Letter

Spelling out leptonic *CP* violation in the language of invariant theory

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(Received 8 March 2022; accepted 24 August 2022; published 9 September 2022)

In terms of flavor invariants, we establish the intimate connection between leptonic *CP* violation in the canonical seesaw model for neutrino masses and that in the seesaw effective field theory (SEFT). For the first time, we calculate the Hilbert series and explicitly construct the primary flavor invariants in the SEFT by considering both the dimension-five Weinberg operator $\mathcal{O}_5^{\alpha\beta} = \overline{\ell_{\alpha \rm L}}\,\tilde{H}\,\tilde{H}^{\rm T}\ell^{\rm C}_{\beta \rm L}$ and the dimension-six operator $\mathcal{O}_6^{\alpha\beta} = (\overline{\ell_{\alpha \rm L}}\,\tilde{H}){\rm i}\partial\!\!\!/(\tilde{H}^\dagger \ell_{\beta \rm L})$ at the tree-level matching. The inclusion of only the Wilson coefficients $C_5^{\alpha\beta}$ and $C_6^{\alpha\beta}$ already enables the SEFT to incorporate all physical information about the full seesaw model. Moreover, the minimal sufficient and necessary conditions for *CP* conservation both in the SEFT and in the full theory are clarified, and the matching between the flavor invariants in both theories is accomplished. Through the matching of flavor invariants, the *CP* asymmetries necessary for successful leptogenesis are directly linked to those in neutrino-neutrino and neutrino-antineutrino oscillations at low energies. Surprisingly, it is revealed that the precise measurements of $C_5^{\alpha\beta}$ and $C_6^{\alpha\beta}$ in low-energy experiments are powerful enough to probe the full seesaw model, including *CP* violation for cosmological matter-antimatter asymmetry.

DOI: 10.1103/PhysRevD.106.L051701

I. INTRODUCTION

The violation of charge-parity (*CP*) symmetry should have played a crucially important role in the dynamical generation of matter-antimatter asymmetry in our Universe [1,2]. While *CP* violation has been discovered in the quark sector [3–5], a number of ongoing and forthcoming long-baseline accelerator neutrino oscillation experiments [6–9] aim to probe *CP* violation in the leptonic sector [10].

In the standard model (SM), it is well known that the *CP*-violating phase in the Cabibbo-Kobayashi-Maskawa (CKM) matrix [11] appearing in the charged-current interaction of quarks accounts for the phenomena of *CP* violation observed in the meson systems. Though the standard parametrization of the CKM matrix [12] in terms of three flavor mixing angles and one Dirac-type *CP*-violating phase is given in the physical basis and thus widely adopted in flavor physics, the observables should be independent of both flavor bases and the specific parametrization of the flavor mixing matrix. In a series of papers [13–15], Jarlskog was the first to construct a basis- and parametrization-independent quantity to characterize *CP* violation [14], namely,

$$\mathrm{Det}\{[H_{\rm u}, H_{\rm d}]\} = 2{\rm i}\Delta_{uc}\Delta_{ct}\Delta_{tu}\Delta_{ds}\Delta_{sb}\Delta_{bd}\mathcal{J}, \qquad (1)$$

where $H_{\rm u} \equiv M_{\rm u}M_{\rm u}^\dagger$ and $H_{\rm d} \equiv M_{\rm d}M_{\rm d}^\dagger$ with $M_{\rm u}$ and $M_{\rm d}$ being the up- and down-type quark mass matrices, respectively. In Eq. (1), $\Delta_{qq'} \equiv m_q^2 - m_{q'}^2$ denotes the quark mass-squared difference, and $\mathcal{J}$ is the Jarlskog rephasing invariant composed of the CKM matrix elements [13,16]. Since $H_{\rm u}$ and $H_{\rm d}$ transform adjointly under the unitary transformations in the quark flavor basis, the determinant of their commutator is a flavor invariant. The vanishing of such a flavor invariant serves as the necessary and sufficient condition for *CP* conservation in the SM.

