-medium photon self energy tensor in absence of any external field is given by

$$\Pi_{\mu\nu}(k) = \Pi_T R_{\mu\nu} + \Pi_L Q_{\mu\nu}$$
$$\text{where} \begin{cases} R_{\mu\nu} = \tilde{g}_{\mu\nu} - Q_{\mu\nu} \\ \tilde{g}_{\mu\nu} = (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) \\ Q_{\mu\nu} = \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2} \\ \tilde{u}_\mu = \tilde{g}_{\mu\nu} u^\nu \end{cases}. \quad (2.12)$$

The dispersive part of $\Pi_{\mu\nu}(k)$ satisfies (2.11) in a charge ($\mathbf{C}$) symmetric ($\mu_{\bar{f}} = \mu_f$) or asymmetric ($\mu_{\bar{f}} \neq \mu_f$) medium automatically. This in turn dictates the functional form of the form factors Π_L and Π_T on the scalars made out of k^2 (k.u.) etc. The scalar form factor $\Pi_L(k)$, corresponding to the longitudinal degree of freedom, is given by [91]

$$\Pi_L(k) = -\frac{k^2}{|\vec{k}|^2}\Pi_{\mu\nu}(k) u^\mu u^\nu,$$
$$\text{where } u^\mu u^\nu \Pi_{\mu\nu}(k) = \omega_p^2\left(\frac{|\vec{k}|^2}{\omega^2} + 3\frac{|\vec{k}|^4}{\omega^4}\frac{T}{m}\right). \quad (2.13)$$

Similarly the transverse form factor Π_T is given by the expressions

$$\Pi_T(k) = R^{\mu\nu}\Pi_{\mu\nu}(k) \quad \text{and} \quad \mathrm{R}^{\mu\nu}\Pi_{\mu\nu}(k) = \omega_p^2\left(1 + \frac{|\vec{k}|^2}{\omega^2}\frac{T}{m}\right). \quad (2.14)$$

In the expressions above ω_p denotes the plasma frequency. In the classical limit, to leading order in $\frac{T}{m}$, it is given by

$$\omega_p = \sqrt{\frac{4\pi\alpha n_e}{m}\left(1 - \frac{5T}{2m}\right)}, \quad (2.15)$$

where n_e is the number density of electrons. A detailed structure of photon polarization tensor $\Pi_{\mu\nu}(k)$ in vacuum, in medium, and in magnetized medium has been discussed in [92]. For more insight into it, one can consult to this reference.

Recalling the transformation properties of the background EM field $\bar{F}^{\mu\nu}$, and the four-vectors u^μ and k^μ under $\mathbf{C}$ conjugation and $\mathbf{P}$ reversal operations [84], it is easy to figure out that under parity transformation the two orthogonal polarization eigenstates of a photon, described in the notation of [93] by $[(\tilde{\bar{F}}_{\mu\nu} f^{\mu\nu}(k)) \triangleq |\perp\rangle]$ and $[(\bar{F}_{\mu\nu} f^{\mu\nu}(k)) \triangleq |\|\rangle]$, are at odds with each other.

Now in the light of Eq. (2.13) $\Pi_{\mu\nu}(k)$ is $\mathbf{C}$ and $\mathbf{P}$ even, when the respective transformations are considered individually or together, i.e.,

$$\mathbf{P}^{-1}\Pi_{\mu\nu}(k)\mathbf{P} = \Pi_{\mu\nu}(k), \quad (2.16)$$

$$\mathbf{C}^{-1}\Pi_{\mu\nu}(k)\mathbf{C} = \Pi_{\mu\nu}(k), \quad (2.17)$$

$$(\mathbf{CP})^{-1}\Pi_{\mu\nu}(k)(\mathbf{CP}) = \Pi_{\mu\nu}(k). \quad (2.18)$$

Hence, recalling the issue of coupling of degrees of freedom in a unmagnetized material medium that was initiated in the introduction, it is easy to realize that in an unmagnetized medium the parity violating state (i.e., $|\perp\rangle$) would propagate freely but not the parity preserving state ($|\|\rangle$). This one would couple to $\phi(k)$ because of Eq. (2.16).[3] Thereby, in a material medium, the dynamics of the system remains as the same as it was in a magnetized vacuum. The kinematics, however, changes. In a plain material medium, the magnitude of the oscillation probability undergoes modification *vis-á-vis* the same in a magnetized vacuum.

In the presence of an external EM field, the photon polarization tensor in a magnetized media can be expressed in terms of the rank two basis tensors constructed out of the field strength tensor $\bar{F}^{\mu\nu}$, the Levi-Civita tensor $\epsilon_{\mu\nu\lambda\sigma}$ along with the other four-vectors and tensors mentioned before and the form factors those are Lorentz scalars constructed using these four-vectors and tensors. We deliberate on this in the next subsection.

B. Photon polarization tensor in a magnetized medium: All orders in (eB)

Photon polarization tensor in a magnetized media in configuration space would follow from Eq. (2.2), where one needs to use the corresponding expressions for in-medium Fermion propagators in an external magnetic field (2.3). The expression for the same is provided below. The Fermionic propagator in magnetized vacuum is [95]

$$\begin{aligned} &iS_F^v(x', x'') \\ &= -\frac{i\Phi(x', x'')}{(4\pi)^2}\int_0^\infty ds \frac{e\mathrm{B}}{\sin(eBs)} \\ &\quad \times \exp\left[-is\left(-\frac{e\sigma_{\mu\nu}F^{\mu\nu}}{2} + m^2 - i\epsilon\right)\right] \\ &\quad \times \exp\left[-\frac{i}{4}((x' - x'')^\alpha (eF\coth(eFs))_{\alpha\beta}(x' - x'')^\beta)\right] \\ &\quad \times \left[\frac{\gamma^\lambda}{2}(eF\coth(eFs) + eF)_{\lambda\rho}(x' - x'')^\rho + m\mathbf{I}\right]. \end{aligned} \quad (2.19)$$

In Eq. (2.19), above the phase factor, $\Phi(x', x'')$ is given by

$$\Phi(x', x'') = \exp\left[ie\int_{x''}^{x'} dx^\mu \left(A_\mu + \frac{1}{2}F_{\mu\nu}(x' - x'')^\nu\right)\right], \quad (2.20)$$

[3] The CP invariant background medium cannot compensate for the parity odd property of $|\perp\rangle$ so that it can couple to $\phi(k)$.

The symbol F in Eq. (2.20) stands for the field strength tensor $F^{\mu\nu}$ (suitably contracted when they appear in combination with functions of the same or Dirac gamma matrices), and quantity $\mathbf{I}$ in Eq. (2.19) stands for a 4×4 unit matrix. Finally the in-medium propagator in a magnetized medium can be obtained by using the propagator (2.19) for $S_F^v(x', x'')$ in Eq. (2.3).

C. Contribution to photon self-energy from a magnetized medium: All odd orders in (eB)

In a magnetized medium, magnetic field (eB) dependent extra contributions appear in the expression for the photon polarization tensor. They are of two types: One of them is even the other one is odd in powers of the field strength eB. The one odd in eB turns out to be also odd in μ (the chemical potential), so that this term remains even under the operation of charge conjugation $\mathbf{C}$. When apart, this term satisfies the conditions given by Eqs. (2.9)–(2.11). This odd eB odd μ contribution to the polarization tensor violates parity. The exact expression of the same is given by

$$\begin{aligned}\Pi^p_{\mu\nu}(k,\mu,T,e\mathrm{B}) \\ = 4ie^2\varepsilon_{\mu\nu\alpha_\parallel\beta}k^\beta \int \frac{d^4p}{(2\pi)^4}\eta_-(p)\int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \\ \times \int_0^\infty ds' e^{\Phi(p',s')}\Big[p^{\tilde{\alpha}_\parallel}\tan e\mathrm{B}s + p'^{\tilde{\alpha}_\parallel}\tan e\mathrm{B}s' \\ - \frac{\tan e\mathrm{B}s\tan e\mathrm{B}s'}{\tan e\mathrm{B}(s+s')}(p+p')^{\tilde{\alpha}_\parallel}\Big]. \qquad (2.21)\end{aligned}$$

Here e is the coupling constant for the U(1) gauge theory, $\eta_-(p) = \eta_F(p) - \eta_F(-p)$ [with $\eta_F(p)$ being the statistical factor involving Fermions and their antiparticles [84]], and B, as mentioned before, is the background magnetic field. The functions $\Phi(p,s)$ and $\Phi(p',s)$ are the contributions from Schwinger propagator [96], having loop momentum p and external momentum k, Fermion mass m and parametric integrating variable s and s'. The symbol p' on the right-hand side stands for $(p+k)$, and the loop momentum four-vector $p^{\tilde{\alpha}_\parallel}$, appearing in (2.21), happens to be components of momentum p, which take only the values 0 and 3 (called the $\parallel$ components) but with a difference. When $\alpha_\parallel$ and $\tilde{\alpha}_\parallel$ appear together in any term and are summed up, then, for $\alpha_\parallel$ equal to zero, $\tilde{\alpha}_\parallel$ would take the value of 3 and for $\alpha_\parallel$ equal to 3, $\tilde{\alpha}_\parallel$ would take the value 0. In the same equation, the symbol $\varepsilon_{\mu\nu\alpha_\parallel\beta}$ is the completely antisymmetric Levi-Civita tensor that takes the values 1 and -1, for even and odd permutations of the indices and vanishes when any two indices are the same:

$$\Phi(p,s) \equiv is\left(p_\parallel^2 - \frac{\tan(e\mathrm{B}s)}{e\mathrm{B}s}p_\perp^2 - m^2\right) - \epsilon|s|, \qquad (2.22)$$

$$\Phi(p',s') \equiv is'\left(p'^2_\parallel - \frac{\tan(e\mathrm{B}s')}{e\mathrm{B}s'}p'^2_\perp - m^2\right) - \epsilon|s'|. \qquad (2.23)$$

This expression is exact to all odd orders in eB; however, performing the integrals and arriving at compact form from this expression is very difficult.[4]

1. *In-medium contribution to photon self energy to O (eB)*

The integral in Eq. (2.21) describing parity violating PSET is a little difficult to evaluate exactly using analytical techniques. But a perturbative evaluation, to leading order in eB, is possible [84]. The perturbative expression can be expressed in terms of a scalar form factor $\Pi^p(k,\mu,T,e\mathrm{B})$ and a projection operator $P_{\mu\nu}$, so $\Pi^p_{\mu\nu}(k,\mu,T,e\mathrm{B})$ to order (eB) can be expressed in the following form [84,85]:

$$\Pi^p_{\mu\nu}(k) = \Pi^p(k)\left(i\epsilon_{\mu\nu\alpha_\parallel\beta}\frac{k^\beta}{|K|}u^{\tilde{\alpha}}\right) = \Pi^p(k)P_{\mu\nu}, \qquad (2.24)$$

where

$$P_{\mu\nu} = i\epsilon_{\mu\nu\alpha\beta_\parallel}\frac{k^\alpha}{|K|}u^{\tilde{\beta}_\parallel}. \qquad (2.25)$$

The tensor $P_{\mu\nu}$ given by (2.25) is Hermitian but it is odd under parity transformation. The superscripts with $\parallel$ means that they can take only values between 0 and 3. Furthermore, in our notation,

$$|K| = \left(\sum_{i=1}^{3}k_i^2\right)^{1/2}. \qquad (2.26)$$

The limit $k\to 0$ in Eq. (2.24) should be taken in such a way that

$$\lim_{|k|\to 0}\left(\frac{k^i}{|k|}\right) \to 1. \qquad (2.27)$$

The scalar form factor $\Pi^p(k)$ appearing in Eq. (2.24) is given by

$$\Pi^p(k) = \frac{\omega\omega_B\omega_p^2}{\omega^2 - \omega_B^2}, \quad \text{where } \omega_B = \frac{\mathrm{eB}}{m} \qquad (2.28)$$

is the gyration frequency.

[4]One can perform the integration with the following substitution: Express the four-vector P^μ as $p^\mu = \alpha_{(1)}u^\mu + \alpha_{(2)}n_2\cdot(k.F)^\mu + \alpha_{(3)}n_3\cdot B^\mu + \alpha_{(4)}n_3\cdot q^\mu$, when $q^\mu = \epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}u_\nu$ and n_is are normalization constants, so as to make the basis vectors orthonormal.

III. EFFECTIVE LAGRANGIAN WITH MAGNETIZED MEDIUM EFFECTS

To summarize the observations of the last section, we note that the effects of a medium to the propagation of excitations of interest can be considered by evaluating the polarization tensor $\Pi_{\mu\nu}(k)$, following the methods of finite temperature quantum field theory [85]. This takes into account the correction arising out of temperature and density effects due to the interactions amongst the particles fields that constitute the media.

The action in momentum space, as the quantum corrections due to ambient medium and an external magnetic field to $\mathcal{O}(e\text{B})$ are taken into account, can be obtained upon taking the Fourier transform of Eq. (2.1). The same turns out to be

$$S = \int d^4k \left[\frac{1}{2}A^\nu(-k)(-k^2\tilde{g}_{\mu\nu} + \Pi_{\mu\nu}(k,\mu,T) + \Pi^p_{\mu\nu}(k,\mu,T,e\text{B}))A^\mu(k) + ig_{\phi\gamma\gamma}\phi(-k)\bar{F}_{\mu\nu}k^\mu A^\nu(k) + \frac{1}{2}\phi(-k)[k^2-m^2]\phi(k)\right]. \tag{3.1}$$

Here $\Pi_{\mu\nu}(k,\mu,T)$ is an in-medium polarization tensor and $\Pi^p_{\mu\nu}(k,\mu,T,e\text{B})$ is the correction due to magnetized medium effects PSET as explained before. For the sake of compactness we would be denoting $\Pi_{\mu\nu}(k,\mu,T)$ as $\Pi_{\mu\nu}(k)$ and $\Pi^p_{\mu\nu}(k,\mu,T,e\text{B})$ as $\Pi^p_{\mu\nu}(k)$ in subsequent sections. We can find the equations of motion in momentum space by standard variational principle. The equation of motion for photons is

$$[-k^2\tilde{g}_{\alpha\nu} + \Pi_{\alpha\nu}(k) + \Pi^p_{\alpha\nu}(k)]A^\nu(k) = -ig_{\phi\gamma\gamma}\bar{F}_{\mu\alpha}k^\mu\phi(k). \tag{3.2}$$

It can be simplified further in Lorentz gauge to

$$k^2A_\alpha(k) - \Pi_{\alpha\nu}(k)A^\nu(k) + \Pi^p_{\alpha\nu}(k)A^\nu(k) = ig_{\phi\gamma\gamma}\bar{F}_{\mu\alpha}k^\mu\phi(k), \tag{3.3}$$

and other equation of motion for $\phi(k)$ is given by

$$(k^2-m^2)\phi(k) = ig_{\phi\gamma\gamma}\bar{F}_{\mu\alpha}k^\mu A^\alpha(k). \tag{3.4}$$

A. Expanding the gauge potential $A_\mu(k)$ in orthogonal basis vectors

In order to capture the dynamics of the available degrees of freedom in a medium, in this subsection, we need to expand the four-vector potential $A_\mu(k)$ in terms of the available four-vectors at our disposal. The available four-vectors as was noted in Sec. II A are k^μ, the four-momentum of the particles and $u^\mu \equiv (1,\mathbf{0})$ the center-of-mass four velocity of the medium and $\epsilon^{\mu\nu\lambda\rho}$. Using these two and the constant external field strength tensor $\bar{F}^{\mu\nu}$ (where the only nonzero component being $\bar{F}^{12} \neq 0$), two other vectors $b^{(1)\mu}$ and $b^{(2)\mu}$ can be constructed. They are given by

$$b^{(1)\nu} = k_\mu\bar{F}^{\mu\nu} \tag{3.5}$$

and

$$b^{(2)\nu} = k_\mu\tilde{\bar{F}}^{\mu\nu}, \tag{3.6}$$

where $\tilde{\bar{F}}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\bar{F}_{\alpha\beta}$. The four-vector defining the longitudinal degree of freedom is usually given by

$$\tilde{u}^\nu = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)u_\mu. \tag{3.7}$$

