Letter

Is there a relativistic Gorini-Kossakowski-Lindblad-Sudarshan master equation?

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The celebrated GKLS master equation, widely called just the Lindblad equation, is the universal dynamical equation of nonrelativistic open quantum systems in their Markovian approximation. It is not necessary and perhaps impossible that GKLS equations possess sensible relativistic forms. In 2017, in a lucid talk on black hole information loss paradox, David Poulin argued for a Lorentz invariant GKLS master equation proposed by Alicki, Fannes, and Verbeure in 1986. The equation is really puzzling. A closer look uncovers a smartly hidden defect that leaves us without Lorentz invariant Markovian master equations. They, in view of the present author, should not exist.

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I. INTRODUCTION

Dissipative relativistic phenomena are real. The simplest example is pions. The pionic quantum state ρ decays toward the pionic vacuum state. The exact dynamics is the reduced dynamics of a unitary quantum field theory (QFT, Standard Model), and as such, it is non-Markovian. The time derivative $d\rho(t)/dt$ depends on the history of ρ before t; the master equation governing the state $\rho(t)$ is called non-Markovian. Its exact form would be cumbersome for pions. (A tractable Lorentz invariant non-Markovian master equation is available for the fermionic subsystem in quantum electrodynamics [1].) Apart from extreme short time scales, the pions decay exponentially, and hence, their effective (not the exact) dynamics is Markovian; i.e., $d\rho(t)/dt$ depends on $\rho(t)$ only. The Lorentz invariant field-theoretic formulation of this effective Markovian dissipative dynamics is missing, and it is not known if it exists at all. Consistency of Lorentz invariant field theory with Markovian dissipation is a general mathematical issue, and a decaying massive scalar field is the simplest case to study [2].

Long ago and far from the context of QFT, a very powerful mathematical theorem [3,4] proved (see also [5,6]) that nonrelativistic Markovian evolution of quantum states can always be expressed by a very specific structure of a number of operators A_n:

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_n \left(A_n\rho A_n^\dagger - \frac{1}{2}\{A_n^\dagger A_n,\rho\}\right). \quad (1)$$

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Popularity of this GKLS master equation, many times referred just as Lindblad master equation after one of the inventors, has been and is remarkably extending to many fields in nonrelativistic quantum physics. It is understood as a Markovian effective equation, not valid at too short of timescales, of open quantum systems [7] whose exact dynamics is non-Markovian.

Still, one may ask if the GKLS dynamics could be extended for relativistic systems or it could not. The old work [2], in terms of rigorous mathematics, seemed to give an affirmative answer, proposing a field-theoretic GKLS equation of decaying scalar particles. An unexpected push came from David Poulin arguing for this relativistic GKLS equation intuitively in his 2017 talk [8]. His reasoning was impressive and has been shaking my firm judgement that relativistic GKLS equations are nonexistent.

II. POULIN'S OBSERVATION

Consider a quantized free scalar field φ of mass m and its canonical momentum π. The Hamiltonian H reads

$$\begin{aligned} H &= \frac{1}{2}\int (\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2)d\mathbf{x} \\ &= \int \omega_\mathbf{k} a_\mathbf{k}^\dagger a_\mathbf{k} d\mathbf{k}. \end{aligned} \quad (2)$$

The state ρ evolves by the Schrödinger (–von-Neumann) equation of motion

$$\frac{d\rho}{dt} = -i[H,\rho]. \quad (3)$$

Lorentz invariance relies simply on the fact that $H = P_0$, where

$$P_\mu = \int k_\mu a_{\mathbf{k}}^\dagger a_{\mathbf{k}} d\mathbf{k} \tag{4}$$

is a four-vector (of total energy momentum).

One can modify the free unitary dynamics by a nonunitary (e.g., dissipative) mechanism represented by a superoperator $\mathcal{D}$:

$$\frac{d\rho}{dt} = -i[H,\rho] + \mathcal{D}\rho, \tag{5}$$

where the dissipator $\mathcal{D}$ has the GKLS structure (1). Poulin's proposal, coinciding with Alicki, Fannes and Verbeure's in [2], is this:

$$\mathcal{D}\rho = \gamma \int (2\pi^- \rho \pi^+ - \{\pi^+\pi^-, \rho\}) d\mathbf{x}$$
$$= \gamma \int \omega_{\mathbf{k}} \left(a_{\mathbf{k}} \rho a_{\mathbf{k}}^\dagger - \frac{1}{2}\{a_{\mathbf{k}}^\dagger a_{\mathbf{k}}, \rho\} \right) d\mathbf{k}, \tag{6}$$

