't Hooft anomalies for staggered fermions

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We show that the phase structure of certain staggered fermion theories can be understood on the basis of exact anomalies. These anomalies arise when staggered fermions are coupled to gravity which can be accomplished by replacing them by discrete Kähler-Dirac fermions. We first show the existence of a perturbative anomaly in even dimensions which breaks an exact $U(1)$ symmetry of the massless theory down to Z_4. If we attempt to gauge this Z_4 symmetry we find a 't Hooft anomaly which can only be canceled for multiples of two Kähler-Dirac fields. This result is consistent with the cancellation of a further mixed nonperturbative 't Hooft anomaly between the global Z_4 and a reflection symmetry. In four-dimensional flat space, theories of two staggered fields yield eight Dirac or 16 Majorana fermions in the continuum limit and this critical number of fermions agrees with results in condensed matter theory literature on the fermion content required to gap boundary fermions in $4+1$-dimensional topological superconductors. It is also consistent with constraints stemming from the cancellation of spin-Z_4 anomalies of Weyl fermions. Indeed, cancellation of 't Hooft anomalies is a necessary requirement for symmetric mass generation and this result gives a theoretical explanation of recent numerical work on the phase diagram of interacting staggered fermions. As an application of these ideas we construct a lattice model whose low-energy continuum limit is conjectured to yield the Pati-Salam grand unified theory.

I. INTRODUCTION

Staggered fermions are a well-known lattice discretization of relativistic fermions. Their use dates from the earliest days of lattice gauge theory [1] and they have since become one of the most popular realizations of lattice fermions employed in large-scale high-precision simulations of QCD—see Ref. [2] and references therein. Given this history one might imagine that there was little new to say concerning theoretical aspects of staggered fermions. In this paper we argue that this is not the case; staggered fermions are subject to certain gravitational 't Hooft anomalies which can be captured exactly in the lattice theory and which play a role in determining the infrared dynamics in these theories.

Our work was driven, in part, by recent observations of new phases of interacting staggered fermions in which the fermions acquire mass without breaking any exact lattice symmetries [3–8]. In this paper we will argue that the existence of such massive symmetric phases requires the cancellation of certain 't Hooft anomalies. To see these

anomalies one needs to generalize staggered fermions to curved space. Fortunately this is possible by exploiting their known connection to Kähler-Dirac fermions. In spite of the need for curved space to expose these anomalies we should stress that they have direct consequences for the possible phases of staggered fermions in flat space.

The plan of the paper is as follows. In the next section we present a lightning summary of the essential features of staggered fermions. This is followed by a review of Kähler-Dirac fermions including their connection to staggered fermions and their discretization on random triangulations which are necessary to describe curved space. We then show how discrete Kähler-Dirac fermions in even dimensions suffer from an anomaly that breaks an exact global $U(1)$ symmetry to Z_4. This global symmetry is a generalization of the usual $U(1)_\epsilon$ of staggered fermions to a general random triangulation.

We then demonstrate how to gauge the Kähler-Dirac action under this Z_4 symmetry and show that in general there is an obstruction to this process indicating the presence of a 't Hooft anomaly. We then show that this anomaly can be canceled and the partition function rendered gauge invariant if the theory contains multiples of two Kähler-Dirac fields. Finally we use a spectral flow argument to expose the presence of a further nonperturbative 't Hooft anomaly in the system corresponding to a mixed anomaly between the Z_4 symmetry and a reflection symmetry. It can be seen by examining the propagation of

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Kähler-Dirac fermions on nonorientable triangulations. We show that this anomaly can also be canceled if the number of Kähler-Dirac fields is zero mod 2. Finally we use these results to construct a staggered lattice model whose low-energy limit we argue targets the Pati-Salam grand unified theory (GUT).

II. REVIEW OF STAGGERED FERMIONS

In this section we briefly review the aspects of staggered fermions which we will need for our later discussion of Kähler-Dirac fermions. The free staggered fermion action takes the form

$$S = \sum_{x,\mu} \eta_\mu(x)\bar{\chi}(x)\Delta_\mu\chi(x) + \sum_x m\bar{\chi}(x)\chi(x), \quad (1)$$

where Δ_μ is a symmetric difference operator defined by

$$\Delta_\mu f(x) = \frac{1}{2}(f(x+\mu) - f(x-\mu)) \quad (2)$$

and $\eta_\mu(x) = (-1)^{\sum_{i=1}^{\mu-1} x_i}$ are the well-known staggered fermion phases which depend on the coordinates of sites in a hypercubic lattice. The staggered fields $\chi(x)$ and $\bar{\chi}(x)$ are single-component Grassmann fields. In addition to the simple $U(1)$ phase invariance corresponding to $\chi \to e^{i\beta}\chi$ and $\bar{\chi} \to \bar{\chi}e^{-i\beta}$ the staggered action at $m = 0$ is also invariant under an additional $U(1)$ symmetry which transforms the fields according to

