Letter

Divergence anomaly and Schwinger terms: Towards a consistent theory of anomalous classical fluids

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In this Letter, anomaly, which is a generic feature of relativistic quantum field theory (QFT), is shown to be present in non-relativistic classical ideal fluid. Also, in this model we have found the presence of anomalous terms in current algebra, an obvious analogue of Schwinger terms in QFT. We work in the Hamiltonian framework, where Eulerian dynamical variables obey an anomalous algebra (with Schwinger terms) that is inherited from modified Poisson brackets, with Berry curvature corrections, among Lagrangian discrete coordinates. The divergence anomaly appears in the Hamiltonian equations of motion. A generalized form of the fluid velocity field can be identified by the "anomalous velocity" of Bloch band electrons appearing in the quantum Hall effect in condensed matter physics. Finally, we show that the divergence anomaly and Schwinger terms satisfy the well-known Adler consistency condition, and we mention possible scenarios that can be impacted by this new anomalous fluid theory.

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I. INTRODUCTION

Anomaly, a generic feature of relativistic quantum field theories (QFT), appear when classical symmetries and (Noether) conservation laws of a classical field theory are not preserved in their QFT counterpart (see [1]). Applying classical equations of motion yield vector and axial vector current conservation laws in mass-less QED, $\partial^\mu J^V_\mu(x) = \partial^\mu(\bar\psi(x)\gamma_\mu\psi(x)) = 0, \quad \partial^\mu J^A_\mu(x) = \partial^\mu(\bar\psi(x)\gamma_\mu\gamma_5\psi(x)) = 0$, respectively. However in QFT, one has to use regularized composite (current) operators and finally the regularization independent conservation equations can get modified giving rise to anomalies. In massless QED, one can choose to preserve charge conservation on physical grounds, $\partial^\mu J^V_\mu = 0$, and in the process break the axial current conservation law, $\partial^\mu J^A_\mu = (e^2/4\pi^2)\mathbf{E}.\mathbf{B}$, ($E$, $\mathbf{B}$ the electric and magnetic fields), generating the Adler Bell Jackiw anomaly [2]. Another crucial aspect of QFT is the current algebra (see, for example, Ref. [3]), where certain equal-time commutation relations among current density operators define an infinite-dimensional Lie algebra. Originally proposed by Gell-Mann to describe strongly interacting hadron physics, the current algebra led to the Adler-Weisberger formula, Sugawara model, Virasoro algebra, the mathematical theory of affine Kac-Moody algebra, and nonrelativistic current algebra in quantum and statistical physics.

Returning to the present context, anomalies can modify the current algebra, i.e., through the introduction of Schwinger terms [4,5], computed along similar lines as the divergence anomaly. As proved by Adler [6] and studied by others [7], the divergence anomaly and Schwinger terms (or commutator anomalies) are complementary effects—the presence of one type necessitates the existence of the other (in fact, in $1+1$ dimensions the Schwinger terms can uniquely yield the divergence anomaly [7])—and the anomalous extensions have to obey the Adler [6] consistency condition. This will play a major role in our work.

In recent years, the divergence anomaly has generated a huge amount of interest in an unexpected scenario—the hydrodynamic regime of nonrelativistic classical field theory. The effect of anomalous current algebra on Raman scattering in Mott insulators was studied in Ref. [8]. Son and Spivak [9] have shown that the large classical negative magnetoresistance of Weyl metals is connected to the triangle anomaly in the classical regime where the mean free path of the electron is short compared to the magnetic length. Further works in related areas are Refs. [10,11]. Further studies include quantum anomalies for global currents in hydrodynamic limit [10] and gauge anomalies in hydrodynamics in a Hamiltonian framework [11]. After reporting our results [12] we noted that [13] discusses divergence anomaly in classical fluid. Surprisingly there are no attempts to *derive* the divergence anomaly from first principles. Also Schwinger terms does not appear in recent works.