The construction of flavor invariants that are odd under the *CP* transformation has been generalized to an arbitrary number of generations of fermions in the SM in Ref. [17] and to the leptonic sector with massive Majorana neutrinos [18–20]. The minimal number of sufficient and necessary conditions for *CP* conservation in the presence of massive Majorana neutrinos and lepton mass degeneracy have been studied in Refs. [21–23]. Only in Ref. [24] was it first pointed out that the Hilbert series (HS) in the invariant theory is a powerful mathematical tool for a systematic study of flavor invariants and their relationships with

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physical parameters in flavor physics. Moreover, the plethystic program [25] has been implemented in Ref. [26] to calculate the HS for the ring of invariants through the Molien-Weyl (MW) formula [27,28]. It has been clarified in Ref. [24] that the number of primary invariants is equal to that of independent physical parameters in the theory, whereas all the flavor invariants can be expressed as the polynomials of the basic invariants in the generating set.

In the type-I seesaw model [29–33] and its low-energy effective theory with only the dimension-five Weinberg operator [34], the basic flavor invariants have been partly investigated [24] and their renormalization-group equations are calculated in Ref. [35]. In the minimal seesaw model with two right-handed (RH) neutrinos, all the basic flavor invariants have been explicitly constructed and connected to the flavor invariants in the effective theory by a proper matching procedure [36]. Recently, the CP-odd flavor invariants have been examined in Ref. [37] in the standard model effective field theory with nonrenormalizable operators of mass dimension up to six [38–40]. However, the flavor mixing and CP violation in the leptonic sector have been switched off in Ref. [37], as the Weinberg operator is ignored and thus no lepton flavor mixing occurs.

In this Letter, we explore the flavor invariants in the type-I seesaw model and those in the seesaw effective field theory (SEFT) at the tree-level matching, where both the Weinberg operator $\mathcal{O}_5^{\alpha\beta} = \overline{\ell_{\alpha \rm L}}\,\tilde{H}\,\tilde{H}^{\rm T}\ell^{\rm C}_{\beta \rm L}$ and the dimension-six operator $\mathcal{O}_6^{\alpha\beta} = (\overline{\ell_{\alpha \rm L}}\,\tilde{H}){\rm i}\partial\!\!\!/(\tilde{H}^\dagger \ell_{\beta \rm L})$ [41,42] are present (here, $\tilde{H} \equiv {\rm i}\sigma_2 H^*$ denotes the Higgs doublet). In the language of invariant theory, we are able to draw a number of interesting conclusions. First, the inclusion of only two Wilson coefficients $C_5^{\alpha\beta}$ and $C_6^{\alpha\beta}$ in the SEFT reproduces the same number of physical parameters as in the full seesaw model. Second, in connection with the previous observation, we demonstrate that the absence of CP violation in the SEFT guarantees CP conservation in the full theory and vice versa. The minimal sufficient and necessary conditions for CP conservation are given. In addition, we show that all physical parameters in the SEFT can be extracted using primary flavor invariants, so any low-energy physical observables can be expressed as functions of flavor invariants. Finally, the matching between the flavor invariants in the effective and full theories is accomplished. As a consequence, the CP asymmetries necessary for a successful leptogenesis for cosmological matter-antimatter asymmetry [43] can be directly related to those in neutrino-neutrino and neutrino-antineutrino oscillations at low energies.

II. FRAMEWORK

To accommodate nonzero neutrino masses, we work in the type-I seesaw model with n RH neutrinos $N_{\rm R}$. Apart from the SM Lagrangian, the RH neutrino part of the full theory is given by

$$\mathcal{L} = \overline{N_{\rm R}}{\rm i}\partial\!\!\!/ N_{\rm R} - \left[\overline{\ell_{\rm L}}Y_\nu \tilde{H} N_{\rm R} + \frac{1}{2}\overline{N^{\rm C}_{\rm R}}M_{\rm R}N_{\rm R} + {\rm H.c.}\right], \quad (2)$$

where $\ell_{\rm L}$ stands for the left-handed lepton doublet. In Eq. (2), Y_ν denotes the Dirac neutrino Yukawa coupling matrix, and $M_{\rm R}$ is the Majorana mass matrix of RH neutrinos.

