Letter

Anyonic spin-Hall effect on the black hole horizon

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Using the fact that the horizon of black holes is a Carroll manifold, we show that an "exotic photon," i.e., a particle without mass and charge but with anyonic spin, magnetic moment, and exotic charges associated with the two-parameter central extension of the two-dimensional Carroll group moves on the horizon of a Kerr-Newman black hole consistently with the Hall law.

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I. INTRODUCTION

The horizon of black holes is a genuine laboratory to explore gravitational physics. Several interesting and non-intuitive effects, linked to key questions such as the information paradox [1,2], are expected to take place on it. In this paper we add one more item to the list by showing that exotic photons (to be introduced below) exhibit the spin-Hall effect on the horizon.

The clue is "Carroll symmetry." The Carroll group, a "degenerate cousin" of the Galilei group (as put by Lévy-Leblond [3]), is obtained by contracting, in the Poincaré group, the velocity of light to zero, instead of letting it go to infinity, as in the usual Galilean limit [3,4]. Alternatively, Carroll symmetry is found by restricting a Lorentzian spacetime to a null hypersurface [5–9].

Recent attention in the subject arose when Carroll symmetry was found to be relevant, for instance, for physics on a black hole horizon [10–13]: the celebrated Bondi-Metzner-Sachs (BMS) group is indeed conformal Carroll [14–16]. Interest in Carroll dynamics has long been limited, though, by the fact that Carroll particles (other than tachyons [16,17]) were believed not to move [3–5,9].

As will be explained elsewhere [18], "no motion" is understood by studying deviations from null geodesics. The "time" coordinate of Carroll geometries is in fact a null coordinate from the ambient spacetime and "not moving" means following the corresponding ambient null geodesic. Another approach [18,19] relates the immobility of quasi-particles called fractons [20] to Carrollian boost invariance.

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The anomalous Hall effect observed in ferromagnetic crystals had been attributed to an anomalous current [21]. Later it was argued that spinning particles (including light) exhibit a spin Hall effect [22,23] for which a semiclassical explanation was proposed using a Berry phase-extended framework [24–28]. The clue is the anomalous velocity relation,

$$\frac{d\boldsymbol{x}}{dt} = \frac{\partial \mathcal{E}(\boldsymbol{p})}{\partial \boldsymbol{p}} - e\mathbf{E} \times \boldsymbol{\Theta}, \tag{1.1}$$

where $\mathcal{E}(\boldsymbol{p})$ is the band energy, $\mathbf{E}(\boldsymbol{x})$ is the electric field, and the three-vector $\boldsymbol{\Theta}(\boldsymbol{p})$ represents the Berry curvature. The anomalous velocity term here is clearly the mechanical counterpart of the anomalous current. Choosing $\mathbf{E}$ in the x-y plane and $\boldsymbol{\Theta}$ perpendicular to it, Eq. (1.1) reduces to the exotic Galilean model based on a two-parameter central extension of the planar Galilei group [29,30]. The physical relevance of central extensions was recognized by Bargmann [31], followed by [32]. See [33] for another recent application. Extensions hint at deviations from null geodesics. For instance, the deviations of light from geodesic motions can be attributed to the coupling of photon spin to the gravitational field including gravitational waves [8,34–41].

It has recently been recognized that in $2+1$ dimensions the Carroll group admits a two-parameter central extension [42,43] inducing an extended dynamics [44].

The aim of this paper is to study the Carrollian analog of the Galilean case, illustrated by motion on a specific Carroll geometry: the Kerr-Newman black hole horizon.

II. DOUBLY EXTENDED CARROLL PARTICLE

The two-parameter nontrivial central extension of the planar Carroll algebra [42,43] is given by

$$
\begin{aligned}
&[J_3,K_i]=\epsilon_{ij}K_j, \quad [K_i,K_j]=\epsilon_{ij}A_{\rm exo},\\
&[J_3,P_i]=\epsilon_{ij}P_j, \quad [K_i,P_j]=\delta_{ij}P_0,\\
&[J_3,P_0]=0, \quad [K_i,P_0]=0,[P_i,P_j]=\epsilon_{ij}A_{\rm mag}, \quad [P_i,P_0]=0,
\end{aligned}
$$

where J_3 is the rotation, the (K_i) are boosts, the (P_i) are spatial translations, (P_0) is time translation, and $A_{\rm exo}$ and $A_{\rm mag}$ are the exotic and "magnetic" extensions, respectively. These extensions allow us to endow planar Carroll particles canonically with two additional central charges we shall call, correspondingly, exotic and magnetic and denote by $\kappa_{\rm exo}$ and $\kappa_{\rm mag}$, respectively [18,44].