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II. SIGNIFICANCE AND NEW RESULTS OF OUR WORK

In this article we will shed some light on the two untouched problems mentioned above in the following format:

(1) First principle derivation of anomalous current algebra (∼ Schwinger terms) in classical fluid dynamics.
(2) Derivation of the classic chiral anomaly form $e^2\mathbf{E}.\mathbf{B}$ in the helicity conservation equation using the above-mentioned Schwinger terms.
(3) Establishing an analogy between the generalized velocity field defined here and the well-known "anomalous velocity" appearing in condensed matter physics [14].
(4) Schwinger terms and the divergence anomaly satisfying the Adler consistency condition [6].
(5) Construction of Casimir operators (in the weak field approximation) in the anomalous system via the Darboux prescription.
(6) *Topical interest of the work*: In our work, Schwinger (or anomalous) terms in algebra among dynamical variables play the primary role and divergence anomaly appears naturally as a derived quantity. We develop the anomalous fluid model systematically in Hamiltonian formalism in Eulerian approach.

The quantum input comes from a basic generalized Poisson bracket structure satisfied by the discrete (Lagrangian) fluid particle coordinates. This phase space characterizes the semiclassical electron dynamics in a magnetic Bloch band in the presence of a periodic potential with an external magnetic field and Berry curvature [15–18]. Berry curvature and an induced magnetic field in momentum space are responsible for the anomalous velocity in the quantum Hall effect [14]. Electron hydrodynamics in condensed matter, i.e., situations where electron flow is influenced by hydrodynamic laws instead of being fully was studied in old [19]. In normal circumstances, electrons in metals behave as a nearly free Fermi gas since the effective mean free path for electron-electron collision is quite large, allowing impurities and lattice thermal vibrations (phonons) to destroy a collective viscous fluidlike electron motion. But, in recent years the hydrodynamic regime has been achieved in extremely pure, high quality, electronic materials—especially graphene [20], layered materials with very high electrical conductivity such as metallic delafossites $PdCoO_2$ and $PtCoO_2$ [21], among others.

We exploit the well-known map (constitutive relation) that expresses the continuous Euler fluid variables in terms of the discrete Lagrangian particle coordinates. Through this map the fluid field algebra inherits the Schwinger terms from the quantum corrected Poisson brackets of Lagrangian coordinates. Divergence anomaly follows from Hamiltonian equations of motion. The sequence of our scheme is as follows: generalized phase-space algebra with Berry curvature corrections [15,16,18] → (via constitutive relations) extended fluid variable algebra → extended fluid equations with divergence anomaly and Schwinger terms → the consistency condition connecting the divergence anomaly and Schwinger terms.

III. DERIVATION OF ANOMALOUS FLUID ALGEBRA (SCHWINGER TERMS)

Berry phase corrected (∼ anomalous or noncommutative) phase-space algebra of the degrees of freedom (d.o.f.), $X_j(\mathbf{x})$, $P_j(\mathbf{x}) = M\dot{X}_j(\mathbf{x})$ (M being the point particle mass), is identified with discrete particle phase-space coordinates [15–18],

$$\{X_i(\mathbf{x}), X_j(\mathbf{x}')\} = -\frac{1}{\rho_0}\epsilon_{ijk}\mathcal{F}_k\delta(\mathbf{x}-\mathbf{x}'); \{X_i(\mathbf{x}), P_j(\mathbf{x}')\} = \frac{M(\delta_{ij}+eB_i\Omega_j)}{\rho_0\mathcal{A}}\delta(\mathbf{x}-\mathbf{x}');$$
$$\{P_i(\mathbf{x}), P_j(\mathbf{x}')\} = e\frac{\epsilon_{ijk}M^2B_k}{\rho_0\mathcal{A}}\delta(\mathbf{x}-\mathbf{x}') \tag{1}$$
$$\mathcal{F}_i(\mathbf{x},\mathbf{k}) = \frac{\Omega_i}{1+e\mathbf{B}(\mathbf{x}).\mathbf{\Omega}(\mathbf{k})}, \quad \mathcal{A}(\mathbf{x},\mathbf{k}) = 1+e\mathbf{B}(\mathbf{x}).\mathbf{\Omega}(\mathbf{k}).$$

In the above Lagrangian d.o.f., $X_{(n)i}, P_{(n)i}$, the discrete particle index n is replaced by $\mathbf{x}$ in the continuum limit. The parameters are; ρ_0 for proper dimension, $e\equiv$ electronic charge, $\mathbf{B}\equiv$ external magnetic field, $W\mathbf{\Omega}(\mathbf{k}) = \nabla_\mathbf{k}\times\mathbf{A}(\mathbf{k})\equiv$ Berry curvature in momentum space. For $\mathbf{\Omega}=0, \mathbf{B}=0$, one recovers the canonical Poisson brackets. $\mathbf{\Omega}$ appears as a result of electron motion in a periodic lattice potential. These are equal time brackets.