$$\chi(x) \to e^{i\alpha\epsilon(x)}\chi(x),$$
$$\bar{\chi}(x) \to \bar{\chi}(x)e^{i\alpha\epsilon(x)}, \quad (3)$$

where $\epsilon(x) = (-1)^{\sum_{i=1}^d}$ is the site parity. Indeed in this limit the action decomposes into two independent pieces:

$$S = \sum_{x,\mu} \eta_\mu(x)\bar{\chi}_-(x)\Delta_\mu\chi_+(x) + \eta_\mu(x)\bar{\chi}_+(x)\Delta_\mu\chi_-(x), \quad (4)$$

where

$$\chi_\pm(x) = \frac{1}{2}(1 \pm \epsilon(x))\chi(x). \quad (5)$$

If we retain just one of these pieces—say the first—we obtain a *reduced* staggered action in which each lattice site contains a single reduced fermion field—χ_+ on even lattice sites and $\bar{\chi}_-$ for odd-parity sites. Renaming $\bar{\chi}_- \to \chi_-$ we can then write the reduced action in the simple form

$$S_{\text{red}} = \sum_{x,\mu} \eta_\mu(x)\chi(x)\Delta_\mu\chi(x). \quad (6)$$

The most common derivation of this action arises by spin diagonalizing the continuum Dirac action but it is better for our purposes to think of it as arising from discretization of the continuum action for a Kähler-Dirac field. In the next section we will review Kähler-Dirac fermions and their connection to staggered fermions.

III. KÄHLER-DIRAC FERMIONS ON AND OFF THE LATTICE

The Laplace–de Rham or Kähler-Dirac operator $d - d^\dagger$, where d is the exterior derivative, is a natural square root of the Laplacian and can be used to write down an equation for fermions which is an alternative to the Dirac equation [9]:

$$(d - d^\dagger + m)\Phi = 0. \quad (7)$$

The Kähler-Dirac field $\Phi = (\phi, \phi_\mu, \dots \phi_{\mu_1\dots\mu_D})$ is a collection of antisymmetric tensor (p-form) fields defined on a general D-dimensional Riemannian manifold. In flat space one can use these forms to build a $2^{D/2} \times 2^{D/2}$ matrix Ψ using elements of the Clifford algebra:

$$\Psi = \sum_{n=0}^D \phi_{\mu_1\dots\mu_n}\gamma^{\mu_1}\dots\gamma^{\mu_D}. \quad (8)$$

It is then a straightforward exercise to show that this matrix field satisfies the usual Dirac equation and describes $2^{D/2}$ degenerate Dirac spinors corresponding to the columns of Ψ [10].

The fact that Kähler-Dirac fermions are equivalent to multiplets of Dirac fermions in flat space ceases to be true in curved space. One way to see this is to recognize that the Kähler-Dirac equation [Eq. (7)] is well defined for any smooth manifold—not just those supporting a spin structure. In fact the global properties of Kähler-Dirac fermions are very different from Dirac—for example the zero modes of the Kähler-Dirac equation on a compact space are simply the harmonic forms. This means that there are zero modes on spaces with positive curvature such as the sphere which is not true of Dirac fermions.

The Kähler-Dirac equation possesses another key advantage over the Dirac equation; it can be discretized *without* introducing spurious fermion doubler modes.[1]

We assume that any curved space can be approximated by a suitable oriented triangulation and discretization proceeds by mapping continuum p-forms to fields defined on p-simplices in the triangulation (sometimes called p-cochains in the literature). Each p-simplex is specified by a list of $(p + 1)$ vertex labels $[a_0, \dots a_p]$ and the p-simplex field $\Phi^{(p)}$ is given by the formal sum

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[1]Of course the resulting lattice theory leads to $2^{D/2}$ Dirac spinors in the continuum limit but this is also a feature of the continuum theory—discretization does not add to this existing degeneracy.

$$\Phi^{(p)} = \sum_{p\text{-simplices}} [a_0, ..., a_p] \phi^{(p)}_{[a_0,...,a_p]}. \tag{9}$$