where $\pi^\pm$ are the positive and negative frequency parts of π. The argument of Lorentz invariance is the same as above. One can write $\mathcal{D}$ in the form

$$\mathcal{D} = \gamma \int \omega_{\mathbf{k}} \left(a_{\mathbf{k}} \otimes a_{\mathbf{k}}^\dagger - \frac{1}{2}(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \otimes I + I \otimes a_{\mathbf{k}}^\dagger a_{\mathbf{k}}) \right) d\mathbf{k}, \tag{7}$$

and argue that $\mathcal{D} = \mathcal{P}_0$, where

$$\mathcal{P}_\mu = g \int k_\mu \left(a_{\mathbf{k}} \otimes a_{\mathbf{k}}^\dagger - \frac{1}{2}(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \otimes I + I \otimes a_{\mathbf{k}}^\dagger a_{\mathbf{k}}) \right) d\mathbf{k}, \tag{8}$$

is a four-vector.

With the new dissipative mechanism, the bosons are decaying, and for a long time, the system's state becomes the vacuum. The stable equilibrium vacuum state is supposed to be approached along a relativistic invariant Markovian evolution by construction. Poulin notes that the dynamics, unlike in standard QFT, is nonlocal on range $1/m$. The resulting acausality is of short range, provided that m is large. This can, in certain theories, be a bearable anomaly.

However, the forthcoming analysis uncovers that the Eq. (5) is not Lorentz invariant. The next section formulates the condition of boost invariance in Markovian dissipative quantum fields, like the proposed one. A lapse of Poulin's argument is detected.

III. CONDITION OF BOOST INVARIANCE

Let us recapitulate the condition of invariance under Lorentz boosts in standard QFT, with interaction V. Let us evolve the system dynamically for a short time δt and perform a boost with small velocity $\delta\mathbf{v}$. Or, apply the boost first and let the system evolve after it. If the dynamics is Lorentz invariant, then the resulting two states must coincide apart from the spatial shift $\delta\mathbf{v}\delta t$ in the second state. The mathematical condition of this invariance (i.e., interchangeability of dynamical evolution and boost) is the following:

$$[\mathbf{K}, H + V] = i\mathbf{P}, \tag{9}$$

where $\mathbf{K}$ is the generator of boosts, and $\mathbf{P}$ is the spatial part of P_μ in (4). The closed expression of $\mathbf{K}$ exists [9], but in practice, we use the boost action on the operator basis $a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger$. The small boost acts like this:

$$a_{\mathbf{k}} + i\delta\mathbf{v}[\mathbf{K}, \sqrt{\omega_{\mathbf{k}}} a_{\mathbf{k}}] = \sqrt{\omega_{\mathbf{k}'}} a_{\mathbf{k}'}, \tag{10}$$

and similarly for $a_{\mathbf{k}}^\dagger$, where $\mathbf{k}' = \mathbf{k} - \delta\mathbf{v}\omega_{\mathbf{k}}$ is the boosted $\mathbf{k}$. Hence, the boost of any operator is equivalent with the boost of the (covariant) creation or annihilation operators. We have $[\mathbf{K}, H] = i\mathbf{P}$ and $[\mathbf{K}, V] = 0$ for nonderivative interaction; the condition (9) is satisfied.

In the proposed Eq. (5), the Hamiltonian interaction term $-i[V,\rho]$ is replaced by the dissipative term $\mathcal{D}\rho$. The second term $[\mathbf{K}, V]$ of the condition (9) becomes nonvanishing:

$$(\mathbf{K} \otimes I)\mathcal{D} - \mathcal{D}(I \otimes \mathbf{K}) = i\boldsymbol{\mathcal{P}}, \tag{11}$$

where $\boldsymbol{\mathcal{P}}$ is the spatial part of $\mathcal{P}_\mu$ in (8). The condition (9) of boost invariance becomes violated.

Now we put the argument of Sec. II under scrutiny. The proposal assumes that the boost generator is the standard Hermitian generator $\mathbf{K}$, acting as in Eq. (10). This cannot be true. Since the time evolution is not unitary, the boosts cannot be unitary either (Fig. 1).

