


Symmetry preserving contact interaction treatment of the kaon

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A symmetry preserving regularization procedure for dealing with the contact interaction model is proposed in this work. This regularization procedure follows a series of consistency conditions that are necessary to maintain gauge symmetry. Under this regularization, proofs for the preservation of the Ward-Takahashi identities are given and the loop integrals in the contact interaction model are systematically computed. As an application example, the kaon electromagnetic form factor and K_{l3} transition form factor are computed, and self-consistent results are obtained. Since the proposed regularization properly handles the divergence, one is freed from the inconsistencies caused by the regularization and can concentrate more on the physical discussion.

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I. INTRODUCTION

Over the years, the Dyson-Schwinger equations (DSEs) have proved to be a powerful tool for studying the non-perturbative nature of quantum chromodynamics (QCD) in the continuum [1–3]. Self-consistent treatments of the quark gap equation and bound state equations, such as the Bethe-Salpeter equation (BSE) and the Faddeev equation, create a bridge connecting the hadrons to the fundamental degrees of freedom of QCD, quarks, and gluons [4,5].

Within the framework of DSEs, a vector-vector contact interaction approximation was proposed in Ref. [6]. Despite the simplicity of the contact interaction model in describing the real world, it has been used to calculate a wide range of hadron properties, including mass spectrum, various decay processes, electromagnetic form factors, transition form factors, and the parton distributions (see Refs. [6–21]). However, because of the nonrenormalizable nature of the contact interaction, the regularization scheme becomes a crucial part in the practical calculation, and a good regularization scheme should properly characterize the divergence structure of the theory. It is worth noting that symmetries, and in particular the vector and axial-vector Ward-Takahashi identities (WTIs), which correspond to the gauge symmetry and chiral symmetry, respectively, provide

a strong constraint that must be preserved during the regularization process.

The regularization procedures performed in previous studies of the contact model have been more or less unsatisfactory. First, both the vector and axial-vector WTIs do not naturally hold under the previous regularization procedure [6]. Moreover, there are cases where inconsistent results occur in the calculation of the kaon electromagnetic form factor, when the principle of charge conservation is violated [10]. The main reason for these problems is that previous regularization procedure fails to properly deal with the quadratic and logarithmic divergent integrals. Another regularization procedure based on a subtraction method has been proposed [15,17], wherein the divergences associated with different tensor structures have been isolated and the WTIs are satisfied subsequently. However, in the calculation of form factors, the method is cumbersome and not easy to be implemented.

Inspired by Ref. [22] we presented a new regularization procedure in this work. This proper regularization meets many interesting properties of the dimensional regularization without changing the spacetime dimension. One of the most fascinating properties is that gauge symmetries are preserved under this regularization.

This paper is organized as follows: Section II introduces the new symmetry preserving regularization that properly handles the divergent integrals. Section III discusses this new regularization in the contact interaction model, incorporating the preserving of (axial-)vector WTIs, and gives steps for the systematic calculations with the contact model. Section IV provides results of the kaon electromagnetic form factor and K_{l3} form factor under the new regularization, and the last section gives a brief summary.

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II. SYMMETRY PRESERVING REGULARIZATION

Before discussing the regularization procedures in detail, it is helpful to introduce the so-called one-fold irreducible loop integrals (ILIs) in Ref. [22]:

$$\begin{aligned} I_{-2\alpha}(\mathcal{M}^2) &= \int_q \frac{1}{(q^2 + \mathcal{M}^2)^{\alpha+2}}, \\ I_{-2\alpha}^{\mu\nu}(\mathcal{M}^2) &= \int_q \frac{q_\mu q_\nu}{(q^2 + \mathcal{M}^2)^{\alpha+3}}, \\ I_{-2\alpha}^{\mu\nu\rho\sigma}(\mathcal{M}^2) &= \int_q \frac{q_\mu q_\nu q_\rho q_\sigma}{(q^2 + \mathcal{M}^2)^{\alpha+4}} \end{aligned} \quad (1)$$

with $\int_q \doteq \int \frac{d^4q}{(2\pi)^4}$ and $\alpha = -1, 0, 1, \dots$. Here $\alpha = -1$ represents quadratically divergent integrals, and $\alpha = 0$ represents logarithmically divergent integrals. With the help of Feynman parametrization, it is straightforward to conclude that all one loop integrals can be expressed in terms of these integrals, where \mathcal{M} is a function of Feynman parameters, external momenta, and the corresponding mass scales. A regularization procedure can be implemented after rearranging one loop integrals into these ILIs.

In Ref. [22], a loop regularization is proposed that simulates in many interesting features to the momentum cutoff, Pauli-Villars and dimensional regularization without modifying the original Lagrangian formalism, and it is directly performed in the spacetime dimension of the original theory. The loop regularization is equivalent to introducing a weight function to regularize the proper-time variable τ integration [23], for example,

$$\begin{aligned} I_{-2\alpha}^{LR}(\mathcal{M}^2) &= \lim_{N \rightarrow \infty} \int_q \int_0^\infty d\tau \mathcal{W}_N(\tau, M_c, \mu_s) \\ &\quad \times \frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)} e^{-\tau(q^2 + \mathcal{M}^2)}, \end{aligned} \quad (2)$$

where the superscript *LR* denotes the loop regularization and $\Gamma(n)$ is the gamma function. An explicit form of the weight function is

$$\mathcal{W}_N(\tau, M_c, \mu_s) = e^{-\tau\mu_s^2} (1 - e^{-\tau M_c^2})^N, \quad (3)$$

with $M_R^2 = M_c^2 h_w(N) \ln N$, $h_w(N) \gtrsim 1$ and $h_w(N \rightarrow \infty) = 1$. The two energy scales M_c and μ_s serve as ultraviolet (UV) and infrared (IR) cutoff, respectively. It is worth noting that when $N \rightarrow \infty$ and $\mu_s = 0$ the weight function becomes

$$\lim_{N \rightarrow \infty} \mathcal{W}_N(\tau, M_c, \mu_s = 0) = \theta(\tau M_c^2 - 1). \quad (4)$$

Thus the weight function regularizes the proper-time integral just as it regularizes it with a hard UV cutoff $1/M_c^2$.