Discrete analogs of d and its adjoint $d^\dagger$ exist—the coboundary $\bar{\delta}$ and boundary δ operators [11–14] with the action of δ on a p-simplex field being given by

$$\delta_p \Phi^{(p)} = \sum_p \sum_{k=0}^p (-1)^k [a_0, ...\widehat{a_k}, ..., a_p] \phi^{(p-1)}_{[a_0,...\widehat{a_k},...,a_p]}, \tag{10}$$

where $\widehat{a}_k$ denotes the vertex that is removed to get the kth boundary $(p-1)$-simplex. Using the abbreviated notation

$$C_p \equiv [a_0, ..., a_p], \tag{11}$$

we can write this as

$$\delta_p \phi(C_p) = \sum_{C_{p-1}} I(C_p, C_{p-1}) \phi(C_{p-1}), \tag{12}$$

where $I(C_p, C_{p-1})$ is a $N_p \times N_{p-1}$ incidence matrix whose matrix elements are $+1$ if C_{p-1} is contained in the boundary of C_p with the correct orientation, -1 if it occurs with opposite orientation and zero otherwise. Similarly

$$\bar{\delta}_p \phi(C_p) = \sum_{C_{p+1}} I(C_{p+1}, C_p)^\dagger \phi(C_{p+1}). \tag{13}$$

The Laplacian operator which maps p-simplex fields to p-simplex fields is then

$$I(C_p, C_{p-1}) I(C_p, C_{p-1})^T + I(C_{p+1}, C_p)^T I(C_{p+1}, C_p). \tag{14}$$

Zero modes of the Kähler-Dirac operator $\delta - \bar{\delta}$ are simultaneously (discrete) harmonic forms.

Finally an oriented triangulation is one in which each D-simplex is equipped with a additional Z_2 element $\tau(C_D)$ which represents the orientation inside the simplex. One can think of it as classifying whether a given vertex ordering is an odd or even permutation of some fixed vertex ordering such as $a_0 < a_1 < ...a_D$. The boundary operator is modified to $\delta_D \to \tau \delta_D$. For an orientable triangulation $\tau(C_D)$ can be chosen in such a way that each face is held with opposite orientation in the two D-simplices that share it.

The discrete Kähler-Dirac equation is simply

$$(\delta - \bar{\delta} + m)\Psi = 0, \tag{15}$$

where $\delta = \sum_p \delta_p$.

Both the continuum and discrete Kähler-Dirac operators anticommute with an operator Γ which acts on a given p-form field by multiplying it by $(-1)^p$. This implies that the associated *massless* Kähler-Dirac action is invariant under a $U(1)$ symmetry which acts as

$$\Phi \to e^{i\alpha\Gamma}\Phi,$$
$$\bar{\Phi} \to \bar{\Phi}e^{i\alpha\Gamma}. \tag{16}$$

In the continuum its action on the matrix fermion Ψ corresponds to $\gamma_5 \Psi \gamma_5$ and hence in flat space it is sometimes called a "twisted" chiral symmetry. Using Γ one can build projectors and reduced Kähler-Dirac fields $\Phi_\pm$ in complete analogy to our earlier discussion of reduced staggered fields. The reduced Kähler-Dirac action for four reduced fields takes the form

$$S_{\text{RKD}} = \int \bar{\Phi}_+ K \Phi_- \quad \text{with} \quad K = \delta - \bar{\delta}. \tag{17}$$

We can rewrite this in a useful form by introducing the reduced field $\Psi^T = (\bar{\Phi}_+ \Psi_-^T)$ as

$$S_{\text{RKD}} = \int \Psi^T \mathcal{K} \Psi, \quad \text{where} \quad \mathcal{K} = \begin{pmatrix} 0 & K \\ -K^T & 0 \end{pmatrix}. \tag{18}$$

In flat space one can map the discrete p-form fields into fields defined on the p-cells of a regular hypercubic lattice rather than a triangulation. By introducing a new lattice with half the lattice spacing one can replace these p-cell fields with single-component fields on this finer lattice according to the prescription

$$\chi(\mathbf{x} + \hat{\mu}_1 + \hat{\mu}_2 + \cdots + \hat{\mu}_p) = \phi_{[\mu_1...\mu_p]}(x). \tag{19}$$

From these we can form a discrete Kähler-Dirac matrix field using

$$\Psi(x) = \sum \chi(\mathbf{x} + \hat{\mu}_1 + \cdots + \hat{\mu}_p) \gamma^{\mu_1}...\gamma^{\mu_D}$$
$$= \sum_b \chi(x+b)\gamma^{x+b}, \tag{20}$$

where b is a vector whose components are either zero or one with the sum extending over the unit hypercube of the lattice associated with lattice site x. We now plug this into the continuum matrix action replacing integrals by lattice sums and the derivative operator by a symmetric difference operator to obtain

$$S = \sum_x \sum_{b,b'} \bar{\chi}(x+b)\chi(x+b'+\mu)\text{Tr}[(\gamma^{x+b})^\dagger \gamma_\mu \gamma^{x+b'+\mu}]$$
$$- (\mu \to -\mu). \tag{21}$$

Evaluating the trace we find a factor of $\eta_\mu(x+b)\delta_{b,b'}$ and the action collapses to the usual one for staggered fermions.

