First-order QCD transition in a primordial magnetic field

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Recalling the expectation of an extremely strong primordial magnetic field H, we recheck transitions among the phases of chiral symmetry restoration (χSR), chiral symmetry breaking (χSB), and pion superfluidity (πSF) in the QCD epoch of the early universe. For homogeneous phases in a finite H, a sensible scheme is adopted to determine the phase boundaries of πSF , which is also the superconductivity phase itself. In the first part, the QCD phase diagrams are studied in detail within the chiral effective Polyakov-Nambu–Jona-Lasinio model, and the transitions involving πSF are found to be of first order at relatively small H. As expected from the Meissner effect, the regime of πSF shrinks with increasing H and completely vanishes beyond a threshold value. In the second part, the bubble dynamics is illuminated for the stronger first-order transition, $\chi SR \rightarrow \pi SF$, in the more convenient Polyakov-quark-meson model. The coupled equations of motion of pion condensate and magnetic field are solved consistently to give the bubble structure. Then, based on bubble collisions, we explore gravitational wave (GW) emission by developing a simple toy model in advance; and the characteristic frequency of the relic GW is estimated to be of the order 0.1–1 K or 10^9-10^{10} Hz in our galaxy.

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I. INTRODUCTION

The exploration of possible phases in quantum chromodynamics (QCD) systems is a renewing topic in both low and high energy nuclear physics. In low energy nuclear physics, quarks are confined and the color singlet hadrons, mainly nucleons and pions, are expected to be effective degrees of freedom in nuclear matter. Because of strong attractive interactions, a dilute nucleonic system was found to prefer self-clustering to the saturation density with almost constant energy per nucleon [1,2]. Such a property implies a first-order gas-liquid transition for isospin symmetric nuclear matter at low temperature [3]. Moreover, celestial neutron stars were expected to be realistic correspondences of isospin asymmetric nuclear matter [4], and physicists had proposed many relevant phases for neutron stars, such as the one with the presence of Cooper pairing of neutrons [5], pasta structure [6,7], hyperon degrees of freedom [8], pion condensation [9], or Kaon condensation [10]; see also the review [11].

The high energy nuclear physics is characterized by considering quark degrees of freedom in the many-body QCD system [12]. Relativistic heavy ion collisions [13] were proposed to look for quark-gluon plasma, the phase where quarks and gluons are released from the interiors of nucleons. Such a phase was justified, and its properties were also well studied in heavy ion colliders (HICs) [14]. In the early stage, HICs usually explored a high temperature and low chemical potential region where no sign of ordered phase transition was ever found [15–18]. Recently, several

experiments focus on the low temperature and high chemical potential region in order to look for the critical end point (CEP) of chiral transition [19]. Similarly, deconfined quarks were also proposed to exist in celestial bodies, mainly the inner cores of neutron stars [20] and quark stars [21], with color superconductors of several kinds [22] and quarkyonic matter [23–26] possible phases.

Actually, the early universe is full of phase transitions [27] with electroweak (EW) and QCD transitions among the earliest ones. The EW transition was sometimes taken to account for baryogenesis [28] and a strong primordial magnetic field was also assumed to be seeded there [29–31]. Though nonvanishing, the baryon density is very small in the early universe [32], so the QCD chiral transition was expected to be a trivial crossover, the same as that in HICs. However, the QCD epoch became more and more interesting when charged pion superfluidity was found to be favored for relatively large light lepton densities [33–35]. The work is an extension of our previous work [35] by taking the primordial magnetic field into account. Two changes are expected: the second-order phase boundary of pion superfluidity might become of first order due to the Meissner effect induced by the magnetic field [36], and consequently the first-order transition would induce a generation of gravitational wave [21,37] even without the inflation effect. Though gravitational wave (GW) emission has been widely studied in the first-order EW transition [38–41], this might be the first time that a reliable firstorder transition is applied to generate GW directly in the QCD epoch.

It is subtle to explore the possibility of a superconductor phase when an external magnetic field is present. For a type-II superconductor, vortical structure can be obtained by consistently solving the coupled equations of motion (EoMs) of the charged condensate and magnetic field, and the finite magnetic field could penetrate through the vortices [42]. However, for a type-I superconductor, the magnetic field can only be present at the surface of the superconductor, known as the Meissner effect [43]. Then, how could we consistently explore the transition between phases with and without a magnetic field in the bulk for a type-I superconductor? The problem had been well addressed in the textbook Ref. [36]: the external magnetic field (H), rather than the total magnetic field (B), is the same for both phases and thus serves the correct variable of free energy for the study of superconductivity. That means we have to transform the Helmholtz free energy with B the variable to Gibbs free energy with H the variable, and the Gibbs free energy of the superconductor can be shown to be the same as that with H = 0.

The paper is mainly composed of two parts. In the first part, Sec. II, we explore the QCD phase diagrams of the early universe by accounting for a primordial magnetic field within the Polyakov-Nambu-Jona-Lasinio (PNJL) model. Formalisms are developed for chiral phases and pion superfluidity in Secs. II A and II B, separately, where we derive free energy, gap equations, and relevant thermodynamic quantities. The most important sectors of a strong interaction are studied in detail in Secs. II A 1 and II B 1, and the sectors of electroweak interaction are briefly summarized in Secs. II A 2 and II B 2. The numerical results of this part are shown in Sec. II C together with relevant discussions. In the second part, Sec. III, we first study bubble dynamics during the first-order QCD transition of the early universe by adopting the two-flavor Polyakov-quark-meson (PQM) model in Sec. III A. And then in Sec. III B, the results are applied to briefly explore the features of the gravitational wave generated by bubble collisions within a toy model. Finally, an overall summary is given in Sec. IV.

II. PART I: PHASE DIAGRAMS WITH THE THREE-FLAVOR PNJL MODEL

In this section, we adopt the three-flavor PNJL model [44–46] for the QCD sector and explore in detail the phase diagrams with the presences of lepton flavor asymmetries and a primordial magnetic field.

A. Chiral phases in the magnetic field

Usually, chiral symmetry restoration and breaking are related to the expectation value of one order parameter, i.e., the scalar field condensate. In the following, we specifically refer to the phases with only scalar field condensation as *chiral phases* to distinguish from the superconducting pion superfluidity phase where chiral symmetry is actually also broken. Because of the Meissner effect, the chiral phases and pion superfluidity should be treated separately in a background magnetic field.

1. The strong interaction sector

In a primordial magnetic field, the Lagrangian of the PNJL model can be modified from the previous one [35] by adopting the covariant derivative $D_{\mu} = \partial_{\mu} + iQ_{q}eA_{\mu}$ to

$$\begin{aligned} \mathcal{L}_{\text{PNJL}} &= -\frac{B^2}{2} + \bar{\psi} \left[i \not\!\!D - i \gamma^4 \left(i g \mathcal{A}^4 + Q_{\text{q}} \mu_{\text{Q}} + \frac{\mu_{\text{B}}}{3} \right) - m_0 \right] \psi \\ &+ G \sum_{a=0}^8 \left[(\bar{\psi} \lambda^a \psi)^2 + (\bar{\psi} i \gamma_5 \lambda^a \psi)^2 \right] + \mathcal{L}_{\text{tH}} - V(L). \end{aligned}$$

$$\tag{1}$$

Here, the field variables are defined as the following: *B* is the magnetic field, A_{μ} is the corresponding vector potential, $\psi = (u, d, s)^T$ is the three-flavor quark field, and the Polyakov loop is $L = \frac{1}{N_c} \operatorname{tr} e^{ig \int dx_4 \mathcal{A}^4}$ with $\mathcal{A}^4 = A^{4c} T^c / 2$ the non-Abelian gauge field. For the quarks, the current mass and electric charge number matrices are, respectively,

$$m_0 \equiv \operatorname{diag}(m_{0u}, m_{0d}, m_{0s}),$$

$$Q_q \equiv \operatorname{diag}(q_u, q_d, q_s) = \frac{1}{3}\operatorname{diag}(2, -1, -1); \qquad (2)$$

and the interaction index $\lambda^0 = \sqrt{2/3} \mathbb{1}_3$ and $\lambda^i (i = 1, ..., 8)$ are Gell-Mann matrices in flavor space. For later use, the 't Hooft term, $\mathcal{L}_{tH} \equiv -K \sum_{t=\pm} \text{Det } \bar{\psi} \Gamma^t \psi$, can be represented as

$$\mathcal{L}_{tH} = -\frac{K}{2} \sum_{t=\pm} \epsilon_{ijk} \epsilon_{imn} (\bar{\psi}^i \Gamma^t \psi^i) (\bar{\psi}^j \Gamma^t \psi^m) (\bar{\psi}^k \Gamma^t \psi^n)$$
(3)

with the interaction vertices $\Gamma^{\pm} = \mathbb{1}_4 \pm \gamma_5$ for right- and left-handed channels, respectively. Here, one should note the Einstein summation convention for the flavor indices *i*, *j*, *k*, *m*, *n* and the correspondences between 1,2,3 and *u*, *d*, *s*. The pure gluon potential was usually obtained by fitting to the lattice QCD data, and we have in saddle point approximation

$$\frac{V(L)}{T^4} = -\frac{1}{2} \left(3.51 - \frac{2.47}{\tilde{T}} + \frac{15.2}{\tilde{T}^2} \right) L^2 - \frac{1.75}{\tilde{T}^3} \\ \times \ln\left[1 - 6L^2 + 8L^3 - 3L^4 \right], \tag{4}$$

where $\tilde{T} \equiv T/T_0$ is the reduced temperature with $T_0 = 0.27$ GeV [44].

Now, in the chiral phases, we consider only nonzero chiral condensations $\sigma_i \equiv \langle \bar{\psi}^i \psi^i \rangle$ with *i* flavor index, and then the 't Hooft term \mathcal{L}_6 can be reduced to effective four fermion interaction forms in the Hartree approximation [45]:

$$\mathcal{L}_{6}^{4} = -\frac{K}{2} \sum_{s=\pm} \epsilon_{ijk} \epsilon_{imn} \langle \bar{\psi}^{i} \Gamma^{s} \psi^{i} \rangle \langle \bar{\psi}^{j} \Gamma^{s} \psi^{m} \rangle \langle \bar{\psi}^{k} \Gamma^{s} \psi^{n} \rangle$$

$$= -\frac{K}{6} \left\{ 2 \sum_{f=u,d,s} \sigma_{f} (\bar{\psi} \lambda^{0} \psi)^{2} - 3\sigma_{s} \sum_{i=1}^{3} (\bar{\psi} \lambda^{i} \psi)^{2} - 3\sigma_{d} \sum_{i=4}^{5} (\bar{\psi} \lambda^{i} \psi)^{2} - 3\sigma_{u} \sum_{i=6}^{7} (\bar{\psi} \lambda^{i} \psi)^{2} + (\sigma_{s} - 2\sigma_{u} - 2\sigma_{d}) (\bar{\psi} \lambda^{8} \psi)^{2} + \sqrt{2} (2\sigma_{s} - \sigma_{u} - \sigma_{d}) (\bar{\psi} \lambda^{0} \psi) - \sqrt{6} (\sigma_{u} - \sigma_{d}) (\bar{\psi} \lambda^{3} \psi) (\bar{\psi} \lambda^{0} \psi - \sqrt{2} \bar{\psi} \lambda^{8} \psi) \right\} - (\lambda^{a} \rightarrow i \lambda^{a} \gamma^{5})$$

$$(5)$$

with ϵ_{ijk} the Levi-Civita symbol. So the reduced three-flavor Lagrangian density with only four fermion effective interactions is

$$\mathcal{L}_{PNJL}^{4} = -\frac{B^{2}}{2} - V(L) + \bar{\psi} \left[i \not\!\!\!D - i \gamma^{4} \left(i g \mathcal{A}^{4} + Q_{q} \mu_{Q} + \frac{\mu_{B}}{3} \right) - m_{0} \right] \psi + \sum_{a,b=0}^{8} \left[G_{ab}^{-} (\bar{\psi} \lambda^{a} \psi) (\bar{\psi} \lambda^{b} \psi) + G_{ab}^{+} (\bar{\psi} i \gamma_{5} \lambda^{a} \psi) (\bar{\psi} i \gamma_{5} \lambda^{b} \psi) \right],$$
(6)

where the nonvanishing elements of the symmetric coupling matrices G^{\pm} are given by [45]

$$G_{00}^{\mp} = G \mp \frac{K}{3} \sum_{f=u,d,s} \sigma_{\rm f}, \qquad G_{11}^{\mp} = G_{22}^{\mp} = G_{33}^{\mp} = G \pm \frac{K}{2} \sigma_{s}, \qquad G_{44}^{\mp} = G_{55}^{\mp} = G \pm \frac{K}{2} \sigma_{\rm d}, \qquad G_{66}^{\mp} = G_{77}^{\mp} = G \pm \frac{K}{2} \sigma_{\rm u}, \qquad G_{88}^{\mp} = G \mp \frac{K}{6} (\sigma_{s} - 2\sigma_{\rm u} - 2\sigma_{\rm d}), \qquad G_{08}^{\mp} = \mp \frac{\sqrt{2}K}{12} (2\sigma_{s} - \sigma_{\rm u} - \sigma_{\rm d}), \qquad G_{38}^{\mp} = -\sqrt{2}G_{03}^{\mp} = \mp \frac{\sqrt{3}K}{6} (\sigma_{\rm u} - \sigma_{\rm d}). \qquad (7)$$