For the mass scale $\Lambda = \mathcal{O}(M_{\rm R})$ of RH neutrinos much higher than the electroweak scale $v \approx 246$ GeV, the low-energy phenomena are described by the SEFT with

$$\mathcal{L}_{\rm SEFT} = \mathcal{L}_{\rm SM} - \left[\frac{C_5}{2\Lambda}\mathcal{O}_5 + {\rm H.c.}\right] + \frac{C_6}{\Lambda^2}\mathcal{O}_6, \quad (3)$$

where $\mathcal{L}_{\rm SM}$ stands for the SM Lagrangian, and Λ is the cutoff scale. At the tree-level matching, the relevant Wilson coefficients can be identified as

$$C_5 = -Y_\nu Y_{\rm R}^{-1} Y_\nu^{\rm T}, \qquad C_6 = Y_\nu (Y_{\rm R}^\dagger Y_{\rm R})^{-1} Y_\nu^\dagger, \quad (4)$$

with $Y_{\rm R} \equiv M_{\rm R}/\Lambda$. Taking account of the charged-lepton part from the SM, we consider the most general flavor-basis transformations in the leptonic sector

$$\ell_{\rm L} \to U_{\rm L}\ell_L, \qquad l_{\rm R} \to V_{\rm R} l_{\rm R}, \qquad N_{\rm R} \to U_{\rm R} N_{\rm R}, \quad (5)$$

where $l_{\rm R}$ represents the RH charged-lepton fields, and $U_{\rm L}, V_{\rm R} \in {\rm U}(m)$, and $U_{\rm R} \in {\rm U}(n)$ are three arbitrary unitary matrices (for m generations of lepton doublets and n generations of RH neutrinos). Then, Eq. (2) is unchanged if we treat the Yukawa coupling matrices as spurions, namely, taking them as spurious fields that transform as

$$Y_l \to U_{\rm L} Y_l V_{\rm R}^\dagger, \qquad Y_\nu \to U_{\rm L} Y_\nu U_{\rm R}^\dagger, \qquad Y_{\rm R} \to U_{\rm R}^* Y_{\rm R} U_{\rm R}^\dagger, \quad (6)$$

where Y_l is the charged-lepton Yukawa coupling matrix. At the matching scale, such transformations in the lepton flavor space in the full theory induce those of the Wilson coefficients in the SEFT, i.e.,

$$C_5 \to U_{\rm L} C_5 U_{\rm L}^{\rm T}, \qquad C_6 \to U_{\rm L} C_6 U_{\rm L}^\dagger. \quad (7)$$

From Eqs. (6) and (7) we can take the matrices $(X_l \equiv Y_l Y_l^\dagger, C_5, C_6)$ in the flavor space as the building blocks for the flavor invariants in the SEFT with the symmetry group ${\rm U}(m)$, whereas $(Y_l, Y_\nu, Y_{\rm R})$ as the building blocks in the full seesaw model with the symmetry group ${\rm U}(m) \otimes {\rm U}(n)$.

Throughout this Letter, we use $\mathcal{I}_{abc}$ to label the flavor invariant with the degrees (a, b, c) of the building blocks (X_l, C_5, C_6) in the SEFT. Similarly, I_{abc} refers to the flavor invariant with the degrees (a, b, c) of the building blocks $(Y_l, Y_\nu, Y_\mathrm{R})$ in the full seesaw model. Here, a, b, c are nonnegative integers. By flavor invariants, we mean the polynomial matrix invariants composed of building blocks that keep unchanged under flavor transformation.

III. TWO-GENERATION SEFT

We begin with the case of only two generations of leptons. Although this is not realistic, it is very instructive for the study of the three-generation case. All the basic flavor invariants in both effective and full theories in the two-generation case can be explicitly constructed and related to the physical observables in an apparent way.

