Letter

Area operator and fixed area states in conformal field theories

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The fixed area states are previously discussed in the quantum error-correction codes interpretation of AdS/CFT. The dual bulk geometry is constructed by gravitational path integrals. In this paper we show the fixed area states correspondence in conformal field theories (CFTs), which are associated with the spectrum decomposition of reduced density matrix ρ_A for a subsystem A. For two-dimensional CFTs we directly build the bulk metric, which is consistent with the expected geometry of the fixed area states. For arbitrary pure state $|\psi\rangle$ with a geometric dual in the bulk we also find the consistency by using the gravity dual of Rényi entropy. We also obtain the parameters relation between the bulk geometry and boundary state. The pure state $|\psi\rangle$ can be expanded as a superposition of the fixed area states. Motivated by this, we propose an area operator $\hat{A}^\psi$. The fixed area state is the eigenstate of $\hat{A}^\psi$, the associated eigenvalue is related to the Rényi entropy of subsystem A in this state. The Ryu-Takayanagi formula can be expressed as the expectation value $\langle\psi|\hat{A}^\psi|\psi\rangle$ divided by $4G$, where G is the Newton constant. We further show the fluctuation of the area operator in the geometric state $|\psi\rangle$ is suppressed in the semiclassical limit $G \to 0$.

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I. INTRODUCTION

AdS/CFT correspondence provides for us a way to understand the nature of the bulk spacetime by the CFT living on the boundary [1–3]. One of interesting topics in AdS/CFT is the exact duality relation between quantum states in the Hilbert space of the boundary CFT and the ones in the bulk. Some states in the CFTs can be effectively described by the classical geometries in the limit $G \to 0$. In this paper we will call them geometric states for short.

The geometry is associated with the entanglement entropy (EE) $S(\rho_A)$ of a boundary subregion A by the well-known Ryu-Takayanagi (RT) formula [4] for the bulk metric with time reflection symmetry,

$$S(\rho_A) = \frac{\text{Area}(\gamma_A)}{4G}, \tag{1}$$

where γ_A is the minimal surface in the bulk that is homology to A, ρ_A denotes the reduced density matrix of A. For general bulk spacetime one should take γ_A to be the Hubeny-Rangamani-Takayanagi surface [5]. The RT formula shows the secret relation between spacetime and intrinsic entanglement of underlying degrees of freedom of quantum gravity [6].

The area lawlike relation is generalized to the holographic Rényi entropy by Dong [7]. The Rényi entropy, defined as $S_n(\rho_A) \coloneqq \frac{\log \text{tr}\rho_A^n}{1-n}$, is one parameter generalization of entanglement entropy. The gravity dual of Rényi entropy is given by

$$n^2\partial_n\left(\frac{n-1}{n}S_n(\rho_A)\right) = \frac{\text{Area}(\mathcal{B}_n)}{4G}, \tag{2}$$

where $\mathcal{B}_n$ denotes the cosmic brane with the tension $\mu_n = \frac{n-1}{4nG}$. The cosmic brane backreacts on the geometry by creating a conical defect with opening angle $\theta = \frac{2\pi}{n}$.

In [8] the authors find the connections between the quantum error-correction (QEC) code and AdS/CFT correspondence. It has led to a better understanding of radial commutativity and subregion duality in the correspondence [9]. The RT formula (1) naturally appears in the QEC code as shown in [10]. To explain the Rényi entropy formula (2) in the same framework the code should satisfy certain special properties [11,12]. It leads to new kinds of geometric states named fixed area states, for which the Rényi entropy are independent with n to the leading order in G. Previous approaches are based on the holographic QEC code and gravitational path integral. The geometry dual of the fixed area state can be obtained by inserting a cosmic brane fixed to be on the RT surface. More discussions on the fixed area states can be found in [13–17]. Therefore, the

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fixed area states are very important to understand the holographic QEC code as well as the AdS/CFT correspondence. However, most of the studies are based on the abstract QEC code or some simple tensor network models [18,19]. In this paper we will work in the framework of AdS/CFT and construct the fixed area states in the CFTs. Our results provide a new way to investigate the holographic QEC code in the language of quantum field theories.