In general $\mathbf{P}$ and $\mathbf{k}$ (the crystal momentum) are related but as in [22], in a toy model Ω can be constant. In realistic models Ω is $\sim(lattice\,constant)^2$ [22]. We will restrict Ω to be independent of dynamical variables. At the end we will comment on some other non-trivial possibilities.

Since M is a constant parameter, we will drop it to express formally the above brackets (1) using $X_i(x)$, $\dot{\mathbf{X}}_\mathbf{i}(\mathbf{x})$. Euler variables density $\rho(\mathbf{r})$ and velocity fields $v_i(\mathbf{r})$ are defined as [23],

$$\rho(\mathbf{r}) = \rho_0\int dx\delta(X(x)-\mathbf{r}),$$
$$v_i(\mathbf{r}) = \frac{\int dx\dot{X}_i(x)\delta(X(x)-\mathbf{r})}{\int dx\delta(X(x)-\mathbf{r})}. \tag{2}$$

$j_i = \rho v_i$ constitutes the momentum density of the fluid. Even though we are not considering relativistic fluid dynamics and our system is non-dissipative, note that this definition pertains to the Landau frame where $j_i(\vec{r},t)$ refers to the values at a fixed spacetime position $\vec{r}, t$. The Landau frame is chosen in the direction of the total energy where the directions of the eigenvector of the energy-momentum tensor and the conserved current match. This is generally true in nonrelativistic hydrodynamic flow, defined as a local particle flux.

Using Eq. (1) it is straightforward to compute the anomalous fluid brackets,

$$\{\rho(\mathbf{r}),\rho(\mathbf{r}')\} = \epsilon_{ijk}\partial_i^{\mathbf{r}}(\rho(\mathbf{r})\mathcal{F}_k(\mathbf{r}))\partial_j^{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}'), \tag{3}$$

$$\{\rho(\mathbf{r}),v_i(\mathbf{r}')\} = \frac{\partial_i^{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}')}{\mathcal{A}(\mathbf{r}')} + eB_j(\mathbf{r}')\mathcal{F}_i(\mathbf{r}')\partial_j^{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}') + \epsilon_{ljk}\mathcal{F}_l(\mathbf{r}')\partial_j^{\mathbf{r}'}v_i\partial_k^{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}'), \tag{4}$$

$$\begin{aligned}\{v_i(\mathbf{r}),v_j(\mathbf{r}')\} = \Big\{&\frac{\partial_j v_i-\partial_i v_j}{\rho\mathcal{A}(\mathbf{r})} + e\epsilon_{ijk}\frac{B_k(\mathbf{r})}{\rho\mathcal{A}(\mathbf{r})}\\ &-\frac{eB_l(\mathbf{r})}{\rho}(\mathcal{F}_i(\mathbf{r})\partial_l v_j-\mathcal{F}_j(\mathbf{r})\partial_l v_i)-2\epsilon_{lmn}\frac{v_i}{\rho^2}\partial_m v_j\partial_l(\mathcal{F}_n(\mathbf{r})\rho)\\ &-\epsilon_{lmn}\frac{1}{\rho}\partial_n v_i\partial_m v_j\mathcal{F}_l(\mathbf{r})-2\epsilon_{lmn}\frac{\partial_m\rho}{\rho^2}\partial_n(v_iv_j)\mathcal{F}_l(\mathbf{r})\Big\}\delta(\mathbf{r}-\mathbf{r}')\\ &+2\epsilon_{lmn}\frac{\partial_m^{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}')}{\rho(\mathbf{r})}\partial_n^{\mathbf{r}}(v_iv_j)\mathcal{F}_l(\mathbf{r}).\end{aligned} \tag{5}$$