By contracting a pair of field and conjugate field operators further in \mathcal{L}_6^4 in Hartree approximation, we find

$$\mathcal{L}_6^2 = -\sum_{s=\pm}^{i(\neq j \neq k)} K \langle \bar{\psi}^j \Gamma^s \psi^j \rangle \langle \bar{\psi}^k \Gamma^s \psi^k \rangle [\bar{\psi}^i \Gamma^s \psi^i] = -2K\sigma_j \sigma_k \bar{\psi}^i \psi^i \quad (i \neq j \neq k, j < k),$$
(8)

which then, together with the contributions from the initial four-quark interactions, gives the effective quark masses as

$$m_i = m_{0i} - 4G\sigma_i + 2K\sigma_i\sigma_k. \tag{9}$$

To evaluate quark masses numerically, we should be equipped with the gap equations directly following the definitions of chiral condensations:

$$\sigma_i \equiv \langle \bar{\psi}^i \psi^i \rangle = -\frac{i}{V_4} \operatorname{Tr} \mathcal{S}_i, \tag{10}$$

where the effective quark propagators in a constant magnetic field are given by [47]

$$\hat{\mathcal{S}}_{i}(k) = i \int ds \, \exp\left\{-i(m_{i}^{2} + k_{4}^{2} + k_{3}^{2})s - i\frac{\tan(q_{i}eBs)}{q_{i}eB}(k_{1}^{2} + k_{2}^{2})\right\} [m_{i} - \gamma^{4}k_{4} - \gamma^{3}k_{3} - \gamma^{2}(k_{2} + \tan(q_{i}eBs)k_{1}) - \gamma^{1}(k_{1} - \tan(q_{i}eBs)k_{2})][1 + \gamma^{1}\gamma^{2}\tan(q_{i}eBs)].$$
(11)

Then, by adopting vacuum regularization, the gap equations are [47]

$$-\sigma_{\rm f} = N_c \frac{m_{\rm f}^3}{2\pi^2} \left[\tilde{\Lambda}_{\rm f} (1 + \tilde{\Lambda}_{\rm f}^2)^{\frac{1}{2}} - \ln\left(\tilde{\Lambda}_{\rm f} + (1 + \tilde{\Lambda}_{\rm f}^2)^{\frac{1}{2}}\right) \right] + N_c \frac{m_{\rm f}}{4\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-m_{\rm f}^2 s} \left(\frac{q_{\rm f} eBs}{\tanh(q_{\rm f} eBs)} - 1 \right) \\ - 6 \sum_{\rm u=\pm} \frac{|q_{\rm f} eB|}{2\pi} \sum_{n=0}^\infty \alpha_{\rm n} \int_{-\infty}^\infty \frac{dk_3}{2\pi} \frac{m_{\rm f}}{E_{\rm f}^{\rm n}} F_{\rm f}^{\rm u}(E_{\rm f}^{\rm n}, L, T, \mu_{\rm Q}, \mu_{\rm B}),$$
(12)

where the reduced cutoff $\tilde{\Lambda}_{\rm f} = \Lambda/m_{\rm f}$, the Landau level factor $\alpha_{\rm n} = 1 - \delta_{\rm n0}/2$, the particle energy $E_{\rm f}^{\rm n}(k_3, m_{\rm f}) = (2n|q_{\rm f}eB| + k_3^2 + m_{\rm f}^2)^{1/2}$, and the fermion distribution function

$$F_{\rm f}^{\rm u}(E_{\rm f}^{\rm n},L,T,\mu_{\rm Q},\mu_{\rm B}) \equiv \frac{Le^{-\frac{1}{T}(E_{\rm f}^{\rm n}-u(q_{\rm f}\mu_{\rm Q}+\frac{\mu_{\rm B}}{3}))} + 2Le^{-\frac{2}{T}(E_{\rm f}^{\rm n}-u(q_{\rm f}\mu_{\rm Q}+\frac{\mu_{\rm B}}{3}))} + e^{-\frac{3}{T}(E_{\rm f}^{\rm n}-u(q_{\rm f}\mu_{\rm Q}+\frac{\mu_{\rm B}}{3}))} + 3Le^{-\frac{2}{T}(E_{\rm f}^{\rm n}-u(q_{\rm f}\mu_{\rm Q}+\frac{\mu_{\rm B}}{3}))} + e^{-\frac{3}{T}(E_{\rm f}^{\rm n}-u(q_{\rm f}\mu_{\rm Q}+\frac{\mu_{\rm B}}{3}))} + e^$$

In advance, the quark part of the thermodynamic potential can be obtained consistently by combining the definitions of effective masses in Eq. (9) and the integrations over m_f of Eq. (12) as [48]

$$\begin{split} \Omega_{q}(B) &= 2G \sum_{f=u,d,s} \sigma_{f}^{2} - 4K \prod_{f=u,d,s} \sigma_{f} - N_{c} \sum_{f=u,d,s} \left\{ \frac{m_{f}^{4}}{8\pi^{2}} \left[\tilde{\Lambda}_{f}(1+2\tilde{\Lambda}_{f}^{2})(1+\tilde{\Lambda}_{f}^{2})^{\frac{1}{2}} - \ln\left(\tilde{\Lambda}_{f} + (1+\tilde{\Lambda}_{f}^{2})^{\frac{1}{2}}\right) \right] \\ &- \frac{1}{8\pi^{2}} \int_{0}^{\infty} \frac{ds}{s^{3}} \left(e^{-m_{f}^{2}s} - e^{-m_{f}^{v2}s} \right) \left(\frac{q_{f}eBs}{\tanh(q_{f}eBs)} - 1 \right) - \frac{1}{8\pi^{2}} \int_{0}^{\infty} \frac{ds}{s^{3}} e^{-m_{f}^{v2}s} \left(\frac{q_{f}eBs}{\tanh(q_{f}eBs)} - 1 - \frac{1}{3}(q_{f}eBs)^{2} \right) \\ &+ 2T \sum_{u=\pm} \frac{|q_{f}eB|}{2\pi} \sum_{n=0}^{\infty} \alpha_{n} \int_{-\infty}^{\infty} \frac{dk_{3}}{2\pi} K_{f}^{u}(E_{f}^{n}, L, T, \mu_{Q}, \mu_{B}) \bigg\}, \end{split}$$
(13)

with

$$K_{\rm f}^{\rm u}(E_{\rm f}^{\rm n},L,T,\mu_{\rm Q},\mu_{\rm B}) = \frac{1}{N_{\rm c}} \ln \left[1 + 3Le^{-\frac{1}{T}(E_{\rm f}^{\rm n} - u(q_{\rm f}\mu_{\rm Q} + \frac{\mu_{\rm B}}{3}))} + 3Le^{-\frac{2}{T}(E_{\rm f}^{\rm n} - u(q_{\rm f}\mu_{\rm Q} + \frac{\mu_{\rm B}}{3}))} + e^{-\frac{3}{T}(E_{\rm f}^{\rm n} - u(q_{\rm f}\mu_{\rm Q} + \frac{\mu_{\rm B}}{3}))} \right].$$

Here, one notes that the terms depending on the quark vacuum mass $m_{\rm f}^{\rm v}$ are introduced for the correct renormalizations of electric charges and magnetic field in the vacuum.

Hence, the Helmholtz free energy for the PNJL model is $\Omega_{\rm H}^{\rm M} = \frac{B^2}{2} + V(L) + \Omega_{\rm q}(B)$ and the *external* magnetic field can be obtained through $H = \frac{\partial \Omega_{\rm H}^{\rm M}}{\partial B}$. In classical words, the magnetic intensity *H* equals the magnetic induction intensity *B* minus magnetization intensity $\mathcal{M} = -\frac{\partial \Omega_{\rm q}}{\partial B}$. Usually, we control the external magnetic field *H* for the exploration of phase transitions [36]; that is, *H* must be a variable of the free energy. So, the right state function is the Gibbs free energy that can be obtained by taking the Legendre transformation of $\Omega_{\rm H}^{\rm M}$ as [36]

$$\Omega_{\chi}^{\rm M} = \Omega_{\rm H}^{\rm M} - BH = -\frac{H^2}{2} + \frac{\mathcal{M}^2}{2} + V(L) + \Omega_{\rm q}(H + \mathcal{M}).$$
(14)

As the magnetizations from quarks and leptons are relatively small for the considered magnetic field in the chiral phases, we could simply take the Gibbs free energy to be $\Omega_{\chi}^{M} = -\frac{H^2}{2} + V(L) + \Omega_{q}(H)$ to the order $o(\mathcal{M}^2)$. The general formula Eq. (14) even consistently applies to the superconducting pion superfluidity, where $B = H + \mathcal{M} = 0$ in the bulk due to the Meissner effect; and we find $\Omega_{\pi SF}^{M} = V(L) + \Omega_{q}(0)$, just the same as the case without an external magnetic field [36].