Our construction of the fixed area states is related to the spectrum decomposition of reduced density matrix ρ_A. Actually one could obtain the decomposition once knowing the Rényi entropy for all the indices n. For a given pure state $|\psi\rangle$ the reduced density matrix of subsystem A is ρ_A^ψ. The modular Hamiltonian $H_A^\psi \coloneqq -\log \rho_A^\psi$ has the same eigenvalues as ρ_A^ψ. In general, assume H_A^ψ has degenerate eigenbasis $|k,\alpha\rangle^\psi$ with $H_A^\psi|k,\alpha\rangle^\psi = (t_k + b^\psi)|k,\alpha\rangle^\psi$ with $t_k \in [0,+\infty)$, where α are degeneracy labels, b^ψ is the minimal eigenvalue of H_A^ψ. It is not hard to show that $b^\psi = \lim_{n\to\infty} S_n(\rho_A^\psi)$ by using the definition of Rényi entropy.

For the pure state $|\psi\rangle$ by Schmidt decomposition we have

$$|\psi\rangle = \sum_t e^{-\frac{b^\psi+t}{2}} \sum_\alpha |k,\alpha\rangle^\psi |\bar{k},\bar{\alpha}\rangle^\psi \delta_{t_{k,\alpha},t}, \tag{3}$$

where $|\bar{k},\bar{\alpha}\rangle^\psi$ are the basis of $H_{\bar{A}}^\psi$. We could construct the state in the subspace associated with a fixed t,

$$|\Phi\rangle_t^\psi \coloneqq \frac{1}{\sqrt{\mathcal{P}^\psi(t)}} \sum_\alpha |k,\alpha\rangle^\psi |\bar{k},\bar{\alpha}\rangle^\psi \delta_{t_{k,\alpha},t}, \tag{4}$$

where $\mathcal{P}^\psi(t) \coloneqq \sum_\alpha \delta_{t_{k,\alpha},t}$ is the dimension of this subspace. Therefore, $|\psi\rangle$ can be expressed as a superposition of the states $|\Phi\rangle_t^\psi$.

By definition of Rényi entropy we have the relation $\mathrm{tr}\rho_A^n \coloneqq \sum_{k,\alpha} e^{-n(b^\psi+t_{k,\alpha})} = \sum_t e^{-n(b^\psi+t)} \sum_\alpha \delta_{t_{k,\alpha},t} = \sum_t e^{-n(b^\psi+t)} \mathcal{P}^\psi(t)$. For quantum field theory the spectra of H_A^ψ are expected to be continuous. We expect $\mathcal{P}^\psi(t)$ has a well-defined continuous limit, which can be taken as the density of eigenstates at t. One could evaluate $\mathcal{P}^\psi(t)$ by approximating the summation over t by an integral, that is

$$\int_0^\infty dt \mathcal{P}^\psi(t) e^{-n(b^\psi+t)} = e^{(1-n)S_n(\rho_A^\psi)}. \tag{5}$$

By an inverse Laplace transformation in the variable n we can obtain $\mathcal{P}^\psi(t)$ [20], see also [21,22].

The reduced density matrix of A is $\rho_{t,A}^\psi = \frac{1}{\mathcal{P}^\psi(t)} \sum_\alpha |k,\alpha\rangle^\psi\, {}^\psi\langle k,\alpha| \delta_{t_{k,\alpha},t}$. It is obvious that $\rho_{t,A}^\psi$ has flat spectra, thus the Rényi entropy is independent with n, which is the key property of the fixed area states constructed in [11,12]. For any operator $\mathcal{O}_A$ located in region A we have

$${}_t^\psi\langle\Phi|\mathcal{O}_A|\Phi\rangle_t^\psi = tr(\rho_A^\psi \mathcal{O}_A) = \mathcal{P}_{\mathcal{O}_A}(t)/\mathcal{P}^\psi(t), \tag{6}$$

where $\mathcal{P}_{\mathcal{O}_A}(t) \coloneqq \sum_\alpha \langle k,\alpha|\mathcal{O}_A|k,\alpha\rangle \delta_{t_{k,\alpha},t}$. One could also evaluate $\mathcal{P}_{\mathcal{O}_A}(t)$ by the method in [22].

