Letter

Natural F-theory constructions of standard model structure from E_7 flux breaking

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We describe a broad class of 4D F-theory models in which an E_7 gauge group is broken through fluxes to the standard model gauge group. These models are ubiquitous in the 4D F-theory landscape and can arise from flux breaking of most models with E_7 factors. While in many cases the E_7 breaking leads to exotic matter, there are large families of models in which the standard model gauge group and chiral matter representations are obtained through an intermediate SU(5) group. The number of generations of matter appearing in these models can easily be small. We demonstrate the possibility of getting three generations of chiral matter as the preferred matter content.

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I. INTRODUCTION

To describe the real world with string theory as a unified theory, it has been a long-standing and primary goal to find the structure of the standard model (SM) of particle physics in string theory. In particular, we would like to identify the SM as a *natural* solution to string theory. F theory [1–3], a strongly coupled version of type IIB string theory, is a particularly promising framework for this purpose as it gives a global description of a large connected class of theories (see [4] for a review). F theory gives 4D low-energy supergravity models when compactified on elliptically fibered Calabi-Yau (CY) fourfolds, which conveniently encode nonperturbative brane physics into geometrical language. Combined with flux data, the gauge symmetries and chiral matter content of any model can be easily determined. Moreover, F theory is dual to many other types of string compactifications (such as heterotic). We focus here on a novel class of F-theory models that naturally give the SM gauge group and chiral matter content.

There have been many attempts to build models with the SM gauge group $G_{\mathrm{SM}} = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)/\mathbb{Z}_6$ in F theory. Starting from [5–8], F-theory grand unified theories (GUTs) have been constructed, using gauge groups of SU(5) [9–13], SO(10) [14], etc. (see [15] for review). Recently, 10^{15} explicit solutions of directly tuned G_{SM} were found in the string landscape [16]. It has also been argued that the SM matter representations generically appear when G_{SM} is directly tuned [17,18]. These results signal that a considerable portion of the landscape may contain SM-like models.

These models cannot be the most generic or natural SMs in the landscape, however. All the preceding gauge groups arise from fine-tuning many moduli. In contrast, most F-theory compactification bases have strong curvature that enforces rigid (also known as "geometrically non-Higgsable" [19]) gauge symmetries, which are present throughout the whole branch of moduli space [20–22]. Furthermore, on many bases, these rigid gauge factors forbid tuning additional factors like G_{SM}.

A generic SM in the landscape can arise more naturally from the geometric rigid gauge symmetries than through tuning moduli. The rigid gauge groups containing G_{SM} are E_8, E_7, and E_6, but not most other traditional GUT groups [19]. [While the non-Abelian $\mathrm{SU}(3) \times \mathrm{SU}(2)$ of G_{SM} can arise as a rigid structure [23], including the Abelian factor is much more subtle [24,25]]. In 4D, it seems that, of these rigid GUT groups, E_8 appears the most frequently in the landscape, while E_7 and E_6 are also quite abundant [20–22]. While E_6 has been one of the traditional GUT groups, little attention has been paid to E_7 since it does not support chiral matter. We find here that, nevertheless, SM-like solutions can be realized in F theory by breaking rigid (or even nonrigid, tuned) E_7 models.

An economic way to tackle the above issues in F theory is to turn on G_4 flux inside a larger rigid group. This can break the larger group down to G_{SM}, while inducing chiral matter in the broken gauge group. In this paper, we describe F-theory models with rigid E_7 and G_4 flux that leads to SM gauge group and chiral matter spectrum with minimal

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supersymmetry. Compared with other SM-like constructions in the past, our models have the following novelties:

(i) These models can be built using generic bases. Little fine-tuning is required to get the desired gauge group and chiral matter spectrum. They are thus more natural in the landscape.

(ii) Gauge groups with no chiral matter like E_7 can also be used as GUT groups. Noncomplex representations in the unbroken gauge group can contribute to chiral matter in the broken group.

(iii) Chiral exotics are easily avoided, even when we start with large GUT groups, although the models we have identified without exotics involve an intermediate SU(5).