We introduce a regularization procedure that is based on the Schwinger's proper-time method. The regularization procedure for the scalar type ILIs is

$$\begin{aligned} I_{-2\alpha}(\mathcal{M}^2) &= \int_q \frac{1}{(q^2 + \mathcal{M}^2)^{\alpha+2}} \\ &= \int_q \int_0^\infty d\tau \frac{\tau^{\alpha+1}}{\Gamma(\alpha+2)} e^{-\tau(q^2 + \mathcal{M}^2)} \\ &= \int_0^\infty d\tau \frac{\tau^{\alpha-1}}{\Gamma(\alpha+2)} \frac{e^{-\tau\mathcal{M}^2}}{16\pi^2} \\ \rightarrow I_{-2\alpha R}(\mathcal{M}^2) &= \int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau \frac{\tau^{\alpha-1}}{\Gamma(\alpha+2)} \frac{e^{-\tau\mathcal{M}^2}}{16\pi^2}. \end{aligned} \quad (5)$$

The label *R* in the subscript denotes the regularized integrals. It is already seen that a hard UV cutoff $\tau_{uv} = 1/M_c$ is equivalent to the loop regularization with $\mu_s = 0$. However, instead of the sliding energy scale μ_s in the loop regularization, we introduce a hard IR cutoff τ_{ir} to implement confinement, as proposed in Ref. [24]. This way of dealing with the IR cutoff matches the regulators in previous contact model studies, which, as we shall see, can also maintain gauge symmetries if the tensor type ILIs are properly regularized. Before proceeding, it is noted that when integer $\alpha < -1$, the loop integral vanishes under Eq. (5), which happens to be the same property of dimensional regularization.

Turning now to the regularization of the tensor type ILIs, the regularization procedure is

$$\begin{aligned} I_{-2\alpha}^{\mu\nu}(\mathcal{M}^2) &= \int_q \frac{q_\mu q_\nu}{(q^2 + \mathcal{M}^2)^{\alpha+3}} \\ &= \int_q \int_0^\infty d\tau q_\mu q_\nu \frac{\tau^{\alpha+2}}{\Gamma(\alpha+3)} e^{-\tau(q^2 + \mathcal{M}^2)} \\ &= \int_q \int_0^\infty d\tau \delta_{\mu\nu} \frac{q^2}{4} \frac{\tau^{\alpha+2}}{\Gamma(\alpha+3)} e^{-\tau(q^2 + \mathcal{M}^2)} \\ &= \delta_{\mu\nu} \int_0^\infty d\tau \frac{\tau^{\alpha-1}}{\Gamma(\alpha+3)} \frac{e^{-\tau\mathcal{M}^2}}{32\pi^2} \\ \rightarrow I_{-2\alpha R}^{\mu\nu}(\mathcal{M}^2) &= \delta_{\mu\nu} \int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau \frac{\tau^{\alpha-1}}{\Gamma(\alpha+3)} \frac{e^{-\tau\mathcal{M}^2}}{32\pi^2} \end{aligned} \quad (6)$$

and

$$\begin{aligned} I_{-2\alpha}^{\mu\nu\rho\sigma}(\mathcal{M}^2) &= \int_q \frac{q_\mu q_\nu q_\rho q_\sigma}{(q^2 + \mathcal{M}^2)^{\alpha+4}} \\ &= \int_q \int_0^\infty d\tau q_\mu q_\nu q_\rho q_\sigma \frac{\tau^{\alpha+3}}{\Gamma(\alpha+4)} e^{-\tau(q^2 + \mathcal{M}^2)} \\ &= \int_q \int_0^\infty d\tau S_{\mu\nu\rho\sigma} \frac{q^4}{24} \frac{\tau^{\alpha+3}}{\Gamma(\alpha+4)} e^{-\tau(q^2 + \mathcal{M}^2)} \\ &= S_{\mu\nu\rho\sigma} \int_0^\infty d\tau \frac{\tau^{\alpha-1}}{\Gamma(\alpha+4)} \frac{e^{-\tau\mathcal{M}^2}}{64\pi^2} \\ \rightarrow I_{-2\alpha R}^{\mu\nu\rho\sigma}(\mathcal{M}^2) &= S_{\mu\nu\rho\sigma} \int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau \frac{\tau^{\alpha-1}}{\Gamma(\alpha+4)} \frac{e^{-\tau\mathcal{M}^2}}{64\pi^2}, \end{aligned} \quad (7)$$

where $S_{\mu\nu\rho\sigma} = \delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\sigma\nu} + \delta_{\mu\sigma}\delta_{\nu\rho}$ is the total symmetric tensor. It is obvious that the regularized tensor type ILIs and scalar type ILIs are related as follows:

$$I_{-2\alpha R}^{\mu\nu}(\mathcal{M}^2) = \frac{\Gamma(\alpha+2)}{2\Gamma(\alpha+3)} \delta_{\mu\nu} I_{-2\alpha R}(\mathcal{M}^2), \quad (8)$$

$$I_{-2\alpha R}^{\mu\nu\rho\sigma}(\mathcal{M}^2) = \frac{\Gamma(\alpha+2)}{4\Gamma(\alpha+4)} S_{\mu\nu\rho\sigma} I_{-2\alpha R}(\mathcal{M}^2). \quad (9)$$

These relations are precisely the so-called consistency conditions of gauge symmetry in Refs. [22,23], which are independent of regularization and are necessary for preserving the gauge invariance of theories. It is noted that ILIs under dimensional regularization also satisfy these conditions. These consistency conditions connect tensor type ILIs and scalar type ILIs, and then any gauge invariant theories can be properly described in terms of the regularized scalar type ILIs. In fact, there are a series of consistency conditions for ILIs with even more Lorentz index which are rarely encountered and therefore are not presented here.