Then we show that massless uncharged particles (we call "exotic Carrollian photon") move on the horizon of a Kerr-Newman black hole by following the Hall law, providing us with a Carrollian version of the spin-Hall effect for anyons and contradicting the no-motion statement [44].

Classical particle models associated with the transitive action of a given symmetry group are conveniently constructed using the Kirillov-Kostant-Souriau (KKS) orbit method [32,45]. In Souriau's version that we follow here, the motions are determined by the "Souriau 2-form" σ, which is closed and has constant rank. Then $\sigma = d\varpi$ (locally). The "Cartan form" ϖ could be used for a variational calculus [32,46]. Splitting the Souriau form as $\sigma = \Omega - d\mathscr{H} \wedge ds$ provides us, moreover, with a symplectic form Ω whose inverse defines, in turn, commutation relations [46]. Applied to the doubly centrally extended Carroll group, the construction yields for a free particle with mass m the Poisson brackets and Hamiltonian,[1]

$$
\begin{aligned}
&\{x_i,x_j\} = \frac{\kappa_{\rm exo}}{mm^*}\epsilon_{ij}, \qquad \{x_i,p_j\} = \frac{m}{m^*}\delta_{ij},\\
&\{p_i,p_j\} = \frac{m}{m^*}\kappa_{\rm mag}\epsilon_{ij}, \qquad \mathscr{H}_0 \equiv 0,
\end{aligned} \tag{2.1}
$$

where

$$
m^* = m\left(1-\frac{\kappa_{\rm exo}}{m^2}\kappa_{\rm mag}\right) \tag{2.2}
$$

is an effective mass, assumed not to vanish [18,44].

Let us underline that (i) the coordinates do not commute and (ii) the second extension parameter $\kappa_{\rm mag}$ behaves as an internal magnetic field carried by the particle.[2]

Note that the free Hamiltonian has no kinetic term. The Hamiltonian equations of motion are thus trivially that a (doubly extended) Carroll particle with nonzero effective mass $m^* \neq 0$ does not move.

The free system (2.1) is by construction invariant with respect to the action of the doubly extended Carroll group

$$
\boldsymbol{x} \to A\boldsymbol{x}+\mathbf{c} \quad s \to s-\mathbf{b}\cdot A\boldsymbol{x}+f, \quad \boldsymbol{p}\to A\boldsymbol{p}+m\mathbf{b}, \tag{2.3}
$$

where $A \in \mathrm{SO}(2)$, $\mathbf{c},\mathbf{b}\in\mathbb{R}^2$, $f\in\mathbb{R}$, cf. no. (3.15) in [44]. Here s is Carrollian time. All this follows from the structure of the Carroll group upon applying the KKS algorithm.

Coupling such a particle to an electromagnetic field modifies both the symplectic structure and the Hamiltonian [32,46,49]. For simplicity, we restrict our attention at an uncharged doubly extended Carroll particle[3] with magnetic moment μ and anyonic spin χ in a static electromagnetic field $(B,\mathbf{E})$. The Poisson brackets are as in (2.1), but the modified Hamiltonian $\mathscr{H} = -\mu\chi B$ yields, for $m\neq 0$,

$$
\begin{aligned}
(x^i)' &= \mu\chi\frac{\kappa_{\rm exo}}{\kappa_{\rm exo}\kappa_{\rm mag}-m^2}\epsilon^{ij}\partial_j B,\\
p_i' &= m^2\frac{\mu\chi\partial_i B}{m^2-\kappa_{\rm exo}\kappa_{\rm mag}},
\end{aligned} \tag{2.4}
$$

where the prime means d/ds.

These equations are both of the first order. The one for $\boldsymbol{x}$ is fully decoupled and can be solved on its own, but the one for $\boldsymbol{p}$ depends on the result for $\boldsymbol{x}$.