It can be seen that the staggered fermion phases $\eta_\mu(x)$ just reflect the antisymmetry of the forms while the operator Γ becomes just the usual site parity $\epsilon(x)$. Notice that this Kähler-Dirac description makes clear how to assemble the staggered fermion fields to rebuild the continuum Dirac spinors—they are once again given by columns of the matrix field Ψ in the given basis of Dirac matrices. At this point it should be clear that the staggered fermion action is merely a form of the discrete Kähler-Dirac action specialized to a flat hypercubic lattice.

IV. A PERTURBATIVE ANOMALY

The existence of Γ ensures that the spectrum of the massless Kähler-Dirac operator on a general random triangulation pairs a state with eigenvalue λ to another with eigenvalue $-\lambda$ and ensures that zero modes of the Kähler-Dirac operator are eigenstates of Γ. Indeed, the Kähler-Dirac operator on a general triangulation obeys an index theorem

$$n_+ - n_- = \chi, \qquad (22)$$

where $n_\pm$ denotes the numbers of zero modes with $\Gamma = \pm 1$ and χ is the Euler characteristic of the space.

This index theorem can be related to an anomaly for the $U(1)$ symmetry given in Eq. (16) by examining the variation of the lattice Kähler-Dirac fermion measure under the symmetry [13]. The latter is given by

$$D\Phi D\bar{\Phi} = \prod_{p=0}^{D} \prod_{i=1}^{N_p} d\phi_p(i) d\bar{\phi}_p(i). \qquad (23)$$

Under $e^{i\alpha\Gamma}$ this transforms as

$$D\Phi D\bar{\Phi} \rightarrow e^{2i\alpha(N_0 - N_1 + \cdots + N_D)} D\Phi D\bar{\Phi} = e^{2i\alpha\chi} D\Phi D\bar{\Phi}, \quad (24)$$

where χ is the Euler character of the space. Compactifying R^D to S^D where $\chi = 2$ one discovers that this gravitational anomaly breaks $U(1) \rightarrow Z_4$. Notice that the appearance of an anomaly in this lattice theory requires only the existence of exact zero modes carrying well-defined charge under the generator of the symmetry—it does *not* require an infinite numbers of degrees of freedom—and the anomaly can be computed precisely on the coarsest triangulation of a manifold with a fixed topology. Ultimately this results from the fact that the number and identity of zero modes of the Kähler-Dirac operator are given by the ranks of the homology groups of the space which are determined by its triangulation [11]. Notice that this remaining Z_4 symmetry is sufficient to prohibit fermion bilinears from arising as quantum corrections to the effective action but allows for four-fermion terms.

One can think of this anomaly as a form of mixed 't Hooft anomaly corresponding to the breaking of the global $U(1)$ symmetry of staggered fermions in a nontrivial background gauge field corresponding to curved space.

The existence of this anomaly has important consequences for theories of *reduced* Kähler-Dirac fermions coupled to a $U(1)$ gauge field and propagating on a curved background. The anomaly in this case now reflects a violation of gauge invariance and ruins the consistency of the theory. This is analogous to the fact that the usual Adler-Bell-Jackiw (ABJ) anomaly of Dirac fermions implies that it is not possible to couple a single Weyl field to a $U(1)$ gauge field. For a set of reduced fields it can canceled if the sum of their Γ charges vanishes.

Alternatively the anomaly can be canceled if the theory lives on the boundary of a higher-dimensional space where the bulk theory contains a topological Chern-Simons term whose anomalous gauge variation cancels that of the boundary fermions. Details of such an anomaly inflow mechanism for the case of a three-dimensional bulk with two-dimensional massless Kähler-Dirac boundary fermions were given in [15]. This construction generalizes straightforwardly to yield a $(4 + 1)$ topological insulator model for Kähler-Dirac fermions. The bulk theory consists of a topological gravity theory of Chern-Simons type [16–18] whose boundary contains a massless four-dimensional reduced Kähler-Dirac field gauged under a $U(1)$ symmetry which is inherited from the de Sitter gauge symmetry of the bulk gravity theory.