Furthermore, the vector that is orthogonal to the vectors given in Eqs. (3.5) and (3.7) is

$$I^\nu = \left(b^{(2)\nu} - \frac{(\tilde{u}^\mu b^{(2)}_\mu)}{\tilde{u}^2}\tilde{u}^\nu\right). \tag{3.8}$$

It should be noted that since the vectors given by Eqs. (3.5)–(3.7) are spacelike, they have to be normalized without compromising the Hermitian character of the gauge fields. The explicit form of the normalization constants are given by

$$N_1 = \frac{1}{\sqrt{-b^{(1)}_\mu b^{(1)^\mu}}} = \frac{1}{K_\perp \text{B}}, \tag{3.9}$$

$$N_2 = \frac{1}{\sqrt{-I_\mu I^\mu}} = \frac{K}{\omega K_\perp \text{B}}, \quad \text{and lastly} \tag{3.10}$$

$$N_L = \frac{1}{\sqrt{-\tilde{u}_\mu\tilde{u}^\mu}} = \frac{\sqrt{k^\mu k_\mu}}{|\vec{k}|}. \tag{3.11}$$

With these definitions, the gauge potential $A^\mu(k)$ can be expressed in terms of these basis vectors and associated form factors as [93,107]

$$A^\nu(k) = N_1A_\parallel(k)b^{(1)\nu} + N_2A_\perp(k)I^\nu + N_LA_L(k)\tilde{u}^\nu + A_{gf}(k)\frac{k^\nu}{k^2}. \tag{3.12}$$

In order to get rid of the redundant degree of freedom of the gauge field, we choose $A_{gf}(k) = 0$. The form factors $A_\parallel(k)$, $A_\perp(k)$, and $A_L(k)$ in Eq. (3.12) are Lorentz scalars made out of linear or nonlinear combinations of the tensor and the four-vectors discussed above, i.e., $\omega = k.u.$, $|\vec{k}| = \sqrt{\omega^2 - k^2}$, $k_\alpha\tilde{\bar{F}}^{\alpha\beta}u_\beta$, $k_\alpha\bar{F}^{\alpha\beta}\bar{F}_{\beta\sigma}k^\sigma$ etc. It is important to note that the magnetic field $\mathcal{B}_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}u^\nu \bar{F}^{\lambda\rho}$ is actually orthogonal to $b^{(1)}_\nu$, that is to say $\mathcal{B}^\nu \cdot b^{(1)}_\nu = 0$, implying that the direction of the polarization vector $b^{(1)}_\nu$ is orthogonal to the external magnetic field. Similar consideration would show that the direction of the polarization vector I_ν is along the external magnetic field. However, to maintain consistency with our previous work [93], we have denoted the associated form factors for the corresponding polarization directions as $A_\parallel(k)$, $A_\perp(k)$.

In order to gain an insight of the equations of motions, it is instructive to know the transformation properties of the vectors, tensors, and the form factors used for Eq. (3.12), under **C**, **P**, and **T** transformations. We briefly discuss them next.

B. Properties of the basis vectors under C, P, and T transformations

The nonzero components of the four vectors under deliberation are $b^{(1)}_\mu = (0, b^{(1)}_1, b^{(1)}_2, 0)$, $b^{(2)}_\mu = (b^{(2)}_0, 0, 0, b^{(2)}_3)$, and $I_\mu = (I_0, I_i)$ when $i = 1, 2, 3$. In order to find out the transformation properties of the same, we need to first identify the **C**, **P**, and **T** transformations of the four-vectors k^μ, u^μ and the tensor $\bar{F}^{\mu\nu}$. For the sake of brevity we do not show the momentum dependence of the form factors here.

The transformation properties of time (k^0) and space components (k^i, for $i = 1, 2, 3$) of wave propagation vector $k^\mu = (k^0, k^i)$ under **C**, **P**, and **T** transformations are given by

$$\mathbf{C}k^0\mathbf{C}^{-1} = k^0, \qquad \mathbf{C}k^i\mathbf{C}^{-1} = +k^i, \tag{3.13}$$

$$\mathbf{P}k^0\mathbf{P}^{-1} = k^0, \qquad \mathbf{P}k^i\mathbf{P}^{-1} = -k^i, \tag{3.14}$$

$$\mathbf{T}k^0\mathbf{T}^{-1} = k^0, \qquad \mathbf{T}k^i\mathbf{T}^{-1} = -k^i. \tag{3.15}$$

The center-of-mass four velocity of the medium, defined as $u^\mu = \frac{dx^\mu}{d\tau}$ (when $d\tau$ is the differential proper time interval), has the following transformation properties under time reversal, parity, and charge conjugation transformation:

$$\mathbf{C}u^0\mathbf{C}^{-1} = -u^0, \qquad \mathbf{C}u^i\mathbf{C}^{-1} = -u^i, \tag{3.16}$$

$$\mathbf{P}u^0\mathbf{P}^{-1} = +u^0, \qquad \mathbf{P}u^i\mathbf{P}^{-1} = -u^i, \tag{3.17}$$

$$\mathbf{T}u^0\mathbf{T}^{-1} = -u^0, \qquad \mathbf{T}u^i\mathbf{T}^{-1} = +u^i. \tag{3.18}$$

The first property (3.16) follows from the observation that the statistical part of the thermal propagator in real time thermal quantum field theory should remain invariant under the operation of charge conjugation, as explained in [84]. The same (u^μ) in the rest frame of the medium is given by $u^\mu = (1, 0, 0, 0)$; for the remaining part of this paper we shall assume this to be true.

The background field strength tensor $\bar{F}^{ij}$, transforms in the following way under **CPT** separately as

$$\mathbf{C}\bar{F}^{0i}\mathbf{C}^{-1} = -\bar{F}^{0i}, \qquad \mathbf{C}\bar{F}^{ij}\mathbf{C}^{-1} = -\bar{F}^{ij}, \tag{3.19}$$

$$\mathbf{P}\bar{F}^{0i}\mathbf{P}^{-1} = -\bar{F}^{0i}, \qquad \mathbf{P}\bar{F}^{ij}\mathbf{P}^{-1} = +\bar{F}^{ij}, \tag{3.20}$$

$$\mathbf{T}\bar{F}^{0i}\mathbf{T}^{-1} = -\bar{F}^{0i}, \qquad \mathbf{T}\bar{F}^{ij}\mathbf{T}^{-1} = -\bar{F}^{ij}. \tag{3.21}$$

With this information we can look into the **C**, **P** and **T** transformation properties of the basis vectors. To begin with, we start with vector $b^{(1)}_\mu$. It has only two nonzero components, and their transformation properties are

$$\mathbf{C}b^{(1)0}\mathbf{C}^{-1} = -b^{(1)0}, \qquad \mathbf{C}b^{(1)i}\mathbf{C}^{-1} = -b^{(1)i}, \tag{3.22}$$

$$\mathbf{P}b^{(1)0}\mathbf{P}^{-1} = +b^{(1)0}, \qquad \mathbf{P}b^{(1)i}\mathbf{P}^{-1} = -b^{(1)i}, \tag{3.23}$$

$$\mathbf{T}b^{(1)0}\mathbf{T}^{-1} = +b^{(1)0}, \qquad \mathbf{T}b^{(1)i}\mathbf{T}^{-1} = +b^{(1)i}. \tag{3.24}$$

Similarly, one can write the transformation properties of $b^{(2)}_\mu$ that has one timelike and one spacelike nonzero component. The timelike component of $b^{(2)}_\mu$ under **C**, **P**, and **T** operations transform as

$$\mathbf{C}b^{(2)0}\mathbf{C}^{-1} = -b^{(2)0}, \qquad \mathbf{C}b^{(2)i}\mathbf{C}^{-1} = -b^{(2)i}, \tag{3.25}$$

$$\mathbf{P}b^{(2)0}\mathbf{P}^{-1} = +b^{(2)0}, \qquad \mathbf{P}b^{(2)i}\mathbf{P}^{-1} = -b^{(2)i}, \tag{3.26}$$

$$\mathbf{T}b^{(2)0}\mathbf{T}^{-1} = -b^{(2)0}, \qquad \mathbf{T}b^{(2)i}\mathbf{T}^{-1} = -b^{(2)i}. \tag{3.27}$$

Now using above equations, it is easy to establish that

$$\mathbf{P}(\tilde{u} \cdot b^{(2)})\mathbf{P}^{-1} = (\tilde{u} \cdot b^{(2)}). \tag{3.28}$$

Recalling, the four-vector I^ν to be given by

$$I^\nu = \left(b^{(2)\nu} - \frac{(\tilde{u}^\mu b^{(2)}_\mu)}{\tilde{u}^2}\tilde{u}^\nu \right); \tag{3.29}$$

the timelike and spacelike components of the same are found to be given by

$$I^0 = \left(b^{(2)0} - \frac{(\tilde{u}^\mu b^{(2)}_\mu)}{\tilde{u}^2}\tilde{u}^0 \right) \quad \text{and}$$
$$I^i = \left(b^{(2)i} - \frac{(\tilde{u}^\mu b^{(2)}_\mu)}{\tilde{u}^2}\tilde{u}^i \right) \quad (\text{for } i = 1, 2, 3). \tag{3.30}$$

Using the **C**, **P**, and **T** transformation properties of the individual components of I^μ, I^0, and I^i would transform under the same as

TABLE I. Transformation properties for the vectors, tensors, and the EM form factors used to construct all the vectors, tensors, and form factors used in this work to expand $A^{\nu}(k)$, under **C**, **P**, and **T**.

	k_μ	u_μ	$\tilde{u}_\mu$	$b^{(1)}_\mu$	$b^{(2)}_\mu$	I_μ	$A_\parallel$	$A_\perp$	A_L	i	$\epsilon_{\mu\nu\rho\sigma}$	$\bar{F}_{\mu\nu}$
C	$+k_\mu$	$-u_\mu$	$-\tilde{u}_\mu$	$-b^{(1)}_\mu$	$-b^{(2)}_\mu$	$-I_\mu$	$+A_\parallel$	$+A_\perp$	$+A_L$	$+i$	$+\epsilon_{\mu\nu\rho\sigma}$	$-\bar{F}_{\mu\nu}$
P	$+k^\mu$	$+u^\mu$	$+\tilde{u}^\mu$	$+b^{(1)\mu}$	$+b^{(2)\mu}$	$+I^\mu$	$+A_\parallel$	$+A_\perp$	$+A_L$	$+i$	$-\epsilon_{\mu\nu\rho\sigma}$	$+\bar{F}^{\mu\nu}$
T	$+k^\mu$	$-u^\mu$	$-\tilde{u}^\mu$	$-b^{(1)\mu}$	$-b^{(2)\mu}$	$-I^\mu$	$-A_\parallel$	$-A_\perp$	$-A_L$	$-i$	$-\epsilon_{\mu\nu\rho\sigma}$	$-\bar{F}^{\mu\nu}$

$$\mathbf{C}I^0\mathbf{C}^{-1} = -I^0, \qquad \mathbf{C}I^i\mathbf{C}^{-1} = -I^i, \tag{3.31}$$

$$\mathbf{P}I^0\mathbf{P}^{-1} = +I^0, \qquad \mathbf{P}I^i\mathbf{P}^{-1} = -I^i, \tag{3.32}$$

$$\mathbf{T}I^0\mathbf{T}^{-1} = -I^0, \qquad \mathbf{T}I^i\mathbf{T}^{-1} = +I^i. \tag{3.33}$$

Since the timelike components of the gauge field under time reversal transformation, remains the same, i.e.,

$$\mathbf{T}A^0\mathbf{T}^{-1} = A^0, \tag{3.34}$$

therefore when the time reversal transformation is imposed on A^0, following the definition of A^0 one should get

$$\begin{aligned}\mathbf{T}^{-1}A^0\mathbf{T}^{-1} &= \mathbf{T}(N_1 I^0 A_\parallel + N_L \tilde{u}^0 A_L)\mathbf{T}^{-1}\\ &= (N_1 I^0 A_\parallel + N_L \tilde{u}^0 A_L);\end{aligned} \tag{3.35}$$

that is, the right-hand side remains invariant. Now under time reversal transformation I^0 and $\tilde{u}^0$, picks up a $-ve$ sign. So to maintain overall neutrality, $A_\parallel$ and A_L must change sign under **T** transformation. Hence,

$$(\mathbf{T}A_\parallel\mathbf{T}^{-1}) = -A_\parallel \quad \text{and} \quad (\mathbf{T}A_L\mathbf{T}^{-1}) = -A_L. \tag{3.36}$$

We recall that the spacelike component of A^μ can be expressed as

$$A_i = N_1 b^{(1)}_i A_\parallel + N_2 I_i A_\perp + N_L \tilde{u}_i A_L. \tag{3.37}$$

The same, that is A^i, under **T** transformation pickup a $-ve$ sign, i.e.,

$$(\mathbf{T}A_i\mathbf{T}^{-1}) = -A_i. \tag{3.38}$$

It then follows that the right-hand side of Eq. (3.37) should respect the transformation Eq. (3.38). Since $b^{(1)}_i$, I_i, and $\tilde{u}_i$ remain invariant under time reversal symmetry transformation, therefore $A_\parallel$, $A_\perp$, and A_L should change sign under time reversal transformation symmetry. The transformation rules of the form factors can now be summarized as

$$\mathbf{C}A_\parallel\mathbf{C}^{-1} = +A_\parallel, \quad \mathbf{C}A_\perp\mathbf{C}^{-1} = +A_\perp, \quad \mathbf{C}A_L\mathbf{C}^{-1} = +A_L, \tag{3.39}$$

$$\mathbf{P}A_\parallel\mathbf{P}^{-1} = +A_\parallel, \quad \mathbf{P}A_\perp\mathbf{P}^{-1} = +A_\perp, \quad \mathbf{P}A_L\mathbf{P}^{-1} = +A_L, \tag{3.40}$$

$$\mathbf{T}A_\parallel\mathbf{T}^{-1} = -A_\parallel, \quad \mathbf{T}A_\perp\mathbf{T}^{-1} = -A_\perp, \quad \mathbf{T}A_L\mathbf{T}^{-1} = -A_L. \tag{3.41}$$

So the **C**, **P**, and **T** transformations properties of the vectors, tensors and the form factors can further be put in tabular form (see Table [I]).

It can be shown that each term appearing in the equations of motion of this article when subjected to these transformations, transforms identically. We will demonstrate the same later.

C. Dynamics of the degrees of freedom

In this section, we discuss the dynamics of the independent degrees of freedom of the system. Recalling the fact that in a medium photon acquire an extra degree of freedom in addition to the two transverse degrees of freedom, therefore for the photon scalar interacting system, there should be four degrees of freedom. Moreover, out of the three degrees of freedom of the photon, the longitudinal degree of freedom has spin zero, and the other two would be having spin one and minus one, respectively. Therefore though naively one may expect that, the longitudinal mode of photon would couple with the scalar degree of freedom, because they both have same spin assignments; however, we would demonstrate in this subsection, by analyzing the equations of motion, that this naive expectation does not hold good.