Then, the gap equation for L can be given through $\partial_L \Omega^M_{\chi} = 0$ as

$$T^{3}\left[-\left(3.51 - \frac{2.47}{\tilde{T}} + \frac{15.2}{\tilde{T}^{2}}\right)L + \frac{1.75}{\tilde{T}^{3}}\frac{12L(1-L)^{2}}{1-6L^{2}+8L^{3}-3L^{4}}\right]$$

= $6\sum_{f=u,d,s}\sum_{u=\pm}\frac{|q_{f}eH|}{2\pi}\sum_{n=0}^{\infty}\alpha_{n}\int_{-\infty}^{\infty}\frac{dk_{3}}{2\pi}\frac{e^{-\frac{1}{T}(E_{f}^{n}-u(q_{f}\mu_{Q}+\frac{\mu_{B}}{3}))} + e^{-\frac{2}{T}(E_{f}^{n}-u(q_{f}\mu_{Q}+\frac{\mu_{B}}{3}))}}{1+3Le^{-\frac{1}{T}(E_{f}^{n}-u(q_{f}\mu_{Q}+\frac{\mu_{B}}{3}))} + 3Le^{-\frac{2}{T}(E_{f}^{n}-u(q_{f}\mu_{Q}+\frac{\mu_{B}}{3}))} + e^{-\frac{3}{T}(E_{f}^{n}-u(q_{f}\mu_{Q}+\frac{\mu_{B}}{3}))}}.$ (15)

And the entropy, electric charge number and baryon number densities follow the thermodynamic relations as

$$s_{\chi}^{M} = 2N_{c} \sum_{f=u,d,s} \sum_{u=\pm} \frac{|q_{f}eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_{n} \int_{-\infty}^{\infty} \frac{dk_{3}}{2\pi} \left[K_{f}^{u}(E_{f}^{n},L,T,\mu_{Q},\mu_{B}) + \frac{1}{T} \left(E_{f}^{n} - u \left(q_{f}\mu_{Q} + \frac{\mu_{B}}{3} \right) \right) F_{f}^{u}(E_{f}^{n},L,T,\mu_{Q},\mu_{B}) \right] + T^{3} \left\{ \frac{1}{2} \left(4 \times 3.51 - 3 \times \frac{2.47}{\tilde{T}} + 2 \times \frac{15.2}{\tilde{T}^{2}} \right) L^{2} + \frac{1.75}{\tilde{T}^{3}} \ln \left[1 - 6L^{2} + 8L^{3} - 3L^{4} \right] \right\},$$
(16)

$$n_{\rm Q}^{\rm q,M} = 2N_{\rm c} \sum_{f=u,d,s} \sum_{\rm u=\pm} \frac{|q_{\rm f}eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_{\rm n} \int_{-\infty}^{\infty} \frac{{\rm d}k_3}{2\pi} u q_{\rm f} F_{\rm f}^{\rm u}(E_{\rm f}^{\rm n},L,T,\mu_{\rm Q},\mu_{\rm B}),\tag{17}$$

$$n_{\rm B}^{\rm M} = 2 \sum_{f=u,d,s} \sum_{\rm u=\pm} \frac{|q_{\rm f}eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_{\rm n} \int_{-\infty}^{\infty} \frac{{\rm d}k_3}{2\pi} u F_{\rm f}^{\rm u}(E_{\rm f}^{\rm n},L,T,\mu_{\rm Q},\mu_{\rm B}).$$
(18)

2. The electroweak interaction sector

In the free gas approximation, the thermodynamic potentials for the quantum electroweak dynamics (QEWD) sector can easily be given by [49,50]

$$\Omega_{\gamma} = 2T \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \log\left(1 - e^{-k/T}\right),\tag{19}$$

$$\Omega_{l}^{M} = \sum_{u=\pm}^{i=e,\mu,\tau} \left\{ -T \sum_{u=\pm} \int \frac{d^{3}k}{(2\pi)^{3}} \log\left[1 + e^{-(k-u\mu_{i})/T}\right] + \frac{1}{8\pi^{2}} \int_{0}^{\infty} \frac{ds}{s^{3}} e^{-m_{1}^{2}s} \left[\frac{eHs}{\tanh(eHs)} - 1 - \frac{1}{3}(eHs)^{2}\right] - 2T \sum_{u=\pm} \frac{|eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_{n} \int_{-\infty}^{\infty} \frac{dk_{3}}{2\pi} \log\left[1 + e^{-(\epsilon_{i}^{n}(k_{3},eH) - u(-\mu_{Q} + \mu_{i}))/T}\right] \right\},$$
(20)

where the degeneracy is one for neutrinos and antineutrinos due to their definite chiralities and $\epsilon_i^n(k_3, eH) = (k_3^2 + 2n|eH| + m_i^2)^{1/2}$. Note that we have approximated *B* by *H* here. Then, the corresponding entropy, electric charge number, and lepton flavor number densities can be derived directly as

$$s_{\gamma} = 2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[-\log\left(1 - e^{-k/T}\right) + \frac{k/T}{e^{k/T} - 1} \right],\tag{21}$$

$$s_{1}^{M} = \sum_{u=\pm}^{i=e,\mu,\tau} \left\{ \int \frac{d^{3}k}{(2\pi)^{3}} \left\{ \log\left[1 + e^{-(k-u\mu_{i})/T}\right] + \frac{(k-u\mu_{i})/T}{1 + e^{(k-u\mu_{i})/T}} \right\} + 2\sum_{u=\pm} \frac{|eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_{n} \int_{-\infty}^{\infty} \frac{dk_{3}}{2\pi} \times \left\{ \log\left[1 + e^{-(\epsilon_{i}^{n}(k_{3},eH) - u(-\mu_{Q} + \mu_{i}))/T}\right] + \frac{(\epsilon_{i}^{n}(k_{3},eH) - u(-\mu_{Q} + \mu_{i}))/T}{1 + e^{(\epsilon_{i}^{n}(k_{3},eH) - u(-\mu_{Q} + \mu_{i}))/T}} \right\} \right\},$$
(22)

$$n_{\rm Q}^{\rm l,M} = 2T \sum_{u=\pm} \frac{|eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_{\rm n} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_3}{2\pi} \frac{-u}{1 + e^{(\epsilon_{\rm i}^{\rm n}(k_3, eH) - u(-\mu_{\rm Q} + \mu_{\rm i}))/T}},\tag{23}$$

$$n_{\rm i}^{\rm M} = T \sum_{u=\pm} \int \frac{{\rm d}^3 k}{(2\pi)^3} \frac{u}{1 + e^{(k-u\mu_{\rm i})/T}} + 2T \sum_{u=\pm} \frac{|eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{{\rm d}k_3}{2\pi} \frac{u}{1 + e^{(\epsilon_{\rm i}^{\rm n}(k_3, eH) - u(-\mu_{\rm Q} + \mu_{\rm i}))/T}}.$$
(24)

Now, collecting contributions from both the QEWD and the QCD sectors, the total thermodynamic potential, entropy, electric charge number and lepton number densities are, respectively,

$$\Omega_{\rm M} = \Omega_{\gamma} + \Omega_{\rm l}^{\rm M} + \Omega_{\chi}^{\rm M}, \qquad s_{\rm M} = s_{\gamma} + s_{\rm l}^{\rm M} + s_{\chi}^{\rm M}, n_{\rm Q}^{\rm M} = n_{\rm Q}^{\rm l,M} + n_{\rm Q}^{\rm q,M}, \qquad n_{\rm l}^{\rm M} = \sum_{i=e,\mu,\tau} n_{\rm i}^{\rm M}$$
(25)

in the QCD epoch. To better catch the expansion nature of the early universe, we define several reduced quantities:

$$b^{\rm M} = n_{\rm B}^{\rm M}/s_{\rm M}, \qquad l^{\rm M} = n_{\rm l}^{\rm M}/s_{\rm M}, \qquad l_{\rm i}^{\rm M} = n_{\rm i}^{\rm M}/s_{\rm M} \quad (26)$$

by following the conventions.

B. The superconducting pion superfluidity

Since the pion superfluid is also an electric superconductor, the magnetic field will be screened from the bulk due to the Meissner effect. Then, the free energy of the bulk must be the same as the one with the same temperature and chemical potentials but without a background magnetic field. So the formalism is the same as the one we presented in our previous work [35] where the magnetic effect was not taken into account.

1. The strong interaction sector

Without a magnetic field, the Lagrangian is given by [44–46]

$$\mathcal{L}_{\text{PNJL}} = -V(L) + \bar{\psi} \left[i \partial \!\!\!/ - i \gamma^4 \left(i g \mathcal{A}^4 + Q_q \mu_Q + \frac{\mu_B}{3} \right) - m_0 \right] \psi + G \sum_{a=0}^8 \left[(\bar{\psi} \lambda^a \psi)^2 + (\bar{\psi} i \gamma_5 \lambda^a \psi)^2 \right] + \mathcal{L}_{\text{tH}}.$$
(27)

For the pion superfluidity phase, we choose the following scalar and charged pseudoscalar condensates to be nonzero:

$$\sigma_{\rm f} = \langle \bar{\psi}_{\rm f} \psi_{\rm f} \rangle, \qquad \Delta_{\pi} = \langle \bar{u} i \gamma^5 d \rangle, \qquad \Delta_{\pi}^* = \langle \bar{d} i \gamma^5 u \rangle.$$

For brevity, we set $\Delta_{\pi} = \Delta_{\pi}^*$ without loss of generality in the following. To facilitate the study, we would first like to reduce \mathcal{L}_{tH} to an effective form with four fermion interactions at most. By applying the Hartree approximation to contract a pair of quark and antiquark in each six fermion interaction term [45], we immediately find

$$\mathcal{L}_{tH}^{4} = -K\{\epsilon_{ijk}\epsilon_{imn}\sigma_{i}(\bar{\psi}^{j}\psi^{m}\bar{\psi}^{k}\psi^{n} - \bar{\psi}^{j}i\gamma^{5}\psi^{m}\bar{\psi}^{k}i\gamma^{5}\psi^{n}) + 2\Delta_{\pi}[\bar{s}s(\bar{u}i\gamma^{5}d + \bar{d}i\gamma^{5}u - \Delta_{\pi}) + \bar{s}i\gamma^{5}s(\bar{u}d + \bar{d}u)]\},$$
(28)

where the second term in the brace is induced by π^{\pm} condensations. Armed with the reduced Lagrangian density

$$\mathcal{L}_{\text{PNJL}} = -V(L,L) + \bar{\psi} \left[i \not{\!\!\!/} - i \gamma^4 \left(i g \mathcal{A}^4 + Q_q \mu_Q + \frac{\mu_B}{3} \right) - m_0 \right] \psi + G \sum_{a=0}^8 [(\bar{\psi} \lambda^a \psi)^2 + (\bar{\psi} i \gamma_5 \lambda^a \psi)^2] + \mathcal{L}_{\text{tH}}^4,$$
(29)

the left calculations can just follow the two-flavor case in principle.

By contracting quark and antiquark pairs once more in the interaction terms of Eq. (29), we find the quark bilinear form as

where the scalar and pseudoscalar masses are, respectively,

$$m_i = m_{0i} - 4G\sigma_i + 2K(\sigma_j\sigma_k + \Delta_{\pi}^2\delta_{i3}),$$

$$\Pi = (-4G + 2K\sigma_3)\Delta_{\pi}$$
(31)

with $i \neq j \neq k$. The *G* and *K* dependent terms in Eq. (31) are from the $U_A(1)$ symmetric and anomalous interactions, respectively. According to Eq. (30), the *s* quark decouples from *u*, *d* quarks, so the gap equation for σ_s can be simply given by [45]

$$\sigma_{\rm s} = \langle \bar{s}s \rangle = \operatorname{tr} \left[i \vec{\varPhi} - i \gamma^4 \left(i g \mathcal{A}^4 + Q_{\rm q} \mu_{\rm Q} + \frac{\mu_{\rm B}}{3} \right) - m_{\rm s} \right]^{-1}.$$
(32)

However, the *u* and *d* light quarks couple with each other through the nondiagonal pseudoscalar mass Π . Since $\mu_{\rm B}$ is usually small in the early universe, we can simply set

$$m_{0u} = m_{0d} \equiv m_{0l}, \qquad \sigma_u = \sigma_d \equiv \sigma_l$$

in order to further carry out analytic derivations. Then, by following a similar procedure as the previous section, the explicit thermodynamic potential can be worked out for the bilinear terms as

$$\Omega_{\rm bl} = -2N_c \int^{\Lambda} \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\sum_{t=\pm} E_1^{\rm t}(k) + \epsilon_{\rm s}(k) \right] - 2T \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{\rm u=\pm} \left[\sum_{t=\pm} Fl\left(L, u, E_1^{\rm t}(k), \frac{\mu_{\rm Q} + 2\mu_{\rm B}}{6} \right) + Fl\left(L, u, \epsilon_{\rm s}(k), \frac{-\mu_{\rm Q} + \mu_{\rm B}}{3} \right) \right] \left\{ Fl(L, u, x, y) = \log[1 + 3Le^{-\frac{1}{T}(x-uy)} + 3Le^{-\frac{2}{T}(x-uy)} + e^{-\frac{3}{T}(x-uy)}] \right\}, \quad (33)$$

with the particle energy functions defined by

$$\epsilon_{\rm i}(k) = \sqrt{k^2 + m_{\rm i}^2}, \qquad E_{\rm l}^{\rm t}(k) = \sqrt{\left[\epsilon_{\rm l}(k) + t \frac{\mu_{\rm Q}}{2}\right]^2 + \Pi^2}.$$
 (34)