V. GAUGING Z_4

Quite generally the presence of a 't Hooft anomaly can be understood as representing an obstruction to gauging global symmetries. With this in mind it is instructive to see how one would go about gauging the residual Z_4 global symmetry discussed in the last section. Again, we will carry out this procedure directly in the lattice theory. To do this we must first generalize the boundary operator given in Eq. (10) in such a way the action is invariant under a local Z_4 transformation of the Kähler-Dirac field corresponding to

$$\phi(C_p) \rightarrow e^{i\frac{\pi}{2}(-1)^p n(C_p)} \phi(C_p), \quad n(C_p) = 0, 1, \ldots, 3. \quad (25)$$

This can be done if the incidence matrices $I(C_p, C_{p-1})$ introduced in Eq. (12) are generalized to Z_4 gauge fields $U(C_p, C_{p-1})$ that transform under gauge transformations as

$$U(C_p, C_{p-1}) \rightarrow e^{-i\frac{\pi}{2}(-1)^p n(C_p)} U(C_p, C_{p-1}) e^{i\frac{\pi}{2}(-1)^p n(C_{p-1})}. \qquad (26)$$

In this way a locally Z_4 invariant massless action can be constructed.[2] However, to check for gauge invariance of the full quantum theory we also need to examine the

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[2]It is not hard to take this prescription applied to a cubical cell decomposition of a regular hypercubic lattice and show that the fields $U(C_p, U_{p-1})$ become the usual gauge links of a staggered fermion.

measure. For a single Kähler-Dirac field there are two fermion integrations per p-simplex $\int d\phi(C_p)d\bar{\phi}(C_p)$ and in general this changes by an element of Z_2 under a local Z_4 transformation. However, it should be clear that the measure can be made locally invariant if the system contains multiples of two Kähler-Dirac fields.

Since 't Hooft anomalies are RG invariants a nonzero UV anomaly generically requires either massless composite fermions or Goldstone bosons to be present in the spectrum of the low-energy theory.[3] This implies that only theories with vanishing UV 't Hooft anomalies can have a trivial gapped state in the IR that characterizes a massive symmetric phase.

VI. A NONPERTURBATIVE MIXED ANOMALY

We have shown that theories of Kähler-Dirac fermions in even dimensions possess a discrete global Z_4 symmetry on coupling to gravity. Furthermore we inferred that the system possesses a 't Hooft anomaly which arises if we try to gauge this Z_4 symmetry. Cancellation of this anomaly dictates that the theory must contain multiples of two Kähler-Dirac fields. We can now ask whether this theory possesses any additional anomalies. In particular we are interested in possible mixed 't Hooft anomalies that arise if we attempt to gauge any additional global symmetries.

The staggered fermion theory in flat space is clearly invariant under a reflection symmetry that inverts one lattice direction.[4] In the continuum a reflection operation corresponds to changing from a right- to a left-handed coordinate system and hence reverses the orientation.

We can now ask what happens if we now consider gauging the reflection symmetry. This means allowing for the freedom to choose the orientation locally on the space. On an orientable space it is possible to choose a single orientation for the entire space and this corresponds to a constant (trivial) background for the corresponding Z_2 orientation gauge field. However, on a nonorientable space this is not possible. Propagation on such a space then corresponds to choosing a nontrivial background for the associated orientation gauge field. Notice that these statements can be applied equally to a triangulation of the space. One can ask whether the use of such a background breaks any global symmetries of the theory. To answer this we will consider a triangulation of a particular nonorientable space —the real projective plane RP^D. This has Euler characteristic $\chi = 1$ corresponding to the fact that on RP^D the Kähler-Dirac operator possesses just a single zero mode (a 0-form). In fact the real projective plane is obtained by identifying antipodal points on the sphere $RP^D \sim S^D/Z_2$. The question is, what happens to the partition function of a

discrete Kähler-Dirac fermion propagating on such a triangulation?

Let us decompose the Kähler-Dirac field into two reduced fields Φ^1 and Φ^2. To allow for possible Z_4 invariant four-fermion terms we couple these fields to a scalar field σ. To generate four-fermion terms one would need to add additional terms quadratic in σ to the action but the argument we give below holds robustly for *any* action which is even in σ including actions containing scalar kinetic terms. The fermion operator is given by

$$M = \delta^{ab}\mathcal{K} + \sigma(x)\epsilon^{ab}. \tag{27}$$

We will assume that the total action is invariant under a discrete symmetry which extends the fermionic Z_4 discussed in the previous section:

$$\Phi^a \to i\Gamma\Phi^a, \tag{28}$$

$$\sigma \to -\sigma. \tag{29}$$

Notice that this fermion operator is antisymmetric and real and hence all eigenvalues of M lie on the imaginary axis. The partition function is then given by the Pfaffian $\mathrm{Pf}(M(\sigma))$ where we will define the latter as the product of the eigenvalues of $M(\sigma)$ in the upper half plane in the background of some reference configuration $\sigma = \sigma_0$. By continuity we define the Pfaffian to be the product of these same eigenvalues under fluctuations of σ and, in particular, under the Z_4 transformation $\sigma \to -\sigma$. The question that then arises is whether the Pfaffian is invariant under Z_4.

