Letter

Stationary black holes and stars in the Brans-Dicke theory with $\Lambda > 0$ revisited

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It was shown a few years back that for a stationary regular black hole or star solution in the Brans-Dicke theory with a positive cosmological constant Λ, endowed with a de Sitter or cosmological event horizon in the asymptotic region, not only there exists no nontrivial field configurations, but also the inverse Brans-Dicke parameter ω^{-1} must be vanishing. This essentially reduces the theory to Einstein's general relativity. The assumption of the existence of the cosmological horizon was crucial for this proof. However, since the Brans-Dicke field ϕ, couples directly to the Λ-term in the energy-momentum tensor as well as Λ acts as a source in ϕ's equation of motion, it seems reasonable to ask: can ϕ become strong instead and screen the effect of Λ, at very large scales, so that the asymptotic de Sitter structure is replaced by some alternative, yet still acceptable boundary condition? In this work we analytically argue that no such alternative exists, as long as the spacetime is assumed to be free of any naked curvature singularity. We further support this result by providing explicit numerical computations. Thus we conclude that in the presence of a positive Λ, irrespective of whether the asymptotic de Sitter boundary condition is imposed or not, a regular stationary black hole or even a star solution in the Brans-Dicke theory always necessitates $\omega^{-1} = 0$, and thereby reducing the theory to general relativity. The qualitative differences of this result with that of the standard no hair theorems are also pointed out.

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I. INTRODUCTION

The classical black hole no hair theorem [1] states that there exists no nontrivial field configuration at the exterior of a stationary black hole spacetime, save the long range gauge fields, related to the uniqueness of stationary black hole spacetimes, see [2] for a vast review and list of references. The no hair theorems, their violations, and associated uniqueness properties are much well studied, see [3] and references therein. Most of these efforts have been devoted to the asymptotically flat spacetimes. However, the overwhelming observational evidences of accelerated expansion of our current universe suggests that there is a strong possibility that it is endowed with some form of the dark energy, an exotic matter field with negative isotropic pressure. A positive cosmological constant Λ is the simplest and phenomenologically very successful model of the same [4]. An important feature of such spacetimes, in addition to the black hole horizon, is certainly the existence of a de Sitter or cosmological event horizon as the outer causal boundary, thereby making the asymptotic structure very different compared to that of $\Lambda \leq 0$, e.g., [5]. Can this horizon bring into nontrivial boundary effects? We refer our reader to [6–9] and references therein for discussion on black hole no hair theorems and their violations in such scenario.

Despite their phenomenological successes, the actual nature of dark energy and dark matter remain elusive so far. This has lead the community to plunge into research in various gravity and dark energy theories alternative to Einstein's general relativity in recent times, see, e.g., [10] for a vast review and references therein. Such alternative models mimic the dark sector chiefly via some dynamical matter fields or modification of the Einstein-Hilbert action. The Brans-Dicke theory in particular, is the prototype of the scalar-tensor class of theories [11,12],

$$S = \int \sqrt{-g} d^4x \left[\phi R - 2\Lambda - \frac{\omega}{\phi}(\nabla\phi)^2 + \mathcal{L}_M\right] \qquad (1)$$

where the scalar ϕ is the Brans-Dicke field, whose inverse acts as a local and dynamical gravitational "constant" and

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$\mathcal{L}_M$ stands collectively for the matter Lagrangian density. ω is the Brans-Dicke parameter and as $\omega \to \infty$, the theory reduces to Einstein's general relativity.

The proof of no hair theorem for Eq. (1) in the asymptotically flat spacetime can be seen in [13]. However, one can have nontrivial configurations for ϕ in asymptotically flat nonblack hole spacetimes like the sun [11]. We further refer our reader to, e.g., [14–32] and references therein for various aspects of black holes and large scale structures in the Brans-Dicke and some other viable alternative gravity models. The extension of the no Brans-Dicke hair theorem of [13] with a positive Λ was considered a few years back in [19], with an asymptotic de Sitter boundary condition. It was shown that not only any nontrivial field configuration for ϕ is excluded, but also the existence of the cosmological horizon reduces the theory to Einstein's General Relativity (i.e., $\omega^{-1} = 0$). Moreover, similar conclusion was shown to exist for a stationary star spacetime. Being a theory getting constrained, clearly these results are in stark contrast to *any* existing no hair theorems, which predict *only* about the field configurations.