Moreover, letting here $m\to 0$, $\kappa_{\rm exo}$ drops out as long as it does not vanish, leaving us with

$$
\boxed{(x^i)' = (\mu\chi)\epsilon^{ij}\frac{\partial_j B}{\kappa_{\rm mag}} \quad \text{and} \quad p_i' = 0.} \tag{2.5}
$$

Note that the usual electromagnetic terms were switched off by choosing $e=0$, but the magnetic field plays a new role—that of an electric potential.

Motion in a curved Carroll manifold was considered in [9,44], however, the gravitational field does not couple to the Carrollian equations of motion. An intuitive justification is that gravitational minimal coupling impacts, through the covariant derivative, the equation for the momentum, however, not that for the position. However (as noticed above), the momentum has no impact on the $\boldsymbol{x}$ motion.

III. CARROLL STRUCTURE OF THE KERR-NEWMAN HORIZON

A Kerr-Newman black hole characterized by its mass M, angular momentum J, and charge Q can be described by using the Eddington-like coordinates (u,r,ϑ,ϕ) [50]. In these coordinates, the metric

[1]The commutation relations (2.1) have an overlap with those of the so-called Maxwell algebra [47]. The respective Hamiltonians are substantially different, though: the doubly extended Carroll and the Maxwell or "enlarged" [48] systems are fundamentally different, as they are built from different ingredients: constant external electromagnetic fields for Maxwell and intrinsic central extension parameters for our doubly extended Carroll, respectively.

[2]In the Galilean theory [30], the mass is part of the moment map.

[3]Charged particles are studied in [18].

$$g = -\frac{\Delta}{\Sigma}\left(du + \frac{\Sigma}{\Delta}dr - a\sin^2\vartheta d\phi\right)^2 + \frac{\sin^2\vartheta}{\Sigma}(adu - (r^2+a^2)d\phi)^2 + \Sigma d\vartheta^2 + \frac{\Sigma}{\Delta}dr^2, \quad (3.1a)$$

$$\Sigma = r^2 + a^2\cos\vartheta, \quad \Delta = r^2 + a^2 + Q^2 - 2Mr, \quad (3.1b)$$

where $a = J/M$, and its inverse is regular on the (outer) horizon $\mathcal{H}$ of a Kerr-Newman black hole defined by $r = r_+ = M + \sqrt{M^2 - (a^2 + Q^2)} = \text{const}$ hypersurface defined by $\Delta = 0$. Note that the seemingly problematic dr^2 terms in (3.1a) containing Δ in their denominator cancel one another out. Then we consider the $2+1$-dimensional structure [11,12] whose ingredients are the induced metric and a vector, In what follows r will denote its value above fixed on the horizon.

$$\tilde{g} = g|_{\Delta=0} = \frac{\sin^2\vartheta}{\Sigma}(adu - (r^2+a^2)d\phi)^2 + \Sigma d\vartheta^2, \quad (3.2a)$$

$$\xi = \partial_u + \Omega_H \partial_\phi \quad \text{where } \Omega_H = \frac{a}{r^2+a^2}, \quad (3.2b)$$

respectively. Here Ω_H is the angular velocity of the horizon. The restricted metric (3.2a) is singular as made manifest by the coordinate change $(\vartheta, \phi, u) \mapsto (\vartheta, \varphi = \phi - \Omega_H u, u)$, which leads to the metric

$$\tilde{g} = \frac{(r^2+a^2)\sin^2\vartheta}{\Sigma} d\varphi^2 + \Sigma d\vartheta^2 \quad \text{and} \quad \xi = \partial_u. \quad (3.3)$$

The kernel is generated by the vector ξ, $\tilde{g}(\xi) = 0$. Thus we have a degenerate metric and a vector field in its kernel, allowing us to conclude that the horizon $\mathcal{H}$ of a Kerr-Newman black hole carries a Carroll structure $(\mathbb{S}^2 \times \mathbb{R}, \tilde{g}, \xi)$ [5]. The degenerate "metric" $\tilde{g}$ carries the geometric information of the $\mathbb{S}^2$ part of the black hole, while ξ generates the $\mathbb{R}$ part.