Clearly the $\mathbf{\Omega}$-dependent terms are Schwinger terms. We stress, although Berry curvature $\mathbf{\Omega}$ behaves as an effective magnetic field, its effect is distinct from external magnetic field $\mathbf{B}$. Also, note that this anomalous fluid algebra is different from the general structure derived from the results in Ref. [24], where $\mathbf{B}$, $\mathbf{E}$ are nonvanishing but $\mathbf{\Omega}=0$. Our algebra matches with the brackets in Ref. [11] for $\mathbf{\Omega}=0$. We emphasize that even if the (canonical) vorticity $\omega_{ij}=\partial_i v_j-\partial_j v_i$ vanishes, the effective vorticity can reappear anomalously, as seen in Eq. (5). Similar types of extended fluid brackets in different contexts have appeared in Ref. [25]. The canonical brackets for $\rho_c, v_{(c)i}$ are recovered for $\mathbf{B}=\mathbf{\Omega}=0$,

$$\{\rho_c(\mathbf{r}),\rho_c(\mathbf{r}')\}=0, \qquad \{\rho_c(\mathbf{r}),v_{(c)i}(\mathbf{r}')\}=\partial_i^{\mathbf{r}}\delta(\mathbf{r}-\mathbf{r}'),$$
$$\{v_{(c)i}(\mathbf{r}),v_{(c)j}(\mathbf{r}')\} = -\frac{(\partial_i v_{(c)j}-\partial_j v_{(c)i})}{\rho_c}\delta(\mathbf{r}-\mathbf{r}'). \tag{6}$$

The anomalous fluid algebra in Eqs. (3)–(5) constitutes the first part of our work.

$j_i=\rho v_i$, the momentum density, acts as translation generator for $B=0$, as shown below for arbitrary functions $\alpha(\rho)$, $A_l(v_i)$ (computational details are in Supplementary Material [26]):

$$\left\{\alpha(\rho(x)),\int d^3y\rho v_l\right\} = \frac{d\alpha}{d\rho}\partial_l\rho = \partial_l\alpha(\rho(x)), \tag{7}$$

$$\left\{A_l(v_i(x)),\int d^3y\rho v_j\right\} = \frac{dA_l}{dv_i}\partial_j v_i = \partial_j A_l(y). \tag{8}$$

IV. CONSERVATION LAW AND HELICITY ANOMALY

For a barometric fluid Hamiltonian (pressure $P=\rho(dV)/(d\rho)-V$ depending only on density ρ)

$$\mathcal{H}_0 = \int d\mathbf{x}\left(\frac{1}{2}\rho v^2+V(\rho)\right) \tag{9}$$

and the brackets in Eq. (6), the canonical continuity and Euler equations are obtained as

$$\dot\rho(\mathbf{x}) = \{\rho(\mathbf{x}),\mathcal{H}_0\} = -\nabla(\rho\mathbf{v}),$$
$$\dot{\mathbf{v}}(\mathbf{x}) = \{\mathbf{v}(\mathbf{x}),\mathcal{H}_0\} = -(\mathbf{v}.\nabla)\mathbf{v}-\frac{\nabla P}{\rho}. \tag{10}$$

The fluid Hamiltonian in the external electromagnetic field is given by

$$\mathcal{H} = \int d\mathbf{x}\left(\frac{1}{2}\rho v^2+V(\rho)-e\rho\Phi\right) \tag{11}$$

where the electric field is $\mathbf{E}=-\nabla\Phi$ in a time-independent scenario. The effect of magnetic field $\mathbf{B}$ has already been taken into account through the extended symplectic structure (3)–(5). The continuity equation is modified as

$$\dot\rho+\nabla.\mathbf{J}^{an} = e\rho\mathcal{F}.(\nabla\times\mathbf{E}) \tag{12}$$

where the anomalous current $\mathbf{J}^{an}$ is

$$\mathbf{J}^{an} = \left(\frac{\rho\mathbf{v}}{\mathcal{A}}\right)+e\rho(\mathcal{F}.\mathbf{v})\mathbf{B}+e\rho(\mathbf{E}\times\mathcal{F})+\mathcal{F}\times\nabla P. \tag{13}$$

However, in the present time-independent case, the conservation law $\dot\rho+\nabla.\mathbf{J}^{an}=0$ survives, albeit with an anomalous current since now Maxwell's equation yields $\nabla\times\mathbf{E}=-(\partial\mathbf{B})/\partial t=0$.