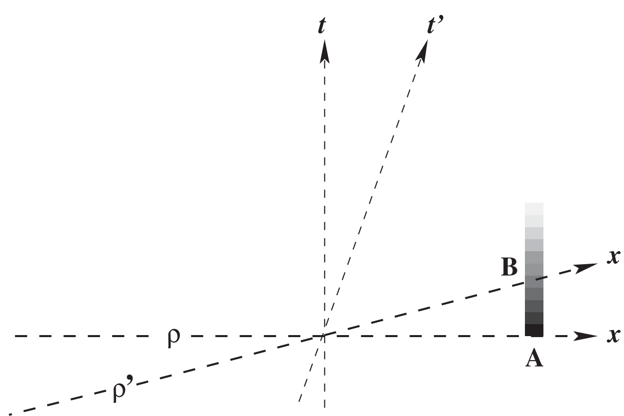

FIG. 1. In frame (t, x), a single-boson nonrelativistic localized state is prepared at location A $(t = 0, x > 0)$ at rest. For $t > 0$, the boson is starting to decay. The initial local system at A reaches B in an irreversible process. If the initial state ρ defined at $t = 0$ were unitary equivalent with ρ' defined at $t' = 0$—where (t', x') is a different Lorentz frame—then the evolution of our local boson should be reversible, which is not the case.

The boost generator might become a superoperator $\mathcal{K}$ to satisfy the condition of invariance, i.e., the interchangeability between dynamical evolution and boost. The superoperator counterpart of the mathematical condition (9) of boost invariance is straightforward. However, in the next section, we show that it is useless to search for the covariant boost. The Eq. (5) cannot be Lorentz invariant.

IV. DISPROOF OF LORENTZ INVARIANCE

The dissipative term does not prevent us from using an interaction picture. We use an unconventional interaction picture where H evolves the state and $\mathcal{D}^\dagger$ evolves the field:

$$\frac{d\rho}{dt} = -i[H,\rho], \tag{12}$$

$$\partial_t \varphi(t,\mathbf{x}) = \mathcal{D}^\dagger \varphi(t,\mathbf{x}). \tag{13}$$

The generator H of the unitary evolution and the generator $\mathcal{D}$ of the dissipative evolution are commuting; hence, the constant H governs the state evolution. Now, the evolution of the state is standard Lorentz invariant. What about the evolution of the field? The initial condition reads:

$$\varphi(0,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2\omega_\mathbf{k}}} a_\mathbf{k} e^{i\mathbf{k}\mathbf{x}} d\mathbf{k} + \text{H.c.} \tag{14}$$

From the relationships $\mathcal{D}^\dagger a = -\gamma a$ and $\mathcal{D}^\dagger a^\dagger = -\gamma a^\dagger$, the solution follows easily:

$$\varphi(t,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2\omega_\mathbf{k}}} a_\mathbf{k} e^{i\mathbf{k}\mathbf{x}-\gamma\omega_\mathbf{k} t} d\mathbf{k} + \text{H.c.} \tag{15}$$

One would prove or disprove the boost invariance of the solutions, but we have a simpler tool, the field equation:

$$\partial_t^2 \varphi(t,\mathbf{x}) = \gamma^2(m^2-\nabla^2)\varphi(t,\mathbf{x}), \tag{16}$$

which is manifest noninvariant. This is not surprising since Sec. III already found a flaw in the argument supporting the Lorentz invariance of the proposal in Sec. II.

V. DIGRESSION: CLASSICAL AND QUANTUM WHITE NOISE

A naive Lorentz invariant field theory appeared in [10] first, where

$$\mathcal{D}\rho = g^2 \int \left(\varphi\rho\varphi - \frac{1}{2}\{\varphi^2,\rho\}\right) d\mathbf{x}. \tag{17}$$

This is a Lindblad form (1), and the corresponding dynamics is Lorentz invariant indeed. It can be derived from the coupling $g\phi\xi$ to an external Lorentz invariant classical white noise field of ultralocal correlation

$$\langle \xi(x)\xi(y)\rangle = g\delta(x-y), \tag{18}$$

after taking the average over this random field. The features of $\mathcal{D}$ are unphysical; it is creating bosons at an infinite rate, which is a trivial consequence of the white noise. Unfortunately, $\xi(x)$ is the only possible Lorentz invariant white noise, or, in other words, the only Lorentz invariant classical Markovian process on the continuum.