Now, even though the asymptotic de Sitter boundary condition seems to be reasonable, as we argue in Sec. II, perhaps it cannot be unique, chiefly owing to the fact that Λ acts as an omnipresent source to the Brans-Dicke field via a Poisson equation, Eq. (2). Can we have hairy black hole and star solutions with some alternative asymptotic structure? Or at least, is it possible to just have the field configuration constrained as of [13], and leave ω unaffected? The answers to both these questions are negative, as no such nonsingular alternative asymptotic structure exists, shown below in Secs. II, II A, and III, both analytically and numerically. In order to prove this, we fix the boundary conditions on the black hole event horizon, as described and argued in Sec. II. Accordingly we conclude that, in the presence of a positive Λ and irrespective of whether the asymptotic de Sitter boundary condition is assumed to hold or not, a stationary black hole or star solution in the Brans-Dicke theory essentially necessitates $\omega^{-1} = 0$, thereby reducing the theory to general relativity, as long as there is no naked curvature singularity in the spacetime.

II. NONEXISTENCE OF BLACK HOLES WITH GENERIC ASYMPTOTIC CONDITION

The equations of motion corresponding to Eq. (1) are given by

$$R_{\mu\nu} = \frac{\Lambda(2\omega+1)}{\phi(2\omega+3)} g_{\mu\nu} + \frac{T_{\mu\nu}}{\phi} - \frac{T(\omega+1)}{\phi(2\omega+3)} g_{\mu\nu} + \frac{\omega}{\phi^2}(\nabla_\mu\phi)(\nabla_\nu\phi) + \frac{\nabla_\mu\nabla_\nu\phi}{\phi}$$
$$\Box\phi = \frac{T-4\Lambda}{2\omega+3} \qquad (\omega \neq -3/2) \tag{2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor corresponding to $\mathcal{L}_M$ and T is its trace. From Eq. (2), the Ricci scalar is found to be

$$R = \frac{2\omega(4\Lambda - T)}{\phi(2\omega+3)} + \frac{\omega}{\phi^2}(\nabla_\mu\phi)(\nabla^\mu\phi) \tag{3}$$

Let us look for regular stationary black hole solutions admitted by Eq. (1). We take the ansatz for a static and spherically symmetric metric

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{4}$$

Since the geometry is static and spherically symmetric, we shall take, by the virtue of Eq. (2), that ϕ is explicitly independent of time and is a function of the radial coordinate only. From the staticity of ϕ, the equation of motion for ϕ becomes

$$D_\mu(\sqrt{f}D^\mu\phi) = -\frac{\sqrt{f}(4\Lambda - T)}{2\omega+3} \tag{5}$$

where D_μ is the spacelike covariant derivative on the (r,θ,ϕ)-hypersurface of Eq. (4). Setting $T_{\mu\nu} = 0$, and multiplying the above equation by $e^{\epsilon\phi}$ ($\epsilon = \pm 1$), and integrating it by parts on that spacelike hypersurface (say, Σ), we have

$$\int_{\partial\Sigma}\sqrt{\frac{f}{h}}e^{\epsilon\phi}\partial_r\phi = \int_\Sigma \sqrt{f}e^{\epsilon\phi}\left[\epsilon(D_\mu\phi)(D^\mu\phi) - \frac{4\Lambda}{2\omega+3}\right] \tag{6}$$