The anomalous Euler equation is derived as

$$\begin{aligned}\dot{\mathbf{v}}+\frac{(\mathbf{v}.\nabla)\mathbf{v}}{\mathcal{A}} = &-\frac{\nabla P}{\rho\mathcal{A}}+e\frac{\rho\mathbf{v}\times\mathbf{B}}{\rho\mathcal{A}}-e\frac{\mathbf{B}.\nabla\mathbf{P}}{\rho}\boldsymbol{\mathcal{F}}-e(\mathbf{v}.\boldsymbol{\mathcal{F}})(\mathbf{B}.\nabla)\mathbf{v}+\left\{\left(\frac{\nabla P}{\rho}\times\mathcal{F}\right).\nabla\right.\\ &\left.-\frac{1}{\rho}\nabla v^2.\left(\nabla\times(\boldsymbol{\mathcal{F}}\rho))+2v^2\left(\boldsymbol{\mathcal{F}}\times\frac{\nabla\rho}{\rho}\right).\nabla-\boldsymbol{\mathcal{F}}.\left(\frac{\nabla\rho}{\rho}\times\nabla v^2\right)\right)\right\}\mathbf{v}-e\left\{\frac{\mathbf{E}}{\mathcal{A}}+e(\mathbf{E}.\mathbf{B})\boldsymbol{\mathcal{F}}-(\mathbf{E}\times\boldsymbol{\mathcal{F}}).\nabla\mathbf{v}\right\}.\end{aligned} \tag{14}$$

We stress that in RHS of (14), $e^2(\mathbf{E}.\mathbf{B})\mathcal{F}$ has the classic chiral anomaly form. In Newtonian fluid, an important pseudoscalar quantity, helicity $\Sigma=\int d\mathbf{x}\,h=\int d\mathbf{x}\,v.\omega$, with $\omega=\nabla\times v$ being vorticity, is conserved. Time-evolution of h is

$$\dot h = \dot{\mathbf{v}}.\omega+\mathbf{v}.\dot\omega = -\nabla.(\mathbf{v}\times\dot{\mathbf{v}})+2\dot{\mathbf{v}}.\omega. \tag{15}$$

The divergence term will not contribute in Σ and using (14) we find

$$\dot{h}/2 = -e^2(\mathbf{E}.\mathbf{B})(\omega.\mathcal{F}) - e\frac{(\omega.\mathbf{E})}{\mathcal{A}} - e\frac{(\mathbf{B}.\nabla P)}{\rho}(\omega.\mathcal{F}) - e\frac{\omega.(\mathbf{v}\times\mathbf{B})}{\mathcal{A}} - e(\mathbf{v}.\mathcal{F})[\omega.(\mathbf{B}.\nabla)\mathbf{v}] + \Psi h + \omega.[\mathcal{F}.(\mathbf{C}\times\nabla)]\mathbf{v} \quad (16)$$

$$\Psi = -\nabla v^2.(\nabla\times\mathcal{F}) + \nabla v^2.\left(\mathcal{F}\times\frac{\nabla\rho}{\rho}\right);$$

$$\mathbf{C} = -\frac{\nabla P}{\rho} + \nabla v^2 - e\mathbf{E}.$$

We consider special cases: (a) In a low energy regime, we keep only first three $O(v)$ terms in RHS and ignore higher order terms. (b) Consider pressure $P = 0$. (c) Rewrite $\omega.\mathbf{E} = (\nabla\times\mathbf{v}).\mathbf{E} = \nabla.(\mathbf{v}\times\mathbf{E}) + \mathbf{v}.(\nabla\times\mathbf{E})$. For time independent external $\mathbf{E}$, $\mathbf{B}$, $\nabla\times\mathbf{E} = 0$ and only the chiral anomaly term survives,

$$\dot{\Sigma} = -2e^2(\mathbf{E}.\mathbf{B})\int d\mathbf{X}(\omega.\mathcal{F}). \quad (17)$$

This is the cherished form of the anomaly in Eulerian fluid (in the low energy limit), and constitutes our principal result.