Now we begin with Eq. (3.3), and substitute the expression for the gauge potential from Eq. (3.12) in the same. The resulting equation [considering the Faraday contribution to be $\Pi^p_{\mu\nu}(k) = -\Pi_p(k)P_{\mu\nu}$, when the projection operator is defined as $P_{\mu\nu} = i\epsilon_{\mu\nu\beta\delta_\parallel}\frac{k^\beta}{|k|}u^{\tilde{\delta}_\parallel}$][5] turns out to be

$$\begin{aligned}&(k^2 - \Pi_T(k))[A_\parallel(k)N_1 b^{(1)}_\alpha + A_\perp(k)N_2 I_\alpha + A_L(k)N_L\tilde{u}_\alpha]\\ &\quad + \Pi_T(k)A_L(k)N_L\tilde{u}_\alpha - \Pi_L(k)A_L(k)N_L\tilde{u}_\alpha\\ &\quad - i\Pi^p(k)\epsilon_{\alpha\nu\beta\delta}\frac{k^\beta}{|k|}u^{\tilde{\delta}_\parallel}[A_\parallel(k)N_1 b^{(1)\nu} + A_\perp(k)N_2 I^\nu\\ &\quad + A_L(k)N_L\tilde{u}^\nu] = ig_{\phi\gamma\gamma}b^{(1)}_\alpha\phi(k),\end{aligned} \tag{3.42}$$

[5]Where $\tilde{\delta}_\parallel$ can takes the value 0 or 3. When $\delta_\parallel = 0$ then $\tilde{\delta}_\parallel = 3$ and vice versa.

Next as we multiply Eq. (3.42) by the normalized basis vectors, we get the equations of motion for different components of the form factors. For example, if we multiply Eq. (3.42) by $b^{(1)\alpha}$, then we find the following equation:

$$\begin{aligned}(k^2 - \Pi_T(k))A_\parallel(k)N_1 b_\alpha^{(1)} b^{(1)\alpha} \\ - i\Pi^p(k)N_2\left[\epsilon_{\alpha\nu\beta\delta_\parallel}\frac{k^\beta}{|k|}u^{\tilde{\delta}_\parallel}b^{(1)\alpha}I^\nu\right]A_\perp(k) \\ = ig_{\phi\gamma\gamma}b_\alpha^{(1)}b^{(1)\alpha}\phi(k).\end{aligned} \tag{3.43}$$

Using the expression of the normalization constants, the same becomes

$$\begin{aligned}(k^2 - \Pi_T(k))A_\parallel(k) + i\Pi^p(k)N_1N_2 \\ \times\left[\epsilon_{\alpha\nu\beta\delta_\parallel}\frac{k^\beta}{|k|}u^{\tilde{\delta}_\parallel}b^{(1)\alpha}I^\nu\right]A_\perp(k) = \frac{ig_{\phi\gamma\gamma}\phi(k)}{N_1}.\end{aligned} \tag{3.44}$$

Similarly, multiplying Eq. (3.42) by I^ν and $\tilde{u}^\alpha$, respectively, we get the following two equations:

$$\begin{aligned}(k^2 - \Pi_T(k))A_\perp(k) - i\Pi^p(k)N_1N_2 \\ \times\left[\epsilon_{\alpha\nu\beta\delta_\parallel}\frac{k^\beta}{|k|}u^{\tilde{\delta}_\parallel}b^{(1)\alpha}I^\nu\right]A_\parallel(k) = 0,\end{aligned} \tag{3.45}$$

$$(k^2 - \Pi_L(k))A_L(k) = 0, \tag{3.46}$$

those describe the dynamics of the three degrees of freedom of the photon. Last, the equation of motion for the scalar field turns out to be

$$(k^2 - m^2)\phi(k) = -\frac{ig_{\phi\gamma\gamma}A_\parallel(k)}{N_1}. \tag{3.47}$$

Equations (3.44)–(3.47) describe the dynamics of the $\gamma\phi$ interaction in a magnetized medium. The correctness of the above equations of motion can be established by performing **PT** transformation on these equations.

To demonstrate it, let us choose Eq. (3.45) and operate (**PT**) from left and $(\mathbf{PT})^{-1}$ from right sides of the equation. Following the transformation rules of table [1], under **PT**, the first term of the equation will pick up a negative sign due to presence of $A_\perp(k)$, which is odd under the same. In the second term, the factors $i, u^{\tilde{\delta}_\parallel}, b^{(1)\alpha}, I^\nu$, and $A_\parallel(k)$ are **PT** odd and rest are **PT** even as shown below:

$$\begin{aligned}(\mathbf{PT})A_\parallel(\mathbf{PT})^{-1} = -A_\parallel, \quad (\mathbf{PT})A_\perp(\mathbf{PT})^{-1} = -A_\perp, \\ (\mathbf{PT})i(\mathbf{PT})^{-1} = -i, \quad (\mathbf{PT})k^\beta(\mathbf{PT})^{-1} = +k^\beta, \\ (\mathbf{PT})u^{\tilde{\delta}_\parallel}(\mathbf{PT})^{-1} = -u^{\tilde{\delta}_\parallel}, \quad (\mathbf{PT})b^{(1)\mu}(\mathbf{PT})^{-1} = -b^{(1)\mu}, \\ (\mathbf{PT})I^\nu(\mathbf{PT})^{-1} = -I^\nu, \quad (\mathbf{PT})\epsilon_{\mu\nu\delta\beta}(\mathbf{PT})^{-1} = +\epsilon_{\mu\nu\delta\beta}.\end{aligned} \tag{3.48}$$

The transformation properties of form factor $\Pi^p(k)$ under **C**, **P** and **T** can be figured out from the following expression:

$$\Pi^p(k) = \frac{(k.u.)(e\mathrm{B}_\parallel/m_e)}{\omega^2 - (e\mathrm{B}_\parallel/m_e)^2}\left(\frac{n_e}{m_e}\right). \tag{3.49}$$

It can be seen from Eq. (3.49) that $\Pi^p(k)$ is invariant under the **PT** transformation, i.e.,

$$(\mathbf{PT})\Pi^p(k)(\mathbf{PT})^{-1} = \Pi^p(k); \tag{3.50}$$

so collectively, the second term of Eq. (3.45) will also pick up a negative sign. Therefore this equation remains invariant under **PT** transformation. The same can be established for other equations also using similar logic.

The **PT** symmetric part of PSET, when included in the effective Lagrangian, can in principle compensate for the **P** violation of $|A_\perp\rangle$, when both appear as a product, as they do in the equations of motion. Thus resulting product of the two becomes **P** even, making mixing between $\phi(k)$ and $|\perp\rangle$ possible.

Now looking at the problem of mixing, we note that the initial mixing between $\phi(k)$ and $|\|\rangle$, due to ϕFF coupling, remains intact, but the introduction of the PSET term causes further mixing between the two orthogonal transverse states, i.e., $|\|\rangle$ and $|\perp\rangle$. Lastly the combination of the product of PSET term and $|\perp\rangle$, as noted in the last paragraph, causes mixing between $|\perp>$ and $\phi(k)$. Thus the system reduces to a system of three mutually coupled degrees of freedom, which evolves following their respective equations of motion.

It may not be quite out of place, to mention here that, instead of $\phi(k)F^{\mu\nu}F^{\mu\nu}$, if the $a(k)\tilde{F}^{\mu\nu}F^{\mu\nu}$ interaction is considered in a magnetized vacuum or an unmagnetized medium, then the role of scalar $\phi(k)$ gets interchanged with that of the pseudoscalar [97]; consequently, the role of $|\|\rangle$ would get interchanged with $|\perp\rangle$ so that the symmetry remains intact. So the prediction from one can be obtained from the prediction of the other.

This simple interrelation between the two, as noted already, however gets modified as one includes the effect of PSET to axion photon or scalar photon systems.

Introducing, $F = \Pi^p(k)N_1N_2[\epsilon_{\alpha\nu\beta\delta}\frac{k^\beta}{|k|}u^{\tilde{\delta}_\parallel}b^{(1)\alpha}I^\nu]$ and $G = \frac{g_{\phi\gamma\gamma}A_\parallel(k)}{N_1}$ for the sake of brevity, the coupled set of equations (of motion) can be presented in matrix form as

$$\begin{bmatrix} (k^2-\Pi_T(k)) & iF & 0 & -iG \\ -iF & (k^2-\Pi_T(k)) & 0 & 0 \\ 0 & 0 & (k^2-\Pi_L(k)) & 0 \\ iG & 0 & 0 & (k^2-m_\phi^2) \end{bmatrix} \begin{bmatrix} A_\parallel(k) \\ A_\perp(k) \\ A_L(k) \\ \phi(k) \end{bmatrix} = 0. \tag{3.51}$$

The longitudinal degree of freedom of photon $[A_L(k)]$ as can be seen, has decoupled from the rest of the degrees of freedom.

D. Background dependent mixing pattern

In this section we try to understand the background's influence on the mixing dynamics of the transverse and longitudinal degrees of freedom of photon with scalar or pseudoscalar as the interaction term in the Lagrangian changes from $g_{\phi\gamma\gamma}\phi FF$ for scalar-photon system to $g_{a\gamma\gamma}a\tilde{F}F$ for pseudoscalar-photon system. As was noted in the introduction, that dynamics of the degrees of freedom of photon in terms of the form factors $A_\parallel(k)$ and $A_\perp(k)$ in a magnetized vacuum with $g_{\phi\gamma\gamma}\phi FF$ interaction can be anticipated from the evolution of these two form factors in pseudoscalar photon system $(a\gamma\gamma)$, with the identification of the role of $A_\parallel(k)$ with $A_\perp(k)$ and $A_\perp$ with $A_\parallel$ when the interaction Lagrangian is $g_{a\gamma\gamma}a\tilde{F}F$.

So one can conclude that these systems seem to have a symmetry that remains invariant under the exchange of the scalar field with pseudoscalar and parallel polarized state $A_\parallel$ with the perpendicular polarized state $A_\perp$ of the photon. Even in material medium, this symmetry behavior remains the same.

In presence of a magnetized medium, however, contributions from the PSET $\Pi_{\mu\nu}(k,\mu,T,e\mathrm{B})$, which is odd in powers of eB, lifts this apparent degeneracy. Like the discrete symmetries of nature, like charge **C**, **T**, and **P**, transformations play a major role in removing this apparent degeneracy. We would like to come back to this issue in future publications.

Since the longitudinal degree of freedom $A_L(k)$ (with dimension-five scalar-photon interaction) in a magnetized medium is decoupled from the rest of the degrees of freedom, we will not be considering it any further. Therefore, the resulting matrix Eq. (3.51) can be cast in the following form:

$$\left[k^2\mathbf{I} - \begin{pmatrix} \Pi_T(k) & iF & -iG \\ -iF & \Pi_T(k) & 0 \\ iG & 0 & m_\phi^2 \end{pmatrix}\right] \begin{bmatrix} A_\parallel(k) \\ A_\perp(k) \\ \phi(k) \end{bmatrix} = 0. \tag{3.52}$$

Here **I** is 3×3 identity matrix. For the sake of compactness, we would further like to denote the matrix, inside bracket, on the left-hand side in Eq. (3.52) as

$$\begin{pmatrix} \Pi_T(k) & iF & -iG \\ -iF & \Pi_T(k) & 0 \\ iG & 0 & m_\phi^2 \end{pmatrix} = \mathbf{M}, \tag{3.53}$$

and would like to discuss about the elements of the matrix **M** for the physical situation under consideration in next few lines.

In the long wavelength limit, one can take $\Pi_T = \omega_p^2$, where ω_p is the plasma frequency. With this identification, the other two parameters F and G are given by $F = \frac{\omega_p^2 e\mathrm{B}\cos\bar{\theta}}{\omega m_e}$ and $G = -g_{\phi\gamma\gamma}\mathrm{B}\sin\bar{\theta}\omega$. The angle $\bar{\theta}$ here corresponds to the angle between the photon propagation vector $\vec{k}$ and the magnetic field B. Lastly the parameter m_e is the electron mass. Now identifying $\mathrm{B}\cos\bar{\theta} = B_\parallel$ and $\mathrm{B}\sin\bar{\theta} = B_\perp$, the equation of motion can further be written as

$$\left[k^2\mathbf{I} - \begin{pmatrix} \omega_p^2 & i\frac{\omega_p^2 eB_\parallel}{(\omega m_e)} & -ig_{\phi\gamma\gamma}B_\perp\omega \\ -i\frac{\omega_p^2 eB_\parallel}{(\omega m_e)} & \omega_p^2 & 0 \\ ig_{\phi\gamma\gamma}B_\perp\omega & 0 & m_\phi^2 \end{pmatrix}\right] \times \begin{bmatrix} A_\parallel(k) \\ A_\perp(k) \\ \phi(k) \end{bmatrix} = 0. \tag{3.54}$$

Although, in this work, we will assume angle $\bar{\theta}$, to be $\pi/4$, however depending on spatial geometry of the emission region this angle can vary between zero to 2π.

When $\bar{\theta} = n\pi$, for any integer n, the dilatons decouples from the system and the resulting equation describe electrodynamics of magnetized medium. This can be used to study the propagation of EM wave in magnetosphere of pulsars, as was done in [98]. In the other limit, when $\bar{\theta} = \pi/2$, the situation reduces to that of the $\gamma - \phi$ interaction in an unmagnetized medium.

To obtain the solutions of Eq. (3.54), first we need to have the knowledge of the eigenvalues of the matrix on the left-hand side of Eq. (3.54) inside the square bracket. Once that is done, one can find the corresponding eigenvectors and from there obtain the unitary matrix, which can reduce the matrix **M**, to diagonal form.

The eigenvalues of matrix **M** can be obtained from the following cubic equation, which follows from the characteristic equation

$$c_1\mathbf{E}_i^3 + c_2\mathbf{E}_i^2 + c_3\mathbf{E}_i + c_4 = 0. \tag{3.55}$$

The variables c_2, c_3, and c_4 appearing in (3.55) are further related to the elements of the mixing matrix $\mathbf{M}$, through the following relations:

$$c_1 = 1, \qquad c_2 = -(2\omega_p^2 + m_\phi^2), \tag{3.56}$$

$$c_3 = \omega_p^4 + 2\omega_p^2 m_\phi^2 - \left(\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}\right)^2 - (g_{\phi\gamma\gamma}\mathrm{B}_\perp\omega)^2, \tag{3.57}$$

$$c_4 = -\omega_p^4 m_\phi^2 + \left(\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}\right)^2 m_\phi^2 + \omega_p^2(g_{\phi\gamma\gamma}\mathrm{B}_\perp\omega)^2. \tag{3.58}$$

More details about the roots and their properties are provided in the Appendices. Here we just state the result. The roots are

$$\mathbf{E}_1 = 2\mathcal{R}\cos(\alpha - \pi/3) - \frac{c_2}{3}, \tag{3.59}$$

$$\mathbf{E}_2 = 2\mathcal{R}\cos(\alpha + \pi/3) - \frac{c_2}{3}, \tag{3.60}$$

$$\mathbf{E}_3 = -2\mathcal{R}\cos(\alpha) - \frac{c_2}{3}. \tag{3.61}$$

Where the variables $\alpha = \frac{1}{3}\cos^{-1}(\frac{\mathcal{Q}}{\mathcal{R}^3})$ and $\mathcal{R} = \sqrt{(-\mathcal{P})}\mathbf{sgn}(\mathcal{Q})$, when $\mathcal{P} = \frac{(3c_3 - {c_2}^2)}{9}$ and $\mathcal{Q} = (\frac{{c_2}^3}{27} - \frac{c_2 c_3}{6} + \frac{c_4}{2})$. This completes our knowledge of the roots.