(iv) The resulting chiral multiplicities can easily be very small. It is natural, and sometimes preferred, to have three generations of chiral matter.

Although we focus here on E_7, there is a similar construction for the similarly abundant rigid E_6. The generalization is nontrivial since E_6 itself already supports chiral matter.

The rest of this paper is organized as follows. We first discuss some general features of 4D F-theory compactifications. We describe the geometry of the rigid E_7 model without assuming a specific base. We then review vertical and remainder fluxes and the flux constraints that lead to consistent solutions. We show how models with SM gauge group and chiral matter spectrum can arise from a combination of vertical breaking to SU(5) and hypercharge breaking from remainder flux. We give a simple explicit example of vertical flux breaking to SU(5) with three generations of chiral matter as the preferred matter content. We conclude with some remarks and future directions.

The arguments presented in this paper are minimal and aim at describing our new class of SM-like models succinctly. We leave the more general formalism and various technical subtleties to a longer follow-up [26].

II. E_7 GAUGE GROUPS IN F THEORY

A 4D F-theory model is defined by an elliptically fibered CY fourfold Y over a threefold base B; this can be considered as a nonperturbative type IIB string compactification on B. An E_7 gauge factor arises in the 4D supergravity theory when Y is described by a certain form of Weierstrass model [27–29]. Treating the elliptic curve as the CY hypersurface in $\mathbb{P}^{2,3,1}$ with homogeneous coordinates $[x:y:z]$, Y is given by the locus of

$$y^2 = x^3 + s^3 f_3 x z^4 + s^5 g_5 z^6, \tag{1}$$

where s, f_3, and g_5 are functions on B [more technically, sections of line bundles $\mathcal{O}(\Sigma), \mathcal{O}(-4K_B - 3\Sigma)$, and $\mathcal{O}(-6K_B - 5\Sigma)$, with K_B the canonical class of B] and the seven-brane locus Σ supporting the E_7 factor is given by $s = 0$. There is adjoint matter **133** on the bulk of Σ. There is also fundamental matter **56** localized on the curve $s = f_3 = 0$, or $C_{\mathbf{56}} = -\Sigma \cdot (4K_B + 3\Sigma)$ in terms of the intersection product, when the curve is nontrivial in homology.

E_7 gauge factors can either be tuned by hand in the Weierstrass model (1) or can be forced from the geometry of B. When a divisor (algebraic codimension-1 locus) Σ on B has a sufficiently negative normal bundle N_Σ, singularities of the elliptic fibration are forced to appear on Σ so that any elliptic fibration over B automatically takes the restricted form (1), and there is a rigid (geometrically non-Higgsable) E_7 gauge factor supported on Σ [19].

The conditions for a rigid E_7 factor are satisfied for a large set of typical F-theory bases. For 6D F-theory models, the toric bases have been completely classified [30] and 60% of the 61539 allowed F-theory bases have rigid E_7 factors. The total number of toric bases for 4D F-theory models is $\mathcal{O}(10^{3000})$ [21,22], which is too large for explicit analysis. A Monte Carlo estimate on a subset of these bases [those without E_8 factors or codimension-2 (4,6) singularities] gives roughly 18% with rigid E_7's [20], although the fraction for all bases may be smaller (the analogous subset for 6D bases contains 24483 bases of which 75% have rigid E_7 factors). Similar statistics may also apply to nontoric bases, but this question has not been addressed in the literature.

III. FLUXES AND GAUGE SYMMETRY BREAKING

The elliptic fibration Y with an E_7 factor over Σ is singular. We need to consider its resolution $\hat{Y}$ to study flux breaking using cohomology and intersection theory on $\hat{Y}$. The resolution results in exceptional divisors D_i, $1 \leq i \leq 7$, corresponding to the Dynkin nodes of E_7 (Fig. 1). The divisors D_I on $\hat{Y}$ are spanned by the zero section ($z = 0$) D_0, pullbacks of the base divisors $\pi^* D_\alpha$ (which we also call D_α depending on context), and exceptional divisors D_i [31,32]. Note that, while the choice of resolution is not unique, our analysis and results are manifestly resolution independent [33].