One of the results of this paper is that the state $|\Phi\rangle_t^\psi$ (4) is exactly dual to the fixed area state for any $t \sim O(c)$ or $O(1/G)$. We make this claim by using the information of $\mathcal{P}^\psi(t)$ and $\mathcal{P}^\psi_{\mathcal{O}_A}(t)$ with $\mathcal{O}_A$ being the stress energy tensor T. We also obtain the relation between the parameter t and the area of the minimal surface associated with A. We should stress most of our calculations are working in CFTs with large central charge c. To find the parameters relation we should use the holographic Rényi entropy formula (2). Another important result of our paper is constructing the area operator $\hat{A}^\psi$ in CFTs. The expectation value of $\hat{A}^\psi$ in the geometric state $|\psi\rangle$ divided by $4G$ in the geometric state gives the holographic EE. This can be seen as a quantum version of the RT formula.

II. FIXED AREA STATES IN AdS_3

Consider a two-dimensional CFT with central charge c on a complex plane with the coordinate $(w,\bar{w}) \coloneqq (x+i\tau, x-i\tau)$. In this section we will remove the superscript "ψ" to indicate the quantities are defined for a vacuum state. For an interval $A = [-R,R]$ in the vacuum state the Rényi entropy is universal for 2D CFTs [23], given by $S_n(\rho_A) = (1+\frac{1}{n})b$ with $b \coloneqq \lim_{n\to\infty} S_n(\rho_A) = \frac{c}{6}\log\frac{2R}{\epsilon}$. We can obtain the density of eigenstates with respect to t [20]:

$$\mathcal{P}(t) = \delta(t) + \sqrt{\frac{b}{t}} I_1(2\sqrt{bt}) H(t), \tag{7}$$

where $I_n(z)$ is the modified Bessel function of the first kind, $H(t)$ is the Heaviside step function.

For the holographic CFT $b \sim O(c) \gg 1$, taking t to be the order of c. The density of state $\mathcal{P}(t) \simeq \frac{be^{2\sqrt{bt}}}{\sqrt{4\pi}(bt)^{3/4}}$. In the CFT side the EE of the state $\rho_{t,A}$ is given by $\log\mathcal{P}(t) \simeq 2\sqrt{bt} + O(\log c)$.

By construction, the Rényi entropy of the state $\rho_{t,A}$ is the same as the EE, which is an important feature of the fixed area states [11]. In the following we would like to show the state $|\Phi\rangle_t$ is a fixed area state by explicitly constructing the bulk geometry.

Using a similar method as in [24], we can get the expectation value of stress energy tensor $T(w)$ in the state $|\Phi\rangle_t$ [22]:

$$\langle T(w)\rangle_t = \frac{cR^2}{6(R^2-w^2)^2}\left(1-\frac{t}{b}\right). \tag{8}$$

Similarly, one could get $\langle\bar{T}(\bar{w})\rangle_t$ by replacing w with $\bar{w}$ in the above expression. The singularity at the ending points of interval A is associated with the conical defect as we will show soon. The bulk geometry is fixed by the one-point function of $T(w)$, that is

$$ds^2 = \frac{dy^2}{y^2} + \frac{L_t}{2}dw^2 + \frac{\bar{L}_t}{2}d\bar{w}^2 + \left(\frac{1}{y^2} + \frac{y^2}{4}L_t\bar{L}_t\right)dwd\bar{w}, \quad (9)$$

where $L_t \coloneqq -\frac{12}{c}\langle T(w)\rangle_t$, $\bar{L}_t \coloneqq -\frac{12}{c}\langle \bar{T}(w)\rangle_t$. The above solution has singularity in the coordinate $(y, w, \bar{w})$. By a conformal transformation $\xi = (\frac{R+w}{R-w})^\alpha$, $\bar{\xi} = (\frac{R+\bar{w}}{R-\bar{w}})^\alpha$ with $\alpha \coloneqq \sqrt{\frac{t}{b}}$, we have $\langle T(\xi)\rangle = \langle \bar{T}(\bar{\xi})\rangle = 0$. At the points $\xi = 0, \infty$ has conical defect with opening angle $\theta = 2\pi\alpha$. The dual bulk solution is the Poincaré coordinate $ds^2 = \frac{d\eta^2 + d\xi d\bar{\xi}}{\eta^2}$ with a conical defect line γ.