The horizon of a Kerr-Newman black hole carries a magnetic field

$$B = (2aQr(r_+^2 + a^2))\frac{\cos\vartheta}{(r_+^2 + a^2\cos^2\vartheta)^3}, \quad (3.4)$$

while in comoving coordinates the electric field vanishes [18].

IV. MOTION ON THE KERR-NEWMAN HORIZON

A massive particle associated with the unextended Carroll group can stay fixed, but cannot move [5,16]. However, the horizon is a $2+1$-dimensional Carroll manifold; therefore, the particle may have an extended dynamics associated with the double central extension with parameters κ_{exo} and κ_{mag} [18,42–44].

Now we show that the extended dynamics *can* lead to motion, namely, on the black hole horizon. Remember first that geodesics on the horizon are necessarily massless [5].[4] An exotic photon, i.e., one with no mass and charge, $m = 0$ and $e = 0$, but with nonvanishing magnetic moment μ, anyonic spin χ, and double central extension can be coupled to the electromagnetic field through a spin-field term $\mathscr{H} = -\mu\chi B$, where B is the magnetic field (3.4) on the horizon. Then the equations of motion (2.5) describe an anomalous spin-Hall effect with $\boldsymbol{\nabla} B$ behaving as an effective electric field, $\mu\chi$ as an effective electric charge, and κ_{mag} as an effective magnetic field.

Coupling to the gravitational field amounts to replacing the derivative on p_i by a covariant derivative [44]. However, this does not change the velocity equation, which is indeed the only relevant one for the poor Carrollian dynamics: the momentum equation remains decoupled.

Having a nonzero gradient for the magnetic field (3.4) requires nonzero electric charge Q and angular momentum J (since $a = J/M$).

Using (comoving) angular coordinates (ϑ, φ, u) we see that the electric field induced on the horizon vanishes. The radial component would survive, but disappears in the $2+1$ restriction. The gradient of B is tangent to the longitudinal great circles $\varphi = \text{const}$. By (2.5), the motion is governed by

$$\boxed{\begin{aligned} &(x^\vartheta)' = 0, \\ &(x^\varphi)' = \left(2aQr(r_+^2 + a^2)\frac{\mu\chi}{\kappa_{\text{mag}}}\right)\frac{(r_+^2 - 5a^2\cos^2\vartheta)}{(r_+^2 + a^2\cos^2\vartheta)^4}\sin\vartheta, \end{aligned}} \quad (4.1)$$

which is shown in Fig. 1. Thus our exotic photon performs azimuthal circular motion with $\vartheta = \text{const}$, parameterized by φ. Consistent with the Hall behavior, the motion is perpendicular to the (longitudinal) effective electric field $\boldsymbol{\nabla} B$ (which vanishes at the poles and takes its maximum on the equator). The direction of the rotation is correlated with the angular momentum J and the charge Q which should not vanish—and this is precisely the reason that we consider Kerr-Newman black holes. The angular velocity goes smoothly to zero as we approach the poles and depends on the radius of the horizon roughly as r, implying that the rotation would be more important for smaller black holes.

The rotation we have just found, although reminiscent of the frame dragging by a rotating black hole, is, however, unrelated to it: frame dragging is hidden in the coordinates, which are comoving with the horizon.

[4] Such a photon trajectory could be created by turning on a lamp and then throwing it into a stationary black hole in such a way that when the lamp crosses the horizon, the photons are emitted in the direction of the horizon's null generator.

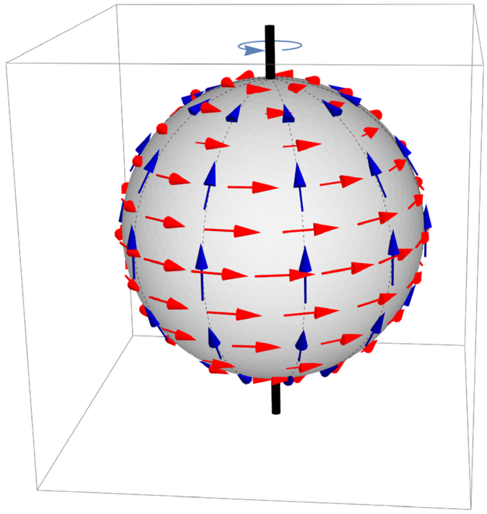

FIG. 1. On the horizon of a Kerr-Newman black hole the velocity field (4.1) is perpendicular to the axis of rotation and obeys the anomalous Hall law with an effective electric field $\mathbf{E}^* = \boldsymbol{\nabla} B$, which is tangent to the longitudinal great circles with $\mu\chi$ playing the role of an effective electric charge. The arrows indicate the directions and norms.