where the left-hand side consists of the boundary integrals over 2-spheres. One of the boundaries is the black hole event horizon where $f, h^{-1} = 0$. If in addition we assume the existence of a cosmological event horizon as a boundary in the asymptotic region, we have $f, h^{-1} = 0$ there as well. The regularity of the field and its derivative on the horizon, as it turns out to be necessary for non-naked singular horizons then implies that the boundary integrals vanish. The derivative term appearing on the right-hand side of Eq. (6), being spacelike, is positive definite. We then set $\epsilon = -1(\epsilon = +1)$ for $2\omega + 3 > 0(2\omega + 3 < 0)$, yielding not only a constant ϕ, but also $\omega^{-1} = 0$. This essentially rules out the theory, provided a cosmological event horizon, in addition to the black hole event horizon exists, as was shown in [19]. Similar result was shown to exist for stationary star solutions as well.

Thus the existence of a cosmological event horizon plays a crucial role above. Such existence seems to be plausible, as intuitively it seems that in the presence of positive Λ's repulsive effects, the Brans-Dicke field will become very diluted at large scales. This assumption is strengthened by the solar system constraint showing the weakness of the Brans-Dicke parameter, $|\omega| \gtrsim 40\,000$, e.g., [10]. However, the field equation for ϕ, Eq. (2), is basically a Poisson equation with an omnipresent source, Λ. Accordingly, even though ω is very large, perhaps we cannot rule out the possibility of existence of certain alternative boundary condition(s) that instead permits a strong ϕ at very large scales. Since Λ couples directly to ϕ, the first of Eq. (2), this could indicate a possible screening of the former. Under such circumstances, a cosmological event horizon, corresponding to the asymptotic de Sitter boundary condition may not exist, replaced by some suitable alternative asymptotic structure. What are these alternative boundary conditions? Can we have a regular stationary black hole solution in this scenario?

One might try conceiving asymptotic(s) alternative to de Sitter or flat spacetimes, keeping only in mind the spacetime must be non-naked singular. However, it seems that any such particular choice will be highly nonunique, being devoid of any symmetry argument in the asymptotic region. In order to tackle this difficulty, and to accommodate sufficient generality, we shall *not* at all impose any boundary condition as $r \to \infty$, and instead, we shall impose the same on the black hole event horizon.

The recent discovery of gravity waves from the black hole mergers suggests that the near horizon geometry matches exceedingly well with the prediction of general relativity [33–36]. This, along with the bound $|\omega| \gtrsim 40\,000$ suggests that the Brans-Dicke field must be weak at small scales such as a black hole. Hence we shall take the near black hole horizon geometry to be the Schwarzschild-de Sitter at the leading order,

$$ds^2|_{\rm BH} \to -\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)dt^2+\left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\varphi^2) \tag{7}$$

Clearly, this necessitates that the field ϕ is close to unity and very slowly varying near the black hole horizon. The general solution of the second of Eq. (2) with $T_{\mu\nu}=0$ in the background of Eq. (7) is given by [19],

$$\phi(r)|_{\rm BH} \to C_2 + \frac{1}{\omega+3/2}\left[\frac{C_1}{r_H}\ln\left(1-\frac{r_H}{r}\right) + \left(1-\frac{C_1}{2r_C}\right)\ln\left(1-\frac{r}{r_C}\right) + \left(1+\frac{C_1}{2r_C}\right)\ln\left(1+\frac{r}{r_C}\right)\right] \tag{8}$$

where C_1, C_2 are integration constants and r_H is the black hole horizon radius (i.e., the smallest positive root of $1-2M/r-\Lambda r^2/3=0$) and $r_C=\sqrt{3/\Lambda}$. For the sake of computational simplicity, in the above derivation we have assumed that the black hole horizon is much small compared to r_C, owing to the observed current tiny value of $\Lambda \sim 10^{-52}\ \mathrm{m}^{-2}$, implying $M\sqrt{\Lambda} \ll 1$. Since as $\omega \to \infty$, we must have $\phi \to 1$, we set $C_2 = 1$. The above solution is divergent on $r = r_H$. Thus a regular solution on it must correspond to $C_1 = 0$, giving

$$\phi(r\to r_H) \to 1 + \frac{1}{\omega+3/2}\ln\left(1-\frac{r^2}{r_C^2}\right) \tag{9}$$