At first glance it seems all is well since it is easy to prove that

$$\Gamma M(\sigma)\Gamma = -M(-\sigma). \tag{30}$$

This result shows that the spectrum and hence the determinant are indeed invariant under the Z_4 transformation $\sigma \to -\sigma$. But this is not enough to show the Pfaffian itself is unchanged since there remains the possibility that an odd number of eigenvalues flow through the origin as σ_0 is deformed smoothly to $-\sigma_0$ leading to a sign change. To understand what happens we consider a smooth interpolation of σ:

$$\sigma(s) = s\sigma_0 \quad \text{with} \quad s \in (-1, +1). \tag{31}$$

The question of eigenvalue flow can be decided by focusing on the behavior of the eigenvalues of the fermion operator closest to the origin at small s. In this region the eigenvalues of smallest magnitude correspond to zero modes of the Kähler-Dirac operator. There is a single such mode on RP^D which satisfies the eigenvalue equation:

$$\sigma_0 s\epsilon^{ab}v^b = \mu v^a. \tag{32}$$

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[3]Another possibility is that the IR phase corresponds to some sort of nontrivial topological field theory.

[4]Note that in Euclidean space such a reflection symmetry and time reversal invariance are equivalent.

The two eigenvalues $\mu = \pm i\sigma_0 s$. Clearly these eigenvalues change sign as s varies from positive to negative values which indeed leads to a Pfaffian sign change. This can also be seen explicitly from Eq. (30) since

$$\mathrm{Pf}[M(-\sigma)] = \det[\Gamma]\mathrm{Pf}[M(\sigma)] = -\mathrm{Pf}[M(\sigma)]. \quad (33)$$

We thus learn that the Pfaffian corresponding to two reduced Kähler-Dirac fields or equivalently one full Kähler-Dirac field indeed changes sign under the Z_4 transformation. On integration over σ the value of any Z_4 invariant function of σ, including the partition function itself, would then yield zero rendering expectation values of such operators ill defined. This corresponds to a non-perturbative mixed anomaly between the Z_4 and reflection symmetries. Our method of derivation is similar to that used by Witten in discussing a nonperturbative anomaly for odd numbers of Weyl fermions in $SU(2)$ [19].

Again, we see that this anomaly can be canceled for systems possessing multiples of two Kähler-Dirac fields since then eigenvalues flow through the origin in pairs and the sign of the partition function does not change. This is true for a variety of four-fermion interactions since one can always perform orthogonal rotations on $2N$ reduced fermions to put the Yukawa interaction in the canonical form $(\lambda_1 i\sigma_2 \oplus \lambda_2 i\sigma_2 \oplus \cdots \oplus \lambda_N i\sigma_2)$.

For staggered fermions in flat space one should think of this breaking of Z_4 as the manifestation of a mixed 't Hooft anomaly arising as a result of gauging the reflection symmetry.

We learn from these arguments that the minimal model of staggered fermions with no 't Hooft anomalies, which is hence capable of symmetric mass generation, contains two staggered or equivalently four *reduced* staggered fields. In the continuum limit such a lattice theory gives rise to four or eight Dirac fermions in two and four dimensions, respectively, which matches the number of fermions which are needed to cancel off the discrete fermion parity and spin-Z_4 anomalies of Weyl fermions in two and four dimensions, respectively [20].

VII. PATI-SALAM MODEL ON THE LATTICE

As an application of these results we will discuss a possible construction of the Pati-Salam GUT model using staggered fermions. We start by considering a four-dimensional continuum theory of four massless Kähler-Dirac fields in which the Z_4 't Hooft anomaly discussed in the previous section vanishes. The flat space action separates into two independent pieces $S = S_1 + S_2$, where

$$S_1 = \int d^4x \mathrm{Tr}[\bar{\Psi}_+ \gamma_\mu \partial_\mu \Psi_-],$$

$$S_2 = \int d^4x \mathrm{Tr}[\bar{\Psi}_- \gamma_\mu \partial_\mu \Psi_+], \quad (34)$$

where each term depends on four *reduced* Kähler-Dirac fields that transform under separate global $SU(4)$ symmetries and the trace in this expression extends over both the $SU(4)$ and internal matrix indices for each reduced field. Suppressing the $SU(4)$ indices and adopting a Euclidean chiral basis for the Dirac matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \quad (35)$$

with $\sigma_\mu = (I, i\sigma_i)$ and $\bar{\sigma}_\mu = (I, -i\sigma_i)$ we find that the reduced fermion field $(\bar{\Psi}_+, \Psi_-)$ appearing in S_1 can be written in the block form

$$\Psi_- = \begin{pmatrix} 0 & \psi_R \\ \psi_L & 0 \end{pmatrix}, \qquad \bar{\Psi}_+ = \begin{pmatrix} \bar{\psi}_L & 0 \\ 0 & \bar{\psi}_R \end{pmatrix}.$$