As has been stated above, the HS is a powerful tool in studying the flavor invariants and the algebraic structure of the invariant ring. In the SEFT with two generations, using the MW formula, one can calculate the HS

$$\mathscr{H}^{(2\,\mathrm{g})}_{\mathrm{SEFT}}(q) = \frac{1 + 3q^4 + 2q^5 + 3q^6 + q^{10}}{(1-q)^2(1-q^2)^4(1-q^3)^2(1-q^4)^2}, \quad (8)$$

where q is an arbitrary complex number that labels the degrees of the invariants. The denominator of the HS carries the information about the primary invariants, i.e., those invariants that are algebraically independent. There are ten factors in the denominator of the HS in Eq. (8), which means there are totally ten primary flavor invariants in the invariant ring. The nontrivial point is that this number also equals the number of the independent physical parameters in the two-generation SEFT (i.e., two charged-lepton masses, two neutrino masses, one mixing angle and one phase in the leptonic flavor mixing matrix, three moduli, and one phase in C_6). As we will show below, all ten physical parameters can be extracted as the functions of ten primary invariants.

Although other invariants in the ring are not algebraically independent of the primary ones, not all of them can be written as the polynomials of the primary invariants. However, for the unitary groups under consideration, one can always find a finite number of invariants known as basic invariants, such that any invariant in the ring can be decomposed as the polynomial of the basic invariants [35,36,44,45]. In general, the number of basic invariants is no smaller than that of primary invariants. This is because there may exist nontrivial polynomial identities among the basic invariants (i.e., the syzygies).

The construction of all the basic invariants can be accomplished by calculating the plethystic logarithm (PL) function of the HS

$$\mathrm{PL}[\mathscr{H}^{(2\mathrm{g})}_{\mathrm{SEFT}}(q)] = 2q + 4q^2 + 2q^3 + 5q^4 + 2q^5 + 3q^6 - 6q^8 - \mathcal{O}(q^9), \quad (9)$$

whose leading positive terms encode the information about the numbers and degrees of the basic invariants [25].

TABLE I. Summary of the basic flavor invariants along with their degrees and CP parities in the case of two-generation leptons in the SEFT, where the subscripts of the invariants denote the degrees of $X_l \equiv Y_l Y_l^\dagger$, C_5, and C_6, respectively. We have also defined $X_5 \equiv C_5 C_5^\dagger$, $G_{l5} \equiv C_5 X_l^* C_5^\dagger$, and $G_{56} \equiv C_5 C_6^* C_5^\dagger$ that transform adjointly under the flavor transformation. There are in total 12 CP-even basic invariants and six CP-odd basic invariants. Note that the ten primary invariants are labeled with $(*)$ in the first column.

Flavor invariants	Degree	CP parity
$\mathcal{I}_{100} \equiv \mathrm{Tr}(X_l)$ $(*)$	1	+
$\mathcal{I}_{001} \equiv \mathrm{Tr}(C_6)$ $(*)$	1	+
$\mathcal{I}_{200} \equiv \mathrm{Tr}(X_l^2)$ $(*)$	2	+
$\mathcal{I}_{101} \equiv \mathrm{Tr}(X_l C_6)$	2	+
$\mathcal{I}_{020} \equiv \mathrm{Tr}(X_5)$ $(*)$	2	+
$\mathcal{I}_{002} \equiv \mathrm{Tr}(C_6^2)$ $(*)$	2	+
$\mathcal{I}_{120} \equiv \mathrm{Tr}(X_l X_5)$ $(*)$	3	+
$\mathcal{I}_{021} \equiv \mathrm{Tr}(C_6 X_5)$ $(*)$	3	+
$\mathcal{I}_{220} \equiv \mathrm{Tr}(X_l G_{l5})$ $(*)$	4	+
$\mathcal{I}^{(1)}_{121} \equiv \mathrm{Tr}(G_{l5} C_6)$	4	+
$\mathcal{I}^{(2)}_{121} \equiv \mathrm{ImTr}(X_l X_5 C_6)$	4	−
$\mathcal{I}_{040} \equiv \mathrm{Tr}(X_5^2)$ $(*)$	4	+
$\mathcal{I}_{022} \equiv \mathrm{Tr}(C_6 G_{56})$ $(*)$	4	+
$\mathcal{I}_{221} \equiv \mathrm{ImTr}(X_l G_{l5} C_6)$	5	−
$\mathcal{I}_{122} \equiv \mathrm{ImTr}(C_6 G_{56} X_l)$	5	−
$\mathcal{I}_{240} \equiv \mathrm{ImTr}(X_l X_5 G_{l5})$	6	−
$\mathcal{I}_{141} \equiv \mathrm{ImTr}(X_5 C_6 G_{l5})$	6	−
$\mathcal{I}_{042} \equiv \mathrm{ImTr}(C_6 X_5 G_{56})$	6	−