We now consider the regularization procedure in previous contact model studies, such as in Ref. [6]. For example, the quadratic divergent tensor type ILI is regularized as follows:

$$\begin{aligned} I_{2R'}^{\mu\nu}(\mathcal{M}^2) &= \int_q \frac{q^\mu q^\nu}{(q^2 + \mathcal{M}^2)^2} \\ &\rightarrow \int_q \frac{\delta_{\mu\nu}}{4} \left(\frac{1}{q^2 + \mathcal{M}^2} - \frac{\mathcal{M}^2}{(q^2 + \mathcal{M}^2)^2} \right) \\ &= \int_q \int_0^\infty d\tau \frac{\delta_{\mu\nu}}{4} e^{-\tau(q^2 + \mathcal{M}^2)} (1 - \tau\mathcal{M}^2) \\ &= \delta_{\mu\nu} \int_0^\infty d\tau \frac{e^{-\tau\mathcal{M}^2}}{64\pi^2 \tau^2} (1 - \tau\mathcal{M}^2) \\ \rightarrow I_{2R'}^{\mu\nu}(\mathcal{M}^2) &= \delta_{\mu\nu} \int_{\tau_{uv}^2}^{\tau_{ir}^2} d\tau \frac{e^{-\tau\mathcal{M}^2}}{64\pi^2 \tau^2} (1 - \tau\mathcal{M}^2), \end{aligned} \quad (10)$$

where R' denotes the regularization in previous contact model studies. Apparently, this way of regularization does not satisfy the consistency conditions and hence breaks the gauge symmetries, especially the WTIs. This inappropriate regularization of the tensor type ILIs is also responsible for the inconsistent results that appear in the more complex computations as the case of the triangle diagrams. The key difference between the two regularizations R and R' is which step to do the symmetry analysis, i.e., the substitution such as

$$q_\mu q_\nu \rightarrow \delta_{\mu\nu} \frac{q^2}{4}. \quad (11)$$

In Eq. (6), one first performs the proper-time method so that the divergent momentum integrals become well-defined,

followed by the substitution. In this way, the tensor type integrals are properly regularized and gauge symmetry is preserved. In contrast, in Eq. (10) the substitution is made while the integrals are still divergent, and then the divergent structures of theories are destroyed and lead to possible inconsistency in further calculations. It is well known that the substitution is valid only when the loop integral is convergent. In principle, if we push the regulators $\tau_{uv} \rightarrow 0$ and $\tau_{ir} \rightarrow \infty$, which restores the original ILIs, the two regularizations R and R' lead to the same results for convergent integrals. However, we keep the two regulators τ_{uv} and τ_{ir} even when the ILIs are convergent, so that the regularized ILIs still retain the following relation to the original ILIs:

$$I_{-2(\alpha+1)R}(\mathcal{M}^2) = -\frac{1}{\alpha+2} \frac{d}{d\mathcal{M}^2} I_{-2\alpha R}(\mathcal{M}^2). \quad (12)$$

In the following sections, we will adopt the new regularization R to illustrate how these consistency conditions keep gauge symmetries and then recalculate the kaon electromagnetic and transition form factors. For simplicity, the label R is suppressed thereafter.

III. SYMMETRY PRESERVING REGULARIZATION IN THE CONTACT MODEL

The quark gap equation in the contact model can be written as

$$S_f^{-1}(p) = S_{f0}^{-1}(p) + \frac{4}{3m_g^2} \int_q \gamma_\alpha S_f(q) \gamma_\alpha, \quad (13)$$

where $S_f^{-1}(p)$ is the inverse of the dressed quark propagator with flavor f and momentum p , which has the general form

$$S_f^{-1}(p) = i\not{p} + M_f, \quad (14)$$

where M_f is independent of momentum due to the features of contact interaction. $S_{f0}^{-1} = i\not{p} + m_f$ is the inverse of the bare quark propagator with m_f being the current quark mass.

In terms of the ILIs, the gap equation becomes

$$\hat{M}_f = \frac{16M_f}{3m_g^2} I_2(M_f^2), \quad (15)$$

where $\hat{M}_f = M_f - m_f$.

In the contact model, the meson's Bethe-Salpeter amplitude (BSA) depends only on the meson's total momentum Q ; thus the most general form of the pseudoscalar (PS) meson BSA with an outgoing a quark and an incoming b quark can be expressed as

$$\Gamma_{\text{PS}}^{ab}(Q) = i\gamma_5 E_{\text{PS}}(Q) + \frac{\gamma_5 \mathcal{Q}}{M_{ab}} F_{\text{PS}}(Q), \quad (16)$$

where we have introduced the usual reduced mass $M_{ab} = \frac{2M_a M_b}{M_a + M_b}$. The BSAs satisfy the following homogeneous Bethe-Salpeter equation:

$$\Gamma_{\text{PS}}^{ab}(Q) = -\frac{4}{3m_G^2} \int_q \gamma_\alpha \chi_{\text{PS}}^{ab}(q, q_-) \gamma_\alpha, \quad (17)$$

where $q_- = q - Q$ and $\chi_{\text{PS}}^{ab}(q, q_-) = S_a(q) \Gamma_{\text{PS}}^{ab}(Q) S_b(q_-)$ is the corresponding Bethe-Salpeter wave function.

In the following, we will show how the new regularization procedure preserves the axial-vector WTI exactly. The axial-vector WTI states

$$\begin{aligned} Q_\mu \Gamma_{5\mu}^{ab}(Q) &= S_a^{-1}(q) i\gamma_5 + i\gamma_5 S_b^{-1}(q_-) \\ &\quad - i(m_a + m_b) \Gamma_5^{ab}(Q). \end{aligned} \quad (18)$$