V. ANALOGY BETWEEN GENERALIZED AND "ANOMALOUS" VELOCITY [14–16,18]

Let us rewrite $\mathbf{J}^{an}$ from Eq. (13) in a more suggestive form as

$$\mathbf{J}^{an} = \left(\frac{\rho}{1+e\mathbf{B}.\mathbf{\Omega}}\right)(\mathbf{v} + e\rho(\mathbf{\Omega}.\mathbf{v})\mathbf{B} + e\rho(\mathbf{E}\times\mathbf{\Omega}) + \left(\frac{\mathbf{\Omega}}{1+e\mathbf{B}.\mathbf{\Omega}}\right)\times\nabla P. \quad (18)$$

Keeping terms of $O(e)$ only, we write $\mathbf{J}^{an} \approx \rho\mathbf{v}^{gen}$ where, ignoring the pressure term, the generalized velocity is

$$\mathbf{v}^{gen} = (1 - e\mathbf{B}.\mathbf{\Omega})\mathbf{v} + e\rho(\mathbf{\Omega}.\mathbf{v})\mathbf{B} + e\rho(\mathbf{E}\times\mathbf{\Omega}). \quad (19)$$

On the other hand, following Refs. [14–16,18], from the Bloch electron dynamics in a magnetic band, with $\epsilon_n(\mathbf{k})$ being the nth band energy,

$$\dot{\mathbf{r}} = \frac{\partial\epsilon_n(\mathbf{k})}{\partial\mathbf{k}} - \dot{\mathbf{k}}\times\mathbf{\Omega}, \qquad \dot{\mathbf{k}} = -e\mathbf{E} - e\dot{\mathbf{r}}\times\mathbf{B}, \quad (20)$$

we obtain

$$\dot{\mathbf{r}} = \frac{\partial\epsilon_n(\mathbf{k})}{\partial\mathbf{k}} + e\mathbf{E}\times\mathbf{\Omega} + e(\dot{\mathbf{r}}\times\mathbf{B})\times\mathbf{\Omega}, \quad (21)$$

where $\dot{\mathbf{r}}$ in Eq. (21) is referred as the anomalous velocity [14–16,18]. Note that the e-dependent terms in Eqs. (19) and (21) are identical. This matching clearly shows that the anomaly inherited by the ideal classical fluid (from the Bloch band electron dynamics) has a deeper significance. This will be further strengthened in the next section where we prove consistency of the full anomalous structure developed in this Letter.

VI. CONSISTENCY CONDITION INVOLVING DIVERGENCE ANOMALY AND SCHWINGER TERM

In this section we restrict ourselves to constant external fields $\mathbf{E}$, $\mathbf{B}$, $\mathbf{\Omega}$, but the results can easily be extended to nonconstant external fields. Let us rewrite Eqs. (12), (3), and (4) schematically as

$$\dot{\rho} + \nabla.\mathbf{J}^{an} = 0, \quad (22)$$

$$\{\rho(\mathbf{r}),\rho(\mathbf{r}')\} = S(\mathbf{r},\mathbf{r}'); \quad \{\rho(\mathbf{r}),J_i^{an}(\mathbf{r}')\} = S_i(\mathbf{r},\mathbf{r}'). \quad (23)$$

Taking the time derivative of both sides of the generic bracket $\{\rho,\rho\} = S$ equation in Eq. (23), we get

$$\partial_0\{\rho(\mathbf{r}),\rho(\mathbf{r}')\} = \{\partial_0\rho(\mathbf{r}),\rho(\mathbf{r}')\} + \{\rho(\mathbf{r}),\partial_0\rho(\mathbf{r}')\} = \partial_0 S(\mathbf{r},\mathbf{r}'). \quad (24)$$