IV. THE UNITARY DIAGONALIZING MATRIX U

Before we go on to discuss the oscillation between γ and ϕ, we need to evaluate the phase space evolution of the individual fields. To achieve that, we need to have the solutions for each one of them. In order to obtain the same, the mixing matrix $\mathbf{M}$ (3.53) has to be transformed to its diagonal form $\mathbf{M}_D$, by unitary matrices $\mathbf{U}$, i.e., $\mathbf{U}^\dagger\mathbf{M}\mathbf{U} = \mathbf{M}_D$, having eigenvalues $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ as the diagonal elements. The construction of the unitary matrix $\mathbf{U}$ had been outlined in the Appendices. Therefore we provide the final result here. Introducing the quantities $u_i = (\omega_p^2 - \mathbf{E}_i)(m_\phi^2 - \mathbf{E}_i)$, $v_i = i\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}(m_\phi^2 - \mathbf{E}_i)$ and $w_i = ig_{\phi\gamma\gamma}\mathrm{B}_\perp\omega(\omega_p^2 - \mathbf{E}_i)$ related to the eigenvectors of matrix $\mathbf{M}$, the unitary matrix $\mathbf{U}$ can be written as

$$\mathbf{U} = \begin{pmatrix} \frac{u_1}{\sqrt{u_1^2+v_1^2+w_1^2}} & \frac{u_2}{\sqrt{u_2^2+v_2^2+w_2^2}} & \frac{u_3}{\sqrt{u_3^2+v_3^2+w_3^2}} \\ \frac{v_1}{\sqrt{u_1^2+v_1^2+w_1^2}} & \frac{v_2}{\sqrt{u_2^2+v_2^2+w_2^2}} & \frac{v_3}{\sqrt{u_3^2+v_3^2+w_3^2}} \\ \frac{w_1}{\sqrt{u_1^2+v_1^2+w_1^2}} & \frac{w_2}{\sqrt{u_2^2+v_2^2+w_2^2}} & \frac{w_3}{\sqrt{u_3^2+v_3^2+w_3^2}} \end{pmatrix}. \tag{4.1}$$

Introducing, $\mathcal{N}_{vn}^{(i)} = \frac{1}{\sqrt{u_i^2+v_i^2+w_i^2}}$ as the normalization constant for the ith eigenvector, we can further express Eq. (4.1) in the following form:

$$\mathbf{U} = \begin{pmatrix} (\omega_p^2 - \mathbf{E}_1)(m_\phi^2 - \mathbf{E}_1)\mathcal{N}_{\mathrm{vn}}^{(1)} & (\omega_p^2 - \mathbf{E}_2)(m_\phi^2 - \mathbf{E}_2)\mathcal{N}_{\mathrm{vn}}^{(2)} & (\omega_p^2 - \mathbf{E}_3)(m_\phi^2 - \mathbf{E}_3)\mathcal{N}_{\mathrm{vn}}^{(3)} \\ i\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}(m_\phi^2 - \mathbf{E}_1)\mathcal{N}_{\mathrm{vn}}^{(1)} & i\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}(m_\phi^2 - \mathbf{E}_2)\mathcal{N}_{\mathrm{vn}}^{(2)} & i\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}(m_\phi^2 - \mathbf{E}_3)\mathcal{N}_{\mathrm{vn}}^{(3)} \\ ig_{\phi\gamma\gamma}\mathrm{B}_\perp\omega(\omega_p^2 - \mathbf{E}_1)\mathcal{N}_{\mathrm{vn}}^{(1)} & ig_{\phi\gamma\gamma}\mathrm{B}_\perp\omega(\omega_p^2 - \mathbf{E}_2)\mathcal{N}_{\mathrm{vn}}^{(2)} & ig_{\phi\gamma\gamma}\mathrm{B}_\perp\omega(\omega_p^2 - \mathbf{E}_3)\mathcal{N}_{\mathrm{vn}}^{(3)} \end{pmatrix}. \tag{4.2}$$

The explicit expression for the $\mathcal{N}_{\mathrm{vn}}^{(i)}$ is given by

$$\mathcal{N}_{\mathrm{vn}}^{(i)} = \frac{1}{\sqrt{(\omega_p^2 - \mathbf{E}_i)^2(m_\phi^2 - \mathbf{E}_i)^2 + (\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega})^2(m_\phi^2 - \mathbf{E}_i)^2 + (g_{\phi\gamma\gamma}\mathrm{B}_\perp\omega)^2(\omega_p^2 - \mathbf{E}_i)^2}}, \tag{4.3}$$

where $\mathbf{E}_i$ stands for the corresponding eigenvalue. The Hermitian conjugate matrix $\mathbf{U}^\dagger$ would follow from (4.2). The unitarity relations, $\mathbf{U}^\dagger\mathbf{U} = \mathbf{1}$, have been verified numerically as well as analytically, $\mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$. Analytical verification of $\mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$ is cumbersome, so we have taken recourse to numerical verification, and checked that they are satisfied.

There are few relations those are satisfied by the elements of $\mathbf{U}$, and use of them makes it convenient to express the probability amplitudes in compact notation. In order to derive them we first, rewrite Eq. (4.2) with respective identifications of the elements, as follows:

$$\mathbf{U} = \begin{pmatrix} \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ \hat{v}_1 & \hat{v}_2 & \hat{v}_3 \\ \hat{w}_1 & \hat{w}_2 & \hat{w}_3 \end{pmatrix}. \tag{4.4}$$

The condition $\mathbf{U}^\dagger\mathbf{U} = \mathbf{1}$ now implies, for i, $j = 1, 2, 3$,

$$\hat{u}_i\hat{u}_j^* + \hat{v}_i\hat{v}_j^* + \hat{w}_i\hat{w}_j^* = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{4.5}$$

A similar exercise for $\mathbf{U}\mathbf{U}^\dagger = \mathbf{1}$, further establishes the following relations amongst the elements of $\mathbf{U}$. The off-diagonal terms of $\mathbf{U}\mathbf{U}^\dagger$ will give

$$\sum_{i=1}^{3} \hat{u}_i \hat{v}_i^* = \sum_{i=1}^{3} \hat{u}_i^* \hat{v}_i = \sum_{i=1}^{3} |\hat{u}_i||\hat{v}_i| = 0, \quad (4.6)$$

$$\sum_{i=1}^{3} \hat{v}_i \hat{w}_i^* = \sum_{i=1}^{3} \hat{v}_i^* \hat{w}_i = \sum_{i=1}^{3} |\hat{v}_i||\hat{w}_i| = 0, \quad (4.7)$$

$$\sum_{i=1}^{3} \hat{w}_i \hat{u}_i^* = \sum_{i=1}^{3} \hat{w}_i^* \hat{u}_i = \sum_{i=1}^{3} |\hat{w}_i||\hat{u}_i| = 0, \quad (4.8)$$

and the diagonal entries of the same will yield

$$\sum_{i=1}^{3} |\hat{u}_i||\hat{u}_i| = \sum_{i=1}^{3} |\hat{v}_i||\hat{v}_i| = \sum_{i=1}^{3} |\hat{w}_i||\hat{w}_i| = 1. \quad (4.9)$$

The variables $\hat{u}_i^*$, $\hat{v}_i^*$, and $\hat{w}_i^*$, appearing in above expressions, represent the conjugates of the corresponding elements of $\mathbf{U}$.

V. CONVERSION PROBABILITY

The conversion probability, of a photon of a particular polarization state to the same of a different polarization state or scalars can be estimated from the evolution (quantum evolution) equation of the corresponding polarized states of the photon and the scalars. One can perform the same by promoting the momentum variables to corresponding operators and components of the vector potential $A^\nu(k)$ and $\phi(k)$ to the corresponding quantum states, following [38,108]. In order to follow the evolution of individual quantum states, one needs to decouple them from each other by the following way. One can multiply Eq. (3.54) by $\mathbf{U}^\dagger$ from the left[6] to reduce it to the following form:

$$[k^2\mathbf{I} - \mathbf{U}^\dagger\mathbf{M}\mathbf{U}] \begin{bmatrix} |A'_\parallel(k)\rangle \\ |A'_\perp(k)\rangle \\ |\phi'(k)\rangle \end{bmatrix} = 0, \quad (5.1)$$

when $\mathbf{U}^\dagger\mathbf{M}\mathbf{U}$ is a diagonal matrix. In the diagonal representation, the propagating states are the diagonal states and they allow principle of superposition. The matrix $\mathbf{U}$ is given in Eq. (4.2), and $\mathbf{U}^\dagger$ is the Hermitian conjugate of the same. Here we have denoted[7]

$$\begin{bmatrix} |A'_\parallel(k)\rangle \\ |A'_\perp(k)\rangle \\ |\phi'(k)\rangle \end{bmatrix} = \mathbf{U}^\dagger \begin{bmatrix} |A_\parallel(k)\rangle \\ |A_\perp(k)\rangle \\ |\phi(k)\rangle \end{bmatrix}. \quad (5.2)$$

The primed states corresponds to the propagating states and unprimed ones are the physical states; they are related to each other by the unitary transformation by $\mathbf{U}$ introduced earlier. For a beam of photon, propagating in the z direction, following the principles stated already, one can promote the momentum k_3 to the corresponding operator in z space and write (using natural units $\hbar = c = 1$) $k^2 \approx 2\omega(\omega - i\partial_z)$. With these manipulations, the resulting equations get transformed from Klien-Gordon to the form used by [38]. Recalling that $\mathbf{U}^\dagger\mathbf{M}\mathbf{U} = \mathbf{M}_D$, where $\mathbf{M}_D$ is the diagonal matrix, Eq. (5.2) can further be cast in the form

$$\left[(\omega - i\partial_z)\mathbf{I} - \begin{bmatrix} \frac{\mathbf{E}_1}{2\omega} & 0 & 0 \\ 0 & \frac{\mathbf{E}_2}{2\omega} & 0 \\ 0 & 0 & \frac{\mathbf{E}_3}{2\omega} \end{bmatrix}\right] \begin{bmatrix} |A'_\parallel(z)\rangle \\ |A'_\perp(z)\rangle \\ |\phi'(z)\rangle \end{bmatrix} = 0. \quad (5.3)$$

The matrix evolution Eq. (5.3) is now easy to solve. Introducing the variables, $\Omega_\parallel = (\omega - \frac{\mathbf{E}_1}{2\omega})$, $\Omega_\perp = (\omega - \frac{\mathbf{E}_2}{2\omega})$, and $\Omega_\phi = (\omega - \frac{\mathbf{E}_3}{2\omega})$, we can now directly write down the solutions for the states vector $[|\mathbf{A}(z)\rangle]$ [where $\mathbf{A}(z) \equiv \mathbf{A}(\omega, k_\perp, z)$] in the following form:

$$\begin{bmatrix} |A_\parallel(z)\rangle \\ |A_\perp(z)\rangle \\ |\phi(z)\rangle \end{bmatrix} = \mathbf{U} \begin{bmatrix} e^{-i\Omega_\parallel z} & 0 & 0 \\ 0 & e^{-i\Omega_\perp z} & 0 \\ 0 & 0 & e^{-i\Omega_\phi z} \end{bmatrix} \mathbf{U}^\dagger \begin{bmatrix} |A_\parallel(0)\rangle \\ |A_\perp(0)\rangle \\ |\phi(0)\rangle \end{bmatrix}. \quad (5.4)$$

The elements of column vector $[|\mathbf{A}(0)\rangle]$ in (5.4) and $[|\mathbf{A_L}(0)\rangle]$ are normalized such that $\langle A_\parallel(0)|A_\parallel(0)\rangle = \langle A_\perp(0)|A_\perp(0)\rangle = \langle\phi(0)|\phi(0)\rangle = \langle A_L(0)|A_L(0)\rangle = 1$. With the help of Eq. (5.4), one can write down the solutions. And they are as follows:

$$\begin{aligned} &|A_\parallel(\omega, z)\rangle \\ &= (e^{-i\Omega_\parallel z}\hat{u}_1\hat{u}_1^* + e^{-i\Omega_\perp z}\hat{u}_2\hat{u}_2^* + e^{-i\Omega_\phi z}\hat{u}_3\hat{u}_3^*)|A_\parallel(\omega, 0)\rangle \\ &\quad + (e^{-i\Omega_\parallel z}\hat{u}_1\hat{v}_1^* + e^{-i\Omega_\perp z}\hat{u}_2\hat{v}_2^* + e^{-i\Omega_\phi z}\hat{u}_3\hat{v}_3^*)|A_\perp(\omega, 0)\rangle \\ &\quad + (e^{-i\Omega_\parallel z}\hat{u}_1\hat{w}_1^* + e^{-i\Omega_\perp z}\hat{u}_2\hat{w}_2^* + e^{-i\Omega_\phi z}\hat{u}_3\hat{w}_3^*)|\phi(\omega, 0)\rangle, \end{aligned} \quad (5.5)$$

the perpendicular $|A_\perp(\omega, z)\rangle$ component is given by

[6] Since $\mathbf{U}^{-1} = \mathbf{U}^\dagger$.

[7] The set of unprimed and primed column vectors, at places, may be defined collectively as $[A(k)] = (A_\parallel(k), A_\perp(k), \phi(k))^{\mathbf{T}}$ and $[A(k)'] = (A'_\parallel(k), A'_\perp(k), \phi(k))^{\mathbf{T}}$, here the superscript $\mathbf{T}$ stands for transpose.

$$
\begin{aligned}
&|A_\perp(\omega,z)\rangle \\
&= (e^{-i\Omega_\parallel z}\hat{v}_1\hat{u}_1^* + e^{-i\Omega_\perp z}\hat{v}_2\hat{u}_2^* + e^{-i\Omega_\phi z}\hat{v}_3\hat{u}_3^*)|A_\parallel(\omega,0)\rangle \\
&\quad + (e^{-i\Omega_\parallel z}\hat{v}_1\hat{v}_1^* + e^{-i\Omega_\perp z}\hat{v}_2\hat{v}_2^* + e^{-i\Omega_\phi z}\hat{v}_3\hat{v}_3^*)|A_\perp(\omega,0)\rangle \\
&\quad + (e^{-i\Omega_\parallel z}\hat{v}_1\hat{w}_1^* + e^{-i\Omega_\perp z}\hat{v}_2\hat{w}_2^* + e^{-i\Omega_\phi z}\hat{v}_3\hat{w}_3^*)|\phi(\omega,0)\rangle,
\end{aligned} \tag{5.6}
$$

and lastly the evolution of the state $|\phi(\omega,z)\rangle$ is given by

$$
\begin{aligned}
&|\phi(\omega,z)\rangle \\
&= (e^{-i\Omega_\parallel z}\hat{w}_1\hat{u}_1^* + e^{-i\Omega_\perp z}\hat{w}_2\hat{u}_2^* + e^{-i\Omega_\phi z}\hat{w}_3\hat{u}_3^*)|A_\parallel(\omega,0)\rangle \\
&\quad + (e^{-i\Omega_\parallel z}\hat{w}_1\hat{v}_1^* + e^{-i\Omega_\perp z}\hat{w}_2\hat{v}_2^* + e^{-i\Omega_\phi z}\hat{w}_3\hat{v}_3^*)|A_\perp(\omega,0)\rangle \\
&\quad + (e^{-i\Omega_\parallel z}\hat{w}_1\hat{w}_1^* + e^{-i\Omega_\perp z}\hat{w}_2\hat{w}_2^* + e^{-i\Omega_\phi z}\hat{w}_3\hat{w}_3^*)|\phi(\omega,0)\rangle.
\end{aligned} \tag{5.7}
$$

The ways to arrive at these results can be found in [86]. We would like to end this subsection with the following observation, that the states defined by $|A_\parallel(\omega,0)\rangle$, $|A_\perp(\omega,0)\rangle$ and $|\phi(\omega,0)\rangle$ are pure states. The corresponding states denoted by $|A_\parallel(\omega,z)\rangle$, $|A_\perp(\omega,z)\rangle$, and $|\phi(\omega,z)\rangle$ are the mixed states those evolve from the pure ones through propagation in phase space through mixing. Even if any one of them is absent at the beginning, it can be generated later through mixing much like the neutrinos.