The axial-vector vertex $\Gamma_{5\mu}^{ab}(Q)$ satisfies an inhomogeneous BSE,

$$\Gamma_{5\mu}^{ab}(Q) = \gamma_5 \gamma_\mu - \frac{4}{3m_G^2} \int_q \gamma_\alpha \chi_{5\mu}^{ab}(Q) \gamma_\alpha, \quad (19)$$

with $\chi_{5\mu}^{ab}(Q) = S_a(q) \Gamma_{5\mu}^{ab}(Q) S_b(q_-)$. The pseudoscalar vertex $\Gamma_5(Q)$ satisfies the equation in analogy with the axial-vector vertex by replacing the inhomogeneous term $\gamma_5 \gamma_\mu$ with γ_5 . The axial-vector WTI connects the vertex BSEs and the gap equation, leading to the following two identities:

$$\begin{aligned} \hat{M}_a + \hat{M}_b &= \frac{4}{3m_G^2} \int_q \left(\frac{4M_a}{q^2 + M_a^2} + \frac{4M_b}{q_-^2 + M_b^2} \right), \\ 0 &= \frac{4}{3m_G^2} \int_q \left(\frac{2Q \cdot q}{q^2 + M_a^2} - \frac{2Q \cdot q_-}{q_-^2 + M_b^2} \right). \end{aligned} \quad (20)$$

By analyzing the integrals with Feynman parametrization, one can rearrange the expressions in terms of ILIs and arrive at

$$\begin{aligned} \hat{M}_a + \hat{M}_b &= \frac{16(M_a + M_b)}{3m_G^2} \int_0^1 du \left\{ I_2(\omega_2) + \frac{(uM_b - \bar{u}M_a)}{M_a + M_b} \right. \\ &\quad \left. \times I_0(\omega_2) ((2u-1)Q^2 - M_b^2 + M_a^2) \right\} \end{aligned} \quad (21)$$

and

$$0 = \frac{8}{3m_G^2} \int_0^1 du \{ Q^2 I_2(\omega_2) - 2Q_\mu Q_\nu I_2^{\mu\nu}(\omega_2) \}, \quad (22)$$

where $\bar{u} = 1 - u$ and $\omega_2 = uM_b^2 + \bar{u}M_a^2 + u\bar{u}Q^2$.

Consider the following integral:

$$H_{AV} = \frac{16(M_a + M_b)}{3m_G^2} \int_0^1 g_{AV}(u, M_a, M_b, Q^2) du, \quad (23)$$

with

$$g_{AV}(u, M_a, M_b, Q^2) = \frac{d}{du} \left[I_2(\omega_2) \frac{(uM_b - \bar{u}M_a)}{M_a + M_b} \right]. \quad (24)$$

On the one hand, one can evaluate this integral directly,

$$\begin{aligned} H_{AV} &= \frac{16(M_a + M_b)}{3m_G^2} I_2(\omega_2) \frac{(uM_b - \bar{u}M_a)}{M_a + M_b} \Big|_0^1 \\ &= \frac{16}{3m_G^2} (M_a I_2(M_a^2) + M_b I_2(M_b^2)). \end{aligned} \quad (25)$$

On the other hand, one can split the total derivative before integration, and it is easy to see that [with the help of the properties in Eq. (12)]

$$\begin{aligned} H_{AV} &= \frac{16(M_a + M_b)}{3m_G^2} \int_0^1 du \left\{ I_2(\omega_2) + \frac{(uM_b - \bar{u}M_a)}{M_a + M_b} \right. \\ &\quad \left. \times I_0(\omega_2) ((2u-1)Q^2 - M_b^2 + M_a^2) \right\}, \end{aligned} \quad (26)$$

which is precisely the right-hand side Eq. (21). Then Eq. (21) is nothing but the sum of two quark gap equations.

The right-hand side of Eq. (22) is exactly zero due to the consistent conditions contained in the new regularization R . While in Ref. [6], the vanishing of the right-hand side of Eq. (22) is imposed by hand. The reason is that the previous regularization R' fails to fulfill the consistency conditions, and thus gauge symmetry is explicitly broken.

Let us now focus on the vector vertex which satisfies the vector WTI,

$$Q_\mu i\Gamma_\mu^{ab}(Q) = S_a^{-1}(q) - S_b^{-1}(q_-) - (m_a - m_b) \Gamma_V^{ab}(Q), \quad (27)$$

and follows the inhomogeneous BSE,

$$\Gamma_\mu^{ab}(Q) = \gamma_\mu - \frac{4}{3m_G^2} \int_q \gamma_\alpha \chi_\mu^{ab}(Q) \gamma_\alpha, \quad (28)$$

with $\chi_\mu^{ab}(Q) = S_a(q) \Gamma_\mu^{ab}(Q) S_b(q_-)$. The scalar vertex $\Gamma_I^{ab}(Q)$ follows a similar equation with the inhomogeneous term being the identity matrix I_D . Under the contact interaction, the general form of the vector and scalar vertexes can be written as

$$\Gamma_\mu^{ab}(Q) = \gamma_\mu^L V_L^{ab}(Q^2) + \gamma_\mu^T V_T^{ab}(Q^2) + I_D (-iQ_\mu) V_S^{ab}(Q^2), \quad (29)$$

$$\Gamma_I^{ab}(Q) = I_D S_1^{ab}(Q^2) - \frac{i\mathcal{Q}}{M_{ab}} S_2^{ab}(Q^2), \quad (30)$$

where $\gamma_\mu^L = \frac{Q_\mu \not{Q}}{Q^2}$, $\gamma_\mu^T = \gamma_\mu - \gamma_\mu^L$. In Ref. [21], an additional interaction kernel is proposed that maintains the WTIs. This additional kernel can automatically generate a quark anomalous moment term $\sigma_{\mu\nu} Q_\nu$ in the vector vertex, which has a significant impact on the vector meson. However, since the additional kernel does not influence the discussion in this paper, we will not adopt it for simplicity.