Evaluating the (internal) trace the action becomes

$$S = \int d^4x [\mathrm{tr}(\bar{\psi}_R \bar{\sigma}^\mu \partial_\mu \psi_R) + \mathrm{tr}(\bar{\psi}_L \sigma^\mu \partial_\mu \psi_L)].$$

We see that each 2×2 block corresponds to a doublet of Weyl spinors transforming under a $SU(2) \times SU(2)$ flavor symmetry in addition to the $SU(4)$ symmetry. Thus Ψ_- contains the spinor representations $(4, 2, 1)_L \oplus (4, 1, 2)_R$ which makes explicit the fact that the two blocks transform under separate $SU(2)$ flavor groups. In Minkowski space we can then trade the right-handed fermions in the usual way for left-handed fermions via $\psi_R = i\sigma_2 \psi_L^{\prime *}$. After doing this it is clear that the reduced Kähler-Dirac field decomposes into a set of left-handed Weyl fields in the representations $(4, 2, 1) \oplus (\bar{4}, 1, 2)$. Thus both the symmetries and representations of this reduced Kähler-Dirac field *in flat space* clearly match those of the Pati-Salam GUT model [21].

However this connection to Pati-Salam is lost if we add in the other sector corresponding to the reduced field $(\bar{\Psi}_-, \Psi_+)$. To recover the Pati-Salam model we must ensure that this *mirror* sector decouples from low-energy physics by adding suitable interactions that are capable of gapping just this sector *without* breaking any symmetries either explicitly or spontaneously. To avoid spontaneous symmetry breaking we require that all 't Hooft anomalies vanish—which from our previous arguments will hold here because the fermions come in multiples of four. To avoid explicitly breaking the symmetry we add a Z_4 symmetric four-fermion term of the form

$$\frac{G^2}{2} \int d^4x \epsilon_{abcd}[\mathrm{tr}(\bar{\Psi}_-^a \bar{\Psi}_-^b)\mathrm{tr}(\bar{\Psi}_-^c \bar{\Psi}_-^d)$$
$$+ \mathrm{tr}(\Psi_+^a \Psi_+^b)\mathrm{tr}(\Psi_+^c \Psi_+^d)], \quad (36)$$

where we have made make explicit the $SU(4)$ indices in the mirror sector and retain only a trace tr over internal $SU(2) \times SU(2)$ indices. Equivalently we can introduce a scalar field ϕ that transforms in the real six-dimensional representation of $SU(4)$ and rewrite the four-fermion term as a Yukawa interaction:

$$\delta S = G \int d^4x \hat{\phi}_{ab} \mathrm{tr}[\bar{\Psi}^a_- \bar{\Psi}^b_- + \Psi^a_+ \Psi^b_+], \qquad (37)$$

where $\hat{\phi}$ satisfies

$$\hat{\phi}_{ab} = \frac{1}{2}\left(\phi_{ab} + \frac{1}{2}\epsilon_{abcd}\phi_{cd}\right). \qquad (38)$$

Using this interaction we might hope to generate masses for the mirrors for large coupling G due to the formation of a condensate of mirror $SU(4)$ baryons. One immediate objection to this strategy for gapping the mirror sector is that any four-fermion interaction corresponds to a perturbatively irrelevant operator and hence should not be capable of changing the IR behavior of a theory. However, numerical work on a related Higgs-Yukawa model with $SO(4)$ rather than $SU(4)$ symmetry provides evidence that it this may be possible nonperturbatively with a symmetric four-fermion condensate forming for sufficiently strong coupling [8,22].

Furthermore, it may be possible to evade the question of whether such four-fermion terms are relevant by instead gauging the $SU(4)$ symmetry of the mirror sector. In that case confinement can drive the mirror sector into a symmetric gapped phase with four-fermion condensate even for small Yukawa coupling—a scenario that has been advocated by Razamat and Tong in the context of chiral fermions [23]. Again, there is evidence in favor of this scenario from lattice simulations of a gauged $SO(4)$ Higgs-Yukawa model [24].