Note that setting $\Lambda = 0$ (i.e., $r_C \to \infty$) yields $\phi \to 1$, reproducing the no hair result of [13]. Using now $r_C \sim 10^{26}$ m, it is easy to see that the dynamical part of the above solution is much small compared to unity. For example, even for a few billion solar mass black hole ($r_H \sim 10^{16}$ m), and $\omega \gtrsim 10^4$, it is at most $\mathcal{O}(10^{-24})$ and is further smaller for smaller black holes. Using now Eqs. (9) and (4) into Eq. (2), it is also easy to find out the leading correction to the near horizon metric

$$f(r\to r_H) = h^{-1}(r\to r_H) \to \left(1-\frac{2M}{r}-\frac{\Lambda r^2}{3}-\frac{\Lambda r^4}{5\omega r_C^2}\right)$$

For $r_H \sim 10^{16}$ m, the mass term is $\mathcal{O}(1)$, the Λ term is $\mathcal{O}(10^{-20})$, whereas the Brans-Dicke correction term is $\mathcal{O}(10^{-45})$, or smaller. Thus the backreaction on the spacetime near the black hole horizon due to $\phi(r)$ can safely be ignored. We also note that any other choice of C_1 in Eq. (8) leads to a curvature singularity on the black hole horizon [19]. Thus Eq. (9) is the unique solution regular on the black hole and hence the boundary condition of Eq. (7) is justified.

Note that in our present scenario, if we construct now an integral equation like Eq. (6), although the boundary integral on the black hole horizon vanishes ($f(r_H), h^{-1}(r_H)=0$), the outer boundary integral at $r\to\infty$ does not, as f and h^{-1} is nowhere vanishing except on the black hole horizon, by our assumption. The simplest way to solve this problem seems to be to solve Eqs. (2) and (4) numerically for general radial values, with the boundary conditions of Eqs. (7) and (9). However, we wish to do the same first in a much simpler and perhaps clearer analytic way, as follows.

Let us integrate Eq. (5) over the spacelike hypersurces (r,θ,ϕ) Σ of Eq. (4), and convert the total divergence into surface integrals over S^2's on the black hole horizon and on some larger radial coordinate, say r_0. As explained above, the surface integral on the black hole horizon vanishes to yield,

$$4\pi r^2\sqrt{\frac{f}{h}}\frac{d\phi}{dr}\bigg|_{r=r_0} = -\frac{4\Lambda}{2\omega+3}\int_\Sigma \sqrt{f} \tag{10}$$

The above equation shows that ϕ must be monotonic in r in the entire domain $r_H \le r < \infty$. For otherwise if it had any extremum at some $r = r_0$, we must have $\omega^{-1} = 0$, since $\Lambda \neq 0$ by our assumption. Also, since there is no horizon for $r > r_H$, $f(r)$ cannot be asymptotically vanishing and hence the integral on the right-hand side of the above equation increases with increasing r_0. Thus since Λ is positive, for $2\omega+3>0$, ϕ is monotonically decreasing whereas it is monotonically increasing for $2\omega+3<0$. Can $\phi(r)$ asymptote to some constant value? The answer is no, as this will simply correspond to asymptotic fall off in inverse power in r for the dynamical part of ϕ, leading to de Sitter geometry as dictated by the field equations Eq. (2). However, as we have stated earlier, in the presence of an asymptotic de Sitter geometry, we must have $\omega^{-1} = 0$ [19].