The most straightforward way of solving the BSE equations is to compute the momentum integrals in the BSE directly. However, this method is computationally cumbersome when evaluating the γ matrix. Thus we solve the vector and scalar vertex BSEs by projecting the BSE with the various projecting operators. However, it is stressed that the Lorentz index of vector or tensor objects is important and should be retained before regularization. As an example, we illustrate this requirement through the solution of the vector vertex,

$$V_T^{ab}(Q^2) = \frac{1}{1 - f_T^{ab}(Q^2)}, \quad (31)$$

with

$$f_T^{ab}(Q^2) = -\frac{4}{3m_G^2} \text{tr}_D \delta_{\mu\nu} \int_q P_\nu^T \gamma_\alpha S^a(q) \gamma_\mu^T S^b(q_-) \gamma_\alpha, \quad (32)$$

by using the projection operator $P_\nu^T = \frac{\gamma_\nu^T}{\text{tr}_D(\gamma_\mu^T \gamma_\mu^T)}$. Here we express

$$P_\mu^T \otimes \gamma_\mu^T \rightarrow \delta_{\mu\nu} P_\nu^T \otimes \gamma_\mu^T, \quad (33)$$

which is used to retain the Lorentz index of the vector vertex $\Gamma_\mu^{ab}(Q)$ herein. Analyzing the integral with Feynman parametrization, rearranging it into ILIs, and then regularizing the ILIs with $\delta_{\mu\nu}$ uncontracted yield the following expression:

$$f_T^{ab}(Q^2) = \frac{4}{3m_G^2} \delta_{\mu\nu} \int_0^1 du \frac{2}{3} \left(\frac{Q_\mu Q_\nu}{Q^2} - \delta_{\mu\nu} \right) I_0(\omega_2) \times (uM_b^2 - M_b M_a + \bar{u}M_a^2 + 2u\bar{u}Q^2). \quad (34)$$

It can be seen that the transverse structure is exactly maintained with the index of vector vertex uncontracted before regularization. If the contraction is performed before regularization, the transverse structure may be destroyed.

By solving the corresponding BSE, the rest of the dressing functions in these two vertices are

$$V_L^{ab}(Q^2) = \frac{1 + 4f_1^{ab}(Q^2)}{1 - f_L^{ab}(Q^2)}, \quad (35)$$

$$V_S^{ab}(Q^2) = \frac{-4f_2^{ab}(Q^2)}{1 - f_L^{ab}(Q^2)}, \quad (36)$$

$$S_1^{ab}(Q^2) = \frac{1 + 2(M_b - M_a)f_2^{ab}(Q^2)}{1 - f_L^{ab}(Q^2)}, \quad (37)$$

$$S_2^{ab}(Q^2) = \frac{2M_{ab}f_2^{ab}(Q^2)}{1 - f_L^{ab}(Q^2)}, \quad (38)$$

where

$$f_L^{ab}(Q^2) = 8Q^2(f_2^{ab}(Q^2))^2 - 2(M_b - M_a)f_2^{ab}(Q^2) - 4f_1^{ab}(Q^2)(1 + 2(M_b - M_a)f_2^{ab}(Q^2)), \quad (39)$$

$$f_1^{ab}(Q^2) = \frac{4}{3m_g^2} \int_0^1 [I_0(\omega_2)(uM_b^2 + \bar{u}M_a^2 + M_b M_a + 2u\bar{u}Q^2) - I_2(\omega_2)], \quad (40)$$

$$f_2^{ab}(Q^2) = \frac{4}{3m_g^2} \int_0^1 I_0(\omega_2)(uM_b - \bar{u}M_a). \quad (41)$$

Plugging the solved vector and scalar vertices into the vector WTI Eq. (27), one finds the following equation:

$$\hat{M}_b - \hat{M}_a = -4(M_b - M_a)f_1^{ab}(Q^2) + 4Q^2 f_2^{ab}(Q^2). \quad (42)$$

This equation can be proved in analogy with Eq. (23) by introducing

$$H_V = \frac{16}{3m_G^2} \int_0^1 g_V(u, M_a, M_b, Q^2) du, \quad (43)$$

where

$$g_V(u, M_a, M_b, P^2) = \frac{d}{du} [I_2(\omega_2)(uM_b + \bar{u}M_a)]. \quad (44)$$

One finds Eq. (42) is exactly the difference between the two quark gap equations.

At the end of this section, we list the systematic procedures in regularizing the one loop integrals in the contact model.

- (1) First, one uses the Feynman parametrization to analyze the integral and then rearranges the expressions in terms of the ILIs.
- (2) One regularizes the ILIs through the new regularization and expresses all regularized tensor type ILIs in terms of scalar type ones through the consistent conditions.
- (3) When evaluating formulas that contain vector or tensor objects, their Lorentz index should be retained rather than contracted before the tensor type ILIs are regularized.

The first step is trivial, but provides a convenient way to manage the expressions in calculation. We can then easily implement regularization in step 2. We emphasize again a proper regularization should let the regularized tensor

type ILIs satisfy the consistent conditions; otherwise, gauge symmetries will be broken. The third rule concerns vector and tensor objects, whose Lorentz metrics are important. To be clear, this rule charges not only for solving the BSEs but also for any physical quantity that includes vector and tensor objects. An easy way of implementing this rule can be achieved by, say, Eq. (33), which will generate the following typical integrals:

$$\delta_{\mu\nu} \int_q \frac{q_\mu q_\nu}{(q^2 + \mathcal{M}^2)^{\alpha+2}}, \quad (45)$$

where $\delta_{\mu\nu}$ comes from Eq. (33). It is apparent that different orders of regularization and contraction lead to different results; take $\alpha = -1$ for example,

$$\delta_{\mu\nu} \int_q \frac{q_\mu q_\nu}{(q^2 + \mathcal{M}^2)^2} \rightarrow \delta_{\mu\nu} I_{2R}^{\mu\nu} = 2I_{2R}, \quad (46)$$

$$\delta_{\mu\nu} \int_q \frac{q_\mu q_\nu}{(q^2 + \mathcal{M}^2)^2} = I_2 - \mathcal{M}^2 I_0 \rightarrow I_{2R} - \mathcal{M}^2 I_{0R}. \quad (47)$$

Here arrows represent operations of regularization and equal signs represent operations of contraction. And we explicitly use the label R to distinguish the regularized ILIs from the original ones. It is observed that if one contracts the index before regularization, the unregularized tensor ILIs are turned into the unregularized scalar ILIs, which incorrectly characterize the divergences of theories, and then some fundamental properties are lost, such as the current conservation.