To break the E_7 factor, we first turn on *vertical* G_4 flux (see, e.g., [4]). This lives in the space of (2,2)-forms spanned by products of harmonic (1,1)-forms (which are Poincaré dual to divisors $[D_I]$),

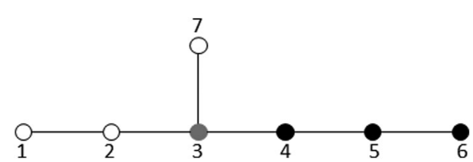

FIG. 1. The Dynkin diagram of E_7. The Dynkin node labeled i corresponds to the exceptional divisor D_i. The solid nodes are the ones we break to get G_{SM}. Node 3 (in gray) is broken by remainder flux, while the others are broken by vertical flux.

$$H^{2,2}_{\mathrm{vert}}(\hat{Y},\mathbb{C}) = \mathrm{span}(H^{1,1}(\hat{Y},\mathbb{C}) \wedge H^{1,1}(\hat{Y},\mathbb{C})). \quad (2)$$

We expand $G_4^{\mathrm{vert}} = \phi_{IJ}[D_I] \wedge [D_J]$ and work with flux parameters ϕ_{IJ}. We denote integrated flux as

$$\Theta_{IJ} = \int_{\hat{Y}} G_4 \wedge [D_I] \wedge [D_J] = \int_{\hat{Y}} G_4^{\mathrm{vert}} \wedge [D_I] \wedge [D_J]. \quad (3)$$

We then have the resolution-independent relation [33,34]

$$\Theta_{i\alpha} = -\Sigma \cdot D_\alpha \cdot D_\beta C^{ij} \phi_{j\beta}, \quad (4)$$

where C^{ij} is the Cartan matrix of E_7.

While E_7 can be broken directly to G_{SM} by vertical flux, this generally produces exotics. To obtain models with only chiral SM matter, we also turn on the following form of *remainder* flux [7,35,36],

$$G_4^{\mathrm{rem}} \in \mathrm{span}([D_i|_{C_{\mathrm{rem}}}]), \quad (5)$$

where C_{rem} is a curve on Σ but becomes homologically trivial on B. Some nontoric bases have rigid Σ with such curves, so that Σ supports both rigid E_7 and the remainder flux [26,37].

G_4 satisfies certain constraints. To preserve Poincaré symmetry, we need $\Theta_{0\alpha} = \Theta_{\alpha\beta} = 0$ [38]; this condition is unaffected by $\phi_{i\alpha} \neq 0$. The flux quantization condition is [39]

$$G_4 + \frac{1}{2} c_2(\hat{Y}) \in H^{2,2}(\hat{Y},\mathbb{R}) \cap H^4(\hat{Y},\mathbb{Z}), \quad (6)$$

where $c_2(\hat{Y})$ is the second Chern class of $\hat{Y}$. We will choose Y with even $c_2(\hat{Y})$ and consider the simple case where ϕ_{IJ} is integral. To preserve supersymmetry, we require primitivity [40,41],

$$J \wedge G_4 = 0, \quad (7)$$

where J is the Kähler form of $\hat{Y}$. This condition stabilizes some Kähler moduli when the gauge group is broken by vertical flux. Not all choices of gauge-breaking flux can stabilize J within the Kähler cone, however. Finally, we have the D3-tadpole condition [42]

$$\frac{\chi(\hat{Y})}{24} - \frac{1}{2}\int_{\hat{Y}} G_4 \wedge G_4 = N_{D3} \in \mathbb{Z}_{\geq 0}, \quad (8)$$

where $\chi(\hat{Y})$ is the Euler characteristic of $\hat{Y}$. In general, $h^{2,2} > 2\chi(\hat{Y})/3 \gg \chi(\hat{Y})/24$. If if we randomly turn on flux in the whole middle cohomology such that the tadpole constraint is satisfied, a generic flux configuration vanishes or has small magnitude in most of the $h^{2,2}$-independent directions.