Let us first consider $2\omega+3>0$, i.e., effectively $\omega>0$, owing to the stringent solar system bound on it. Note that the dynamical part of Eq. (9) is negative, as $r_H < r_C$. Clearly, since $\phi(r)$ must be monotonically decreasing, this dynamical part, as we move toward larger radial values, would decrease to further negative values, no matter what its explicit form is. Eventually thus the dynamical part will reach minus of unity, making $\phi(r)$ vanishing at that point. However, the first term on the right-hand side of Eq. (3), will diverge then. The second term contains a spacelike inner product since ϕ is explicitly independent of time. Thus Eq. (3) is basically the sum of two positive definite quantities for $\phi \geq 0$. Hence no matter whether the second term diverges or not as $\phi \to 0$, the divergence of the first term sufficiently indicates the divergence of the Ricci scalar, $R \to \infty$, i.e., a naked curvature singularity. Thus we must have $\omega^{-1} = 0$, in order to have a regular black hole spacetime, thereby reducing the theory to general relativity, and hence we have the Schwarzschild-de Sitter to be the only solution.

Since $\phi(r)$ is positive on the horizon, Eq. (9), for $2\omega + 3 < 0$ (i.e., $\omega < 0$ effectively) on the other hand, $\phi(r)$ is ever increasing to larger positive values with increasing r, and hence would eventually diverge as $r \to \infty$. We shall show in Sec. II A that this also gives rise to a naked curvature singularity. However, even without this knowledge we wish to argue that having $\phi \to \infty$ is unphysical and unacceptable. This is because since ϕ^{-1} should act as an effective Newton's "constant" in the Brans-Dicke theory, a divergent ϕ in the asymptotic region would make any matter field backreactionless there. For example, let us imagine a static point mass m at some $r = r_0$. We have its energy momentum tensor

$$T_{\mu\nu} = mf(r)\delta_{\mu 0}\delta_{\nu 0}\delta^3(\vec{r} - \vec{r}_0)$$

substituting this into the first of Eq. (2), we may solve for corrections to f and h perturbatively due to m, assuming it is tiny. The corresponding potential due to m will contain a $\phi(r_0)$ in the denominator. Since $\phi(r_0)$ diverges as $r_0 \to \infty$, the potential must vanish. This not only violates Mach's principle upon which the Brans-Dicke theory is based [11,12], but also violates the most fundamental fact that any mass-energy distribution, no matter how tiny it is, must create its own gravity. To the best of our knowledge and understanding, the above scenario is physically unacceptable, thereby necessitating $\omega^{-1} = 0$ in this case as well. Thus we may conclude that there exists no regular or physically acceptable stationary black hole solution in the Brans-Dicke theory with a positive Λ, irrespective of whether asymptotically de Sitter boundary condition holds or not. We must have $\omega^{-1} = 0$ and hence the solution corresponds to that of the general relativity.

The above result can easily be extended to rotating black hole spacetimes, by replacing the near horizon boundary condition, Eq. (7), with the Kerr-de Sitter spacetime. The near horizon solution for ϕ can be found in the form $\phi(r \to r_H) \to 1 + \phi(r) + \phi(\theta)$, where $\phi(r)$ is analogous to Eq. (9), free of any singularity on the black hole horizon and $\phi(\theta) \to 0$ as we take the static limit. Similar argument as earlier then yields the same nonexistence result.

In the next section we shall further reinforce these nonexistence features by explicit numerical computations.

A. Explicit numerical computations

Substituting Eq. (4) into Eq. (2), we now solve for $f(r)$, $h(r)$ and $\phi(r)$ numerically using *Mathematica* with $T = 0$, subject to the boundary conditions Eqs. (7) and (9). We take $2M \sim 10^{16}$ m and $\Lambda \sim 10^{-52}$ m^{-2}. Using these we estimate the above three functions and their first derivatives' numerical values on the black hole event horizon, and further solve the coupled differential equations for general $r > r_H$. Figure 1 shows the variation of the metric functions $f(r)$, $h(r)$ at large scales. Figure 2 shows the variation of $\phi(r)$, whereas Fig. 3 depicts the same with a magnified resolution. Finally, using these results, we compute the Ricci scalar, Eq. (3), in Fig. 4. We have taken $|\omega| \geq 40\,000$, once again to be consistent with the recent observational evidence [10].