IV. KAON FORM FACTORS

As the simplest bound state with strangeness, kaon has attracted much attention since its discovery in the middle of the last century [25] and has led to much important research in the standard model, such as the introduction of strangeness [26], the violation of parity [27], quark mixing and the CKM matrix [28,29], CP violation [30], and the Glashow-Iliopoulos-Maiani (GIM) mechanism [31]. Recently, CERN has approved a world-unique QCD facility where highly intense and energetic kaon beams are used to map out the complete spectrum of excited kaons with an unprecedented precision [32]. This approval of CERN injects vitality nowadays into a wide range studies on low energy kaon phenomenology and high energy processes including kaon excitations, and this opens a completely new horizon in kaon physics.

The kaon electromagnetic and transition form factor have also attracted fruitful studies [33–39]. In the previous contact model computation of these two form factors [10], inconsistency results occurred due to the failure of the regularization to properly handle the divergent integrals.

To calculate the kaon electromagnetic form factor and the K_{13} transition form factor, three elements are required.

The quark propagator and vector vertex are given in Sec. III (in the following, Γ_μ^{ff} denotes the quark-photon vertex and Γ_μ^{su} denotes the quark- W boson vertex). The remaining ingredients are the pion and kaon BSAs, which are solved from the pseudoscalar BSEs. In solving this equation, we follow Ref. [10], the only difference being the regularization. Even though we get the same kernel as in Ref. [10], it is worth nothing that the following relation holds:

$$K_{FF} = -\frac{(M_s + M_u)^2}{2M_s M_u} K_{FE}. \quad (48)$$

This is a natural result since the axial-vector WTI is automatically satisfied under the new regularization. Since the meson BSE is a homogeneous equation, normalization of the BSA is needed for the computation of physical quantities. The canonical normalization of pseudoscalar meson reads

$$2Q_\mu = N_c \text{tr}_D \int_q \Gamma_{PS}^{ab}(-Q) S(q) \Gamma_{PS}^{ab}(Q) \frac{\partial}{\partial Q_\mu} S(q_-), \quad (49)$$

and the leptonic decay constant reads

$$f_{PS}^{ab} Q_\mu = \frac{1}{\sqrt{2}} N_c \text{tr}_D \int_q \gamma_5 \gamma_\mu \chi_{PS}^{ab}(q, q_-). \quad (50)$$

The parameters are fixed to obtain the physical observables in Table I $m_u = 0.007$ GeV, $m_s = 0.17$ GeV, $m_G = 0.132$ GeV, $\tau_{uv} = 1/0.905$ GeV $^{-1}$, and $\tau_{ir} = 1/0.24$ GeV $^{-1}$.

A. Kaon electromagnetic form factor

In the impulse approximation, the kaon electromagnetic form factor is

$$F_{K^+}^{em}(Q^2) = e_u F_{K^+}^u(Q^2) + e_s F_{K^+}^s(Q^2), \quad (51)$$

where

$$2P_\mu F_{K^+}^u(Q^2) = N_c \text{tr}_D \int_q S^s(q) i\Gamma^{K^+}(-p) S^u(q+p) \times i\Gamma_\mu^{uu}(Q) S^u(q+k) i\Gamma^{K^+}(k), \quad (52)$$

$$2P_\mu F_{K^+}^s(Q^2) = N_c \text{tr}_D \int_q S^s(q-p) i\Gamma^{K^+}(-p) S^u(q) \times i\Gamma^{K^+}(k) S^s(q-k) i\Gamma_\mu^{ss}(Q), \quad (53)$$

where $p = P + Q/2$, $k = P - Q/2$ with k the incoming kaon momentum and p the out going kaon momentum. The

TABLE I. Masses and decay constants, with the units GeV.

M_u	M_s	m_π	m_K	f_π	f_K
0.368	0.533	0.140	0.499	0.101	0.106

on-shell condition entails $P \cdot Q = 0$ and $P^2 = -m_K^2 - Q^2/4$. We express the form factor formula as follows:

$$F_{K^+}^u(Q^2) = (F_{EE}^u(Q^2)E_{K^+}E_{K^+} + F_{EF}^u(Q^2)E_{K^+}F_{K^+} + F_{FF}^u(Q^2)F_{K^+}F_{K^+})V_T^{uu}(Q^2), \quad (54)$$

where E_{K^+}, F_{K^+} is kaon's amplitude and the corresponding functions can be found by the new regularization approach as

$$\begin{aligned} F_{EE}^u(Q^2) &= 2N_c \left(\int_x I_0(\omega_{K2}) - \int_{xy} A_{EE} I_{-2}(\omega_{K3}) \right), \\ F_{EF}^u(Q^2) &= -2 \frac{M_u + M_s}{M_{us}} F_{EE}^u(Q^2) + N_c \int_x A_{EF} I_0(\omega_{K2}), \\ F_{FF}^u(Q^2) &= -\frac{M_u + M_s}{2M_{us}} F_{EF}^u(Q^2) + N_c \int_x A_{FF} I_0(\omega_{K2}) \\ &\quad + N_c \int_x B_{FF} I_0(M_u^2 + x(1-x)Q^2), \end{aligned} \quad (55)$$

where

$$\begin{aligned} A_{EE} &= 2((M_s - M_u)^2 - m_K^2)(1-x-y) + Q^2(x+y), \\ A_{EF} &= \frac{4(M_s(1-x) + M_u x)}{M_{us}}, \\ A_{FF} &= \frac{-2(M_s^2(1-x) + M_u^2 x + M_u M_s)}{M_{us}^2}, \\ B_{FF} &= \frac{4Q^2 x(1-x)}{M_{us}^2}, \end{aligned} \quad (56)$$

and $\int_x = \int_0^1 dx$, $\int_{xy} = \int_0^1 dx \int_0^{1-x} dy$, $\omega_{K2} = xM_u^2 + (1-x)M_s^2 - x(1-x)M_K^2$, and $\omega_{K3} = (x+y)M_u^2 + (1-x-y)M_s^2 - m_K^2(x+y)(1-x-y) + Q^2xy$.