With the geometry (9) one could find the geodesic line γ_A connecting the ending points of A and evaluate the holographic EE by using the RT formula (1). The details of the calculations can be found in Supplemental Material [25]. The result is

$$S_A(\rho_{t,A}) = \frac{L_{\gamma_A}}{4G} = \frac{\alpha c}{3}\log\frac{2R}{\epsilon} = 2\sqrt{bt}, \quad (10)$$

where we have used the Brown-Henneaux relation $c = \frac{3}{2G}$ [26]. The result of EE is exactly consistent with the CFT calculation to the leading order in $1/G$.

In [11] the fixed area states are constructed by inserting a cosmic brane (line in AdS_3) and requiring that the location of the cosmic brane coincides with the RT surface. Here we would like to show the conical defect line γ the same as the geodesic line γ_A by using $S_n(\rho_{t,A}) = S(\rho_{t,A})$ for the state $|\Phi\rangle_t$. To show this, we need to evaluate the holographic Rényi entropy.

Consider the n-replica state $\rho^n_{t,A}$, the one-point function $\mathrm{tr}(\rho^n_{t,A}T(w))$ is given by the same formula as (8). Now w is the coordinate on the n-sheet Riemann surface $\mathcal{R}_n$. Adopting polar coordinates near the ending points of A, we have $w - R \simeq re^{i\theta}$ with $\theta \sim \theta + 2n\pi$. Using the same conformal transformation $w \to \xi = (\frac{w+R}{w-R})^\alpha$, $\mathcal{R}_n$ is mapped to the ξ plane with the conical defect with opening angle $\theta_n = 2\pi n\alpha$. Therefore, the dual bulk geometry $\mathcal{M}_n$ for $\mathcal{R}_n$ is the Poincaré coordinate with a conical defect line γ. Moreover, $\mathcal{M}_n$ can be constructed by cyclically gluing n-copy geometry (9) together along the defect line γ. The conical defect line can be realized by inserting codimension-2 cosmic branes (lines in AdS_3). The tension of the cosmic brane μ_n is associated with the parameter α by the relation $\mu_n = \frac{1-n\alpha}{4G}$ [27].

To evaluate the Rényi entropy $S_n(\rho_{t,A})$ we need to know the bulk action $I_{\mathrm{bulk}}(n)$, which includes the on-shell action $I_g(n)$ of the geometry $\mathcal{M}_n$ and the brane action $I_b(n)$. We show the details of the calculations in the Supplemental Material [25]. The result is

$$S_n(\rho_{t,A}) = \frac{I_{\mathrm{bulk}}(n) - nI_{\mathrm{bulk}}(1)}{n-1} = \frac{L_\gamma}{4G}. \quad (11)$$

Comparing with the holographic EE result (10) we have $L_\gamma = L_{\gamma_A}$. This means the defect line γ coincides with the geodesic line γ_A.

We expect the states (4) are dual to the fixed area states only for $t \sim O(c)$ in the holographic CFTs. For $t \sim O(1)$ or $t \ll c \sim b$, the one-point function of T is still given by (8). It seems we could construct the geometry for these states, but the density of state $\mathcal{P}(t)$ no longer scales as $e^{2\sqrt{bt}}$, thus the EE $\log\mathcal{P}(t)$ in these states is not of $O(c)$. We do not expect they have well-defined bulk geometry.

The above results give us a new way to understand the vacuum AdS_3 by decomposing them into fixed area states. To be more precise we have

$$|0\rangle = \sum_t \sqrt{\mathcal{P}(t)}\, e^{-\frac{b+t}{2}}|\Phi\rangle_t. \quad (12)$$

The reduced density matrix of A is

$$\rho_A = \sum_t e^{-b-t}\mathcal{P}(t)\rho_{t,A}. \quad (13)$$

Actually, (13) is just the spectrum decomposition of the operator ρ_A, $P_t \coloneqq \mathcal{P}(t)\rho_{t,A}$ are projections into the Hilbert subspace with respect to the spectrum e^{-b-t}. The states $|\Phi\rangle_t$ are fixed area states if $t \sim O(c)$. However, the contributions from $t \ll c$ are usually exponentially suppressed in the large c limit. We can safely take the vacuum state of a holographic CFT as a superposition of fixed area states by introducing a lower cutoff of the summation (12).