A. Oscillation probability $P_{\gamma_\parallel\to\phi}$

The amplitude for the transition of a photon of energy ω in state $|A_\parallel(\omega,0)\rangle$ to $|\phi(\omega,z)\rangle$ after traversing a distance z is given by $\langle A_\parallel(\omega,0)|\phi(\omega,z)\rangle$. The probability of the same, $P_{\gamma_\parallel\to\phi}(\omega,z)$, can be estimated from the evolution equations obtained above by using the formula $P_{\gamma_\parallel\to\phi}(\omega,z) = |\langle A_\parallel(\omega,0)|\phi(\omega,z)\rangle|^2$. The same turns out to be

$$
P_{\gamma_\parallel\to\phi} = |(e^{-i\Omega_\parallel z}\hat{u}_1\hat{w}_1^* + e^{-i\Omega_\perp z}\hat{u}_2\hat{w}_2^* + e^{-i\Omega_\phi z}\hat{u}_3\hat{w}_3^*)|^2. \tag{5.8}
$$

After performing some lengthy algebra, one can observe that the resulting expression for Eq. (5.8) contains a sum of three quadratic pieces, i.e., $[(|\hat{u}_1||\hat{w}_1|)^2 + (|\hat{u}_2||\hat{w}_2|)^2 + (|\hat{u}_3||\hat{w}_3|)^2]$, plus three other pieces involving the distance parameter z. Upon converting these quadratic pieces into a square of their sum and rearranging the resultant expression, Eq. (5.8) reduces to the following form:

$$
\begin{aligned}
P_{\gamma_\parallel\to\phi} &= [|\hat{u}_1||\hat{w}_1| + |\hat{u}_2||\hat{w}_2| + |\hat{u}_3||\hat{w}_3|]^2 \\
&\quad - 2|\hat{u}_1||\hat{w}_1||\hat{u}_2||\hat{w}_2|[1-\cos((\Omega_\perp - \Omega_\parallel)z)] \\
&\quad - 2|\hat{u}_1||\hat{w}_1||\hat{u}_3||\hat{w}_3|[1-\cos((\Omega_\parallel - \Omega_\phi)z)] \\
&\quad - 2|\hat{u}_3||\hat{w}_3||\hat{u}_2||\hat{w}_2|[1-\cos((\Omega_\phi - \Omega_\perp)z)].
\end{aligned} \tag{5.9}
$$

At this stage, it is convenient to consider defining, $\mathbb{A} = |\hat{u}_1||\hat{w}_1|$, $\mathbb{B} = |\hat{u}_2||\hat{w}_2|$, and $\mathbb{C} = |\hat{u}_3||\hat{w}_3|$. The constraint, $|\hat{u}_1||\hat{w}_1| + |\hat{u}_2||\hat{w}_2| + |\hat{u}_3||\hat{w}_3| = 0$, that follows from Eq. (4.8) can now be recasted as

$$
\mathbb{A} + \mathbb{B} + \mathbb{C} = 0. \tag{5.10}
$$

Now we can make use of Eq. (5.10) in (5.9), to verify that the perfect square term vanishes owing to the constraint equation (5.10); and the remaining z dependent pieces can be manipulated further to provide

$$
\begin{aligned}
P_{\gamma_\parallel\to\phi} &= 4\mathbb{A}(\mathbb{A}+\mathbb{C})\sin^2\left(\frac{(\Omega_\perp - \Omega_\parallel)z}{2}\right) \\
&\quad + 4\mathbb{B}(\mathbb{B}+\mathbb{A})\sin^2\left(\frac{(\Omega_\phi - \Omega_\perp)z}{2}\right) \\
&\quad + 4\mathbb{C}(\mathbb{C}+\mathbb{B})\sin^2\left(\frac{(\Omega_\parallel - \Omega_\phi)z}{2}\right).
\end{aligned} \tag{5.11}
$$

For the sake of completeness, we provide the expressions for the new variables, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$, introduced earlier, in terms of the parameters of the matrix $\mathbf{M}$. And they are

$$
\mathbb{A} = \mathcal{N}_{\rm vn}^{(1)}\mathcal{N}_{\rm vn}^{(1)}(g_{\phi\gamma\gamma}\mathrm{B}_\perp\omega)(\omega_p^2 - \mathbf{E}_1)(\omega_p^2 - \mathbf{E}_1)(m_\phi^2 - \mathbf{E}_1), \tag{5.12}
$$

$$
\mathbb{B} = \mathcal{N}_{\rm vn}^{(2)}\mathcal{N}_{\rm vn}^{(2)}(g_{\phi\gamma\gamma}\mathrm{B}_\perp\omega)(\omega_p^2 - \mathbf{E}_2)(\omega_p^2 - \mathbf{E}_2)(m_\phi^2 - \mathbf{E}_2), \tag{5.13}
$$

$$
\mathbb{C} = \mathcal{N}_{\rm vn}^{(3)}\mathcal{N}_{\rm vn}^{(3)}(g_{\phi\gamma\gamma}\mathrm{B}_\perp\omega)(\omega_p^2 - \mathbf{E}_3)(\omega_p^2 - \mathbf{E}_3)(m_\phi^2 - \mathbf{E}_3). \tag{5.14}
$$

The fact that Eqs. (5.12)–(5.14)) follow Eq. (5.10) to a good accuracy has been verified numerically.

B. Oscillation probability $P_{\gamma_\perp\to\phi}$

The oscillation probability for the $A_\perp$ component of a photon to scalar ϕ can be derived similarly. Therefore, instead of going through the same set of arguments, we would provide the final result. Before we go to the final expression, like before, we introduce the new set of variables, $\mathbb{L}$, $\mathbb{M}$, and $\mathbb{N}$; defined as $\mathbb{L} = |\hat{v}_1||\hat{w}_1|$, $\mathbb{M} = |\hat{v}_2||\hat{w}_2|$, and $\mathbb{N} = |\hat{v}_3||\hat{w}_3|$. Their actual form, in terms of the elements of the mixing matrix, will be provided shortly. The expression for $P_{\gamma_\perp\to\phi}$ is

$$
P_{\gamma_\perp\to\phi} = 4\mathbb{L}(\mathbb{L}+\mathbb{N})\sin^2\left(\frac{(\Omega_\perp - \Omega_\parallel)z}{2}\right) \tag{5.15}
$$

$$
+ 4\mathbb{M}(\mathbb{M}+\mathbb{L})\sin^2\left(\frac{(\Omega_\phi - \Omega_\perp)z}{2}\right) \tag{5.16}
$$

$$+4\mathbb{N}(\mathbb{N}+\mathbb{M})\sin^2\left(\frac{(\Omega_\parallel-\Omega_\phi)z}{2}\right). \tag{5.17}$$

The values of the parameters $\mathbb{L}$, $\mathbb{M}$, and $\mathbb{N}$ are in terms of the parameters of the theory are given by

$$\mathbb{L}=\mathcal{N}_{\rm vn}^{(1)}\mathcal{N}_{\rm vn}^{(1)}(g_{\phi\gamma\gamma}\mathrm{B}_\perp\omega)\left(\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}\right)(\omega_p^2-\mathbf{E}_1)(m_\phi^2-\mathbf{E}_1), \tag{5.18}$$

$$\mathbb{M}=\mathcal{N}_{vn}^{(2)}\mathcal{N}_{vn}^{(2)}(g_{\phi\gamma\gamma}\mathrm{B}_\perp\omega)\left(\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}\right)(\omega_p^2-\mathbf{E}_2)(m_\phi^2-\mathbf{E}_2), \tag{5.19}$$

$$\mathbb{M}=\mathcal{N}_{\rm vn}^{(3)}\mathcal{N}_{\rm vn}^{(3)}(g_{\phi\gamma\gamma}\mathrm{B}_\perp\omega)\left(\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}\right)(\omega_p^2-\mathbf{E}_3)(m_\phi^2-\mathbf{E}_3). \tag{5.20}$$

As before, following Eq. (4.7), Eqs. (5.18)–(5.20) too have to satisfy the constraint $\mathbb{L}+\mathbb{M}+\mathbb{N}=0$. We have tested the same numerically to a good accuracy during the corresponding probability evaluation.

C. Oscillation probability $P_{\gamma_\parallel\to\gamma_\perp}$

Lastly, we provide the conversion probability for parallel component of photon to perpendicular component of photon here, i.e., $P_{\gamma_\parallel\to\gamma_\perp}$. We introduce here, as before, the quantities to be defined later below: $\mathbb{P}=|\hat{u}_1||\hat{v}_1|$, $\mathbb{Q}=|\hat{u}_2||\hat{v}_2|$, and $\mathbb{R}=|\hat{u}_3||\hat{v}_3|$. The corresponding probability for conversion, $P_{\gamma_\parallel\to\gamma_\perp}$, turns out to be

$$P_{\gamma_\parallel\to\gamma_\perp}=4\mathbb{P}(\mathbb{P}+\mathbb{R})\sin^2\left(\frac{(\Omega_\perp-\Omega_\parallel)z}{2}\right) \tag{5.21}$$

$$+4\mathbb{Q}(\mathbb{Q}+\mathbb{P})\sin^2\left(\frac{(\Omega_\phi-\Omega_\perp)z}{2}\right) \tag{5.22}$$

$$+4\mathbb{R}(\mathbb{R}+\mathbb{Q})\sin^2\left(\frac{(\Omega_\parallel-\Omega_\phi)z}{2}\right), \tag{5.23}$$

where

$$\mathbb{P}=\mathcal{N}_{\rm vn}^{(1)}\mathcal{N}_{\rm vn}^{(1)}\left(\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}\right)(\omega_p^2-\mathbf{E}_1)(m_\phi^2-\mathbf{E}_1)^2, \tag{5.24}$$

$$\mathbb{Q}=\mathcal{N}_{\rm vn}^{(2)}\mathcal{N}_{\rm vn}^{(2)}\left(\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}\right)(\omega_p^2-\mathbf{E}_2)(m_\phi^2-\mathbf{E}_2)^2, \tag{5.25}$$

$$\mathbb{R}=\mathcal{N}_{\rm vn}^{(3)}\mathcal{N}_{\rm vn}^{(3)}\left(\frac{e\mathrm{B}_\parallel}{m_e}\frac{\omega_p^2}{\omega}\right)(\omega_p^2-\mathbf{E}_3)(m_\phi^2-\mathbf{E}_3)^2. \tag{5.26}$$

Like before we have verified that the condition, $\mathbb{P}+\mathbb{Q}+\mathbb{R}=0$, is maintained to a good accuracy all along during the course of the computation.

There is one important observation that follows from the probabilities derived above: that is, the probability $P_{\gamma_\parallel\to\gamma_\perp}$ is the only probability out of the three discussed above that survives in the limit $g_{\phi\gamma\gamma}\to 0$ and $m_\phi\to 0$. This can be related to the "rotation measure," which is usually encountered in studies of optical activity relating the angle of rotation of the plane of polarization of a beam of plane polarized light, after traveling some distance L. The rotation measure in this case can be defined by $\pi/\bar{\lambda}$ when $\bar{\lambda}$ is the minimum distance that a plane polarized light beam needs to travel to have $P_{\gamma_\parallel\to\gamma_\perp}(\omega,\bar{\lambda})=1$. The nonlinear dependence of $\bar{\lambda}$ on the parameters of the theory indicates that it is difficult to separate the contributions to $P_{\gamma_\parallel\to\gamma_\perp}$ into parts originating from (i) the magnetized plasma and (ii) the one originating from magnetic field induced dilatonic interactions. Thus it should be considered as a total of the two contributions mentioned above.

Last, the other important aspect of this analysis is that the other probabilities, i.e., $P_{\gamma_\parallel\to\phi}$ and $P_{\phi\to\gamma_\parallel}$, $P_{\gamma_\perp\to\phi}$, and $P_{\phi\to\gamma_\perp}$ along with $P_{\gamma_\parallel\to\gamma_\perp}$ and $P_{\gamma_\perp\to\gamma_\parallel}$, turn out to be same as they should be even otherwise.

D. Conversion probability $P_{\gamma_\parallel\to\phi}$ in an unmagnetized medium

In this subsection we provide the photon scalar conversion rate without magnetized medium effects to compare its size with the same due to magnetized medium induced effects. The mixing matrix for the same is similar to that of chameleon-photon mixing in an unmagnetized media: 2×2. The probability for the photon scalar transition obtained in [28] turns out to be same as that of [38]. This probability of transition without any approximation is given by

$$P_{\gamma_\parallel\to\phi}=\frac{4\mathrm{B}_\perp{}^2\omega^2}{(Mm_{ef}^2)^2+4\mathrm{B}_\perp{}^2\omega^2}\times\sin^2\left(\frac{\sqrt{(Mm_{ef}^2)^2+4\mathrm{B}_\perp{}^2\omega^2}}{4\omega M}z\right). \tag{5.27}$$

In Eq. (5.27) $m_{ef}^2=m_\phi^2-\omega_p^2-\frac{\mathrm{B}_\perp{}^2}{M^2}$ and m_ϕ is mass of scalar field, as stated before. In this case only one degree of freedom of photon (i.e., $A_\parallel$) mix with the scalar, and the other degrees of freedom ($A_\perp$, A_L) propagate freely. It has two special features: One is that, in the limit when the angle between B and $k\to$ zero or π, the probability in (5.27) vanishes, indicating a decoupling of fields. The second one is in a noninteracting limit, i.e., $g_{\phi\gamma\gamma}\to 0$ or $M\to\infty$, $P_{\gamma_\parallel\to\phi}\to 0$ for $\omega_p\neq m_\phi$. But in the same limit if $\omega_p=m_\phi$, then the oscillation length diverges. It is straightforward to verify the same. In the magnetized plasma case, however the $P_{\gamma_\parallel \to \perp}$ and $P_{\gamma_\perp \to \parallel}$ remains finite even if the angle between B and $k \to$ zero or π. These checks are useful to prove the consistency of the results.

VI. ASTROPHYSICAL APPLICATIONS

The presence of a very strong magnetic field and an ambient plasma in the environment of astrophysical compact objects like white dwarves or neutron stars, gamma ray bursters etc. provides an opportunity to look for astrophysical signatures of the kind of particles we have been studying in this article. In this work, we have focussed on the EM signals from the compact objects that may bear possible signatures of dimension-five $\phi F^{\mu\nu} F_{\mu\nu}$ interaction.

In this context we have tried to estimate the ratio of parallel component of the electric flux with the perpendicular component. The estimate of this ratio in dipole magnetic field of an aligned rotor model is known. We have found the same ratio in presence of the scalar field. So with the difference between the two when compared with observations, one might be able to draw some conclusions about the existence of the field ϕ.

Following this point of view, we have taken the plasma frequency ω_p to be of the order 10^{-2} eV. We have further considered the photon path length to be $z = 1.2$ km [109]. For these numbers of ω_p and z, we have estimated various oscillation probabilities in KeV energy range (20–100) KeV as shown in Figs. 1 and 2. The details that led to the choice of these parameters have been provided in Appendix D.