Consider the interchange $s \leftrightarrow u$ and $p \leftrightarrow -k$ in Eq. (53), and comparing it with Eq. (52), one finds that $F_{K^+}^s(Q^2)$ is obtained from $-F_{K^+}^u(Q^2)$ by the interchange of $s \leftrightarrow u$. From the explicit expressions of $F_{EF}^u(Q^2)$, and some symmetries of the Feynman parameters, one gets the following relation:

$$F_{EF}^u(Q^2 = 0) + F_{EF}^s(Q^2 = 0) = 0. \quad (57)$$

Once again we see that the symmetry preserving regularization procedure leads to the desired result, Eq. (57), which should always hold, whereas in previous contact interaction studies this relation is broken because of an improper regularization procedure. The result for the kaon electromagnetic form factor is presented in Fig. 1, which, as expected, is hard due to the nature of contact interaction.

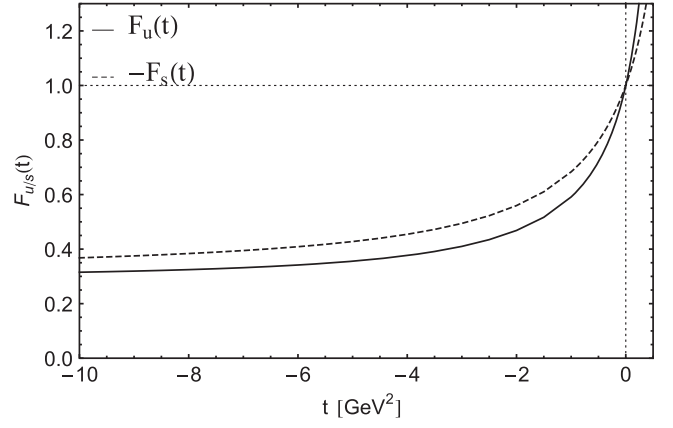


FIG. 1. The momentum dependence of kaon electromagnetic form factors $F_{u/s}(t = -Q^2)$.

B. K_{l3} transition form factors

The impulse approximation result for the K_{l3} transition amplitude is

$$\begin{aligned} M_\mu^{K_{l3}} &= \frac{1}{\sqrt{2}} N_c \text{tr}_D \int_q \{ S^u(q-p) \Gamma^\pi(-p) S^u(q) \\ &\quad \times \Gamma^{K^+}(k) S^s(q-k) i \Gamma_\mu^{su}(Q) \}, \end{aligned} \quad (58)$$

where k is the incoming kaon momentum and p the outgoing pion momentum. The on-shell conditions entails $P \cdot Q = (m_K^2 - m_\pi^2)/2$ and $P^2 = -(m_K^2 + m_\pi^2)/2 - Q^2/4$. The amplitude can be expressed as

$$M_\mu^{K_{l3}} = \frac{1}{\sqrt{2}} (2P_\mu f_+(Q^2) - Q_\mu f_-(Q^2)). \quad (59)$$

In addition to the primary and secondary transition form factors $f_+(Q^2)$ and $f_-(Q^2)$, it is helpful to characterize the transitions by the following function:

$$f_0(Q^2) = f_+(Q^2) - Q^2 f_-(Q^2) / (m_K^2 - m_\pi^2). \quad (60)$$

Since the expressions for $f_+(Q^2)$ and $f_-(Q^2)$ are complicated, we fit these two functions by interpolating in the domain $-2 < t/\text{GeV}^2 < 0.5$,

$$\begin{aligned} f_+(t) &= f_+(0) \frac{1 - 0.7397t + 0.1159t^2}{1 - 1.4312t + 0.4574t^2}, \\ f_-(t) &= f_-(0) \frac{1 - 1.2922t + 0.0933t^2}{1 - 2.0758t + 1.0671t^2}, \end{aligned} \quad (61)$$

with $t = -Q^2$, and in the following we use t as a variable of $f_{+,-,0}$. And in the physical interesting domain $m_l^2 < t/\text{GeV}^2 < t_m$, where $t_m = (M_K - m_\pi)^2 \sim 0.13 \text{ GeV}^2$, a linear fit is performed,

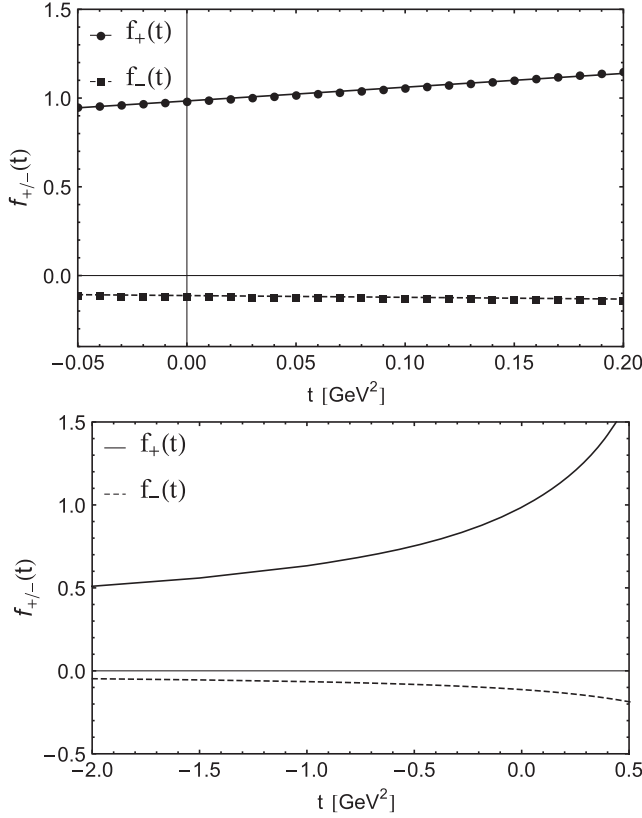


FIG. 2. Results for $f_{+/-}(t)$. In the upper panel, the filled circles and squares are results of the calculation, and the solid and dashed lines are linear fits in Eq. (62). In the lower panel, the solid and dashed lines are the results of the calculation.

$$\begin{aligned} f_+(t) &= 0.9838 + 0.7751t, \\ -f_-(t) &= 0.1128 + 0.0992t. \end{aligned} \quad (62)$$

The slopes of transition form factors are less deep than the results reported in Ref. [36]. The results for the form factors f_+ and f_- are plotted in Fig. 2.