Using the bracket in Eq. (3), the rhs of Eq. (24) is given by

$$\partial_0 S(\mathbf{r},\mathbf{r}') = \partial_j^r(\epsilon_{ijk}\mathcal{F}_k\partial_i(\dot{\rho})\delta(\mathbf{r}-\mathbf{r}')) = -\partial_j^r(\epsilon_{ijk}\mathcal{F}_k\partial_i(\partial_l J_l^{an})\delta(\mathbf{r}-\mathbf{r}')) \quad (25)$$

where J_l^{an} is given by Eq. (13). In the lhs of Eq. (24), we have

$$\{\partial_0\rho(\mathbf{r}),\rho(\mathbf{r}')\} + \{\rho(\mathbf{r}),\partial_0\rho(\mathbf{r}')\} = -\{\partial_i J_i^{an}(\mathbf{r}),\rho(\mathbf{r}')\} - \{\rho(\mathbf{r}),\partial_i J_i^{an}(\mathbf{r}')\}. \quad (26)$$

Again substituting J_i^{an} from Eq. (13) and computing each bracket from the full bracket structure in the Supplemental Material (1,2,3) [26], after a long algebra we recover Eq. (25), thereby ensuring that the consistency condition is satisfied.

VII. DARBOUX TRANSFORMATION AND CASIMIR OPERATORS

As shown here, the anomalous fluid being an extension of the canonical Hamiltonian fluid, we now construct extensions of the two Casimirs of the latter, total ρ-charge and h-helicity, (arising from relabeling symmetry in Euler formulation for fluids), in anomalous fluid. From Darboux theorem it is possible (at least locally) to construct combinations of noncanonical variables that behave canonically. Instead of working directly with the continuous fluid variables, it is easier to construct the Darboux map in the discrete noncommutative variables (1) where the following combinations are canonical, up to $O(e\mathbf{E}, \mathbf{eB}, \mathbf{\Omega})$ for simplicity with $\{q_i, q_j\} = \{p_i, p_j\} = 0, \{q_i, p_j\} = \delta_{ij}$

$$X_i = q_i + \frac{1}{2M}\epsilon_{ijk}p_j\Omega_k, \quad P_i = p_i + \frac{eM}{2}\epsilon_{ilk}q_l B_k, \tag{27}$$

from definition of fluid variables in terms of discrete degrees of freedom (2), we generate the combinations of anomalous variables that behave canonically (subject to the approximation mentioned above):

$$\rho_c(\mathbf{r}) = \rho(\mathbf{r}) - \frac{1}{2}\epsilon_{ijk}\Omega_k\partial_i j_j(\mathbf{r}), \tag{28}$$

$$j_{(c)i}(\mathbf{r}) = j_i(\mathbf{r}) - \rho_0\int dx\left[\frac{1}{2}\epsilon_{ljk}\Omega_k\dot{X}_i\dot{X}_j\partial_l^{\mathbf{r}}\delta(\mathbf{X}(\mathbf{x})-\mathbf{r}) - \frac{e}{2}\epsilon_{ilk}X_l B_k\delta(\mathbf{X}(\mathbf{x})-\mathbf{r})\right]. \tag{29}$$

We derive the map between $\rho_c, v_{(i)c}$ and its anomalous counterpart $\rho, v_{(i)}$ and construct the Casimir operators in anomalous phase space. In (28) we have one of the cherished Casimirs ρ_c and the other Casimir, i.e. helicity can in principle be constructed using (28), (29). Note that rhs of (29) is not closing in terms of ρ, j_i, and higher moments come into play due to the essential nonlinearity in the model.

VIII. NONTRIVIAL FORMS OF Ω

Besides the constant form used here, there are nontrivial forms of $\mathbf{\Omega}$ that are of topical interest in condensed matter physics, such as

(i) Anomalous Hall Effect [14,27] in metallic ferromagnets, in the semiclassical framework, induces an anomalous velocity contribution to the Bloch wave-packet group velocity, generated by momentum-space Berry curvatures. In this case, the anomalous Hall conductivity $\sigma_{ij} = -\epsilon_{ijl}\frac{e^2}{\hbar}\Omega^l$ consists of the Berry phase $\mathbf{\Omega}$ a linear combination of reciprocal lattice vectors $\mathbf{G}$. Here, our formalism can be carried through by simply substituting the explicit expression for $\mathbf{\Omega}$ in density (ρ) and current (j_i) operators.