A. Electric field

In this subsection we provide the expression for the electric field in a frame of reference where the basis vectors are orthogonal to the propagation direction k_μ. We begin by noting that in the momentum space k_μ the electric field in four-component notation can be written as

$$E_\mu = \omega \tilde{A}_\mu - k_\mu(\tilde{A}.u). \tag{6.1}$$

It should be noted that, in Eq. (6.1), the vector potential $\tilde{A}_\mu$ refers to the gauge fields for dynamical photon (in absence of scalar-photon mixing). Expressing the vector potential using Eq. (3.12) in the basis where $A.k = 0$, the electric field E_μ is

$$E_\mu = \omega[\tilde{A}_\parallel(k)\hat{b}^{(1)}_\mu + \tilde{A}_\perp(k)\hat{I}_\mu + \tilde{A}_L(k)\hat{\tilde{u}}_\mu] - k_\mu(\tilde{A}.u). \tag{6.2}$$

The components of the electric fields vector can be written in terms of the form factors of the vector potentials by contracting it with the corresponding polarization vectors as

$$E_\parallel = \hat{b}^{(1)}_\mu E^\mu = \omega \tilde{A}_\parallel(k), \tag{6.3}$$

$$E_\perp = \hat{I}_\mu E^\mu = \omega \tilde{A}_\perp(k), \tag{6.4}$$

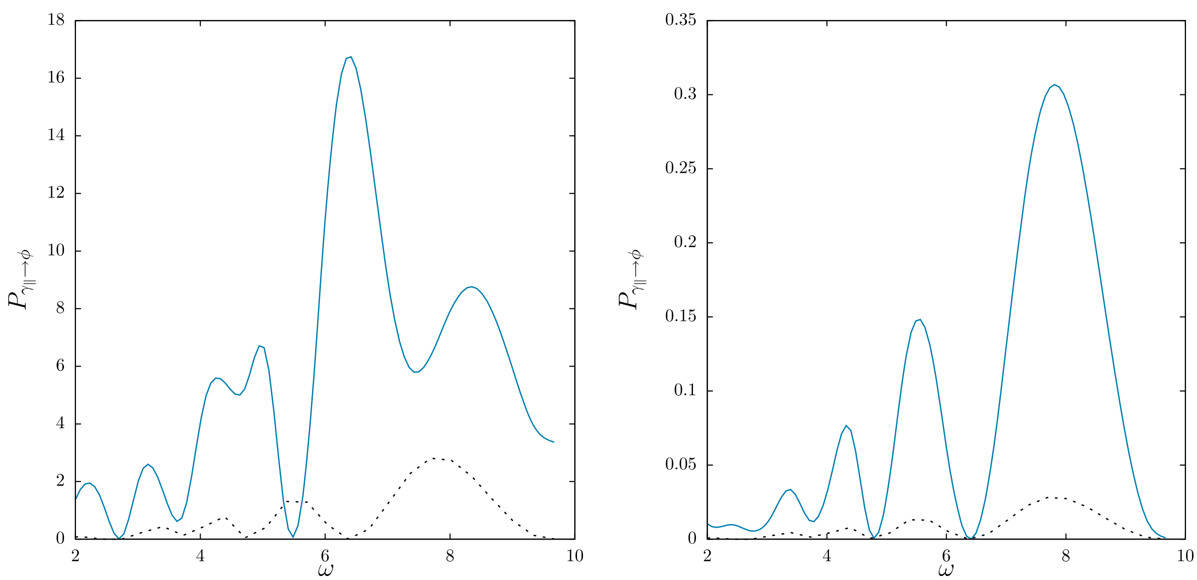

FIG. 1. Plot of conversion probability of a parallel polarized photon into a scalar dilaton in magnetic field $B = 10^{12}$ Gauss (in the left panel) and the same in magnetic field $B = 10^{11}$ Gauss (in the right panel). The solid line is for conversion probability in presence of magnetized media and the dashed curve is the same in absence of self energy correction $\Pi^p_{\mu\nu}$ from magnetized medium effects. The abscissa [energy of photon (ω) in GeV] is in units of 10^{-5} and the ordinates are plotted in the units of 10^{-4}. Here, the parameters used are the mass of dilaton (scalar) particle ϕ: $(m_\phi) = 1.0 \times 10^{-12}$ GeV, coupling constant $(g_{\phi\gamma\gamma}) = 1.0 \times 10^{-11}$ GeV, photon path length $(z) = 1.2$ km, and plasma frequency $(\omega_p) = 1.96 \times 10^{-2}$ eV.

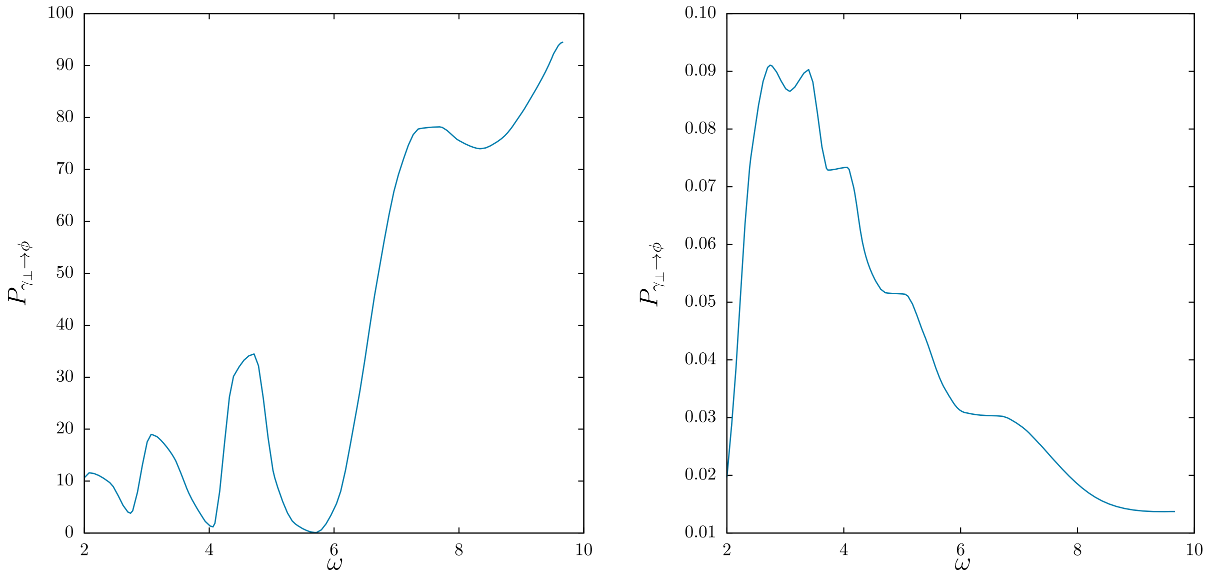

FIG. 2. Plot of conversion probability of perpendicularly polarized photon into scalar in magnetic field B $= 10^{12}$ Gauss (in the left panel) and the same in magnetic field B $= 10^{11}$ Gauss (in the right panel). The abscissa [energy of photon (ω) in units of 10^{-5} GeV] and the ordinate are plotted in units of 10^{-5}. Here, the parameters used are mass of dilaton (scalar) particle ϕ (m_ϕ) $= 1.0 \times 10^{-12}$ GeV, coupling constant ($g_{\phi\gamma\gamma}$) $= 1.0 \times 10^{-11}$ GeV, photon path length (z) $= 1.2$ km, and plasma frequency (ω_p) $= 1.96 \times 10^{-2}$ eV.

$$E_L = \hat{\tilde{u}}_\mu E^\mu = \tilde{A}_L(k), \tag{6.5}$$

$$k^\mu \hat{b}^{(1)}_\mu = k^\mu \hat{I}_\mu = k^\mu \hat{\tilde{u}}_\mu = 0. \tag{6.6}$$

The magnetic field also can be written in terms of these three components. Therefore the total energy stored in the form of EM fields $E_{\text{tot}} = \frac{1}{\mu_0}(E^2 + B^2)$.

B. Intensities of polarization modes

For highly Lorentz boosted electrons, the opening angle for the curvature photons follow the relation, $\Gamma \simeq \frac{1}{\theta_e}$, when θ_e is the opening angle. Therefore in this approximation, the amplitudes of the two orthogonal modes of curvature photon in terms of modified Bessel functions $K_{1/3}(\xi)$ and $K_{2/3}(\xi)$ would be given by [110]

$$|\tilde{A}_\parallel| \simeq \frac{\sqrt{6}\Gamma}{\omega_c} K_{1/3}(\xi), \tag{6.7}$$

$$|\tilde{A}_\perp| \simeq \frac{2\sqrt{3}\Gamma}{\omega_c} K_{2/3}(\xi). \tag{6.8}$$

The argument (ξ) of the modified Bessel functions is defined as

$$\xi = \frac{\omega R_c}{3c}\left(\frac{1}{\Gamma^2} + \theta_e^2\right)^{3/2}. \tag{6.9}$$

Using Eq. (D5) and the approximation considered above (i.e., $\Gamma \simeq \frac{1}{\theta_e}$), the same in terms of ω_c (in the units of $\hbar = c = 1$) turns out be $\xi = \frac{\omega}{0.7\omega_c}$. The modified Bessel functions when $\omega \ll \omega_c$, i.e., for small arguments are given by

$$K_{2/3}\left(\frac{\omega}{\omega_c}\right) \simeq 2^{-1/3}\Gamma_E\left(\frac{2}{3}\right)\xi^{-2/3},$$
$$K_{1/3}\left(\frac{\omega}{\omega_c}\right) \simeq 2^{-2/3}\Gamma_E\left(\frac{1}{3}\right)\xi^{-1/3} \tag{6.10}$$

(here Γ_E is the Euler gamma function); one can express the square of the ratio of the amplitude of the two polarization components as

$$\left|\frac{\tilde{A}_\parallel}{\tilde{A}_\perp}\right|^2 \simeq 2^{-5/3}\left[\frac{(\Gamma_E(1/3))}{(\Gamma_E(2/3))}\right]^2 \xi^{2/3}. \tag{6.11}$$

Since the emitted energy peaks at ω_c, one considers $\omega \simeq 0.7\omega_c$, the ratio of the two amplitudes for this energy comes out as $2^{-5/3}[\frac{(\Gamma_E(1/3))}{(\Gamma_E(2/3))}]^2$. It should be noted that this ratio is independent of the path traversed by the radiation. Ideally, for the kind of situation under consideration, this is what one would expect for the ratio of the intensities of the two polarization states. However in the presence of the $\phi F^{\mu\nu}F_{\mu\nu}$ interaction, the same will be modified. The square of the modified amplitudes ratio turns out to be

$$\left|\frac{A_\parallel}{A_\perp}\right|^2 \simeq 2^{-5/3}\left[\frac{(\Gamma_E(1/3))}{(\Gamma_E(2/3))}\right]^2 \xi^{2/3} \times \left[\frac{1 - P_{\gamma_\parallel \to \phi}(\omega, z)}{1 - P_{\gamma_\perp \to \phi}(\omega, z)}\right]. \tag{6.12}$$

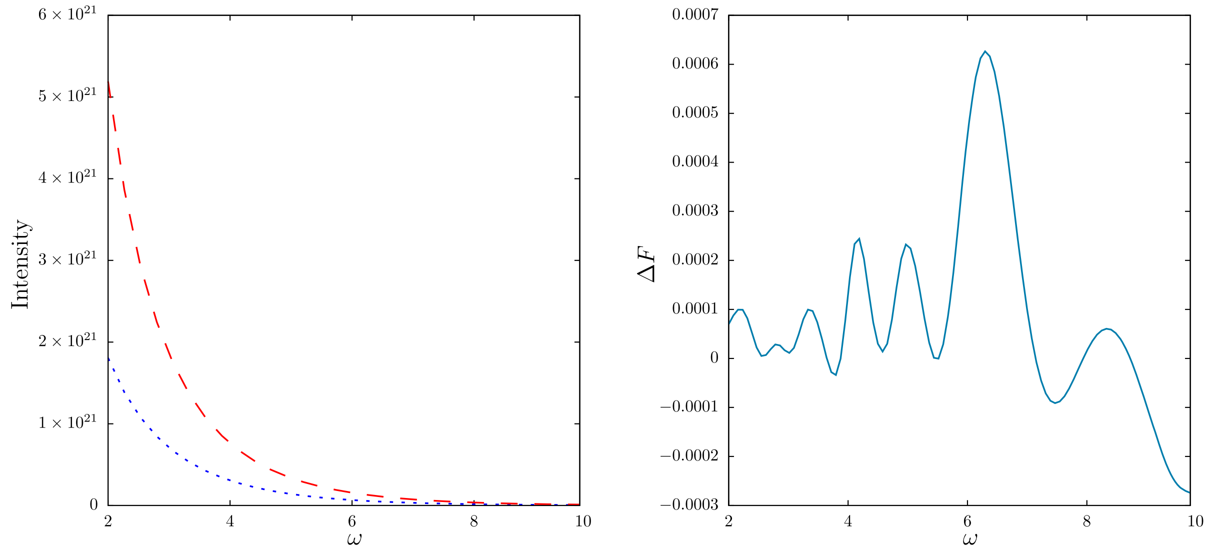

FIG. 3. Left panel: intensity of parallel and perpendicular polarized photons with scalar photon interaction in magnetized media vs energy (ω). The dot curve represents $|A_\parallel|^2$ and the dashed curve represents $|A_\perp|^2$. Right panel: difference between ratios of intensity of parallel and perpendicular polarized photons (in absence and in presence of scalar photon interaction) ΔF. Parameters chosen are Γ (Lorentz boost factor) $= 1.0 \times 10^6$, $\omega_c \sim 5 \times 10^{-5}$ GeV, B $\sim 10^{12}$ Gauss. The abscissa (energy of photon (ω) in GeV) is in units of 10^{-5}.

The last term inside the square bracket in Eq. (6.12) is the modification factor, where we have taken into account the modifications to the intensities in the $\parallel$ ($\perp$) direction due to oscillation of the $\parallel$ ($\perp$) mode into ϕ after traveling a distance z. It is to be noted that the denominator of the modification factor in Eq. (6.12) would remain unified unless the effect of the photon self-energy correction in a magnetized media to order eB, is considered. In this work we have retained this piece consistently in our formalism to find its contribution to the ratio of the two polarization states. And in absence of scalar-photon interaction, the ratio would be the same for pure curvature radiation. In presence of this interaction (dimension-five-scalar photon), the same would be given by Eq. (6.12). The contribution of this deviation in intensities can be observed when we take the difference of the two, i.e.,

$$\Delta F = \left|\frac{\tilde{A}_\parallel}{\tilde{A}_\perp}\right|^2 - \left|\frac{A_\parallel}{A_\perp}\right|^2. \tag{6.13}$$

The same has been estimated numerically in the right panel of Fig. 3.

In absence of PSET in magnetized media, the result would be same as that of mixing with 2×2 mixing matrix.

VII. RESULT AND DISCUSSION

In this work we have explored the behavior of the magnetized matter on the probabilities $P_{\gamma_{\parallel,\perp}\to\phi}$ for a electrodynamic system and estimated their magnitudes numerically. To perform that we need to compare the effects of magnetized matter effects with unmagnetized matter effects in different magnetic field environments. We have numerically estimated various probabilities for the astrophysical parameters given by the plasma frequency of the stellar environment $(\omega_p) = 1.96 \times 10^{-2}$ eV, scalar photon coupling constant $(g_{\phi\gamma\gamma}) = 1.0 \times 10^{-11}$ GeV^{-1}, and the photon path length considered here is $z \sim 1.2$ km. The results are plotted in Figs. 1 and 2. There are three major outcomes of our analysis. First, $P_{\gamma_\parallel\to\phi}$ for magnetized media, turns out to be different from $P_{\gamma_\parallel\to\phi}$ for unmagnetized media, they admit frequency dependent modification over each other's contributions when the polarized photon has propagated over the same distance. Qualitatively, $P_{\gamma_\parallel\to\phi}$ conversion probability was found to be the only nonzero probability when the correction of $\Pi^p_{\mu\nu}(k)$ is absent. Second, the numerical strength of the probabilities of conversion $P_{\gamma_\parallel\to\phi}$ (or $P_{\gamma_\perp\to\phi}$) increases with the increase in strength of the magnetic field. Since an analytical estimate of the same is difficult, we have estimated the same numerically and have shown both of them in the same panel (Figs. 1 and 2). One interesting feature of $P_{\gamma_\parallel\to\phi}$ is that it shows excess enhancement over $P_{\gamma_\parallel\to\phi}$ estimated in an unmagnetized medium at various energies. The third interesting feature is that the intensity of the other orthogonally polarized mode of the photon (denoted as $A_\perp$) no longer remains the same when the effect of magnetized-self-energy correction to photon is considered for studying (photon scalar) oscillation. In the absence of magnetized-self-energy correction, the intensity of an $A_\perp$ mode remains constant, provided that no other absorption or enhancement mechanism is in operation.