Eventually, the coupled gap equations follow directly from the definitions of condensates:

$$\sigma_{\rm s} \equiv \langle \bar{s}s \rangle = \frac{\partial \Omega_{\rm bl}}{\partial m_{\rm s}}, \qquad 2\sigma_{\rm l} \equiv \langle \bar{u}u \rangle + \langle \bar{d}d \rangle = \frac{\partial \Omega_{\rm bl}}{\partial m_{\rm l}}, \qquad 2\Delta_{\pi} \equiv \langle \bar{u}i\gamma^5 d \rangle + \langle \bar{d}i\gamma^5 u \rangle = \frac{\partial \Omega_{\rm bl}}{\partial \Pi}, \tag{35}$$

and the minimal condition $\partial_{\rm L}[V(L,L)+\Omega_{\rm bl}]=0$ as [35]

$$\sigma_{\rm s} = -2N_c \int^{\Lambda} \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{m_{\rm s}}{\epsilon_{\rm s}(k)} + 2N_c \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{m_{\rm s}}{\epsilon_{\rm s}(k)} \sum_{u=\pm} dV_1 \left(L, u, \epsilon_{\rm s}(k), \frac{-\mu_{\rm Q} + \mu_{\rm B}}{3}\right),\tag{36}$$

$$2\sigma_{\rm l} = -2N_c \int^{\Lambda} \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{t=\pm} \frac{m_{\rm l}}{\epsilon_{\rm l}(k)} \frac{\epsilon_{\rm l}(k) + t^{\frac{\mu_{\rm Q}}{2}}}{E_{\rm l}^{\rm t}(k)} + 2N_c \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{t,u=\pm} \frac{m_{\rm l}}{\epsilon_{\rm l}(k)} \frac{\epsilon_{\rm l}(k) + t^{\frac{\mu_{\rm Q}}{2}}}{E_{\rm l}^{\rm t}(k)} dV_1 \left(L, u, E_{\rm l}^{\rm t}(k), \frac{\mu_{\rm Q} + 2\mu_{\rm B}}{6}\right), \quad (37)$$

$$2\Delta_{\pi} = -2N_c \int^{\Lambda} \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{t=\pm} \frac{\Pi}{E_1^{\mathrm{t}}(k)} + 2N_c \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{t,u=\pm} \frac{\Pi}{E_1^{\mathrm{t}}(k)} dV_1 \left(L, u, E_1^{\mathrm{t}}(k), \frac{\mu_{\mathrm{Q}} + 2\mu_{\mathrm{B}}}{6}\right), \tag{38}$$

$$T^{4}\left[-\left(3.51-\frac{2.47}{\tilde{T}}+\frac{15.2}{\tilde{T}^{2}}\right)L+\frac{1.75}{\tilde{T}^{3}}\frac{12L(1-L)^{2}}{1-6L^{2}+8L^{3}-3L^{4}}\right] = 6T\int\frac{\mathrm{d}^{3}k}{(2\pi)^{3}}\sum_{u=\pm}\left[\sum_{t=\pm}dV_{2}\left(L,u,E^{t}(k),\frac{\mu_{\mathrm{Q}}+2\mu_{\mathrm{B}}}{6}\right) +dV_{2}\left(L,u,\epsilon_{\mathrm{s}}(k),\frac{-\mu_{\mathrm{Q}}+\mu_{\mathrm{B}}}{3}\right)\right].$$
(39)

Note that $\Delta_{\pi} = 0$ is a trivial solution of Eq. (38), so Δ_{π} or Π is still a true order parameter for I_3 isospin symmetry [51] in the three-flavor case. The total self-consistent thermodynamic potential can be found to be

$$\Omega_{\pi SF} = V(L,L) + \Omega_{\rm bl} + 2G(\sigma_{\rm s}^2 + 2\sigma_{\rm l}^2 + 2\Delta_{\pi}^2) - 4K(\sigma_{\rm l}^2 + \Delta_{\pi}^2)\sigma_{\rm s}$$
(40)

by utilizing the definitions of condensates and their relations to scalar and pseudoscalar masses [refer to Eqs. (35) and (31)]. And the entropy, electric charge number and baryon number densities can be given according to the thermodynamic relations as [35]

$$s_{\pi SF} = 2 \int \frac{d^3k}{(2\pi)^3} \sum_{t,u=\pm} \left[Fl\left(L, u, E_1^{t}(k), \frac{\mu_Q + 2\mu_B}{6}\right) + \frac{3(E_1^{t}(k) - u\frac{\mu_Q + 2\mu_B}{6})}{T} dV_1\left(L, u, E_1^{t}(k), \frac{\mu_Q + 2\mu_B}{6}\right) \right] + 2 \int \frac{d^3k}{(2\pi)^3} \sum_{u=\pm} \left[Fl\left(L, u, \epsilon_s(k), \frac{-\mu_Q + \mu_B}{3}\right) + \frac{3(E_1^{t}(k) - u\frac{-\mu_Q + \mu_B}{3})}{T} dV_1\left(L, u, \epsilon_s(k), \frac{-\mu_Q + \mu_B}{3}\right) \right] + T^3 \left\{ \frac{1}{2} \left(4 \times 3.51 - 3 \times \frac{2.47}{\tilde{T}} + 2 \times \frac{15.2}{\tilde{T}^2} \right) L^2 + \frac{1.75}{\tilde{T}^3} \ln\left[1 - 6L^2 + 8L^3 - 3L^4\right] \right\},$$
(41)

$$n_{Q}^{\pi SF} = N_{c} \int^{\Lambda} \frac{d^{3}k}{(2\pi)^{3}} \sum_{t=\pm} t \frac{\epsilon_{l}(k) + t\frac{\mu_{Q}}{2}}{E_{1}^{t}(k)} - 3 \int \frac{d^{3}k}{(2\pi)^{3}} \sum_{t,u=\pm} t \frac{\epsilon_{l}(k) + t\frac{\mu_{Q}}{2}}{E_{1}^{t}(k)} dV_{1} \left(L, u, E_{1}^{t}(k), \frac{\mu_{Q} + 2\mu_{B}}{6}\right) + \int \frac{d^{3}k}{(2\pi)^{3}} \sum_{t,u=\pm} u dV_{1} \left(L, u, E_{1}^{t}(k), \frac{\mu_{Q} + 2\mu_{B}}{6}\right) - 2 \int \frac{d^{3}k}{(2\pi)^{3}} \sum_{t,u=\pm} u dV_{1} \left(L, u, \epsilon_{s}(k), \frac{-\mu_{Q} + \mu_{B}}{3}\right), \quad (42)$$

$$n_{\rm B}^{\pi\rm SF} = 2 \int \frac{{\rm d}^3 k}{(2\pi)^3} \sum_{t,u=\pm} u dV_1 \left(L, u, E_1^{\rm t}(k), \frac{\mu_{\rm Q} + 2\mu_{\rm B}}{6} \right) + 2 \int \frac{{\rm d}^3 k}{(2\pi)^3} \sum_{t,u=\pm} u dV_1 \left(L, u, \epsilon_{\rm s}(k), \frac{-\mu_{\rm Q} + \mu_{\rm B}}{3} \right).$$
(43)

2. The electroweak interaction sector

In free gas approximation, the thermodynamic potentials for the QEWD sector can easily be given by [49]

$$\Omega_{\gamma} = 2T \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \log{(1 - e^{-k/T})},\tag{44}$$

$$\Omega_{\rm l} = -T \sum_{u=\pm}^{\rm i=e,\mu,\tau} \int \frac{{\rm d}^3 k}{(2\pi)^3} \{ 2\log\left[1 + e^{-(\epsilon_{\rm i}(k) - u(-\mu_{\rm Q} + \mu_{\rm i}))/T}\right] \\ + \log\left[1 + e^{-(k - u\mu_{\rm i})/T}\right] \}, \tag{45}$$

where the degeneracy is one for neutrinos and antineutrinos due to their definite chiralities. Then, the corresponding entropy, electric charge number and lepton flavor number densities can be derived directly as

$$s_{\gamma} = 2 \int \frac{\mathrm{d}^{3}k}{(2\pi)^{3}} \left[-\log\left(1 - e^{-k/T}\right) + \frac{k/T}{e^{k/T} - 1} \right], \quad (46)$$

$$s_{1} = \sum_{u=\pm}^{\mathrm{i=e},\mu,\tau} \int \frac{\mathrm{d}^{3}k}{(2\pi)^{3}} \left\{ 2\log\left[1 + e^{-(\epsilon_{\mathrm{i}}(k) - u(-\mu_{\mathrm{Q}} + \mu_{\mathrm{i}}))/T}\right] + \log\left[1 + e^{-(k - u\mu_{\mathrm{i}})/T}\right] + \frac{2(\epsilon_{\mathrm{i}}(k) - u(-\mu_{\mathrm{Q}} + \mu_{\mathrm{i}}))/T}{1 + e^{(\epsilon_{\mathrm{i}}(k) - u(-\mu_{\mathrm{Q}} + \mu_{\mathrm{i}}))/T}} + \frac{(k - u\mu_{\mathrm{i}})/T}{1 + e^{(k - u\mu_{\mathrm{i}})/T}} \right\}, \quad (47)$$

$$n_{\rm Q}^{\rm l} = 2T \sum_{u=\pm}^{\rm i=e,\mu,\tau} \int \frac{{\rm d}^3 k}{(2\pi)^3} \frac{-u}{1 + e^{(\epsilon_{\rm i}(k) - u(-\mu_{\rm Q} + \mu_{\rm i}))/T}}, \quad (48)$$

$$n_{i} = -\frac{\partial\Omega_{i}}{\partial\mu_{i}} = T \sum_{u=\pm} \int \frac{d^{3}k}{(2\pi)^{3}} \left[\frac{2u}{1 + e^{(\epsilon_{i}(k) - u(-\mu_{Q} + \mu_{i}))/T}} + \frac{u}{1 + e^{(k - u\mu_{i})/T}} \right], \qquad i = e, \mu, \tau.$$
(49)

Now, collecting contributions from both QEWD and QCD sectors, the total thermodynamic potential, entropy,

electric charge number and lepton number densities are, respectively,

$$\Omega = \Omega_{\gamma} + \Omega_{l} + \Omega_{\pi SF}, \qquad s = s_{\gamma} + s_{l} + s_{\pi SF},$$

$$n_{Q} = n_{Q}^{l} + n_{Q}^{\pi SF}, \qquad n_{l} = \sum_{i=e,\mu,\tau} n_{i} \qquad (50)$$

in the QCD epoch. To better catch the expansion nature of the early universe, we define several reduced quantities:

$$b = n_{\rm B}^{\pi {\rm SF}}/s, \qquad l = n_{\rm l}/s, \qquad l_{\rm i} = n_{\rm i}/s \qquad (51)$$

by following the conventions.