As indicated by Eq. (9), there are totally 18 [obtained from the sum of all the coefficients in Eq. (9) until the first negative term] basic invariants in the ring: two of degree 1, four of degree 2, two of degree 3, five of degree 4, two of degree 5, and three of degree 6. Furthermore, we can explicitly construct all the basic flavor invariants in the two-generation SEFT, and the results are summarized in Table I. The parities of basic flavor invariants under the CP transformation have been listed in the last column. The 18 basic invariants (12 CP-even and six CP-odd) in Table I serve as the generators of the invariant ring in the sense that any flavor invariant can be written as the polynomial of them. For a systematic algorithm of decomposing an arbitrary invariant into the polynomial function of the basic invariants and finding out all the syzygies at a certain degree, see Appendix C of Ref. [35].

The 18 basic flavor invariants in Table I, however, are not algebraically independent. As one can verify, there are six syzygies first appearing at degree 8, corresponding to the first negative term $-6q^8$ in Eq. (9). Among them, four syzygies imply four linear relations among six CP-odd basic invariants and another two involve only CP-even invariants. This is in accordance with the fact that there are only $6 - 4 = 2$ independent phases in the two-generation case of the SEFT.

In Table I, ten primary flavor invariants are labeled by "$(*)$." It can be shown that from them one can extract all the physical parameters in the two-generation SEFT (cf. Supplemental Material [46]). In this sense, the set of primary invariants is actually equivalent to that of independent physical parameters in the theory. Therefore, one can express any low-energy physical observables in an explicit and basis-independent form with only flavor invariants. In particular, any CP-violating observable $\mathcal{A}_{\rm CP}$ can be written as [47]

$$\mathcal{A}_{\rm CP} = \sum_j \mathcal{F}_j[\mathcal{I}_k^{\rm even}]\mathcal{I}_j^{\rm odd} , \tag{10}$$

where $\mathcal{I}_j^{\rm odd}$ refer to CP-odd basic flavor invariants, and $\mathcal{F}_j[\mathcal{I}_k^{\rm even}]$ are some functions of only CP-even basic flavor invariants. Thus, the vanishing of all CP-odd basic invariants in the ring ensures the absence of CP violation in the theory. We shall leave the proof of this general formula for Ref. [47]. Instead we mention that CP asymmetries $\mathcal{A}_{\nu\nu}$ in neutrino oscillations and those $\mathcal{A}_{\nu\bar{\nu}}$ in neutrino-antineutrino oscillations [48–50] can indeed be cast in the form of Eq. (10). After some lengthy calculations, we obtain $\mathcal{A}_{\nu\nu} = \mathcal{F}_{\nu\nu}\mathcal{I}_{121}^{(2)}$ and $\mathcal{A}_{\nu\bar{\nu}} = \mathcal{F}_{\nu\bar{\nu}}\mathcal{I}_{240}$, where $\mathcal{F}_{\nu\nu}$ and $\mathcal{F}_{\nu\bar{\nu}}$ are functions of CP-even primary invariants, while $\mathcal{I}_{121}^{(2)}$ and $\mathcal{I}_{240}$ are two CP-odd basic invariants in Table I.

Finally, we discuss the conditions for CP conservation. Though there are six CP-odd basic invariants in the ring, only two of them are algebraically independent due to the syzygies. On the other hand, there are two independent phases in the leptonic sector. Hence, the minimal conditions to guarantee CP conservation is the vanishing of only two CP-odd invariants. We find that the vanishing of $\mathcal{I}_{121}^{(2)}$ and $\mathcal{I}_{240}$ is sufficient to this end. Therefore, CP asymmetries in neutrino oscillations and neutrino-antineutrino oscillations already contain all the information about CP violation at low energies.