First, Fig. 1 shows that subject to the boundary condition we have chosen on the black hole event horizon, there is indeed no cosmological event horizon (necessitating $f \to 0, h^{-1} \to 0$) in the asymptotic region. Figures 2 and 3 shows $\phi(r)$ indeed monotonically decreases and increases with r, respectively for $2\omega + 3 > 0$ and $2\omega + 3 < 0$, as was argued in the preceding section. Also Fig. 3 shows that $\phi(r)$ respectively passes through zero (diverges) for $2\omega + 3 > 0$ $(2\omega + 3 < 0)$. Finally, Fig. 4 depicts the divergence of Ricci scalar for large radial values, thereby clearly proving that the spacetime we have obtained

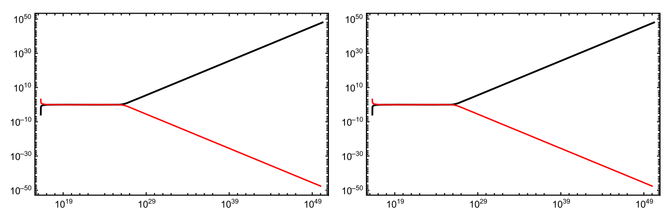

FIG. 1. The variation of the metric functions $f(r)$ (black curve) and $h(r)$ (red curve) vs r. The first plot corresponds to the Brans-Dicke parameter $\omega = 40\,000$, whereas the right one corresponds to $\omega = -40\,000$. We have used the logarithmic scales on both the horizontal and vertical axes, in order to accommodate the large variations.

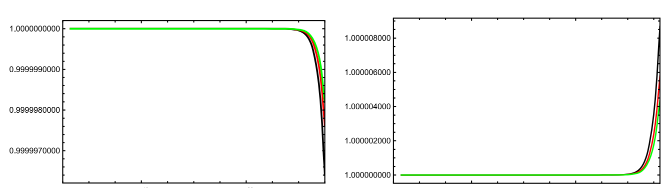

FIG. 2. The Brans-Dicke field $\phi(r)$ vs r, for different values of the Brans-Dicke parameter ω showing its monotonic behavior, as argued in Sec. II. The first set is for $\omega > 0$, and the second is for $\omega < 0$. The different colors correspond to: black ($\omega = \pm 40\,000$), red ($\omega = \pm 60\,000$) and green ($\omega = \pm 80\,000$).

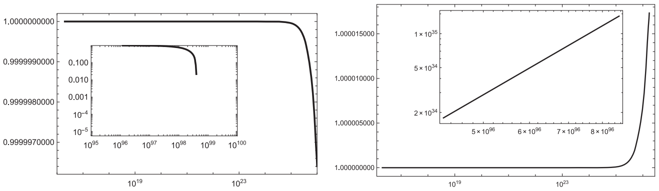

FIG. 3. $\phi(r)$ vs r with a magnified scale, depicting its pathological behavior at large radial distances. The left and right plot respectively corresponds to $\omega = \pm 40\,000$. See main text for discussion.

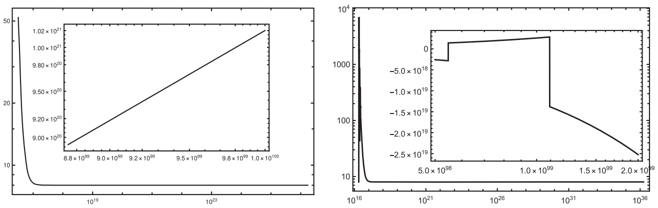

FIG. 4. The variation of the Ricci scalar R, Eq. (3), vs r. The left and right plot respectively corresponds to $\omega = \pm 40\,000$. We have used the logarithmic scales on both the horizontal and vertical axes. The plots in the inset are in the usual scale to show the asymptotic divergences in R. See main text for discussion.

is naked singular and hence unacceptable. Thus the numerical analysis presented here explicitly reconciles with the conclusion reached in the preceding section.