It should be clear at this point how to construct a lattice mirror model that targets the Pati-Salam theory in the naive continuum limit—simply replace the continuum Kähler-Dirac field Ψ by a staggered lattice field χ transforming in the fundamental representation of an $SU(4) \times SU(4)$ symmetry together with suitable Yukawa interactions targeting a reduced component of the full staggered field. In detail the proposed lattice action is

$$S = \sum_{x,\mu} \eta_\mu(x)[\bar{\chi}_+ \Delta_\mu \chi_- + \bar{\chi}_- \Delta^c_\mu \chi_+], \qquad (39)$$

$$G \sum_x \hat{\phi}_{ab}[\bar{\chi}^a_- \bar{\chi}^b_- + \chi^a_+ \chi^b_+] + \frac{1}{2} \sum_x \hat{\phi}^2_{ab}, \qquad (40)$$

and the lattice covariant difference operator in the mirror sector is given by

$$\Delta^c_\mu \chi_+(x) = U_\mu(x)\chi_+(x+\mu) - U^\dagger_\mu(x-\mu)\chi_+(x-\mu) \qquad (41)$$

with $U_\mu(x)$ an $SU(4)$ lattice gauge field. The mirror sector should confine and produce a four-fermion condensate for any nonzero G without breaking symmetries while the low-energy sector should be a free theory whose continuum limit corresponds to 16 chiral fermions transforming in the Pati-Salam representations of an $SU(4) \times SU(2) \times SU(2)$ global symmetry.[5] Notice that it is not possible to write down couplings of the Pati-Salam and mirror sectors that are $SU(4)$ gauge invariant so that the Pati-Salam sector should remain completely decoupled from the mirror sector in flat space.

VIII. CONCLUSIONS

In this paper we have shown that staggered fermions experience both perturbative and nonperturbative gravitational anomalies. To see these anomalies we need to generalize staggered fields to discrete curved spaces by promoting them to Kähler-Dirac fields. The perturbative anomaly breaks an exact global $U(1)$ symmetry of the massless theory down to Z_4 in even dimensions. If we try to gauge this Z_4 symmetry, we detect the presence of a further 't Hooft anomaly. Cancellation of this anomaly requires the theory to contain multiples of two Kähler-Dirac fields. This constraint can also be seen in the presence of a mixed anomaly between the global Z_4 symmetry and a reflection symmetry when the latter is gauged by considering the propagation of Kähler-Dirac fields on nonorientable triangulations. It is quite remarkable that these anomalies are manifest even in these discrete systems and place direct constraints on the continuum infrared behavior of these theories.

Similar exact lattice 't Hooft anomalies have been found for central branch Wilson fermions in two dimensions [26]. In this case it is again an exact staggered $U(1)$ symmetry, similar to the usual staggered $U_\epsilon(1)$ encountered in this analysis that plays a crucial role.

In the continuum limit two staggered fields give rise to four or eight Dirac fermions in two and four dimensions, respectively, which matches the number of fermions which are needed to cancel off the discrete fermion parity and spin-Z_4 anomalies of Weyl fermions in two and four dimensions, respectively [20]. In odd dimensions the story is similar. Staggered fermion theories with four-fermion interactions experience a 't Hooft anomaly associated with the Z_4 symmetry that can be canceled only for multiples of two staggered fields. In one and three dimensions such theories yield eight and 16 Majorana fermions in the continuum limit which again matches the number needed to cancel 't Hooft anomalies arising from time reversal invariance. It appears that the constraints arising from canceling gravitational anomalies of Kähler-Dirac fermions

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[5]The current proposal differs from that given in [25] where the staggered field transforms in the eight-dimensional representation of a spin(7) group with a different set of Yukawa interactions.

match those associated with the vanishing of a variety of discrete anomalies for Weyl or Majorana fermions.

As discussed earlier the cancellation of all 't Hooft anomalies is a necessary condition for symmetric mass generation—the appearance of a trivial gapped phase in theories of interacting fermions and the constraints described in this paper agree with results for gapping boundary states in topological superconductors [27,28]. A general review that summarizes and synthesizes these results can be found in [29]. The cancellation of these anomalies also explains recent numerical studies of symmetric mass generation with staggered fermions [8,22].

Finally we show how to construct a simple continuum Kähler-Dirac theory satisfying these constraints that separates into a mirror sector which can be decoupled from low-energy physics and a light sector whose matter representations and global symmetries match those of the Pati-Salam GUT model. It is interesting that considerations of gravitational anomaly cancellation for Kähler-Dirac fermions yield a well-known GUT model. Furthermore, the Kähler-Dirac construction suggests the existence of a massive mirror or dark sector whose vacuum consists of a condensate of $SU(4)$ baryons together with massive excitations that couple only gravitationally to the Pati-Salam fields in the low-energy sector. Clearly more work is needed to clarify whether these features can be exploited to construct models of relevance to cosmology.

Finally the fact that the anomaly structure survives intact under discretization naturally leads to a proposal for a staggered fermion model that targets the Pati-Salam GUT in the continuum limit.

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