We can now identify fluxes that break the model into G_{SM} with SM chiral matter. If $\Theta_{i\alpha} \neq 0$ for some roots i, the corresponding gauge bosons become massive; similarly, the appropriate linear combinations of Cartan gauge bosons get masses through the Stückelberg mechanism [43,44]. To get the SM gauge group and exact chiral matter spectrum, we proceed in two steps. First we break (uniquely up to E_7 automorphism) E_7 to SU(5) by turning on vertical flux with $\Theta_{i'\alpha} \neq 0$ for $i' = 4, 5, 6$ (see Fig. 1). This can be done by turning on appropriate $\phi_{i\alpha}$ using Eq. (4). In the second step, in parallel with earlier work on tuned SU(5) GUT models [5–13], we also turn on a remainder hypercharge flux

$$G_4^{\mathrm{rem}} = [D_Y|_{C_{\mathrm{rem}}}], \quad (9)$$

where $D_Y = 2D_1 + 4D_2 + 6D_3 + 3D_7$ is the exceptional divisor corresponding to the hypercharge generator. This breaks SU(5) to G_{SM}. Both **56** and **133** are then broken into SM matter,

$$(\mathbf{3},\mathbf{2})_{1/6}, \quad (\mathbf{3},\mathbf{1})_{2/3}, \quad (\mathbf{3},\mathbf{1})_{-1/3}, \quad (\mathbf{1},\mathbf{2})_{1/2}, \quad (\mathbf{1},\mathbf{1})_{1}, \quad (10)$$

along with an exotic $(\mathbf{3},\mathbf{2})_{-5/6}$ from **133**, which is nonchiral since it directly descends from the SU(5) adjoint.

The above vertical flux also induces chiral matter. To calculate the multiplicities, we first need to locate the matter surfaces (the fibration over matter curves). A weight β in a representation R of E_7 can be expressed in the basis of simple roots α_i,

$$\beta = -\sum_i b_i \alpha_i. \quad (11)$$

When localized on a matter curve C_R, we can decompose the matter surface $S(\beta)$ into [45]

$$S(\beta) = S_0(R) + \sum_i b_i D_i|_{C_R}, \quad (12)$$

where $S_0(R)$ only depends on R and does not contain any $D_i|_{C_R}$ components. Since E_7 itself does not support any chiral matter, $S_0(R)$ does not contribute to chiral multiplicities in the broken gauge group.

We now turn to the broken gauge group. Each set of values of $b_{i'}$ for fixed R give an irreducible representation R' in the broken gauge group, while the weights in R' are spanned by α_i for unbroken i. Different R and different sets of $b_{i'}$ can give the same R', however. The chiral multiplicity of R' is then (generalizing [12,34,46,47])

$$\chi_{R'} = \sum_R \sum_{b_{i'}(R)} \int_{S(\beta)} G_4, \quad (13)$$

which can be easily computed using group theory data and $\Theta_{i\alpha}$, thanks to the absence of $S_0(R)$. For $R = \mathbf{133}$, which lives on the bulk of the gauge divisor Σ instead of a matter curve, the chiral multiplicity can be computed by replacing C_R with the canonical class K_Σ [45], where $K_\Sigma = \Sigma \cdot (K_B + \Sigma)$ by adjunction. The above formula ensures that $\chi_{R'}$ computed from different weights in R' are all the same. Moreover, we have $\chi_{R'} = -\chi_{\bar{R}'}$, and anomaly cancellation is guaranteed [26]. In particular, the above flux constraints imply $\chi_{(\mathbf{3},\mathbf{2})_{-5/6}} = 0$ regardless of solutions.

IV. SMALL NUMBER OF GENERATIONS

This class of SM-like models, combining vertical and remainder fluxes, can be realized on a large class of bases with rigid (or tuned) E_7 but cannot be constructed completely in simple toric geometries. Here, for simplicity, to illustrate the multiplicity of generations, we focus on vertical flux breaking to SU(5) and give an oversimplified example, using the Hirzebruch surface $\mathbb{F}_1$ as the gauge divisor Σ.[1] When further breaking to G_{SM} through remainder flux is possible on more complicated surfaces, this further breaking does not affect multiplicities.