That is to say, the millicharge pair production processes in a magnetic field ($\gamma + \gamma \rightarrow e_m^+ + e_m^-$) that modifies the $A_\perp$ spectra is considered absent.

The x-ray sources for compact stars can be classified into three categories: one is due to the bombardment of pair produced flux on the polar cap region of the compact star, the second one is due to the conduction heat coming out of the surface of a star having the surface temperature $T \leq 10^6 K$. And the last one is due to the curvature radiation from the star.

The beams of EM radiations from first two sources follow the blackbody radiation pattern and are unpolarized. However due to the presence of the magnetic field, the isotropic beams of radiation get resolved into two components, one of them becomes polarized parallel and the other one becomes polarized perpendicular to the magnetic field. Total intensity of each of the components becomes equal to the other.

On the other hand the amplitudes of the two polarized components of the curvature radiation that we have discussed in this paper, denoted by $\tilde{A}_\parallel$ and $\tilde{A}_\perp$, are different. As a result, the spectra for their respective polarization would look different from the usual blackbody spectrum. The ratios of $\tilde{A}_\parallel$ and $\tilde{A}_\perp$ would be given by Eqs. (6.7) and (6.8). That yields $\frac{\tilde{A}_\parallel}{\tilde{A}_\perp} \sim (\frac{\omega}{0.7\omega_c})^{1/3}$ for $\omega_c \gg \omega$.

Once the scalar photon interaction is taken into account, and the same ratios are estimated this pattern changes. If we take the difference between the square of these ratios without and with scalar photon modification, i.e., $|\frac{\tilde{A}_\parallel}{\tilde{A}_\perp}|^2 - |\frac{A_\parallel}{A_\perp}|^2$, then the resulting plot looks entirely different. The intensities of both the polarization states (i.e., $A_\parallel$ and $A_\perp$) under the circumstances mentioned above plotted separately can be found in the left panel of Fig. 3. And the differences in their ratios of intensities (without and with scalar photon interaction) can be found in the right panel of Fig. 3. Such an oscillating curve may bear the possible smoking gun signature of dilaton-photon interaction from a compact star.

VIII. OUTLOOK

Existence of DM was postulated to explain some astrophysical observations, such as (i) galaxy rotation curve, (ii) observation of x rays from bullet cluster, (iii) weak microlensing effect, (iv) structure formation etc. The candidate particles proposed to explain the same also predict few additional signatures stemming from ALP photon interaction [38], like supernova dimming, modification to distance duality relation, ALPs from supernova, ALP photon conversion in magnetized domain etc. [108,111–119], that happens to be some of them. So there has been a surge for the search of ALP signals from the magnetized environments of the compact stars [120–132] for their identifications. Therefore to complement this surge in the interest we have explored ALP signal from magnetized media in this work. In the course of this investigation we find that the formalism employed in this analysis interpolates between two extreme ends: one being the effective quantum statistical field theory of electrodynamics in a magnetized environment when the angle between B and k is zero or π; and the other being the interacting theory of ALP with quantum statistical field theory of electrodynamics at finite density when the angle between B and k is $\pi/2$, which follows from Eq. (3.54). These are the checks that can be employed to establish the consistency of our formalism for magnetized media.

It has been found that, in addition to $P_{\gamma_\parallel \rightarrow \phi}$ there are two more additional probabilities of conversion i.e., $P_{\gamma_\perp \rightarrow \phi}$ and $P_{\gamma_\parallel \rightarrow \gamma_\perp}$ possible in a magnetized media according to the number of possible in-medium polarization states of photon. Our analysis showed the existence of normal modes corresponding to the three orthogonal directions of propagation of photon.

Furthermore, we have noted in the body of the text that $P_{\gamma_\parallel \rightarrow \phi}$ in a magnetized medium can be much larger than the contribution of the same in an unmagnetized medium in some energy range for suitable values of other parameters. Since the produced ALPs would leave the production region (of the star) fast because of its weak interaction with the medium, this will lead to anomalous cooling of the star. The indication of presence of ALP in the stellar environment comes from the estimate of ΔF (i.e., $|\frac{\tilde{A}_\parallel}{\tilde{A}_\perp}|^2 - |\frac{A_\parallel}{A_\perp}|^2$) that becomes zero in the absence of an ALP, and it becomes nonzero when ALP interaction is present. These are the novel signatures that a magnetized compact star environment can offer for the verification of ALP.

To conclude, we have shown in this work that the effect of magnetized medium brings nontrivial modifications to the ALP induced signals from compact objects. Though the physics of these objects are complex, however it may be possible to get a tell-tale signature of ALP-like objects from the EM signals of the compact stars. Some of these studies would be considered in separate publications.

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APPENDIX A: TECHNICAL DETAILS OF CONSTRUCTING THE UNITARY MATRIX

In order to describe the dynamics of scalar-photon interaction in magnetized media in terms of different d.o.f. of mixing, the Hermitian mixing matrix M_H obtained from the equations of motion of scalar photon interaction, given by

$$M_H = \begin{bmatrix} \Pi_T(k) & iF & iG \\ -iF & \Pi_T(k) & 0 \\ -iG & 0 & m_\phi^2 \end{bmatrix}, \qquad \text{(A1)}$$

needs to be diagonalized. Therefore, to do the same, we write M_H as

$$M_H = \begin{bmatrix} a & ib & ic \\ -ib & d & 0 \\ -ic & 0 & e \end{bmatrix}, \tag{A2}$$

when $a = \Pi_T(k)$, $b = F$, $c = G$, and $e = m_\phi^2$. It can be shown that an unitary matrix Q and its Hermitian conjugate $Q^\dagger$ defined as

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}, \qquad Q^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix} \tag{A3}$$

can transform M_H to real symmetric matrix M_{RS} by $QM_HQ^\dagger = M_{RS}$, where M_{RS} is given by

$$M_{RS} = \begin{bmatrix} a & b & c \\ b & d & 0 \\ c & 0 & e \end{bmatrix}. \tag{A4}$$

Now, if for a real symmetric matrix M_{RS}, there exists a unitary matrix $\tilde{\mathbf{U}}$ such that $\tilde{\mathbf{U}}^\dagger M_{RS}\tilde{\mathbf{U}} = M_D$, where M_D is a diagonalized matrix having the eigenvalues of matrix M_{RS} as diagonal elements. Now, using this fact, we can subsequently find out the unitary matrix $\mathbf{U}$ that would diagonalize M_H. The argument goes as follows. Since

$$\tilde{\mathbf{U}}^\dagger M_{RS}\tilde{\mathbf{U}} = M_D, \tag{A5}$$

then, using $QM_HQ^\dagger = M_{RS}$ in Eq. (A5), we get

$$\tilde{\mathbf{U}}^\dagger QM_HQ^\dagger\tilde{\mathbf{U}} = M_D, \tag{A6}$$

$$\mathbf{U}^\dagger M_H\mathbf{U} = M_D, \tag{A7}$$

where we have defined $Q^\dagger\tilde{\mathbf{U}} = \mathbf{U}$ and $(Q^\dagger\tilde{\mathbf{U}})^\dagger = \mathbf{U}^\dagger$. The eigenvalues of the unitary related matrices M_H and M_{RS} remain same. To obtain M_D, we need the unitary matrix $\mathbf{U}$ that follows from the unitary matrix $\tilde{\mathbf{U}}$, constructed from the eigenvectors of M_{RS}.

APPENDIX B: EIGENVECTORS OF M_{RS}

1. The characteristics equation

In this section we denote the eigenvalues of M_{RS} as $\mathbf{E}_i$, $i = 1, 2, 3$, then the characteristic equation for the same can be written as

$$|M_{RS} - \mathbf{E}_i\mathbf{I}| = 0. \tag{B1}$$

Where $\mathbf{I}$ is a 3×3 identity matrix. Hence,

$$\begin{vmatrix} a - \mathbf{E}_i & b & c \\ b & d - \mathbf{E}_i & 0 \\ c & 0 & e - \mathbf{E}_i \end{vmatrix} = 0. \tag{B2}$$

The characteristics equation that follows from the determinant is

$$(a - \mathbf{E}_i)(d - \mathbf{E}_i)(e - \mathbf{E}_i) - b^2(e - \mathbf{E}_i) - c^2(d - \mathbf{E}_i) = 0, \tag{B3}$$

which yields, upon simplification,

$$\mathbf{E}_i^3 c_3 + \mathbf{E}_i^2 c_2 + \mathbf{E}_i c_1 + c_0 = 0. \tag{B4}$$

Where the coefficients $c_i's$ are defined as

$$c_3 = 1, \tag{B5}$$

$$c_2 = -(a + d + e), \tag{B6}$$

$$c_1 = (ae + de + ad - b^2 - c^2), \tag{B7}$$

$$c_0 = (b^2e + c^2d - aed). \tag{B8}$$

The nature of the roots of the cubic equation (B4) must satisfy the following properties:

$$\begin{aligned} \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 &= -c_2, \\ \mathbf{E}_1\mathbf{E}_2 + \mathbf{E}_2\mathbf{E}_3 + \mathbf{E}_3\mathbf{E}_1 &= c_1, \\ \mathbf{E}_1\mathbf{E}_2\mathbf{E}_3 &= -c_0. \end{aligned} \tag{B9}$$

The root of Eq. (B4) depends on the discriminant, $\mathcal{D} = \mathcal{Q}^2 + \mathcal{P}^3$, where variables $\mathcal{P}$ and $\mathcal{Q}$ are related to the elements of Eq. (B4) through the relations

$$\mathcal{P} = \frac{(3c_1 - {c_2}^2)}{9} \quad \text{and} \quad \mathcal{Q} = \left(\frac{{c_2}^3}{27} - \frac{c_2c_1}{6} + \frac{c_0}{2}\right). \tag{B10}$$

Furthermore, for Hermitian matrices, roots are real, and we should have $\mathcal{Q}^2 + \mathcal{P}^3 \le 0$. Finally, following [133,134], the roots turns out to be

$$\begin{aligned} \mathbf{E}_1 &= \mathcal{R}\cos\alpha + \sqrt{3}\mathcal{R}\sin\alpha - c_2/3, \\ \mathbf{E}_2 &= \mathcal{R}\cos\alpha - \sqrt{3}\mathcal{R}\sin\alpha - c_2/3, \\ \mathbf{E}_3 &= -2\mathcal{R}\cos\alpha - c_2/3, \\ &\text{with} \begin{cases} \alpha = \frac{1}{3}\cos^{-1}(\frac{\mathcal{Q}}{\mathcal{R}^3}) \\ \mathcal{R} = \sqrt{(-\mathcal{P})}\mathrm{sgn}(\mathcal{Q}) \end{cases}. \end{aligned} \tag{B11}$$

For having real roots, one should have $\mathcal{P} < 0$ and $|\frac{\mathcal{Q}}{\mathcal{R}^3}| \le 1$. Although for analytic evaluation of the roots, this condition may not pose any problem, however during numerical evaluation of the same, maintaining this ratio may become difficult if the magnitudes of elements of the matrix **M** are close to the precision available for the machine in use.

One can scale the matrix **M** by a suitable numerical factor to avoid this difficulty.

2. Eigenfunction and unitary matrix

The normalized eigenvector $\tilde{\mathbf{V}}_i$ of the real symmetric matrix M_{RS} can be represented as

$$\tilde{\mathbf{V}}_i = N_i \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \\ \tilde{w}_i \end{bmatrix}, \tag{B12}$$

where N_i is the normalization constant. Therefore, using the eigenvalue equation, i.e., $(M_{RS} - \mathbf{E}_i \mathbf{I})\tilde{\mathbf{V}}_i = 0$. Hence,

$$\begin{bmatrix} a & b & c \\ b & d & 0 \\ c & 0 & e \end{bmatrix} \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \\ \tilde{w}_i \end{bmatrix} = \mathbf{E}_i \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \\ \tilde{w}_i \end{bmatrix}. \tag{B13}$$

We get

$$(a - \mathbf{E}_i)\tilde{u}_i + b\tilde{v}_i + c\tilde{w}_i = 0, \tag{B14}$$

$$b\tilde{u}_i + (d - \mathbf{E}_i)\tilde{v}_i = 0, \tag{B15}$$

$$c\tilde{u}_i + (e - \mathbf{E}_i)\tilde{w}_i = 0. \tag{B16}$$

Using the method discussed in [135], we can obtain analytical expressions of the elements of the eigenvector $\tilde{\mathbf{V}}_i$. They are

$$\tilde{\mathbf{V}}_i = N_i \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \\ \tilde{w}_i \end{bmatrix} = N_i \begin{bmatrix} (d - \mathbf{E}_i)(e - \mathbf{E}_i) \\ -b(e - \mathbf{E}_i) \\ -c(d - \mathbf{E}_i) \end{bmatrix}. \tag{B17}$$

Here, $N_i = \frac{1}{\sqrt{((d-\mathbf{E}_i)(e-\mathbf{E}_i))^2 + (b(e-\mathbf{E}_i))^2 + (c(d-\mathbf{E}_i))^2}}$, is a normalization constant. We can now construct the unitary matrix $\tilde{\mathbf{U}}$, using (B17), as

$$\tilde{\mathbf{U}} = \begin{bmatrix} N_1\tilde{u}_1 & N_2\tilde{u}_2 & N_3\tilde{u}_3 \\ N_1\tilde{v}_1 & N_2\tilde{v}_2 & N_3\tilde{v}_3 \\ N_1\tilde{w}_1 & N_2\tilde{w}_2 & N_3\tilde{w}_3 \end{bmatrix}. \tag{B18}$$

3. Orthonormality check of $\tilde{\mathbf{V}}$

One can easily check that the vectors are normalized, i.e., $\tilde{\mathbf{V}}_i.\tilde{\mathbf{V}}_j = 1$ when $i, j = 1, 2, 3$ and $i = j$. Furthermore, the verification of the orthogonal properties of vectors are provided below:

$$\begin{aligned} \tilde{\mathbf{V}}_1.\tilde{\mathbf{V}}_2 &= N_1 N_2(\tilde{u}_1\tilde{u}_2 + \tilde{v}_1\tilde{v}_2 + \tilde{w}_1\tilde{w}_2), \\ &= N_1 N_2((d - \mathbf{E}_1)(e - \mathbf{E}_1)(d - \mathbf{E}_2)(e - \mathbf{E}_2) \\ &\quad + b^2(e - \mathbf{E}_1)(e - \mathbf{E}_2) + c^2(d - \mathbf{E}_1)(d - \mathbf{E}_2)). \end{aligned} \tag{B19}$$

Now

$$(e - \mathbf{E}_1)(e - \mathbf{E}_2) = e^2 - e(\mathbf{E}_1 + \mathbf{E}_2) + \mathbf{E}_1\mathbf{E}_2. \tag{B20}$$

We will convert the above equation having two functions $\mathbf{E}_1$ and $\mathbf{E}_2$ into a single function $\mathbf{E}_3$ by using the properties of roots of cubic equation. In order to do that, we will try the following substitutions:

$$\mathbf{E}_1 + \mathbf{E}_2 = [\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3] - \mathbf{E}_3, \tag{B21}$$

$$\mathbf{E}_1\mathbf{E}_2 = [\mathbf{E}_1\mathbf{E}_2 + \mathbf{E}_2\mathbf{E}_3 + \mathbf{E}_3\mathbf{E}_1] - \mathbf{E}_3(\mathbf{E}_2 + \mathbf{E}_1). \tag{B22}$$

Following the identities of Eq. (B9) and make substitutions in Eq. (B19), we obtain

$$\begin{aligned} \tilde{\mathbf{V}}_\mathbf{1}.\tilde{\mathbf{V}}_\mathbf{2} &= N_1 N_2(a - \mathbf{E}_3)[(a - \mathbf{E}_3)(d - \mathbf{E}_3)(e - \mathbf{E}_3) \\ &\quad - b^2(e - \mathbf{E}_3) - c^2(d - \mathbf{E}_3)]. \end{aligned} \tag{B23}$$

The terms inside the square brackets on the rhs of Eq. (B23) happens to be zero due to Eq. (B3), which follows from the characteristic equation. Hence,

$$\tilde{\mathbf{V}}_\mathbf{1}.\tilde{\mathbf{V}}_\mathbf{2} = 0. \tag{B24}$$

Following the same procedure one can verify that $\tilde{\mathbf{V}}_\mathbf{1}.\tilde{\mathbf{V}}_\mathbf{2} = \tilde{\mathbf{V}}_\mathbf{2}.\tilde{\mathbf{V}}_\mathbf{3} = \tilde{\mathbf{V}}_\mathbf{3}.\tilde{\mathbf{V}}_\mathbf{1} = 0$.