III. FIXED AREA STATES IN ANY DIMENSION

For arbitrary pure geometric state $|\psi\rangle$, the reduced density matrix of subsystem A can be expressed as

$$\rho^\psi_A = \sum_t \mathcal{P}^\psi(t)e^{-b^\psi - t}\rho^\psi_{t,A}. \quad (14)$$

The density of eigenstates $\mathcal{P}^\psi(t)$ is given by

$$\begin{aligned}\mathcal{P}^\psi(t) &= \mathcal{L}^{-1}[e^{nb+(1-n)S_n(\rho^\psi_A)}](t)\\ &= \frac{1}{2\pi i}\int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} dn e^{s_n},\end{aligned} \quad (15)$$

with

$$s_n \coloneqq n(t+b) + (1-n)S_n(\rho^\psi_A), \quad (16)$$

where $\mathcal{L}^{-1}[\cdots]$ denotes the inverse Laplace transformation of the expression in the square brackets, γ_0 is chosen for the convergence of the integration, and $S_n(\rho^\psi_A)$ is the Rényi entropy of subsystem A in the state $|\psi\rangle$. In general, it is hard to evaluate the Rényi entropy for arbitrary states. For holographic theories, $S_n(\rho_A^\psi)$ is expected to be of order $O(c)$. For $t \sim O(c)$ we can evaluate the integral (15) by saddle point approximation. That is to solve the equation

$$\partial_n s_n = (t+b) + \partial_n[(1-n)S_n(\rho_A^\psi)] = 0. \tag{17}$$

In general, (17) is a complicated equation for n. Assume the solutions exist. If it gives more than one solution, we should take the one that maximizes s_n. With the solution $n^* = n^*(t)$ we have

$$s_{n^*} = [S_n(\rho_A^\psi) + n(n-1)\partial_n S_n(\rho_A^\psi)]_{n=n^*}. \tag{18}$$

Using Dong's formula of holographic Rényi entropy (2) we have

$$s_{n^*} = \frac{\text{Area}(\mathcal{B}_{n^*})}{4G}. \tag{19}$$

Therefore, the density of eigenstates is given by

$$\mathcal{P}^\psi(t) \propto e^{\frac{\text{Area}(\mathcal{B}_{n^*})}{4G}}. \tag{20}$$

By definition the Rényi entropy of the state $\rho_{t,A}^\psi$ is independent with n, given by

$$S_n(\rho_{t,A}^\psi) = \log \mathcal{P}^\psi(t) \simeq \frac{\text{Area}(\mathcal{B}_{n^*})}{4G}. \tag{21}$$

Our results show the states $|\Phi\rangle_t^\psi$ have the same property as the fixed area state. Equations (17) and (21) give the dual relation between the parameter t and the bulk fixed area, that is the area of the cosmic brane $\text{Area}(\mathcal{B}_{n^*})$. Suppose the geometry dual to $|\psi\rangle$ is $\mathcal{M}_\psi$. According to Dong's formula of Rényi entropy the tension of the codimension-2 cosmic brane $\mathcal{B}_n$ is $\mu_n = \frac{n-1}{4nG}$. To obtain the geometry dual to the fixed area state $|\Phi\rangle_t^\psi$ one should insert a codimension-2 cosmic brane with tension $\mu_t = \frac{n^*-1}{4n^*G}$, where n^* is the solution of the equation (17). If the equation has more than one solution, we should take the one that maximizes the function s_n (16). The cosmic brane backreacts on the geometry $\mathcal{M}_\psi$ and creates a conical defect with opening angle $\theta \coloneqq 2\pi\alpha_t = 2\pi - 8\pi G\mu_{n^*}$. The location of the cosmic brane coincides with the RT surface for subregion A in the backreacted geometry. The role of the cosmic brane is like a sharp projection that maps the original geometry $\mathcal{M}_\psi$ to the fixed area geometry. The above results are consistent with the discussion in [11] by using the gravitational path integral. We illustrate the geometry dual to the fixed area state $|\Phi\rangle_t^\psi$ in Fig. 1.