$\mathbb{F}_1$ is a $\mathbb{P}^1$ bundle over another $\mathbb{P}^1$. We denote S as the section $\mathbb{P}^1$ and F as the fiber $\mathbb{P}^1$. Then the intersection numbers are $F^2 = 0$, $F \cdot S = 1$, and $S^2 = -1$. Its anticanonical class is $-K_\Sigma = 2S + 3F$. Now embed $\mathbb{F}_1$ into B with normal bundle $N_\Sigma = -aS - bF$. Let F_S and F_F be divisors with $\Sigma \cdot F_S$ and $\Sigma \cdot F_F$ being pushforwards of S and F into B, respectively. Without remainder flux, we can assume $\Sigma \cdot F_S$ and $\Sigma \cdot F_F$ are independent. By choosing $N_\Sigma = -8S - 7F$,

$$F_k = -4K_\Sigma + (4-k)N_\Sigma,$$
$$G_l = -6K_\Sigma + (6-l)N_\Sigma \tag{14}$$

are both effective only when $k \geq 3$ and $l \geq 5$, so we have a rigid E_7 supported on Σ [19]. The nonzero intersection numbers are then $\Sigma \cdot F_S \cdot F_F = 1$, $\Sigma^2 \cdot F_F = -8$, $\Sigma \cdot F_S^2 = -1$, $\Sigma^2 \cdot F_S = 1$, and $\Sigma^3 = 48$.

We claim that all the above constraints on vertical flux can be solved inside the Kähler cone by turning on nonzero but sufficiently small integer ϕ_{iF_S} and ϕ_{iF_F} with opposite signs. We require the ratio ϕ_{iF_S}/ϕ_{iF_F} to be the same for all i. To break the gauge group, we turn on integer ϕ_{5F_S}, ϕ_{6F_S} freely and

$$(\phi_{1F_S}, \phi_{2F_S}, \phi_{3F_S}, \phi_{4F_S}, \phi_{7F_S}) = (2, 4, 6, 5, 3)n_S, \tag{15}$$

and similar for ϕ_{iF_F}, where n_S and n_F are integers with opposite signs. Equation (13) then gives a simple formula for the number of generations of SU(5) GUT matter,

$$\chi_{\mathbf{10}} = -\chi_{\mathbf{5}} = -7n_S - 4n_F. \tag{16}$$

This is a linear Diophantine equation and the number of generations can be any sufficiently small integer. As explained above, it is natural to consider small ϕ_{iF_S} and ϕ_{iF_F}. The minimal flux configuration has $n_S = -1$ and $n_F = 1$, hence $\chi = 3$ appears to be preferred. This is an example of an F-theory model with exactly three generations of chiral matter, with minimal fine-tuning.

The above is the most general vertical flux we can turn on given the flux constraints and conditions on $\Theta_{i\alpha}$. All other $\phi_{i\alpha}$ are equivalent to a combination of ϕ_{iF_S} and ϕ_{iF_F} by homology relations.

The above construction can be easily generalized to incorporate hypercharge flux by using more complicated Σ's on nontoric bases [36,37] (with different multiplicities in each case). We provide such explicit constructions in [26].

V. CONCLUSION AND REMARKS

Within the framework of F-theory compactifications, we have described a large class of SM-like models with the right gauge group and chiral matter spectrum. These can originate from rigid E_7 gauge symmetry, which is ubiquitous in the landscape. String theory methods allow us to go beyond the limit of field theories and use E_7 as a GUT group. Remarkably, a subset of these models prefer three generations of SM chiral matter. Although we lack an exact quantification, we believe that these models are more generic than tuned SM-like models in the landscape.