C. Numerical results

To carry out numerical calculations, we get the electron and muon masses from the Particle Data Group as $m_e =$ 0.53 MeV and $m_{\mu} = 113$ MeV and suppress the contribution of heavy τ leptons for the electroweak interaction sector. The model parameters are fixed for the strong interaction sector as the following [52,53]:

$$m_{01} = 5.5 \text{ MeV}, \quad m_{0s} = 140.7 \text{ MeV}, \quad \Lambda = 602.3 \text{ MeV},$$

 $G\Lambda^2 = 1.835, \quad K\Lambda^5 = 12.36.$ (52)

First of all, we have to determine which phase the QCD matter is in, the chiral phases or pion superfluidity, by comparing Ω_{χ}^{M} and $\Omega_{\pi SF}$. We choose the recent constraints $n^{\rm Q} = 0, b^{\rm M} = 8.6 * 10^{-11}$ [32], and $l^{\rm M} = -0.012$ [54] up to the point when the latest first-order transition took place in the QCD epoch of the early universe. The nonconstrained values of l_e^M and l_{μ}^M can be randomly fixed at that point, but we only consider the case with $l_e^M = 0$ for a given $l_e^M + l_\mu^M$. Note that our previous work had showed that the phase boundaries were not sensitive to the fraction of l_e^M [35]. When first-order phase transitions are involved, the total entropy does not change continuously at the transition point due to the latent heat released or absorbed; thus, we should not require $b^{\rm M} = b$ or $l_{\rm i}^{\rm M} = l_{\rm i}, (i = e, \mu, \tau)$ in the bulk. For pion superfluidity, the magnetic field in the bulk is canceled out by the current produced at the surface and the magnetic flux is only present at the surface [36]. So in the sense of total baryon and lepton flavor number conservations, the deficits $n_{\rm B}^{\rm M}V^{\rm M} - n_{\rm b}V$ and $n_{\rm i}^{\rm M}V^{\rm M} - n_{\rm i}V$ should be found at the surface of the pion superfluidity.

Nevertheless, the ratios n_i^M/n_B^M must be the same in the χSR and χSB phases separated by pion superfluidity if it exists. Following the ansatz of isentropic expansion for a given phase, we expect n_B^M/s and n_i^M/s to be the same at the entrance and the exit of pion superfluidity.

For the given values of n^Q , b^M , and l_i^M , the values of chemical potentials μ_Q , μ_B , and μ_i can be uniquely determined in the chiral phases for a fixed temperature. And then the transition points could be pinned down by requiring $\Omega_{\chi}^M = \Omega_{\pi SF}$ at the same temperature and chemical potentials. Now we can obtain two different sets of *b* and l_i at the transition points, but how should they evolve in between in the pion superfluidity phase? For H = 0, we expected them to be the same as the recent constraints since no extra particles are reserved at the surface. Any reasonable scheme must recover the results in the vanishing *H* limit, and hence we simply adopt linear interpolations between the two sets of *b* and l_i for finite *H*. That means the baryons and lepton flavors are gradually deposited into or withdrawn from the surface with the temperature decreasing.

Next, to study the effect of the primordial magnetic field, the order of H should be estimated in the QCD epoch of the early universe. In the Milky Way, the average magnitude of the magnetic field was found to be $H = 10^{-6} - 10^{-5}$ Gs according to the observations of galactic background radio radiation and polarization of star light [55]. In natural units, the magnitude is $eH = 10^{-26} - 10^{-25} \text{ GeV}^2$ since eH = $5.9 \times 10^{-21} \text{ GeV}^2$ for H = 1 Gs. Tracing back to the QCD epoch, the magnetic field would be greatly enhanced due to a very large scaling factor $a = 10^{12} - 10^{12.5}$ [27] and we have $eH = (10^{-26} - 10^{-25})a^2 = 10^{-2} - 1(\text{GeV}^2)$. As we will see, the favored region of pion superfluidity would greatly shrink with the magnetic field increasing. So to explore nontrivial physics, we focus on the lower region of the magnetic field domain estimated, that is, $eH \sim$ 10^{-2} GeV². Note that the magnetic field is not homogeneous all across the Milky Way but only locally; hence the volume where the homogeneous phase transition might happen should not be taken to be infinite.

The phase diagrams with fixed background magnetic fields or lepton flavor densities are illustrated together in Fig. 1. Since the densities are not continuous across the first-order transition point, we take the lepton flavorentropy ratios l_i^M in the chiral phases for reference. From the upper panel, one can tell that the regime of pion superfluidity shrinks quickly with an increasing magnetic field but the tails are never found to end at a large $|l_e^M + l_\mu^M|$. Nevertheless, when *eH* exceeds 0.0222 GeV², the width of the regime of the pion superfluidity is found to vanish and one is left with only chiral phases. This explains our focus on the magnitude of the magnetic field: $eH \sim 10^{-2}$ GeV². As shown in the lower panel, the phase boundaries become of first order and oscillate with finite *H*, which is known as the de Haas–van Alphen effect [47] when both magnetic





FIG. 1. Upper panel: the $T - (l_e^M + l_\mu^M)$ phase diagrams for the magnetic fields eH = 0, 0.01, 0.02, and 0.022 GeV² (colors: blue, red, green, and cyan) with the bullets the critical end points. Lower panel: the T - eH phase diagram for the lepton flavorentropy ratio $l_e^M + l_\mu^M = -0.2$. The shadows correspond to the pion superfluidity phase, and the blue dashed line and other colored lines denote second- and first-order transition boundaries, respectively.

field and chemical potentials are present. For small H, there are actually very special points where the transitions remain first order along the $|l_e^M + l_u^M|$ direction but are of second order along the T direction. These CEPs are actually where the upper and lower boundaries meet each other and thus with the smallest value of $|l_e^M + l_u^M|$; see the bullets in the upper panel of Fig. 1. It is constructive to demonstrate the evolutions of the temperature T and lepton flavor-entropy ratio $l_e^{\rm M} + l_{\mu}^{\rm M}$ with an increasing magnetic field eH at the CEPs in Fig. 2. Consistent with the upper panel of Fig. 1, T_{CEP} decreases gradually with eH (upper panel) but $|l_e^{\rm M} + l_{\mu}^{\rm M}|_{\rm CEP}$ increases abruptly around $eH = 0.022 \ {\rm GeV^2}$ (lower panel), which seem to be related to the transition orders just discussed. Moreover, from the lower panel of Fig. 2, we can easily understand the complete disfavor of pion superfluidity for larger eH.

Finally, we take the case $eH = 0.01 \text{ GeV}^2$, for example, to show the evolution features of the order parameters quark masses m_f , pion condensate Δ_{π} , and Polyakov loop L



FIG. 2. The temperature T_{CEP} (upper panel) and lepton flavorentropy ratio $(l_e^{\text{M}} + l_{\mu}^{\text{M}})_{\text{CEP}}$ (lower panel) as functions of magnetic field *eH* at the critical end points.



FIG. 3. The quark masses $m_{\rm f}$, pion condensate Δ_{π} , and Polyakov loop *L* as functions of the temperature *T*, decreasing with the expansion of the early universe, for the case $eH = 0.01 \text{ GeV}^2$ and $l_{\rm e}^M + l_{\mu}^M = -0.2$. Here, $m_{\rm u}$ and $m_{\rm d}$ are almost the same with each other and thus can be consistently presented as $m_{\rm l}$.

in Fig. 3. Since temperature decreases with time in the early universe, we call in the following the transition points with larger and smaller temperatures as the first and second ones, respectively. Then, the region between the first and second transition points corresponds to the pion superfluidity phase as $\Delta_{\pi} \neq 0$. As we can see, both chiral symmetry and I_3 isospin symmetry get broken through the formation



FIG. 4. Upper panel: the cosmic trajectories of the chemical potentials of electric charge (μ_Q) , baryon (μ_B) , and lepton flavors $(\mu_e, \mu_\mu, \text{ and } \mu_\tau)$ as functions of the temperature *T*; lower panel: the entropy density (*s*) and density-entropy ratios $(q, b, l_e, l_\mu, \text{ and } l)$ as functions of the temperature *T*. The same as that in Fig. 3, we consider the case $eH = 0.01 \text{ GeV}^2$ and $l_e^M + l_\mu^M = -0.2$.

of a large Δ_{π} at the first transition point. Then, the I_3 isospin symmetry becomes restored at the second transition point, where mainly m_1 and Δ_{π} exchange their roles, and thus does not cause many changes to thermodynamical quantities. Compared to that, the first transition is much stronger since m_1 stays small, but there are large gaps of $\Delta_{\pi}, m_{\rm s}$, and L. The corresponding cosmic trajectories of the chemical potentials and the entropy and number densities are demonstrated in the upper and lower panels of Fig. 4, respectively. While the chemical potentials evolve continuously across the transition points as should be, there is a nonmonotonic feature in the spectrum of $\mu_{\rm B}$, stronger than that in the vanishing H limit [35]. According to the lower panel, q and l_e , which vanish in the chiral phases, are more sensitive to the second transition point, but all the left are more sensitive to the first one. This is another indication that the first phase transition is stronger.

III. PART II: FIRST-ORDER PHASE TRANSITION WITH POLYAKOV-QUARK-MESON MODEL

It is hard to study the first-order phase transition dynamics in the framework of the PNJL model that involves four- and six-quark interaction terms. Since it had been shown that the NJL model could be bosonized to a model with only meson degrees of freedom [45], we alternatively adopt the simple two-flavor PQM model [56] to demonstrate the transition process in this section. Note that there are two reasons why we focus on the two-flavor rather than the three-flavor PQM model: first, the effect of strange quarks is small for the exploration of pion super-fluidity [35]; second, we could reduce the number of coupled equations of motions to facilitate numerical evaluations. In the model, the quark degrees of freedom can be integrated

out to introduce the effects of the temperature, chemical potentials, and magnetic field into the mesonic system, and further analysis of bubble dynamics could just follow those given in the pioneer works [57,58] for boson systems.

A. Bubble dynamics

In a background magnetic field, the Lagrangian density of the renormalizable two-flavor PQM model [56] is given by

$$\mathcal{L}_{PQM} = -\frac{B^2}{2} - V(L,L) + \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \pi^0)^2] + \mathcal{D}^{\mu\dagger} \pi^+ \mathcal{D}_\mu \pi^- - \frac{\lambda}{4} [\sigma^2 + (\pi^0)^2 + 2\pi^+ \pi^- - \upsilon^2]^2 + c\sigma + \bar{\psi} \bigg[i \not\!\!\!D - i\gamma^4 \bigg(ig\mathcal{A}^4 + Q_q \mu_Q + \frac{\mu_B}{3} \bigg) - g(\sigma + i\gamma^5 (\tau_3 \pi^0 + \tau_+ \pi^+ + \tau_- \pi^-)) \bigg] \psi,$$
(53)

where $\psi(x) = (u(x), d(x))^T$ denotes the two-flavor quark field, the derivative for π^- is $\mathcal{D}_{\mu} = \partial_{\mu} - ieA_{\mu} + i\mu_Q\delta_{\mu 0}$, and $\tau_{\pm} = \frac{1}{\sqrt{2}}(\tau_1 \pm i\tau_2)$. The energy terms of the background magnetic field and Polyakov loop are the same as those in Eq. (1). The linear term $c\sigma$ violates chiral symmetry explicitly, and we can verify that the Lagrangian has exact $U_3(1)$ chiral symmetry in the chiral limit c = 0. The model parameters of the mesonic sector λ , v, and c are fixed by the sigma mass $m_{\sigma} = 660$ MeV, pion mass $m_{\pi} = 138$ MeV, and pion decay constant $f_{\pi} = 93$ MeV; and the quarkmeson coupling constant is determined by $m_q^v \equiv gf_{\pi} = m_{\sigma}/2$ in the chiral symmetry breaking phase (for the stability of nucleons, $N_c m_q^v > m_N$) [56].