V. CARROLL SYMMETRY ON THE HORIZON

We conclude our paper with a short survey of the symmetries, conveniently studied by looking at the Cartan 1-form [32] ϖ defined by $\sigma = d\varpi$ as mentioned in Sec. II. Switching to $\boldsymbol{v} = \boldsymbol{p}/m$ before letting $m \to 0$ yields

$$\varpi = \frac{\kappa_{\text{exo}}}{2}\epsilon_{ij} v^i dv^j + \frac{\kappa_{\text{mag}}}{2}\epsilon_{ij} x^i dx^j + \mu\chi B ds. \tag{5.1}$$

In presymplectic terms, the Noether theorem says: a vector field X is a symmetry of the dynamics, if $L_X\varpi$ vanishes up to a total derivative, $L_X\varpi = df$. Then Cartan's formula implies that

$$Q_X = i_X\varpi - f \tag{5.2}$$

is conserved [32].

The isometry group of the Kerr-Newman horizon $\mathcal{H}$ is $\mathrm{SO}(2) \ltimes \mathcal{T}$, generated by the vector fields

$$X = \partial_{\tilde\varphi} + \mathscr{T}(\vartheta, \tilde\varphi)\partial_s, \tag{5.3}$$

where the "supertranslation" $\mathscr{T}$ is an arbitrary function of the coordinates $(\vartheta, \tilde\varphi)$ on the horizon. Thus:

(i) Translations of the black hole horizon generated by ∂_φ change ϖ by a surface term, $L_{\partial_\varphi}\varpi = df$ with $f = -(\kappa_{\text{mag}}/2)\vartheta$. Thus (5.2) yields the conserved quantity

$$p_\varphi = \kappa_{\text{mag}}\vartheta. \tag{5.4}$$

This unusual expression is consistent with (3.18c) in [44].

(ii) Now look at the zeroth order expansion of a supertranslation, i.e., a (Carrollian) time translation $X = \partial_s$. We readily have $L_X\varpi = 0$, and so

$$i_X\varpi = \mu\chi B \equiv \mathscr{H}, \tag{5.5}$$

identified with the Carroll Hamiltonian is conserved.

(iii) For a general supertranslation $X = \mathscr{T}(\vartheta, \varphi)\partial_s$ we have, instead,

$$L_X\varpi = (\mu\chi)B\partial_i\mathscr{T}dx^i, \tag{5.6}$$

which is *not* a total derivative, in general due to $d(L_X\varpi) \propto dB \wedge d\mathscr{T} \neq 0$, unless $\mathscr{T} = \mathscr{T}(\vartheta)$—for which (5.2) then yields a conserved quantity. If the supertranslation is, for example, induced by the magnetic field, $\mathscr{T} = \mathscr{T}(B)$, e.g., for $\mathscr{T}_n \propto B^n$ for some positive integer n, then $L_X\varpi = nd\mathscr{H}^{n+1}$ is a total derivative, providing us with an infinite tower of conserved quantities $Q_n = \mathscr{H}^{n+1}$—which are, however, mere powers of the Hamiltonian in (5.5).

(iv) It is instructive to study Carroll boosts in (2.3),

$$\boldsymbol{x} \to \boldsymbol{x} \qquad s \to s - \mathbf{b}\cdot\boldsymbol{x}, \qquad \mathbf{b} \in \mathbb{R}^2, \tag{5.7}$$

characteristic for the Carroll symmetry. They belong to the isometry "bottom" of BMS supertranslations [15]. "Horizontal" boosts along ∂_φ are broken by the magnetic field, however, for "vertical" boosts, $\mathscr{T} = -b_\vartheta\vartheta$, (5.2) provides us with[5]

$$Q = (\mu\chi)\left[-B\vartheta + \int Bd\vartheta\right]. \tag{5.8}$$

So far we proceeded as follows: first we solved the equations of motion and then checked that the associated Noetherian quantities are indeed conserved along the trajectories. The conservation of p_ϑ in (5.4), of $\mathscr{H}$ in (5.5), or of even the weird boost momentum in (5.8) is indeed manifest from the fact that

$$\vartheta = \text{const} \tag{5.9}$$

along the trajectories, as we had found earlier.