IV. TWO-GENERATION SEESAW

In the full seesaw model, the building blocks transform in the flavor space as in Eq. (6). Then, the HS can be computed as [24]

$$\mathcal{H}_{\rm SS}^{(2g)}(q) = \frac{1+q^6+3q^8+2q^{10}+3q^{12}+q^{14}+q^{20}}{(1-q^2)^3(1-q^4)^5(1-q^6)(1-q^{10})},$$

which exhibits the algebraic structure of the flavor space in the full theory. We observe that the denominator of the HS in the full theory and that of Eq (8) have the same number of factors, implying that there are equal numbers of algebraically independent invariants (i.e., primary invariants) in the flavor space of full theory and that of the SEFT. Given the fact that the number of primary invariants is equal to that of independent physical parameters, we reach the conclusion that the inclusion of just one dimension-five and one dimension-six operator in the effective theory is already adequate to incorporate all physical information about the full theory, including the source of CP violation [41,42,51].

TABLE II. Summary of the basic flavor invariants along with their degrees and CP parities in the case of two-generation leptons in the type-I seesaw model. The subscripts of the invariants denote the degrees of Y_l, Y_ν, and $Y_{\rm R}$, respectively. We have also defined some building blocks that transform adjointly under the flavor transformation: $X_l \equiv Y_l Y_l^\dagger$, $X_\nu \equiv Y_\nu Y_\nu^\dagger$, $\tilde{X}_\nu \equiv Y_\nu^\dagger Y_\nu$, $X_{\rm R} \equiv Y_{\rm R}^\dagger Y_{\rm R}$, $G_{l\nu} \equiv Y_\nu^\dagger X_l Y_\nu$, $G_{\nu{\rm R}} \equiv Y_{\rm R}^\dagger \tilde{X}_\nu^* Y_{\rm R}$, and $G_{l\nu{\rm R}} \equiv Y_{\rm R}^\dagger G_{l\nu}^* Y_{\rm R}$. There are in total 12$CP$-even basic invariants and six CP-odd basic invariants. The ten primary invariants are labeled with $(*)$ in the first column.

Flavor invariants	Degree	CP parity
$I_{200} \equiv {\rm Tr}(X_l)$ $(*)$	2	+
$I_{020} \equiv {\rm Tr}(X_\nu)$ $(*)$	2	+
$I_{002} \equiv {\rm Tr}(X_{\rm R})$ $(*)$	2	+
$I_{400} \equiv {\rm Tr}(X_l^2)$ $(*)$	4	+
$I_{220} \equiv {\rm Tr}(X_l X_\nu)$ $(*)$	4	+
$I_{040} \equiv {\rm Tr}(X_\nu^2)$ $(*)$	4	+
$I_{022} \equiv {\rm Tr}(\tilde{X}_\nu X_{\rm R})$ $(*)$	4	+
$I_{004} \equiv {\rm Tr}(X_{\rm R}^2)$ $(*)$	4	+
$I_{222} \equiv {\rm Tr}(X_{\rm R} G_{l\nu})$ $(*)$	6	+
$I_{042} \equiv {\rm Tr}(\tilde{X}_\nu G_{\nu{\rm R}})$	6	+
$I_{242}^{(1)} \equiv {\rm Tr}(G_{l\nu} G_{\nu{\rm R}})$	8	+
$I_{242}^{(2)} \equiv {\rm ImTr}(\tilde{X}_\nu X_{\rm R} G_{l\nu})$	8	−
$I_{044} \equiv {\rm ImTr}(\tilde{X}_\nu X_{\rm R} G_{\nu{\rm R}})$	8	−
$I_{442} \equiv {\rm Tr}(G_{l\nu} G_{l\nu{\rm R}})$ $(*)$	10	+
$I_{262} \equiv {\rm ImTr}(\tilde{X}_\nu G_{l\nu} G_{\nu{\rm R}})$	10	−
$I_{244} \equiv {\rm ImTr}(X_{\rm R} G_{l\nu} G_{\nu{\rm R}})$	10	−
$I_{462} \equiv {\rm ImTr}(\tilde{X}_\nu G_{l\nu} G_{l\nu{\rm R}})$	12	−
$I_{444} \equiv {\rm ImTr}(X_{\rm R} G_{l\nu} G_{l\nu{\rm R}})$	12	−

This point can be seen more clearly from the basic invariants. In the two-generation case, one can explicitly construct all the basic flavor invariants in the full theory, as listed in Table II. To one's surprise, there are exactly equal numbers of CP-odd and CP-even basic invariants in Tables I and II, namely, both are six and 12, respectively. Recalling that the basic invariants serve as the generators of the invariant ring, we conclude that the invariant ring in the SEFT and that in the full theory share an equal number of generators. One can establish a direct link between these two sets of generators by noticing that the building blocks C_5 and C_6 in the SEFT are related to the building blocks Y_ν and $Y_{\rm R}$ in full theory via Eq. (4). Through a proper matching procedure [36,47], we find all flavor invariants in the SEFT can be written as the rational functions of those in the full seesaw model.