III. NONEXISTENCE OF STATIONARY STAR SOLUTIONS

We assume that the trace of the energy momentum-tensor constituting the star is less than or equal to zero. This is a reasonable assumption, indicating that the pressure of the matter field is not "too large" [37]. We also assume that the center of the star is flat, owing to the fact that as we move closer to the center, we have lesser and lesser matter fields to create gravity [37]. There is no event horizon in this spacetime. Using the flatness at the centre, we now integrate Eq. (5) from $r = 0$ up to the star surface to have for $d\phi/dr$,

$$4\pi r^2 \sqrt{\frac{f}{h}} \frac{d\phi}{dr}\bigg|_{r=R_0} = -\frac{1}{2\omega+3}\int_{\Sigma}^{R_0} \sqrt{f}(4\Lambda - T) \qquad (11)$$

where R_0 is radius of the star. We next integrate Eq. (5) from $r = R_0$ up to some $r = r_0$ outside the star and use Eq. (11) into it to have

$$4\pi r^2 \sqrt{\frac{f}{h}} \frac{d\phi}{dr}\bigg|_{r=r_0} = -\frac{1}{2\omega+3}\int_{\Sigma, r=0}^{R_0} \sqrt{f}(4\Lambda - T) - \frac{4\Lambda}{2\omega+3}\int_{\Sigma, R_0}^{r_0} \sqrt{f} \qquad (12)$$

Thus as earlier, $d\phi/dr$ is monotonically decreasing (increasing) for $2\omega + 3 > 0$ ($2\omega + 3 < 0$). Hence essentially the nonexistence result of Sec. II holds for a star as well.

IV. CONCLUSION

We have discussed in this work the nonexistence of regular stationary black hole and star solutions in the Brans-Dicke theory in the presence of a positive cosmological constant, Λ. It was shown earlier in [19] that if a cosmological event horizon exists, we must have the inverse Brans-Dicke parameter $\omega^{-1} = 0$, thereby reducing the theory to Einstein's general relativity. As we have argued in Sec. II, even though it is reasonable to expect a cosmological event horizon in the asymptotic region owing to the repulsive effects of positive Λ, one cannot *a priori* rule out possible alternative boundary conditions where the Brans-Dicke field $\phi(r)$ is strong instead, and thereby screening Λ at large scales. This corresponds to the fact that ϕ is sourced by an omnipresent Λ, and it couples directly with Λ, Eq. (2). Note that once we discard the asymptotic de Sitter structure, we do not have any obvious symmetry argument to explicitly define an alternative one. Hence in order to make our analysis as generic as possible, we did not impose any boundary condition for large r, but did so instead on the black hole event horizon explicitly, inspired by the recent gravity wave data coming from the black hole mergers [33–36]. With this "initial condition," we showed in Secs. II, II A, and III that the existence of any non-naked singular stationary black hole as well as star spacetimes with $\Lambda > 0$ essentially necessitate, $\omega^{-1} = 0$, thereby the theory reduces to Einstein's general relativity in this scenario as well. The black hole and the exterior of the star will then be described by the Kerr- or the Schwarzschild-de Sitter spacetime.

We would like to emphasize the stark contrast that while the standard no hair theorems only talk about the field configurations, e.g., [13], we obtain the parameter characterising a theory is getting constrained, not only for black holes but also for stars. Note also that the nontrivial effects due to a positive Λ reported in [19] (also in, e.g., [7,38]), was chiefly related to the exotic boundary effects due to the cosmological event horizon. This is contrary to our present case, as no such horizon was assumed to be present and hence no boundary effect was possible here. Since the Brans-Dicke theory is considered to be the prototype of the scalar-tensor class of alternative gravity theories, we believe the result we have found to be interesting and important in its own right.

It seems to be an interesting task to check the cosmological anisotropy dissipation/no hair theorem of [39] in the context of the Brans-Dicke theory with a positive Λ. We hope to return to this issue in a future work.

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Correction: The Government of India File number contained an error and has been fixed.