Here we list in Table II a range of quantities that are used to characterize the kaon semileptonic decays. The difference between the results of previous work [10] and the present results comes from the regularization procedure.

Current algebra predicts [43] that

$$f_0(t_\Delta = m_K^2 - m_\pi^2) = f_K/f_\pi + \Delta_{CT}, \quad (63)$$

TABLE II. Quantities derived from the kaon transition form factors.

	$f_+(0)$	$f_+(t_m)$	$-f_-(0)$	$-f_-(t_m)$	$f_0(t_\Delta)$
Here	0.98	1.09	0.11	0.12	1.05
Ref. [10]	0.98	1.07	0.087	0.096	1.06
Ref. [36]	0.96	1.13	0.10	0.11	1.18
Refs. [40–42]	0.96	1.16	0.12		

where $f_{\pi,K}$ is the leptonic decay constant of pion and kaon, and the correction Δ_{CT} is of $\mathcal{O}(m_u, m_d)$, which is usually negligible. In our calculations, we obtain $\Delta_{CT}/f_0(t_\Delta) = 3.7 \times 10^{-4}$. Actually, $\Delta_{CT} = 0$ if one works in the chiral limit, i.e., $m_u = 0 = m_\pi$. It is known that the solution of Eq. (19) has the form in the neighborhood of $P^2 = 0$ (with $a, b = u$),

$$\Gamma_{5\mu}^{uu}(P) = \frac{P_\mu}{P^2} \sqrt{2} f_\pi \Gamma^\pi(P) + \gamma_5 \gamma_\mu F_R(P), \quad (64)$$

where F_R is the regular part of the vertex at $P^2 = 0$. Plugging Eq. (64) into the axial-vector WTI Eq. (18), one obtains the following generalized Goldberger-Treiman relations [44]:

$$\sqrt{2} E_\pi(P) f_\pi = 2M_u, \quad (65)$$

$$\frac{2F_\pi(P)}{E_\pi(P)} + F_R(P) = 1. \quad (66)$$

Then pion BSA can be expressed as follows:

$$\Gamma^\pi(P) = \frac{S^{-1}(q+P) i\gamma_5 + i\gamma_5 S^{-1}(q) - \gamma_5 \not{P} F_R(P)}{\sqrt{2} f_\pi}. \quad (67)$$

Substituting Eq. (67) into the K_{13} transition amplitude Eq. (58), one arrives at the following expressions for the transition form factors:

$$\begin{aligned} f_+^{cl}(t) &= \frac{f_K}{2f_\pi} - \frac{E_\pi - 2F_\pi}{2} (f_{cl+}^{FE}(t) E_K + f_{cl+}^{FF}(t) F_K), \\ f_-^{cl}(t) &= \frac{f_K}{2f_\pi} - \frac{E_\pi - 2F_\pi}{2} (f_{cl-}^{FE}(t) E_K + f_{cl-}^{FF}(t) F_K), \end{aligned} \quad (68)$$

where $f_{+/-}^{FX}(t)$ with $X = E, F$ comes from the following way of expressing the transition form factor in analogy with Eq. (54):

$$\begin{aligned} f_{+/-}(t) &= (f_{+/-}^{EE}(t) E_\pi E_K + f_{+/-}^{EF}(t) E_\pi F_K \\ &\quad + f_{+/-}^{FE}(t) F_\pi E_K + f_{+/-}^{FF}(t) F_\pi F_K), \end{aligned} \quad (69)$$

and $f_{cl+/-}^{FX}(t) = f_{+/-}^{FX}(t, m_\pi = 0)$. Furthermore, one can extract the vector vertex ingredients,

$$f_+^{FX}(t) = f_+^{FX,T}(t) V_T^{su}(t), \quad (70)$$

$$\begin{aligned} f_-^{FX}(t) &= f_-^{FX,T}(t) V_T^{su}(t) + f_-^{FX,L}(t) V_L^{su}(t) \\ &\quad + f_-^{FX,S}(t) V_S^{su}(t). \end{aligned} \quad (71)$$

Our computation indicates the following relations in the chiral limit:

$$f_{cl+}^{FX,T}(t) + \frac{t}{m_K^2} f_{c-}^{FX,T}(t) = 0, \quad (72)$$

$$f_{cl-}^{FX,L}(t) \sim \mathcal{O}(t - m_K^2), \quad (73)$$

$$f_{cl-}^{FX,S}(t) \sim \mathcal{O}(t - m_K^2), \quad (74)$$

which leads to

$$f_0^{cl}(t_\Delta) = f_+^{cl}(t_\Delta) + f_-^{cl}(t_\Delta) = \frac{f_K}{f_\pi}. \quad (75)$$

Hence Δ_{CT} vanishes in the chiral limit.

V. SUMMARY

In this paper, we introduce a symmetry preserving regularization based on Schwinger's proper-time method. This regularization meets many good features of dimensional regularization without changing spacetime dimensions. One of the most important properties is that it ensures the regularized ILIs satisfy a series of consistency conditions which are necessary for preserving gauge symmetries. In particular, we show that gauge symmetries are

preserved by proving that the WTIs hold exactly after the regularization. We also show why the regularization procedure in previous studies of the contact model breaks WTIs. Systematic steps for regularizing the contact model are presented at the end of Sec. III.

As an application example, we recalculated kaon form factors which exhibit inconsistency in Ref. [10] because of the improperly regularized ILIs, whereas under the new regularization the results are self-consistent. The present regularization would also rescue the flaw in the calculation of other form factors, such as heavy-light mesons semi-leptonic transitions (see Ref. [45] for some discussion).

With this symmetry preserving regularization, the hadron properties computed from the contact model are valid, as potential inconsistencies from regularization are eliminated. It is hoped that the contact model studies can provide more inspiring results that may shed light on the realistic DSE studies.

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