As a check of the above statement, let us consider the vacuum state in AdS_3. Taking the Rényi entropy $S_n(\rho_A) = (1+\frac{1}{n})b$ into the equation (17), we have the solution $n^* = \sqrt{b/t}$. The tension of the cosmic line is $\mu_t = \frac{1}{4G}(1-\sqrt{t/b})$ and the opening angle of the conical defect line is $\theta = 2\pi\sqrt{t/b}$. The results are exactly consistent with our direct calculations in the last section.

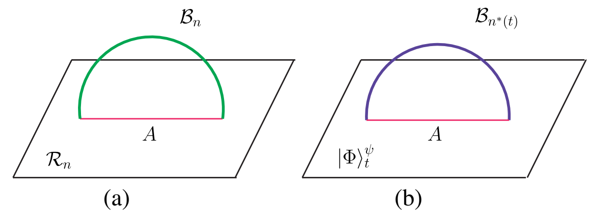

FIG. 1. Illustration of the gravity dual of Rényi entropy and geometry dual of the fixed area state. The field theory lives on the plane and is dual to gravitational theory in the bulk above the plane. A (red) is the subsystem. (a) Dong's formula for computing the Rényi entropy of A. The plane denotes the manifold $\mathcal{R}_n$, defined by n copies of original space on which the theory lives with singularity along the boundary of subsystem A. The bulk geometry is realized by inserting a cosmic brane $\mathcal{B}_n$ (green) with tension $\mu_n = \frac{n-1}{4nG}$. (b) Our proposal of the geometry dual to the fixed area state $|\Phi\rangle_t^\psi$. The cosmic brane $\mathcal{B}_{n^*(t)}$ (blue) is similar as (a) but the tension of the brane is $\mu_t = \frac{n^*-1}{4n^*G}$, where n^* is a function of t determined by the solution of (17). The RT surface of A coincides with the location of the brane.

IV. PROBABILITY OF THE FIXED AREA STATES

By using the expression of $\mathcal{P}^\psi(t)$, arbitrary pure geometric state $|\psi\rangle$ can be seen as a superposition of a series of the fixed area states,

$$|\psi\rangle = \sum_t \sqrt{p_t^\psi}\,|\Phi\rangle_t^\psi, \tag{22}$$

where $p_t^\psi \coloneqq e^{\frac{\text{Area}\mathcal{B}_{n^*}}{4G} - b^\psi - t}$. Like the vacuum case we expect the contributions from small t ($t \ll c$) of the above integration are negligible. The quantum error correction code interpretation of AdS/CFT suggests the coefficients p_t^ψ of (22) can be associated with the on-shell action I_t^ψ of the corresponding fixed area states $|\Phi\rangle_t^\psi$ [11,15]. The expected relation is $p_t^\psi = e^{-I_t^\psi}$. Using the result (22), we have

$$I_t^\psi = b^\psi + t - \frac{\text{Area}\mathcal{B}_{n^*}}{4G}, \tag{23}$$

which depends on the parameter t. p_t^ψ can be explained as the probability for the geometric state $|\psi\rangle$ to be the fixed area state $|\Phi\rangle_t^\psi$.

For the vacuum case $|0\rangle$, $b_\psi = b$ and $\frac{\text{Area}\mathcal{B}_{n^*}}{4G} = 2\sqrt{bt}$, the action $I_t = b(1-\sqrt{\frac{t}{b}})^2 = b(1-\alpha)^2$, which is consistent with $I_{\text{bulk}}(n)$ ([28] and the results in [15]). The probability distribution $p_t \coloneqq e^{-I_t}$ has maximal value at $t = b$. In the semiclassical limit $G \to 0$, the distribution will approach a delta function $\delta(t-b)$. Therefore, ρ_A can be approximated by the fixed area state $\rho_{t=b}$. One could check the EE of ρ_A is the same as $\rho_{t=b}$ in the leading order of G. Taking $t=b$ into (9) we get the same geometry as the vacuum AdS_3. However, we could find other probes that could distinguish the two states, see more discussions in [22]. This means the superposition among the fixed area states is important to understand the full properties of the geometry dual to $|\psi\rangle$. We can also consider the unnormalized n-copy state,

$$(\rho_A)^n = \sum_t p_t^n (\rho_{t,A})^n \simeq \sum_t \sqrt{\frac{b}{t}} e^{-n(b+t)+2\sqrt{bt}} \rho_{t,A}. \tag{24}$$