We have verified that all 18 basic flavor invariants in the SEFT can be explicitly expressed as rational functions of the 18 basic flavor invariants in the full seesaw model. The complete set of matching conditions are given in Supplemental Material [46]. In particular, one can set up a one-to-one correspondence between six CP-odd basic invariants in the SEFT and those in the full theory, namely,

$$\mathcal{I}_{121}^{(2)} = \frac{2}{(I_{002}^2 - I_{004})^2}[I_{242}^{(2)} I_{022} - I_{044} I_{220} + I_{262} I_{002} - I_{244} I_{020}], \tag{11}$$

$$\mathcal{I}_{221} = \frac{2}{(I_{002}^2 - I_{004})^2}[I_{242}^{(2)} I_{222} + I_{244} I_{220} + I_{462} I_{002} - I_{444} I_{020}], \tag{12}$$

$$\mathcal{I}_{122} = \frac{2}{(I_{002}^2 - I_{004})^3}\{I_{242}^{(2)}[3I_{022}^2 + 2I_{040}(I_{002}^2 - I_{004}) - 4I_{020}I_{002}I_{022}] + I_{244}(3I_{020}I_{022} - 2I_{042}) + I_{044}(4I_{020}I_{222} - I_{220}I_{022} - 2I_{242}^{(1)}) + I_{262}[3I_{002}I_{022} - I_{020}(I_{002}^2 + 3I_{004})]\}, \tag{13}$$

$$\mathcal{I}_{240} = \frac{1}{(I_{002}^2 - I_{004})^2}[3I_{242}^{(2)}(I_{022}I_{220} - I_{020}I_{222}) - I_{044}I_{220}^2 + I_{262}(3I_{002}I_{220} - 2I_{222}) - 2I_{244}I_{020}I_{220} + I_{462}(2I_{022} - 3I_{002}I_{020}) + I_{444}I_{020}^2], \tag{14}$$

$$\mathcal{I}_{141} = \frac{2}{(I_{002}^2 - I_{004})^3}\{I_{242}^{(2)}I_{020}I_{022}^2 + I_{044}I_{020}(I_{022}I_{220} - 2I_{242}^{(1)}) + I_{244}I_{020}(I_{020}I_{022} - 2I_{042}) + I_{262}[I_{002}I_{020}I_{022} + I_{040}(I_{004} - I_{002}^2)]\}, \tag{15}$$

$$\mathcal{I}_{042} = \frac{2}{(I_{002}^2 - I_{004})^3} I_{044}(I_{020}^2 - I_{040})^2. \tag{16}$$

Notice that Eqs. (11)–(16) form a system of linear equations for the *CP*-odd invariants, and the determinant of the coefficient matrix in Eqs. (11)–(16) turns out to be nonzero in general. This proves that the vanishing of all the *CP*-odd flavor invariants in the SEFT is equivalent to the vanishing of all *CP*-odd invariants in the full theory. Therefore, the absence of *CP* violation in the low-energy effective theory up to the order of $\mathcal{O}(1/\Lambda^2)$ is equivalent to the *CP* conservation in the full seesaw model. Note that a similar conclusion was also drawn in Ref. [42] but without the language of invariant theory.