By integrating over the quark degrees of freedom, the Lagrangian can be bosonized as

$$\mathcal{L}_{PQM} = -\frac{B^2}{2} - V(L,L) + \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \pi^0)^2] + \mathcal{D}^{\mu \dagger} \pi^+ \mathcal{D}_\mu \pi^- - \frac{\lambda}{4} [\sigma^2 + (\pi^0)^2 + 2\pi^+ \pi^- - v^2]^2 + c\sigma + \operatorname{Tr} \ln \left[i \not\!\!\!D_q - i \gamma^4 \left(i g \mathcal{A}^4 + Q_q \mu_Q + \frac{\mu_B}{3} \right) - g(\sigma + i \gamma^5 (\tau_3 \pi^0 + \tau_+ \pi^+ + \tau_- \pi^-)) \right].$$
(54)

In mean field approximation, the Gibbs free energy for the chiral phases and pion superfluidity can easily be evaluated as

$$\Omega_{PQM}^{M} = -\frac{H^{2}}{2} + V(L,L) + \frac{\lambda}{4} (\sigma_{1}^{2} - v^{2})^{2} - c\sigma_{1} - N_{c} \sum_{f=u,d} \left\{ -\frac{1}{8\pi^{2}} \int_{0}^{\infty} \frac{ds}{s^{3}} (e^{-m_{f}^{2}s} - e^{-m_{f}^{2}s}) \left(\frac{q_{f}eHs}{\tanh(q_{f}eHs)} - 1 \right) - \frac{1}{8\pi^{2}} \int_{0}^{\infty} \frac{ds}{s^{3}} e^{-m_{f}^{2}s} \left(\frac{q_{f}eHs}{\tanh(q_{f}eHs)} - 1 - \frac{1}{3} (q_{f}eHs)^{2} \right) + 2T \sum_{t=\pm} \frac{|q_{f}eH|}{2\pi} \sum_{n=0}^{\infty} \alpha_{n} \int_{-\infty}^{\infty} \frac{dk_{3}}{2\pi} K_{f}^{t}(E_{f}^{n}, L, T, \mu_{Q}, \mu_{B}) \right\}$$

$$(55)$$

and

$$\Omega_{\rm PQM}^{\pi \rm SF} = V(L,L) + \frac{\lambda}{4} [\sigma_1^2 + 2\Delta_{\pi}^2 - v^2]^2 - \mu_{\rm Q}^2 \Delta_{\pi}^2 - c\sigma - 2N_{\rm c} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{t=\pm} \left[\left| \epsilon_1(k) + t \frac{\mu_{\rm Q}}{2} \right| - \epsilon_1(k) \right] - 2T \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{t,u=\pm} Fl\left(L, u, E_1^{\rm t}(k), \frac{\mu_{\rm Q} + 2\mu_{\rm B}}{6}\right)$$
(56)

by following the schemes given in Secs. II A 1 and II B 1, respectively. Note that the quark vacuum terms with cutoff are dropped to avoid double counting and the last but one in $\Omega_{\pi SF}$ guarantees its form for $\Delta_{\pi} = 0$ to be consistent with

that of Ω_{PQM}^{M} in the vanishing *H* limit. Here, we have taken $m_{\rm u} = m_{\rm d} = g\sigma_{\rm l}$ as the involved magnetic field is relatively small. As we can see, the effects of the temperature, magnetic field, and chemical potentials can be conveniently

introduced into the system through the quark loops at the mean field approximation level.

The contributions of the electroweak interaction sector are the same as those given in Secs. II A 2 and II B 2. We compare the contributions of the strong and electroweak interaction sectors to the free energy difference between chiral phases and pion superfluidity in Fig. 5 with the notations $\Delta \Omega_{\rm C} \equiv \Omega_{\rm POM}^{\pi \rm SF} - \Omega_{\rm POM}^{\rm M}$ and $\Delta \Omega_{\rm EW} \equiv \Omega_{\rm l} - \Omega_{\rm l}^{\rm M}$. According to our evaluations, $\Delta \Omega_{EW}$ are two orders suppressed compared to $\Delta \Omega_{\rm C}$. We also present the results where the magnetic field eH is artificially set to 0 for all the charged fermion loops: since $|\Delta\Omega_{\rm C} - \frac{H^2}{2} - \Delta\Omega_{\rm C}|_{H \to 0}| \ll \frac{H^2}{2}$, the magnetization from quark loops is negligible compared to the chosen magnetic field. Therefore, to conveniently study the first-order QCD phase transition, we can simply focus on the strong interaction sector and set H = 0 in the quark loops for the chosen magnetic field. Actually, the latter approximation is consistent with that done in Eq. (48.18) for the study of the superconductor in Ref. [36] and helps to explore bubble dynamics of the first phase transition. The bosonized Lagrangian density is now reduced to



FIG. 5. The free energy differences of strong interaction sector $\Delta\Omega_{\rm C}$ (dotted line) and electroweak interaction sector $\Delta\Omega_{\rm EW}$ (dashed line) as functions of temperature *T* at $eH = 0.01 \,{\rm GeV}^2$. After taking the constraints $n^{\rm Q} = 0, b^{\rm M} = 8.6 \times 10^{-11}, l^{\rm M} = -0.012$, $l_{\rm e}^{\rm M} = 0$, and $l_{\mu}^{\rm M} = -0.2$ to fix the chemical potentials, we compare the cases with *H*-dependent (blue line) and H = 0 (red line) fermion loops. At the low temperature $T = 0.05 \,{\rm GeV}$, the value $\Delta\Omega_{\rm C} = 5.45 \times 10^{-4} \,{\rm GeV}^4$ actually equals the energy density of the background magnetic field $H^2/2$.

$$\mathcal{L}_{\mathrm{PQM}} = -rac{H^2}{2} + rac{1}{2} (\partial_\mu \sigma)^2 + \mathcal{D}^{\mu \dagger} \pi^+ \mathcal{D}_\mu \pi^- - U(L,\sigma,\pi^\pm),$$

where the total potential is given by

$$\begin{split} U(L,\sigma,\pi^{\pm}) &\equiv V(L,L) + \frac{\lambda}{4} [\sigma^2 + 2\pi^+ \pi^- - v^2]^2 - c\sigma - 2N_c \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{t=\pm} \left[|\epsilon(k) + t \frac{\mu_Q}{2}| - \epsilon(k) \right] \\ &- 2T \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_{t,u=\pm} Fl\left(L, u, E^{\mathrm{t}}(k), \frac{\mu_Q + 2\mu_B}{6}\right) \end{split}$$

with the energy functions $\epsilon(k) = \sqrt{k^2 + g^2 \sigma^2}$ and $E^{\pm}(k) = \sqrt{(\epsilon(k) \pm \frac{\mu_0}{2})^2 + 2g^2 \pi^+ \pi^-}$. Note that we have neglected the irrelevant π^0 degree of freedom and assumed local approximation [59] to reserve the spacetime dependence of the fields σ, π^{\pm} , and *L* in $U(L, \sigma, \pi^{\pm})$.

By following the same scheme as in Sec. II C, the first phase transition point is found to be located at

$$\mu_{\rm Q} = -0.704 \text{ GeV}, \qquad \mu_{\rm B} = 0.352 \text{ GeV}, \qquad \mu_{\rm e} = -0.437 \text{ GeV},$$

$$\mu_{\mu} = -1.372 \text{ GeV}, \qquad \mu_{\tau} = 1.510 \text{ GeV}, \qquad T = 0.217 \text{ GeV}$$
(57)

with the homogeneous order parameters

$$\sigma_{l} = 0.0026 \text{ GeV}, \qquad L = 0.509, \qquad \Delta_{\pi} = 0 \text{ GeV};$$

 $\sigma_{l} = 0.0018 \text{ GeV}, \qquad L = 0.154, \qquad \Delta_{\pi} = \pm 0.116 \text{ GeV}$
(58)

for the χSR and pion superfluidity phases, respectively. Compared to the changing ratios of Δ_{π} and *L*, that of σ_{l} is small across the transition point; so we can safely drop σ degrees of freedom, as was done to π^{0} , for the bubble dynamics. Eventually, the Lagrangian density is further reduced to



FIG. 6. For homogeneous phases, the pure QCD part of the free energy Ω_{QCD} is depicted as a function of the order parameters Δ_{π} and *L* at temperature T = 0.2 GeV.

$$\mathcal{L}_{PQM} = -\frac{F_{\mu\nu}F^{\mu\nu}}{4} + \partial^{\mu}\tilde{\pi}\partial_{\mu}\tilde{\pi} + (\mu_{Q}^{2} - e^{2}\mathbf{A}^{2})\tilde{\pi}^{2} - U(L,\tilde{\pi}),$$
(59)

where we have set $\pi^- = \pi^+ \equiv \tilde{\pi}$ without loss of generality and $U(L, \tilde{\pi}) \equiv U(L, 0, \tilde{\pi})$ should be understood. Note that a spacetime dependent phase of π^{\pm} , that is, $\pi^{\pm} \equiv e^{\pm i\alpha(x)}\tilde{\pi}$, can be absorbed by redefinition of the vector potential $eA_{\mu} \rightarrow eA_{\mu} + \partial_{\mu}\alpha(x)$. And since **A** could be time dependent in the bubble dynamics, we resort to a more general expression of the background EM field term by adopting $F_{\mu\nu}$. According to the opposite signs of μ_Q^2 and $e^2 \mathbf{A}^2$ in the quadratic term of $\tilde{\pi}$, the effects of μ_Q and H could be intuitively expected to be opposite for the pion superfluidity—they actually correspond to Bose-Einstein condensation and Meissner effects, respectively.

In the mean field approximation with $\Delta_{\pi} \equiv \langle \tilde{\pi} \rangle$ and L constants, the pure QCD part of the free energy, $\Omega_{\text{QCD}} = -\mu_{\text{Q}}^2 \Delta_{\pi}^2 + U(L, \Delta_{\pi})$, is depicted as a function of Δ_{π} and L in Fig. 6 for T = 0.2 GeV. Note that the temperature is a bit smaller than the first transition point $T_c = 0.217$ GeV and then the bubble dynamics can be well explored in the transition from χSR to pion superfluidity. Here, two kinds of minimal points can be identified, ($\Delta_{\pi} = 0, L \sim 0.4$) and ($|\Delta_{\pi}| \sim 0.1$ GeV, $L \sim 0.1$), which correspond to the χSR and pion superfluidity phases, respectively. The sign of Δ_{π} is irrelevant; hence we take $\Delta_{\pi} \ge 0$ in the following. By taking the magnetic part, $-H^2/2 = -5.45 \times 10^{-4} \text{GeV}^4$, into account in the χSR phase, the gap between the free energies of these phases would reduce to $\varepsilon_{v} = |\Delta\Omega| = 7.28 \times 10^{-4} \text{GeV}^4$.

During the first-order phase transition, bubbles of true vacuum will be formed and then expand against the false vacuum [57]. In such a case, the expectation values of the order parameters (*L* and $\tilde{\pi}$) and vector potential **A** must be inhomogeneous across the spacetime. By following the Euler-Lagrangian equation $\partial^{\mu} \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\varphi)} - \frac{\partial \mathcal{L}}{\partial\varphi} = 0$, the corresponding EoMs can be derived as



FIG. 7. The solution *L* as a function of $\tilde{\pi}$ from Eq. (60) within the range where the homogeneous solutions Eq. (58) (red bullets) are covered.

$$\partial_L U(L, \tilde{\pi}) = 0, \tag{60}$$

$$2\partial^{\mu}\partial_{\mu}\tilde{\pi} + 2(e^{2}\mathbf{A}^{2} - \mu_{Q}^{2})\tilde{\pi} + \partial_{\tilde{\pi}}U(L,\tilde{\pi}) = 0, \quad (61)$$

$$\partial_0^2 \mathbf{A} + \nabla \times \mathbf{B} + 2e^2 \tilde{\pi}^2 \mathbf{A} = 0 \tag{62}$$

with $\mathbf{B} \equiv \nabla \times \mathbf{A}$. Note that we work in Coulomb gauge with $A_0 = 0$. As we can see, the effective pion field $\tilde{\pi}$ is the key to couple all the equations. Equation (60) is actually an algebra equation of L and $\tilde{\pi}$ and can be solved numerically to obtain L as a function of $\tilde{\pi}$, i.e., $L(\tilde{\pi})$; see the numerical results in Fig. 7. Substituting $L(\tilde{\pi})$ into Eq. (61), we are then left with two coupled differential EoMs for $\tilde{\pi}$ and \mathbf{A} . For homogeneous pion superfluidity with $\tilde{\pi}$ a nonzero constant, the static EoM of \mathbf{B} can be obtained by taking curl of Eq. (62) as

$$\nabla^2 \mathbf{B} - 2e^2 \tilde{\pi}^2 \mathbf{B} = 0. \tag{63}$$

The equation is similar to Eq. (49.11) in Ref. [36], so the Meissner effect resumes for $\tilde{\pi} \neq 0$ in the relativistic case. Generally, the gauge coupling term $e^2 \mathbf{A}^2 \tilde{\pi}^2$ in Eq. (59) favors $\tilde{\pi}B = 0$ in the bulk for a homogeneous phase, so one can check that the solutions in Eq. (58) automatically apply to Eqs. (60)–(62) as should be. Actually, with the temperature decreasing in the QCD epoch, the two solutions mainly correspond to the boundaries and centers of the bubbles in the first-order transition process.