4. Analytical check of $\tilde{U}^\dagger M_{RS}\tilde{\mathbf{U}} = M_D$

We have obtained the unitary matrix $\tilde{\mathbf{U}}$, which is supposed to diagonalize the matrix M_{RS}. To check this claim analytically, we first start the unitary transformation of M_{RS} by $\tilde{\mathbf{U}}$, given as follows:

$$\tilde{U}^\dagger M_{RS}\tilde{\mathbf{U}} = M_D, \tag{B25}$$

which implies

$$\begin{bmatrix} N_1\tilde{u}_1 & N_1\tilde{v}_1 & N_1\tilde{w}_1 \\ N_2\tilde{u}_2 & N_2\tilde{v}_2 & N_2\tilde{w}_2 \\ N_3\tilde{u}_3 & N_3\tilde{v}_3 & N_3\tilde{w}_3 \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & 0 \\ c & 0 & e \end{bmatrix} \begin{bmatrix} N_1\tilde{u}_1 & N_2\tilde{u}_2 & N_3\tilde{u}_3 \\ N_1\tilde{v}_1 & N_2\tilde{v}_2 & N_3\tilde{v}_3 \\ N_1\tilde{w}_1 & N_2\tilde{w}_2 & N_3\tilde{w}_3 \end{bmatrix} = M_D, \tag{B26}$$

$$\begin{bmatrix} N_1(\tilde{u}_1 a + \tilde{v}_1 b + \tilde{w}_1 c) & N_1(\tilde{u}_1 b + \tilde{v}_1 d) & N_1(\tilde{u}_1 c + \tilde{w}_1 e) \\ N_2(\tilde{u}_2 a + \tilde{v}_2 b + \tilde{w}_2 c) & N_2(\tilde{u}_2 b + \tilde{v}_2 d) & N_2(\tilde{u}_2 c + \tilde{w}_2 e) \\ N_3(\tilde{u}_3 a + \tilde{v}_3 b + \tilde{w}_3 c) & N_3(\tilde{u}_3 b + \tilde{v}_3 d) & N_3(\tilde{u}_3 c + \tilde{w}_3 e) \end{bmatrix} \tag{B27}$$

$$\times \begin{bmatrix} N_1\tilde{u}_1 & N_2\tilde{u}_2 & N_3\tilde{u}_3 \\ N_1\tilde{v}_1 & N_2\tilde{v}_2 & N_3\tilde{v}_3 \\ N_1\tilde{w}_1 & N_2\tilde{w}_2 & N_3\tilde{w}_3 \end{bmatrix} = M_D. \tag{B28}$$

Using Eqs. (B14)–(B16) in (B28), we get

$$\begin{bmatrix} N_1\tilde{u}_1\mathbf{E}_1 & N_1\tilde{v}_1\mathbf{E}_1 & N_1\tilde{w}_1\mathbf{E}_1 \\ N_2\tilde{u}_2\mathbf{E}_2 & N_2\tilde{v}_2\mathbf{E}_2 & N_2\tilde{w}_2\mathbf{E}_2 \\ N_3\tilde{u}_3\mathbf{E}_3 & N_3\tilde{v}_3\mathbf{E}_3 & N_3\tilde{w}_3\mathbf{E}_3 \end{bmatrix} \begin{bmatrix} N_1\tilde{u}_1 & N_2\tilde{u}_2 & N_3\tilde{u}_3 \\ N_1\tilde{v}_1 & N_2\tilde{v}_2 & N_3\tilde{v}_3 \\ N_1\tilde{w}_1 & N_2\tilde{w}_2 & N_3\tilde{w}_3 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{V}}_1.\tilde{\mathbf{V}}_1\mathbf{E}_1 & \tilde{\mathbf{V}}_1.\tilde{\mathbf{V}}_2\mathbf{E}_1 & \tilde{\mathbf{V}}_1.\tilde{\mathbf{V}}_3\mathbf{E}_1 \\ \tilde{\mathbf{V}}_2.\tilde{\mathbf{V}}_1\mathbf{E}_2 & \tilde{\mathbf{V}}_2.\tilde{\mathbf{V}}_2\mathbf{E}_2 & \tilde{\mathbf{V}}_2.\tilde{\mathbf{V}}_3\mathbf{E}_2 \\ \tilde{\mathbf{V}}_3.\tilde{\mathbf{V}}_1\mathbf{E}_3 & \tilde{\mathbf{V}}_3.\tilde{\mathbf{V}}_2\mathbf{E}_3 & \tilde{\mathbf{V}}_3.\tilde{\mathbf{V}}_3\mathbf{E}_3 \end{bmatrix} = M_D. \tag{B29}$$

Now, if we use the orthonormality of the eigenvectors, we get

$$M_D = \begin{bmatrix} \mathbf{E}_1 & 0 & 0 \\ 0 & \mathbf{E}_2 & 0 \\ 0 & 0 & \mathbf{E}_3 \end{bmatrix}. \tag{B30}$$

APPENDIX C: DATA AVAILABILITY

The data used for plotting Figs. 1 and 2 in this article will be shared on request to the corresponding author.

APPENDIX D: STELLAR ENVIRONMENT AND RADIATION PROCESS

In this appendix, we outline the essentials of emission mechanism of compact objects common to most of the models developed for that purpose. There are various models those have been successful to some extent to describe the emission mechanism from compact objects, like neutron stars or white dwarfs. Some of the most used ones are polar-cap [136], slotgap [137], outergap [138] models etc.

We will consider here a simple picture of the energetic-emission physics from a typical compact object. The specific details can be found in [139,140] and the references provided therein. The basic picture according to these models is that the electric field $\mathbf{E}_{||}$ is produced due to the rotating dipolar magnetic field B of the compact object and is directed parallel to the ambient dipolar field. Due to the action of electric field ($E_{||}$), charged particles (e^+ or e^-) are pulled out of the surface of the compact star. Their number density is n_{GJ} (where $n_{GJ} = \frac{\Omega B}{2\pi ce}$) [141]. The radiation is emitted from the charged particles, those are accelerated by the electric field $\mathbf{E}_{||}$.

For a typical pulsar of period 0.5 sec. and magnetic field $\mathrm{B} \sim 10^{12}$ Gauss, the Goldreich-Juliean number density n_{GJ} turns out to be $3.6 \times 10^{11}/\mathrm{cm}^3$. These charged particles on their way along the curved magnetic field radiate EM radiation that is called curvature radiation. The total energy released in the process is given by

$$\dot{E}_{\mathrm{rot}} = \frac{2}{3}\frac{\Omega^4 \overline{R}^6 \mathrm{B}_s^2}{e^3}. \tag{D1}$$

In the expression (D1) $\bar{R}$ is the radius, B_s is the surface magnetic field, and Ω is the rotational frequency of the compact star. The total observed energy or total luminosity coming from these objects have been observed to lie between 10^{25}–10^{35} erg/ sec. That emitted energy can be as high as 10^7 MeV for a pulsar having magnetic field $\mathrm{B} \sim 10^{12}$ Gauss and rotational period $P = 1$ sec [142]. The radiated photons of this energy may create a secondary plasma of e^+ and e^- pairs through the process, $\gamma + \gamma \to e^+ + e^-$. These secondary particles alter the number density of charge particles in the pulsar magnetosphere.

The energy of the emitted photons due to curvature radiations of the charge particles in the magnetosphere of the compact object can be expressed in terms of the instantaneous Lorentz factor (Γ) of the radiating charged particles. It is given by $\omega = \frac{3}{2}\frac{\Gamma^3}{R_c}$, when R_c is the radius of curvature of the dipolar magnetic field lines.

Evolution of Γ, taking radiation reaction into account and energy gain due to the electric field is described by [139]

$$m\frac{d\Gamma}{dx} = e\mathbf{E}_{||} - \frac{2e^2\Gamma^4}{3R_c^2}. \quad \text{(D2)}$$

When the distance from the center of the star is given by $r = x + \bar{R}$ ($\bar{R}$ is the radius of the compact object and x is the distance from the surface of the star). This equation can be derived from the Lorentz-Abhram-Diraction equation discussed in [110]. When the energy-gain becomes equal to energy loss in Eq. (D2), a quasisteady state is reached, which gives the estimate of Γ in terms of electric field. This is given by

$$\Gamma = \left(\frac{3\mathbf{E}_{||}\mathrm{R}_c^2}{2e}\right)^{\frac{1}{4}}. \quad \text{(D3)}$$

The electric field, $\mathbf{E}_{||}$ at a position r, ($r > \bar{\mathrm{R}}$) from the center of the compact object is given by [142,143] in the space charge limited flow in pulsar emission model:

$$\mathbf{E}_{||} \sim \frac{1}{8\sqrt{3}}(\Omega\bar{\mathrm{R}})^{\frac{5}{2}}\mathrm{B}\sqrt{\frac{2\bar{\mathrm{R}}}{r}}. \quad \text{(D4)}$$

In Eq. (D4), B is the surface magnetic field of the compact object. Since $\mathbf{E}_{||}$ is position dependent, it introduces position dependence to Γ, and thereby to the emitted radiation, ω. Once the emission region for the radiation of energy ω is determined, the distance it would travel in the magnetosphere of the object can be estimated from the knowledge of R_{LC} (when $R_{LC} = 1/\Omega$ is a light cylinder radius).

Though in principle one can solve Eq. (D2), to find out position dependence of the Lorentz factor, but solving it analytically is difficult. However it is possible to solve Eq. (D2) numerically. A numerical solution is provided in Fig. 4. The upward turning point (P) corresponds to the quasistatic limit in Eq. (D2). Similar behavior for the primary particles has also been reported in [144].

Since the peak emitted energy ω_c, of the curvature photon is given by

$$\omega_c = \frac{3}{2}\frac{\Gamma^3}{R_c}. \quad \text{(D5)}$$

Equations (D3)–(D5) can be used to estimate the emission point of photons of a particular energy.[8] The relation between the angular speed Ω, the point of emission$\frac{r}{\mathrm{R}_{LC}}$, radius $\bar{\mathrm{R}}$, magnetic field B, and energy of the emitted photon ω can be expressed in dimensionless quantities as

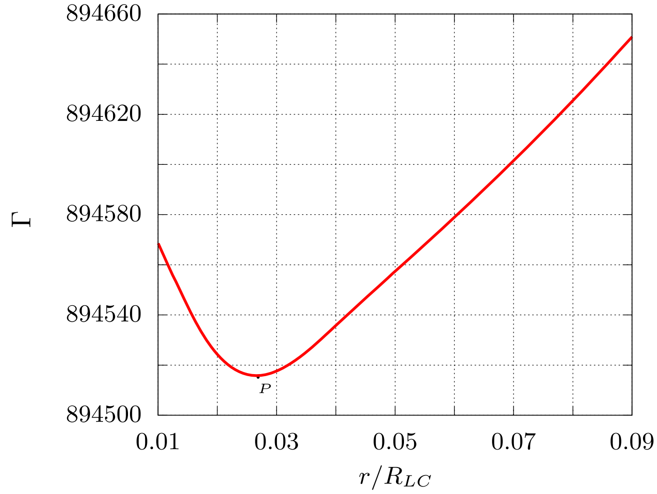

FIG. 4. Evolution of Γ (Lorentz factor) of primary charge particle in the pulsar atmosphere with radiation reaction. The lowest point (P) in the curve represents the situation when energy gain by the charge particle is equal to energy loss due to radiation reaction, i.e., when (D2) equals zero.

$$\Omega = \frac{1}{\bar{\mathrm{R}}}\left[\frac{(\bar{K}\frac{\mathrm{eB}}{m_e^2})^3}{(m_e\bar{\mathrm{R}})^2}\sqrt{\frac{r}{R_{LC}}}\left(\frac{\omega}{m_e}\right)^4\right]^{\frac{1}{7}}. \quad \text{(D6)}$$

In Eq. (D6), we have used a numerical parameter $\bar{K}$, defined as $\bar{K} = 4\pi\alpha\frac{160}{\sqrt{6}}(\frac{2}{3})^{4/3}$. One can estimate the path length $z = (R_{LC} - r)$ for photons produced at point r with energy ω from Eq. (D6) for various values of neutron star parameters like the radius, the magnetic field, and the rotational time period. The estimation of the density of the plasma is provided below.

The net number density of plasma produced in the magnetosphere can be estimated by dividing the luminosity of star by their estimated energy [139]. The combined effect of pair produced plasma and the plasma ejected from the surface of star can create a final plasma density, more than n_{GJ}. Following this point of view, we have taken the plasma frequency ω_p to be of the order 10^{-2} eV. We have further considered the photon path length to be $z = 1.2$ km [109]. For these numbers of ω_p and z, we have estimated various oscillation probabilities in KeV energy range (20–100) KeV as shown in Figs. 1 and 2.

The basic motivation for this choice of parameters was to find out the imprints of $\phi F^{\mu\nu}F_{\mu\nu}$ interaction on the non-thermal spectropolarimetric signals from the star magnetosphere is realized in nature. In the presence of such an interaction, some of these energetic photons may eventually oscillate into scalars and go out of the magnetosphere of the compact object and get detected by oscillating back into the photon. Although the same could be pointed out with the results available in the literature [10] and the references therein; however, in this work, we have considered the effect of magnetized medium to the oscillation probability.

[8]Here we have considered $R_c = \sqrt{r/\Omega}$, which is a good approximation for dipole magnetic fields, close to the light cylinder.

Our numerical estimates show that the emitted photon energy is $\omega \gg \omega_p$; therefore there is no self-absorption in the medium.

However the main process of obstruction, to the propagation of primary photons, comes in the form of absorption. There are few processes those contribute to this phenomenon. They are (i) $\gamma + \mathrm{B} \to e^+e^-$, (ii) $\gamma + \gamma \to e^+e^-$, (iii) scattering with ambient electrons in the medium, (iv) synchrotron self absorption etc.

For subthreshold photon energies (i.e., $\omega < 2m_e$) (as is the kinematics considered here), process (i) is forbidden. In order to have the second process viable, for photons with ~100 KeV energy, the minimum energy required for the second photon $\omega_{th} \geq m_e^2\omega$, which is also above a pair production threshold. It is unlikely that at $r > 0.99\mathrm{R}_{LC}$ the primary electrons will generate photons with energy $\omega = 10m_e$ or more. Therefore the second process also seems unlikely.

For stellar objects of mass M_s, temperature T, the scale height (defined as $h = \frac{k_B T \bar{R}^2}{m_p G M_s}$) is the quantity that determines the density of the atmosphere of the stellar object. Although the proton mass m_p is used in estimating the same, however if one invokes local charge neutrality the scale hight of electrons would also be similar. For a compact object with about one solar mass and temperature around 10^5 Kelvin the ambient atmosphere close to the light cylinder would be extremely thin to contribute to degradation of photon energy through Compton scattering.

Last, the synchrotron self-absorption coefficient follows a power law behavior $\alpha_\omega \propto \frac{1}{\omega^s}$, where the index $s > 2$. Hence the same may also be neglected for consideration towards absorption. Therefore for the kind of physical situation we have considered so far, the main mechanism towards photon depletion would essentially be due to conversion to scalars.

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