It can be shown $(\rho_A)^n \simeq e^{-(n-\frac{1}{n})b} \rho_{t=\frac{b}{n^2},A}$ by approximating the above summation by integral. This means the geometry of the n-copy state is approximated by the fixed area state with $t=\frac{b}{n^2}$, which is the spacetime inserting a cosmic brane with tension $\frac{n-1}{4Gn}$. It is a consistent check with Dong's formula of holographic Rényi entropy.

In general, I_t^ψ is proportional to $1/G$. In the semiclassical limit $G\to 0$, we expect the probability p_t^ψ has maximal value at $\bar{t}$, which is fixed by the equation $\partial_t I_t^\psi|_{t=\bar{t}}=0$. It is not easy to find $\bar{t}$ by solving (23) and (17). Motivated by the vacuum case, we can fix $\bar{t}$ by requiring the EE of $\rho_{t=\bar{t}}^\psi$ is equal to the EE of ρ_A^ψ. This leads to $n^*(\bar{t})=1$. Using (17) we find $\bar{t}=S(\rho_A^\psi)-b^\psi$. In [22] we show the one-point functions of local operators $\mathcal{O}$ in states ρ_A^ψ are equal to the ones in $\rho_{t=\bar{t}}^\psi$ in the semiclassical limit $G\to 0$. This leads to the result

$$\int_0^\infty dt p_t^\psi \to \int_0^\infty dt \delta(t-\bar{t}), \tag{25}$$

in the semiclassical limit $G\to 0$.

V. THE AREA OPERATOR

The fixed area states $|\Phi\rangle_t^\psi$ can be taken as the basis of a given pure geometric state $|\psi\rangle$. We may introduce an operator $\hat{A}^\psi$, which is expected to satisfy the following conditions:

(1) Positive semidefinite Hermitian and state-dependent operator [29].
(2) Fixed area states are its eigenstates.
(3) Located in subsystem A or $\bar{A}$.
(4) Its expectation value in geometric state $|\psi\rangle$ divided by $4G$ gives the RT formula [30] and its fluctuation in $|\psi\rangle$ is suppressed in the semiclassical limit $G\to 0$.

The area operator $\hat{A}^\psi$ can be constructed by spectrum decomposition. The modular Hamiltonian H_A^ψ has the spectrum decomposition as $H_A^\psi=\sum_t (t+b^\psi)P_t^\psi$, where $P_t^\psi \coloneqq \mathcal{P}^\psi(t)\rho_{t,A}^\psi$. According to the operator theory [31], we can define the new operators,

$$F(H_A^\psi) \coloneqq \sum_t F(t+b^\psi) P_t^\psi, \tag{26}$$

where $F(x)$ is the functions of x [32]. The operators satisfy $F(H_A^\psi)|\Phi\rangle_t^\psi = F(t+b^\psi)|\Phi\rangle_A^\psi$. The area operator can be defined as

$$\hat{A}^\psi = s(H_A^\psi - b^\psi) = \sum_t s(t) P_t^\psi, \tag{27}$$

where $s(t)\coloneqq \frac{6}{c} s_{n^*}$, s_{n^*} is given by (18). If we further use (19), the area operator is

$$\hat{A}^\psi = \sum_t \mathrm{Area}(\mathcal{B}_{n^*}) P_t^\psi, \tag{28}$$

where we used (19) and the Brown-Henneaux relation $c=\frac{3}{2G}$. The area operator has the similar structure as the one constructed in the holographic QEC code [10–12]. It is obvious that $\hat{A}^\psi|\Phi\rangle_t^\psi = \mathrm{Area}(\mathcal{B}_{n^*})|\Phi\rangle_t^\psi$, $\mathrm{Area}(\mathcal{B}_{n^*})$ is the area of the bulk RT surface for the geometry dual to the fixed area state $|\Phi\rangle_t^\psi$. The expectation value of $\hat{A}^\psi$ in $|\psi\rangle$ is

$$\langle \hat{A}^\psi \rangle_\psi = \int_0^\infty dt p_t^\psi \mathrm{Area}(\mathcal{B}_{n^*}) = \int_0^\infty dt e^{-I_t^\psi} \mathrm{Area}(\mathcal{B}_{n^*}).$$