Now, the tough mission left is to work out the bubble structure from the coupled differential equations of motion Eqs. (61) and (62), that is, the forms of $\tilde{\pi}(t, \mathbf{r})$ and $\mathbf{A}(t, \mathbf{r})$, and thus $L(t, \mathbf{r}) \equiv L(\tilde{\pi}(t, \mathbf{r}))$. To uniquely fix the bubble structure, we simply require the bubbles to smoothly approach the χSR phase at the boundary $\mathbf{r} \to \infty$. First, as the background magnetic field is homogeneous and along *z* direction, we assume the *B* vortices as well as the $\tilde{\pi}$ bubbles to be cylindrical with their symmetry axes along

the *z* direction. Hence, the bubbles are expected to be *z* independent. Second, rotational symmetry would require $\tilde{\pi}(\mathbf{r}) = \tilde{\pi}(r)$ and $B_z(\mathbf{r}) = B_z(r)$, where *r* is the radius in the x - y plane and the center is chosen to be at x = y = 0. To ensure explicit rotational symmetry of Eq. (61), we choose the symmetric gauge for the vector potential; that is, $\mathbf{A}(t, \mathbf{r}) = -A(t, r)\mathbf{y}\mathbf{i} + A(t, r)x\mathbf{j}$ and the corresponding magnetic field is $B_z(t, r) = \frac{\partial r^2 A(t, r)}{r \partial r}$. Within such a scheme, we are glad that the equations of motion of the A_x and A_y components reduce to the same one, and we have

$$2\left[\partial_{t}^{2} - \partial_{r}^{2} - \frac{1}{r}\partial_{r} + e^{2}A^{2}(r)r^{2} - \mu_{Q}^{2}\right]\tilde{\pi} + \partial_{\tilde{\pi}}U(\tilde{\pi}) = 0, \quad (64)$$

$$\left(\partial_t^2 - \partial_r^2 - \frac{3}{r}\partial_r + 2e^2\tilde{\pi}^2\right)A(r) = 0$$
 (65)

instead of Eqs. (61) and (62). Note that we have used $U(\tilde{\pi}) \equiv U(L(\tilde{\pi}), \tilde{\pi})$ for brevity here.

To solve the coupled differential equations (64) and (65), boundary conditions are needed. Conventionally, to avoid singularities at the origins of space and time, we require the first-order derivatives to be vanishing [57], that is,

$$\frac{\partial \tilde{\pi}}{\partial t}\Big|_{t=0} = \frac{\partial \tilde{\pi}}{\partial r}\Big|_{r=0} = \frac{\partial A(r)}{\partial t}\Big|_{t=0} = \frac{\partial A(r)}{\partial r}\Big|_{r=0} = 0.$$
(66)

As we can see, the magnetic effect is of order $o(r^2)$ around $r \sim 0$ in Eq. (64), so we could neglect such a term and solve the O(3) symmetric equation of motion

$$2\left[\partial_{\rho}^{2} + \frac{2}{\rho}\partial_{\rho} + \mu_{Q}^{2}\right]\tilde{\pi} - \partial_{\tilde{\pi}}U(\tilde{\pi}) = 0$$
 (67)

to get the imaginary time $\tau = it$ dependence of $\tilde{\pi}$ at the origin r = 0. Note that the Lorentz invariant solution was found to be with the lowest energy for a Lorentz invariant equation of motion. Then, it is natural to give the differential equation of motion for $\tilde{\pi}$ at the initial time t = 0 as

$$2\left[\partial_r^2 + \frac{2}{r}\partial_r - e^2 A^2(r)r^2 + \mu_Q^2\right]\tilde{\pi} - \partial_{\bar{\pi}}U(\tilde{\pi}) = 0, \quad (68)$$

which reduces to Eq. (67) around $r \sim 0$. Equation (68) is more useful than Eq. (67) as the solution interpolates between the true and false vacua and could be directly taken as the boundary condition for Eq. (64).

Next, by taking curl of Eq. (62), we have

$$\partial_0^2 \mathbf{B} - \nabla^2 \mathbf{B} + 2e^2 \tilde{\pi}^2 \mathbf{B} + 4e^2 \tilde{\pi} \nabla \tilde{\pi} \times \mathbf{A}$$

= $[\partial_0^2 B - \nabla^2 B + 2e^2 \tilde{\pi}^2 B + 4e^2 \tilde{\pi} \partial_r \tilde{\pi} A(r) r] \hat{z} = 0.$ (69)

Because of the boundary conditions in Eq. (66), the last term on the left-hand side of Eq. (69) is of order $o(r^2)$ and

can be safely dropped around $r \sim 0$. Then, we get an O(3) symmetric equation of motion for the magnetic field B, that is,

$$\partial_{\rho}^2 B + \frac{2}{\rho} \partial_{\rho} B - 2e^2 \tilde{\pi}^2 B = 0.$$
⁽⁷⁰⁾

Recalling the relation $B(t, r) = \frac{\partial r^2 A(t, r)}{r \partial r}$, Eq. (66) implies similar boundary conditions for *B*:

$$\frac{\partial B(t,r)}{\partial t}\Big|_{t=0} = \frac{\partial B(t,r)}{\partial r}\Big|_{r=0} = 0.$$

So by substituting the solution $\tilde{\pi}(\rho)$ from Eq. (67) into Eq. (70), the form of $B(\rho)$ could be worked out and $A(\tau, r)$ follows as

$$A(\tau, r) = \frac{1}{r^2} \int_0^r B(\sqrt{\tau^2 + s^2}) s ds$$

$$\approx \frac{1}{2} \left[B(\tau) + \frac{1}{2} \frac{B'(\tau)}{\tau} r^2 \right]$$
(71)

to order $o(r^2)$. One can substitute $A(\tau, r)$ into Eq. (62) to find it the same as Eq. (70) at r = 0, so $A(\tau, 0) = \frac{1}{2}B(\tau)$ at the spatial origin r = 0. As mentioned, the form of B(0, r)is more important. Following the previous discussions, it is natural to give the differential equation of motion for *B* at the initial time from Eq. (69) as

$$\partial_r^2 B_z + \frac{2}{r} \partial_r B_z - 2e^2 \tilde{\pi}^2 B - 4e^2 \tilde{\pi} \partial_r \tilde{\pi} A(r) r = 0.$$
(72)

So by solving the differential equations $B(r) = \frac{\partial r^2 A(r)}{r \partial r}$ Eqs. (68) and (72) together, we could obtain the spatial structures of the fields $\tilde{\pi}$, *B*, and *A* at the initial time t = 0. The normalized results are presented in Fig. 8. First of all, the bubble structures of $\tilde{\pi}$ and L are almost the same as the case B = 0 with the radius $R_h \approx 9.60 \text{ GeV}^{-1}$, so it is a good approximation to treat the bubble formation with the Lorentz invariant assumption. Second, the characteristic length ($\xi \lesssim 6 \text{ GeV}^{-1}$) of the pion condensate $\tilde{\pi}$ is consistent with that of the Polyakov loop L. Third, the magnetic field does not change much in the whole bubble range, which means that the penetration depth is much larger than ξ , so the bubbles are quite like the vortices in the Type-II superconductor [36]. Fourth, a bump is found in the magnetic field around the bubble wall, which is more obvious when the temperature is closer to the transition point. Such a feature is a consequence of the latent heat released from the phase transition and the coupling between $\tilde{\pi}$ and A. The underlying physics is consistent with the logic of how first-order phase transition could induce an extremely large primordial magnetic field in the early universe [29].



FIG. 8. The solutions $\tilde{\pi}(r)$ (yellow solid line), L(r) (green dotted line), and B(r) (black dashed line) as functions of the radius *r* from Eqs. (60), (68), and (72). The Polyakov loop and pion condensate are normalized by their maximal values to L^r and Δ_{π}^r , and the magnetic field *B* by its background value *H* to B^r , respectively. The red baseline is f(r) = 1.

Now, we are ready to solve the full differential equations (61) and (62) by adopting the initial conditions presented in Fig. 8. The spacetime evolution of the reduced fields $\tilde{\pi}^{r}$ and B^{r} are illustrated in Fig. 9. The



FIG. 9. The spacetime evolution of the reduced fields $\tilde{\pi}^{r}$ (upper panel) and B^{r} (lower panel).

bubble wall of $\tilde{\pi}^r$ expands ~16 GeV⁻¹ within the time interval ~30 GeV⁻¹, so the speed is smaller than the velocity of light as should be. Furthermore, with the development of the superconductor phase, the magnetic field decreases further deep inside the superconductor and would approach zero for a longer time. Meanwhile, the bump structure of the magnetic field becomes more and more sharp with time going, since more and more latent heat is released to the wall as the bubble expand. In this sense, the thin wall approximation could be adopted to calculate the nonvanishing quartic moment caused by bubble collisions, which is closely connected with the radiation of the gravitational wave [60].

B. Gravitational wave within a toy system

To explore bubble collisions, the tunneling probability density Γ has to be estimated. Since our study has shown that the bubble formation is not far from the threedimensional Lorentz invariant, Γ could be evaluated in the O(3) symmetric Euclidean space as $\Gamma = \mathcal{A}(\frac{\mathcal{B}}{2\pi})^{3/2}e^{-\mathcal{B}}$ [58]. In thin wall approximation, the involved coefficients are, respectively,

$$\mathcal{A} = \frac{(\mu R_b)^{7/3}}{R_b^3},$$

$$\mathcal{B} = (2R_b)4\pi \int_0^\infty \rho^2 d\rho \left\{ \left(\frac{d\tilde{\pi}}{d\rho}\right)^2 + (e^2 A^2(\rho)\rho^2 - \mu_Q^2)\tilde{\pi}^2 + U(\tilde{\pi}) - U(0) - \frac{B^2(\rho) - H^2}{2} \right\},$$
(73)

where μ is the renormalization scale for the 2 + 1 dimension [61] and could be set to $\mu = 1$ GeV for QCD. Even though we have assumed the bubble dynamics to be z independent to simplify the interplay between B and $\tilde{\pi}$, the bubble must be bounded in the z direction in order to be consistent with the case B is irrelevant. For the small bubbles, we set the size to be the same as that in the x and y directions for simple estimation since Lorentz invariance is not violated in the t - z space. So the generated bubbles are roughly cylinders with the same height and diameter—that is why $2R_b$ shows up in the expression of \mathcal{B} .