The matching conditions in Eqs. (11)–(16) are useful to build a bridge between the *CP* violation at low energies and that at high energies. For example, if RH neutrino masses are strongly hierarchical, the (unflavored) *CP* asymmetry in the decay of the lightest RH neutrino can simply be written as [36]

$$\epsilon_1 = \frac{3}{16\pi} \frac{I_{044}}{I_{002}(I_{022} - I_{002}I_{020})}. \tag{17}$$

Then, via Eq. (16), ϵ_1 can be related to the *CP*-odd basic flavor invariant $\mathcal{I}_{042}$ in the SEFT. Furthermore, using four syzygies involving *CP*-odd invariants at degree 8, one can express $\mathcal{I}_{042}$ as the linear combination of $\mathcal{I}_{121}^{(2)}$ and $\mathcal{I}_{240}$ [47]. Finally, one arrives at

$$\epsilon_1 = \mathcal{R}_1[I_{\rm even}]\mathcal{I}_{121}^{(2)} + \mathcal{R}_2[I_{\rm even}]\mathcal{I}_{240}, \tag{18}$$

where $\mathcal{R}_1[I_{\rm even}]$ and $\mathcal{R}_2[I_{\rm even}]$ are rational functions of only *CP*-even basic invariants in the full theory listed in Table II. As $\mathcal{I}_{121}^{(2)}$ and $\mathcal{I}_{240}$ are, respectively, responsible for *CP* violation in neutrino oscillations and neutrino-antineutrino oscillations, Eq. (18) establishes a direct link between low- and high-energy *CP* asymmetries in a basis-independent way. If $\mathcal{A}_{\nu\nu} = \mathcal{A}_{\nu\bar{\nu}} = 0$, which means $\mathcal{I}_{121}^{(2)} = \mathcal{I}_{240} = 0$, then ϵ_1 also vanishes. This is obviously in accordance with the conclusion drawn from Eqs. (11)–(16) that *CP* conservation in the SEFT also implies the absence of *CP* violation in the full seesaw model.

V. THREE-GENERATION CASE

All the results obtained in the two-generation SEFT can be generalized to the realistic three-generation scenario in a straightforward way, though the calculations are much more complicated. In this Letter, we just collect the main conclusions and will present the details in a separate work [47].

First, the HS in the three-generation SEFT can be computed by using the MW formula, whose expression is much lengthier than that in Eq. (8). However, as a highly nontrivial result, we find that the denominator of the HS has 21 factors, which exactly matches the number of independent physical parameters in the SEFT. On the other hand, there are also 21 independent physical parameters in the three-generation seesaw. Moreover, the HS in the three-generation seesaw has been calculated in Ref. [26], and its denominator also has 21 factors. This implies there are 21 primary invariants in both the SEFT and the full theory for three generations. Second, those 21 primary invariants in the SEFT can be explicitly constructed, and from them we can extract all the physical parameters. Among them, there are six CP-odd invariants corresponding to six independent phases in the SEFT. In particular, any CP-violating observables can also be cast into the form of Eq. (10). Third, the vanishing of six certain CP-odd flavor invariants serves as the minimal sufficient and necessary condition for CP conservation in the leptonic sector. The absence of CP violation in the SEFT is enough to guarantee CP conservation in the full theory, and vice versa. Finally, any flavor invariants in the SEFT can be written as rational functions of those in the full theory, which as the matching conditions set a connection between low- and high-energy observables.

VI. CONCLUDING REMARKS

The invariant theory is an extremely useful tool for studying CP violation in nature. Any physical observables should be independent of the flavor basis and the specific parametrization of Yukawa matrices that one chooses. This feature is exactly what flavor invariants own. Therefore, it is more natural to express observables in a complete form of flavor invariants.

In this Letter, we demonstrate the intimate connection between the canonical seesaw model and its low-energy effective theory in the language of invariant theory. We show that the inclusion of only one dimension-five and one dimension-six operator in the effective theory is already adequate to contain all physical information about the full theory, including the source of CP violation. The HS of the flavor space in the SEFT is calculated, and all the physical parameters are explicitly extracted using primary invariants, which is helpful for phenomenological studies at low energies. The matching between flavor invariants in the SEFT and those in the full seesaw model is accomplished, offering a basis-independent way to relate CP violation for cosmological matter-antimatter asymmetry to that in low-energy phenomena.

The results in this work prove the usefulness and power of the invariant theory and call for more applications of flavor invariants to flavor puzzles as well as other important topics in particle physics in general.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grant No. 11835013 and the Key Research Program of the Chinese Academy of Sciences under Grant No. XDPB15.

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