However conservation laws are often used conversely, i.e., to derive the motions. Can we proceed in the reversed direction? Remarkably, the answer is yes: their explicit forms manifestly *require* (5.9) for being conserved.

VI. CONCLUSION

The absence of the kinetic term in their Hamiltonian implies that Carroll particles have a purely anomalous velocity relation. Position and momenta are partially decoupled and we end up with first-order equations. The motion of an exotic photon (2.5) is poor but not entirely trivial: it exhibits the anyonic spin-Hall effect [23].

The horizon of a Kerr-Newman black hole realizes these conditions: its magnetic field B (3.4) induces anomalous

[5]For $B = \text{const}$ we would get $Q = 0$ consistent with [16].

Hall motion for our exotic photon. Masslessness is mandatory for getting nontrivial dynamics [18].

Particles of the type of our exotic photons might actually play a role in condensed matter physics as quasiparticles [51]. Here we took them chargeless for simplicity, however, they could, in principle, carry also an electric charge [18].

The double central extension of the Carroll group is a mathematical fact [42,43]. But is it a physical reality? With no experimental data at hand, we just recall what Dirac wrote about his magnetic monopole [52]:

> *"This new development … is merely a generalisation of the possibilities … Under these circumstances one would be surprised if Nature had made no use of it."*

We note also that our gravitational ideas could, in principle, be tested in laboratory by the remarkable analog of a Kerr-Newman black hole, which could be created in condensed matter [53,54].

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Note added.—Recently, we were informed by T. R. Perche [55] that they are also considering similar problems and arrived, by following similar methods, at similar but slightly different conclusions.

[1] S. W. Hawking, The information paradox for black holes, arXiv:1509.01147.
[2] S. W. Hawking, M. J. Perry, and A. Strominger, Soft Hair on Black Holes, Phys. Rev. Lett. **116**, 231301 (2016).
[3] J. M. Lévy-Leblond, Une nouvelle limite non-relativiste du group de Poincaré, Ann. Inst. Henri Poincaré **3**, 1 (1965), http://archive.numdam.org/article/AIHPA_1965__3_1_1_0.pdf.
[4] V. D. Sen Gupta, On an analogue of the Galileo Group, Il Nuovo Cimento **54**, 512 (1966).
[5] C. Duval, G. W. Gibbons, P. A. Horvathy, and P. M. Zhang, Carroll versus Newton and Galilei: Two dual non-Einsteinian concepts of time, Classical Quantum Gravity **31**, 085016 (2014).
[6] C. Duval, G. W. Gibbons, and P. Horvathy, Celestial mechanics, conformal structures and gravitational waves, Phys. Rev. D **43**, 3907 (1991).
[7] X. Bekaert and K. Morand, Embedding nonrelativistic physics inside a gravitational wave, Phys. Rev. D **88**, 063008 (2013); K. Morand, Embedding Galilean and Carrollian geometries I. Gravitational waves, J. Math. Phys. (N.Y.) **61**, 082502 (2020).
[8] C. Duval, G. W. Gibbons, P. A. Horvathy, and P. M. Zhang, Carroll symmetry of plane gravitational waves, Classical Quantum Gravity **34**, 175003 (2017).
[9] E. Bergshoeff, J. Gomis, and G. Longhi, Dynamics of Carroll particles, Classical Quantum Gravity **31**, 205009 (2014).
[10] L. Donnay, G. Giribet, H. A. Gonzalez, and M. Pino, Supertranslations and Superrotations at the Black Hole Horizon, Phys. Rev. Lett. **116**, 091101 (2016).
[11] L. Marsot, Caractérisation géométrique des structures de Bargmann et de Carroll et des groupes de Schrödinger et de Bondi-Metzner-Sachs, Master thesis, Marseille University, 2016 (unpublished).
[12] L. Donnay and C. Marteau, Carrollian physics at the black hole horizon, Classical Quantum Gravity **36**, 165002 (2019).
[13] L. Freidel and P. Jai-akson, Carrollian hydrodynamics from symmetries, arXiv:2209.03328; Carrollian hydrodynamics and symplectic structure on stretched horizons, arXiv:2211.06415.
[14] A. Bagchi, Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories, Phys. Rev. Lett. **105**, 171601 (2010).
[15] C. Duval, G. W. Gibbons, and P. A. Horvathy, Conformal Carroll groups and BMS symmetry, Classical Quantum Gravity **31**, 092001 (2014).
[16] C. Duval, G. W. Gibbons, and P. A. Horvathy, Conformal Carroll groups, J. Phys. A **47**, 335204 (2014).
[17] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma, and S. Vandoren, Carroll symmetry, dark energy and inflation, Front. Phys. **10**, 810405 (2022).
[18] L. Marsot, P.-M. Zhang, M. Chernodub, and P. A. Horvathy, Hall effects in Carroll dynamics, arXiv:2212.02360.
[19] L. Bidussi, J. Hartong, E. Have, J. Musaeus, and S. Prohazka, Fractons, dipole symmetries and curved spacetime, SciPost Phys. **12**, 205 (2022).
[20] M. Pretko, The fracton gauge principle, Phys. Rev. B **98**, 115134 (2018).
[21] R. Karplus and J. M. Luttinger, Hall effect in ferromagnetics, Phys. Rev. **95**, 1154 (1954).
[22] M. I. Dyakonov and V. I. Perel, Possibility of orientating electron spins with current, Sov. Phys. JETP Lett. **13**, 467 (1971); J. E. Hirsch, Spin Hall Effect, Phys. Rev. Lett. **83**, 1834 (1999).
[23] K. Yu. Bliokh and Yu. P. Bliokh, Topological spin transport of photons: The optical Magnus effect and Berry phase, Phys. Lett. A **333**, 181 (2004); M. Onoda, S. Murakami, and N. Nagaosa, Hall Effect for Light, Phys. Rev. Lett. **93**, 083901 (2004).