Then, after inserting the solutions of Eqs. (60), (68), and (72), we find

$$\mathcal{A} = 0.221 \text{ GeV}^{-3}, \qquad \mathcal{B} = 14.2,$$
(74)

and thus $\Gamma = 5.29 \times 10^{-7}$ GeV⁻³. Then, considering a toy cylindrical system with height *h* and diameter *d* of the same size, that is, h = d = 200 GeV⁻¹, we expect around two bubbles to be generated within the timescale $t \sim d/2$. In the following, we assume two bubbles to be generated simultaneously at $x = \pm d/4$, y = z = 0 in such a system and study the features of the gravitational wave observed at the



FIG. 10. A toy model for bubble collision: the system and nucleated bubbles are all chosen to be cylinders with the height equal to the diameter and their axes along the direction of the background magnetic field, and the two bubbles are assumed to be generated simultaneously at the locations $x = \pm d/4$, y = z = 0. The system size is $d = 200 \text{ GeV}^{-1}$ and the bubble radius was found to be $R_{\rm b} = 9.60 \text{ GeV}^{-1}$ according to the calculation in Sec. III A.

origin (see Fig. 10). Compared to the common cases where the GW sources are far away from us [60], we are assumed to be inside the source here as we are studying the magnetic effect in our own galaxy. Inspired by the fact that the characteristic frequency of GW does not depend on the bubble number in the early universe [38], we expect the characteristic frequency to be determined by the energy scale at the QCD epoch and not sensitive to the details of bubble dynamics. In principle, the magnitude of the GW depends on the system size as the GW originated from different locations would overlap with each other. For a larger system, we could divide it to several numbers of the toy system and the boundary of the toy system roughly corresponds to the overlapping region with the bubbles actually outside.

As the two bubbles expand and collide with each other, the variation of stress tensor would induce GW radiation, that is,

$$T_{ij}(\mathbf{x},t) = \partial_i \tilde{\pi} \partial_j \tilde{\pi} - \frac{F_{i\nu} F^{j\nu}}{2}.$$
 (75)

The resultant GW strains are then given by [60]

$$\begin{split} h_{+}(t) &= 2G\Re \int d\omega \int \frac{dz d^{2}\mathbf{r}}{R} e^{-i\omega(t-R)} [T_{xx} - T_{yy}](z,\mathbf{r},\omega), \\ h_{\times}(t) &= 4G\Re \int d\omega \int \frac{dz d^{2}\mathbf{r}}{R} e^{-i\omega(t-R)} T_{xy}(z,\mathbf{r},\omega), \end{split}$$

where \Re means the real part and *R* is the distance of the GW source from the observer at the origin. For rough

estimation, the strain will be evaluated by adopting the envelope approximation, which was shown to be in good agreement with the exact numerical evaluation [62]. The initial approximation is based on two simplifications: (i) the bubbles expand spherically with speed of light and do not interfere with each other; (ii) only the bubble walls that do not overlap with others and are inside the system are taken into account in the integration. Here, we modify (i) by requiring the bubbles to expand with half speed of light in the x - y plane according to the numerical results in Fig. 9. Hence, the GW strains become

$$h_{+}(t) = G\Re \int d\omega \, e^{-i\omega t} H_{+}(\omega), \qquad (76)$$

$$h_{\times}(t) = G\Re \int d\omega \, e^{-i\omega t} H_{\times}(\omega), \qquad (77)$$

where $H_+(\omega)$ and $H_{\times}(\omega)$ depend on the stress tensors and are given in frequency space as

$$H_{+}(\omega) = \frac{\varepsilon_{\rm v}}{2\pi} \int_{0}^{\infty} dt \int_{-l_{\rm z}(t)}^{l_{\rm z}(t)} dz \, r^{2}(t) \int_{S} d\theta \frac{e^{i\omega[t+R(z,t,\theta)]}}{R(z,t,\theta)} \cos(2\theta),$$
(78)

$$H_{\times}(\omega) = \frac{\varepsilon_{\mathrm{v}}}{2\pi} \int_{0}^{\infty} dt \int_{-l_{z}(t)}^{l_{z}(t)} dz r^{2}(t) \int_{S} d\theta \frac{e^{i\omega[t+R(z,t,\theta)]}}{R(z,t,\theta)} \sin(2\theta).$$
(79)

Here, we define $l_z(t) = R_b + t$, $r(t) \approx R_b + 0.5t$, and $R(z, t, \theta) = \sqrt{z^2 + r^2(t) + (\frac{d}{4})^2 - \cos \theta r(t) \frac{d}{2}}$, and *S* means the surface of the nonoverlapping bubble walls.

Since the configuration of bubbles is symmetric under the transformation $\theta \rightarrow -\theta$ at any time, we expect $H_{\times}(\omega) = 0$. For $r(t) > \frac{d}{2}$, the integration region of θ would start to be eaten by the overlapping with both the other bubble and the system boundary. Then, by taking into account the fact that the bubble configuration is reflectional symmetric with respect to all the axial planes, the explicit form of $H_{+}(\omega)$ is

$$\begin{aligned} H_{+}(\omega) \\ &= \frac{2\varepsilon_{\rm v}}{\pi} \int dt r^{2}(t) \vartheta \left(r(t) - \frac{d}{4} \right) \vartheta \left(\frac{d}{2} - l_{\rm z}(t) \right) \\ &\times \int_{-l_{\rm z}(t)}^{l_{\rm z}(t)} dz \int_{\theta_{1}}^{\theta_{2}} d\theta \frac{e^{i\omega[t+R(z,t,\theta)]}}{R(z,t,\theta)} \cos(2\theta) \vartheta(\theta_{2} - \theta_{1}) \\ &= \frac{2\varepsilon_{\rm v}}{\pi} \int dt r^{2}(t) \text{Boole} \left(\frac{d}{4} < r(t) < \sqrt{5} \frac{d}{4} \right) \\ &\times \vartheta \left(\frac{d}{2} - l_{\rm z}(t) \right) \int_{-l_{\rm z}(t)}^{l_{\rm z}(t)} dz \int_{\theta_{1}}^{\theta_{2}} d\theta \frac{e^{i\omega[t+R(z,t,\theta)]}}{R(z,t,\theta)} \cos(2\theta). \end{aligned}$$

$$\end{aligned}$$

$$\tag{80}$$

Here, the integral limits of θ are defined as $\cos \theta_1 = \frac{d}{4r(t)}$ and $\cos \theta_2 = -\frac{3(d/4)^2 - r^2(t)}{r(t)d/2}$ according to the collisions with the other bubble and system boundary, respectively. Note that *Boole (conditions)* is the Boole function which is 1 if the conditions are all true and 0 otherwise.

Of course, after we obtained the explicit form of $H_+(\omega)$ by numerical calculations, h_+ can be evaluated according to the inverse Fourier transformation as shown in Eq. (76). On the other hand, we could also work out an integral form for h_+ by inserting Eq. (80) into Eq. (76) and then carry out numerical calculations directly. Actually, the integral over the frequency ω gives rise to a delta function $\delta(t - t' - R(z, t', \theta))$, which can be rewritten as $\frac{R(z,t',\theta)}{r(t)d/4}\delta(\theta - \theta_3)\vartheta(t - t')$ with $\cos\theta_3 = \frac{z^2 + r^2(t') + (d/4)^2 - (t - t')^2}{r(t')d/2}$. Then, by completing the integration over θ , we have

$$h_{+}(t) = 4G\varepsilon_{v} \int dt' \frac{r(t')}{d/4} \operatorname{Boole}\left(\frac{d}{4} < r(t') < \sqrt{5}\frac{d}{4}\right)$$
$$\times \vartheta\left(\frac{d}{2} - l_{z}(t')\right) \vartheta(t - t') \int_{-l_{z}(t')}^{l_{z}(t')} dz \cos(2\theta_{3})$$
$$\times \operatorname{Boole}(\theta_{1} < \theta_{3} < \theta_{2}), \tag{81}$$



FIG. 11. The spectra of the GW strain: the real (red solid line) and imaginary (blue dashed line) parts of H_+ with frequency ω in the upper panel and h_+ with time t in the lower panel.

where one should keep in mind that θ_3 depends on *t*, *t'*, and *z*.

The numerical results for the GW strain are illustrated in Fig. 11. From the upper panel, we can identify that the real and imaginary parts of H_{+} are even and odd functions of ω , respectively, and the characteristic frequency is of order 0.1 GeV. So the characteristic frequency is consistent with the energy scale of QCD, $\Lambda_{OCD} = 0.2-0.3$ GeV, and then that of the relic GW is of order 10^{-4} – 10^{-5} eV in our recent galaxy after being scaled by the factor $a = 10^{12} - 10^{12.5}$ [27]. In other more familiar units, the relic frequency is of order 0.1–1 K or 10^9 – 10^{10} Hz, comparable to that of the cosmic microwave background. From the lower panel, we find the magnitude of the GW strain to be of order 10^{-38} . Our system size is $5 * 10^{-13}$ fm which corresponds to 0.5–5 m in the recent universe. If the overlapping region of the GW is as large as 5 km, the magnitude can be estimated to be of order $[(10^3)^3 - (10^4)^3] * 10^{-38} = 10^{-29} - 10^{-26}$. quite within the capability of the next generation GW detectors [63]. If the bubble distribution is random, it is reasonable to assume that the characteristic frequency of $H_{ imes}$ and magnitude of $h_{ imes}$ should be of the same order as those of H_+ and h_+ [59].

IV. SUMMARY

In the first part, we have extended our study of the QCD phase diagram of the early universe [35] by taking into account the primordial magnetic field within the PNJL model. By referring to Gibbs free energy, we were able to demonstrate the Meissner effect at mean field level; that is, the finite external magnetic field H tends to reduce the free energy of chiral phases rather than that of superconducting pion superfluidity. Now, the transitions between chiral phases and pion superfluidity were well explored and were found to be of first order at relatively small H, compared to that of second order for vanishing H. With H increasing, the regime of πSF shrinks quickly and vanishes at the threshold value $eH = 0.0222 \text{ GeV}^2$, so first-order transition to πSF is possible only when the primordial magnetic field is around the lower limit of the estimated magnitude 10^{-2} -1 GeV². Then, taking the case with eH = 0.01 GeV² and $l_e^{\rm M} + l_{\mu}^{\rm M} = -0.2$, for example, we showed the evolutions of order parameters, entropy, densities, and the cosmic trajectories of the chemical potentials with temperature, which decreases with time in the early universe. There, two first-order transitions were explicitly demonstrated according to the noncontinuity of the order parameters, entropy, and densities; and the first transition was found to be stronger than the second one.

In the second part, we have adopted instead the twoflavor PQM model to study bubble dynamics during the first-order QCD transition. The equations of motion of mesons could easily be obtained in the PQM model, and then we could conveniently apply the formalism of bubble dynamics [57,58] well developed before to the QCD system. At the first transition point, we reduced the set of EoMs by suppressing π^0 and σ degrees of freedom, since the expectation value of π^0 is zero and that of σ remains small across the point at mean field approximation. After working out the dependence of the Polyakov loop on the charged pion condensate, we could eventually reduce the number of coupled EoMs to two, one for the charged pion and the other for the magnetic field. From that, we could directly reproduce the Meissner effect for the relativistic system, and the expanding features of bubbles were well demonstrated. Then, the generating probability density was estimated for bubbles in the PQM model, and we introduced a toy model by considering a cylindrical system where two bubbles could form. The two bubbles would expand and collide with each other to produce a gravitational wave. By adopting the envelope approximation, the spectra of GW strain were briefly studied with respect to the variations of both frequency and time, and the characteristic frequency of the relic GW was estimated to be of the order 0.1–1 K or 10^9-10^{10} Hz in our galaxy. For comparison, the characteristic frequency of the relic GW produced due to inflation [64] is of order 10^{-10} - 10^{-8} Hz when the early universe passed through the pion superfluidity phase with second-order transitions [33].

We have to admit that the recent study of a gravitational wave generated from first-order QCD transition is quite preliminary; more realistic calculations should be carried out in the future. The dynamics with more bubbles would be considered for a much larger volume, and then the characteristic frequency and magnitude of GW strain can be further constrained. The contribution of domain walls to GW is also an interesting topic to explore just after the firstorder QCD transition [39,41]. On the other hand, the bubble structure shown in Fig. 8 seems to indicate that the Type-II superconducting pion superfluidity is more favored in an external magnetic field. It happens that the study of phonon modes favors such a possibility [65], so we will check that by considering a magnetic vortical structure in the future. Though the transitions involving the Type-II superconductor are usually of second order, direct GW emission is still supposed to be generated from the collisions among vortices presented there.

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