According to (25), we have

$$\langle \hat{A}^\psi \rangle_\psi \to \int_0^\infty dt \delta(t-\bar{t}) \mathrm{Area}(\mathcal{B}_{n^*}) = \mathrm{Area}(\mathcal{B}_1), \tag{29}$$

in the semiclassical limit $G\to 0$. $\mathrm{Area}(\mathcal{B}_1)$ is just the area of the RT surface in the geometry dual to $|\psi\rangle$. The RT formula can be expressed by the area operator as

$$S(\rho_A^\psi) = \frac{\langle \hat{A}^\psi \rangle_\psi}{4G}. \tag{30}$$

By using the definition of the EE $S(\rho_A^\psi) = -tr(\rho_A^\psi \log \rho_A^\psi) = \langle \psi | H_A | \psi \rangle$, we have a nice result:

$$\langle \psi | H_A^\psi - \frac{\hat{A}^\psi}{4G} | \psi \rangle = \int_0^\infty dt e^{-I_t^\psi} \left(t + b^\psi - \frac{\mathrm{Area}(\mathcal{B}_n)}{4G} \right) \to \left(\bar{t} + b^\psi - \frac{\mathrm{Area}(\mathcal{B}_1)}{4G} \right) = 0, \tag{31}$$

in the limit $G\to 0$. This can seen as the bulk dual of the modular Hamiltonian to the leading order in the $1/G$ expansion [33].

To characterize the fluctuation of the area operator in the state $|\psi\rangle$, we can define the uncertainty of the area operator $\langle\Delta\hat{A}^{\psi}\rangle_{\psi} \coloneqq \sqrt{\langle(\hat{A}^{\psi})^2\rangle_{\psi} - \langle\hat{A}^{\psi}\rangle_{\psi}^2}$. By using (25), we can show $\langle\Delta\hat{A}^{\psi}\rangle_{\psi} = 0$ in the limit $G \to 0$. This is the expected feature for the geometric state, for which the quantum fluctuation should be suppressed. This property is similar to the constraints of geometric states [34], which are expressed as conditions for connected correlation functions of stress energy tensor. We show the results for the vacuum state in the Supplemental Material [25].

VI. DISCUSSION

The fixed area state plays a crucial role in the holographic QEC code. Our results serve as a bridge to construct the code by CFT states. With this one could have a more precise playground to better understand QEC interpretation of AdS/CFT correspondence, as well as the deep connection between quantum information theory and holography.

Though we have constructed the fixed area states for a given geometric state $|\psi\rangle$, there are still some unsolved problems on its relation to the holographic QEC code. From our constructions the fixed area states and the area operator are closely associated with the given geometric state $|\psi\rangle$, since our constructions are based on the Schmidt decomposition of $|\psi\rangle$. If one chooses another geometric state, say $|\psi'\rangle$, it seems the corresponding fixed area states and area operator are different from the case of $|\psi\rangle$. It is still unclear what the relation is between different geometric states. For the holographic QEC code we will expect the duality is not only useful for some special states. However, the vacuum state of quantum field theory (QFT) is cyclic [35], which means one could construct any states of QFT by only local operations on the vacuum. The cyclic property of vacuum may help us to understand the relations between different geometric states.

Our constructed area operator is expressed as a superposition of projectors in CFTs. It may be possible to find its bulk dual by reconstruction of the bulk operators in an entanglement wedge [9,36].

In this paper we only focus on the pure geometric state. Some important modifications are necessary to generalize the results to the mixed states. We only consider the leading order result in the expansion of gravitational coupling G. The RT formula would receive correction at higher orders in G [37]. That would be interesting to consider the higher order corrections, which is important to understand the relation between boundary and bulk modular Hamiltonian [33]. Finally, it would be interesting to generalize the quantum version of RT formula (30) with including the higher order G correction. It is probably related to the quantum extremal surface prescription [38], which plays an important role understanding the information paradox of black hole [39,40].

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