[24] M. C. Chang and Q. Niu, Berry phase, hyperorbits, and the Hofstadter spectrum: Semiclassical dynamics in magnetic Bloch bands, Phys. Rev. B **53**, 7010 (1996).

[25] M. C. Chang and Q. Niu, Berry Phase, Hyperorbits, and the Hofstadter Spectrum, Phys. Rev. Lett. **75**, 1348 (1995).

[26] P. A. Horvathy, Anomalous Hall effect in non-commutative mechanics, Phys. Lett. A **359**, 705 (2006).

[27] C. Duval and P. A. Horvathy, Chiral fermions as classical massless spinning particles, Phys. Rev. D **91**, 045013 (2015); P. Zhang and P. A. Horváthy, Anomalous Hall effect for semiclassical chiral fermions, Phys. Lett. A **379**, 507 (2014).

[28] M. Stone, V. Dwivedi, and T. Zhou, Berry phase, Lorentz covariance, and anomalous velocity for Dirac and Weyl particles, Phys. Rev. D **91**, 025004 (2015).

[29] J.-M. Lévy-Leblond, Galilei group and Galilean invariance, in *Group Theory and Applications*, edited by Ernest M. Loebl, Vol. II (Acadamic Press, New York, 1972), p. 222.

[30] C. Duval and P. A. Horvathy, The "Peierls substitution" and the exotic Galilei group, Phys. Lett. B **479**, 284 (2000); Exotic Galilean symmetry in the noncommutative plane, and the Hall effect, J. Phys. A **34**, 10097 (2001).

[31] V. Bargmann, On unitary ray representations of continuous groups, Ann. Math. **59**, 1 (1954).

[32] J.-M. Souriau, *Structure of Dynamical Systems. A Symplectic View of Physics*, translated by C. H. Cushman-de Vries (R. H. Cushman and G. M. Tuynman, Translation Editors), (Birkhäuser, Boston, MA, 1997).

[33] L. Inzunza and M. S. Plyushchay, Dynamics, symmetries, anomaly and vortices in a rotating cosmic string background, J. High Energy Phys. 01 (2022) 179.