Some remarks and future directions are as follows:

(i) Although we only give an oversimplified example of the models, the same construction giving SM gauge group and chiral matter can be done on most bases containing rigid (or tunable) E_7 factors. In general, for other local geometries supporting E_7, the number of generations may be different. In some cases, the chiral multiplicity is a multiple of integers other than 3 and $\chi = 3$ is forbidden. In most cases, $\chi = 3$ is still allowed and generically natural because of small $\phi_{i\alpha}$ and the Diophantine structure, but this may not be the most preferred chiral matter content. In special cases, χ is a multiple of 3 and $\chi = 3$ is both the minimal and preferred matter content.

(ii) A generic base has many other rigid gauge factors apart from E_7's. We can apply our SM-like construction on one of the E_7's, while other gauge factors can serve as hidden sectors such as dark matter [19,48].

(iii) One interesting feature of this construction of E_7 breaking is that it relies intrinsically on nonperturbative physics of F theory and does not have any immediately obvious description in the low-energy field theory. It would be interesting to understand the

[1] This oversimplification also leads to exotic U(1) gauge factors along with the SU(5) [26]. Here, as a mere demonstration, we focus on the SU(5) representations and ignore the U(1) charges when calculating chiral indices.

structure of these models better from the low-energy and/or dual heterotic pictures.

(iv) We have chosen a subset of embeddings of G_{SM} into E_7 which lead to SM chiral matter. The root embedding of $\mathrm{SU}(3) \times \mathrm{SU}(2)$ is unique up to automorphisms; however, there are other embeddings of the U(1) factor that give various kinds of exotic chiral matter.

(v) It is clear that a similar construction as above also works for E_6. To get SM gauge group and chiral matter, we can use the same breaking pattern as in Fig. 1 but without the rightmost node. Calculating chiral multiplicities becomes more nontrivial, however, since E_6 itself supports chiral matter. There are more flux parameters ϕ_{ij} to turn on, and the matter surface $S_0(R)$ also contributes nontrivially. This generalization is done in [26].

(vi) We have been working with E_7, while the most generic rigid gauge group supporting G_{SM} is likely E_8. A rigid E_8 generically contains codimension-2 (4,6) singularities, however, which signal the presence of strongly coupled superconformal sectors [49,50] and cannot be analyzed using our formalism. If we apply the same formalism to rigid E_8 without this kind of singularity, we can break it into G_{SM}, but surprisingly no chiral matter is induced. In particular, the F-theory geometry with the most flux vacua [51] contains E_8 instead of E_7 and does not support our formalism. Meanwhile, there have been similar attempts working with rigid E_8 using other formalisms like E-string theory [52].

(vii) We have been focusing on the chiral matter spectrum, while knowledge of the vectorlike spectrum is required for analyzing the Higgs sector and avoiding the exotic $(\mathbf{3},\mathbf{2})_{-5/6}$. This requires explicit cohomology data from topologically nontrivial 3-form potential backgrounds [53–55]. Such data are much harder to analyze than G_4 flux and are beyond the scope of this paper.

(viii) Comparing with other tuned SM-like or GUT models, the origin of Yukawa couplings in our models is less clear due to several reasons. First, matter on both the bulk of Σ and curve $C_{\mathbf{56}}$ are involved. It was argued that the Yukawa couplings between three bulk fields on Σ always vanish if $-K_\Sigma$ is effective [6]. In contrast, we see no obstruction to having SM Yukawa couplings between the Higgs on Σ and chiral matter on $C_{\mathbf{56}}$ [6], but a rigorous construction is still lacking. Besides, the Yukawa couplings between matter localized on curves are usually extracted from codimension-3 singularities on the base. Instead, the codimension-3 singularity on $C_{\mathbf{56}}$ has degree (4,6), which goes beyond the Kodaira classification and may not be simply interpreted as Yukawa couplings. These (4,6) points are associated with nonflat fibers, which possibly encode extra flux backgrounds and strongly coupled (chiral) degrees of freedom [33,56,57]. These will be studied in a future publication [58].

We hope to address some of these issues in future studies, and some of them will be explored in the follow-up paper [26]. With this large class of SM-like constructions, we hope to shed some light on where our Universe sits in the string theory landscape and whether it is a natural solution in the landscape.

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