We can construct a Lorentz invariant quantum white noise $b(x)$ as well. It is a trivial relativistic generalization of quantum white noise $b(t)$ introduced for damped quantum systems [11] and extensively used, e.g., in quantum optics [12]. The canonical commutator is ultralocal bosonic:

$$[b(x), b^\dagger(y)] = \delta(x-y). \tag{19}$$

We use $b(x)$ as an auxiliary field to construct a unitary QFT. Poulin's impressive proposal corresponds to the coupling

$$\sqrt{2\gamma}(\pi^+ b + \pi^- b^\dagger). \tag{20}$$

Assuming that the initial state of the b-field is the vacuum state, we evolve the composite state $\rho\otimes|\text{vac}\rangle\langle\text{vac}|$ unitarily and trace out the auxiliary field. We mentioned in Sec. I that in standard QFTs, the reduced dynamics are non-Markovian, but the auxiliary b field is exceptional, it is ultralocal, nonpropagating, etc., so we get a Markovian evolution for ρ of the φ field. This is exactly the dissipative dynamics (5) in interaction picture:

$$\frac{d\rho}{dt} = \gamma \int (2\pi^-\rho\pi^+ - \{\pi^+\pi^-,\rho\}) d\mathbf{x}, \tag{21}$$

which is not Lorentz invariant according to Secs. III and IV.

How is it possible? The coupling was Lorentz invariant and the reduction is Lorentz invariant, so then where has Lorentz invariance been lost? Sure, Lorentz invariance of the reduced dynamics is undermined by the nonlocality of $\pi^\pm$ in the otherwise Lorentz invariant coupling (20). Weinberg [9] warns us about the importance of locality condition. *It is this condition that makes the combination of Lorentz invariance and quantum mechanics so restrictive.*

VI. CLOSING REMARKS

For a long time, there has been one only context with the interest and unfulfilled desire for relativistic GKLS equations. The assumption of a tiny fundamental and spontaneous decoherence in massive degrees of freedom was realized by the nonrelativistic GKLS equations [13–15], but the relativistic extensions are missing up till now. Efforts [16–21], mostly related to the structure (17), are always leading to unphysical features, like, e.g., the mentioned vacuum instability or just presence of tachyons.

Poulin's motivation was not different in that he assumed a tiny fundamental dissipative mechanism. He did it directly in the relativistic realm. The proposed GKLS equation is smartly hiding its defect. To point it out took quite a time for the present author initially unaware of the work [2]. Alicki, Fannes, and Verbeure require explicitly that the generator of the GKLS dynamics be covariant under the unitary representation of the Lorentz(-Poincaré) group; this is certainly consistent mathematically, but dissipative physical systems do not transform unitarily under boosts.

Finally, my general arguments and conjecture are the following. Any Markovian irreversible field process—whether quantized or classical—is underlain by instantaneous jumps and they do not exist relativistically. Hence, Lorentz invariant master (kinetik) equations do not exist for such processes.

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[1] L. Diósi, Found. Phys. **20**, 63 (1990).

[2] E. Alicki, M. Fannes, and A. Verbeure, J. Phys. A **19**, 919 (1986).

[3] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).

[4] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. (N.Y.) **17**, 821 (1976).

[5] V. A. Franke, Theor. Mat. Fiz. **27**, 406 (1976).

[6] Proofs [3,4] were based on the important restrictive condition of complete-positivity. In [5] this condition is implicite and its role remains somehow opaque.

[7] H. P. Beuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).

[8] D. Poulin, Information loss in quantum field theories, https://online.kitp.ucsb.edu//online/qinfo_c17/poulin/.

[9] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, England, 1995), Vol. 1, p. 145.

[10] A. Barchielli, L. Lanz, and G. M. Prosperi, in *Proceedings of the International Symposium on the Foundations of Quantum Mechanics*, edited by S. Kamefuchi *et al.* (Physical Society of Japan, Tokyo, 1986).

[11] C. W. Gardiner and M. J. Collett, Phys. Rev. A **31**, 3761 (1985).

[12] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994).

[13] L. Diósi, Phys. Lett. **120A**, 377 (1987).

[14] G. C. Ghirardi, P. Pearle, and A. Rimini, Phys. Rev. A **42**, 78 (1990).

[15] G.C. Ghirardi and A. Bassi, Phys. Rep. **379**, 257 (2003).

[16] L. Diósi, Phys. Rev. A **42**, 5086 (1990).

[17] P. Pearle, in *Sixty-Two Years of Uncertainty*, edited by A. Miller (Plenum, New York, 1990), p. 193.

[18] P. Pearle, Phys. Rev. A **59**, 80 (1999).

[19] M. A. Kurkov and V. A. Franke, Found. Phys. **41**, 820 (2011).

[20] D. Bedingham and P. Pearle, Phys. Rev. Research **1**, 033040 (2019).

[21] C. Jones, G. Gasbarri, and A. Bassi, J. Phys. A **54**, 295306 (2021).