[34] C. Duval, Z. Horvath, and P. A. Horvathy, Fermat principle for spinning light, Phys. Rev. D **74**, 021701(R) (2006).

[35] P. Gosselin, A. Berard, and H. Mohrbach, Spin Hall effect of photons in a static gravitational field, Phys. Rev. D **75**, 084035 (2007).

[36] M. A. Oancea, J. Joudioux, I. Y. Dodin, D. E. Ruiz, C. F. Paganini, and L. Andersson, Gravitational spin Hall effect of light, Phys. Rev. D **102**, 024075 (2020).

[37] C. Duval, L. Marsot, and T. Schücker, Gravitational birefringence of light in Schwarzschild spacetime, Phys. Rev. D **99**, 124037 (2019).

[38] V. P. Frolov, Maxwell equations in a curved spacetime: Spin optics approximation, Phys. Rev. D **102**, 084013 (2020).

[39] N. Yamamoto, Spin Hall effect of gravitational waves, Phys. Rev. D **98**, 061701 (2018).

[40] L. Andersson, J. Joudioux, M. A. Oancea, and A. Raj, Propagation of polarized gravitational waves, Phys. Rev. D **103**, 044053 (2021).

[41] A. I. Harte and M. A. Oancea, Spin Hall effects and the localization of massless spinning particles, Phys. Rev. D **105**, 104061 (2022).

[42] J. A. de Azcarraga, F. J. Herranz, J. C. Perez Bueno, and M. Santander, Central extensions of the quasiorthogonal Lie algebras, J. Phys. A **31**, 1373 (1998).

[43] A. Ngendakumana, Group theoretical construction of planar noncommutative systems, arXiv:1401.5213; A. Ngendakumana, J. Nzotungicimpaye, and L. Todjihounde, Group theoretical construction of planar noncommutative phase spaces, J. Math. Phys. (N.Y.) **55**, 013508 (2014).

[44] L. Marsot, Planar Carrollean dynamics, and the Carroll quantum equation, J. Geom. Phys. **179**, 104574 (2022).

[45] A. A. Kirillov, Unitary representations of nilpotent Lie groups, Russ. Math. Surv. **17**, 53 (1962); B. Kostant, Quantization and Unitary Representations, in *Lectures in Modern Analysis and Applications III*, edited by R. M. Dudley *et al.*, Springer Lecture Notes in Mathematics (Springer-Verlag, New York, 1970), Vol. 170, p. 87.

[46] P. A. Horvathy, The noncommutative Landau problem and the Peierls substitution, Ann. Phys. (Amsterdam) **299**, 128 (2002).

[47] J. Negro, M. A. del Olmo, and J. Tosiek, Anyons, group theory and planar physics, J. Math. Phys. (N.Y.) **47**, 033508 (2006).

[48] P. A. Horvathy, L. Martina, and P. C. Stichel, Enlarged Galilean symmetry of anyons and the Hall effect, Phys. Lett. B **615**, 87 (2005).

[49] S. Sternberg, in *Proceedings of the 2nd Bonn Conference on Differential Geometric Methods in Mathematical Physics*, Springer Lecture Notes in Mathematics Vol. 676 (Springer, Berlin, 1978), pp. 1–80.

[50] E. T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, Metric of a rotating, charged mass, J. Math. Phys. (N.Y.) **6**, 918 (1965).

[51] L. Venema, B. Verberck, I. Georgescu *et al.*, The quasiparticle zoo, Nat. Phys. **12**, 1085 (2016).

[52] P. A. M. Dirac, Quantised singularities in the electromagnetic field, Proc. R. Soc. A **133**, 60 (1931).

[53] R. A. Tinguely and Andrew P. Turner, Optical analogues to the equatorial Kerr-Newman black hole, Commun. Phys **3**, 120 (2020).

[54] P. M. Zhang, C. Duval, G. W. Gibbons, and P. A. Horvathy, Soft gravitons and the memory effect for plane gravitational waves, Phys. Rev. D **96**, 064013 (2017).

[55] F. Gray, D. Kubizňák, T. R. Perche, and J. Redondo-Yuste, Carrollian motion in magnetized black hole horizons, arXiv:2211.13695.