(ii) Another well-known example of a nontrivial Berry phase is in Weyl semimetals [28]: $\Omega_i = \pm p_i/(2|p|^3)$. A Weyl semimetal is a 3D crystal whose low energy excitations are Weyl fermions. The Berry curvature monopoles are located at Weyl points in the Brillouin zone. Unfortunately, a naive adaptation of our formalism for this particular case might be computationally problematic due to the presence of v_i in the rhs of the anomalous brackets in Eqs. (3)–(5).

IX. DISCUSSION

In this paper, we have developed an extended classical fluid model, incorporating Berry phase effects, that generates a divergence anomaly $\sim e^2\mathbf{E}.\mathbf{B}$ (having the form of the Adler-Bell-Jackiw chiral anomaly) in the helicity conservation equation. We have shown a direct analogy between the generalized fluid velocity field defined here and the well-known anomalous velocity appearing in the quantum Hall effect in condensed matter physics. The overall validity of the entire anomalous structure is demonstrated by satisfying the Adler consistency condition. We have developed a systematic program to construct the Casimir operators for the anomalous fluid model. Lastly, we have discussed the applicability of our scheme in some specific Berry curvature structures that are of interest in condensed matter physics.

Our approach is semiclassical, and we have constructed the anomalous fluid bracket structure based on Poisson brackets augmented with the (quantum mechanical) Berry phase effect. The latter is applicable for electrons moving in magnetic Bloch bands. Hence, our anomalous fluid model can have relevance in hydrodynamic equations describing electron gas models subject to spin-orbit-like interactions in condensed matter systems, such as graphene [29]. In condensed matter systems in a semiclassical framework, transport is studied through the Boltzmann equation involving the distribution function $f(\mathbf{x},\mathbf{p},t)$, and the density function appearing here is $\rho(\mathbf{x},t) = \int mf\, d\mathbf{v}$. Thus, the anomalous equations revealed here will alter the Boltzmann equation. Electron transport in Bloch bands in the hydrodynamic limit are given by Eulerian fluid equations, which should be modified appropriately. An interesting recent case is Ref. [30], where the transport of collective excitations, named chiral Berry plasmons, in the hydrodynamic limit is studied in generic interacting metallic systems with nonzero Berry flux. In Ref. [31], chiral liquids, consisting of right-left asymmetric massless fermions, are considered, where the electromagnetic current in the presence of an external magnetic field will carry a chiral anomaly.

Previously, $O(\hbar^2)$ corrections were introduced in classical fluid equations [32] from a moment expansion of the Wigner-Boltzmann equation. Interestingly, in the present work, the Berry curvature plays an essential role in inducing the $O(\hbar)$ correction. This can be seen by comparing a classical model Lagrangian

$$L = \frac{1}{2}m\mathbf{v}^2 - e\Phi + e\mathbf{A}.\mathbf{v}, \tag{30}$$

with e dimensionless, and $[\mathbf{B}] = \frac{M}{t}, [\mathbf{E}] = \frac{ML}{t^2}$, and the definitions of Lorentz force and electromagnetic fields, respectively,

$$\mathbf{F} = \nu(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \qquad \mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A}$$

with a quantum Lagrangian

$$L = \hbar k\dot{\mathbf{r}} - e\dot{\mathbf{r}}\times\mathbf{A} + \hbar\dot{k}\mathbf{A}_\beta + e\Phi - W. \tag{31}$$

Here, $[\hbar k] = \frac{ML}{t}$, $[A_B] = L$, $[\mathcal{F}] = \frac{L}{M\frac{L}{t}} = \frac{t}{M}$, with $[\mathbf{B}.\Omega]$ being dimensionless. In the above equation, A_B is the Berry potential and $\Omega = \nabla_p\times A_B$ the Berry curvature. Coming back to the fluid variables, the dimensions are $[\rho] = \frac{M}{L^3}$, $[\mathbf{v}] = \frac{L}{t}$.

Presence of chiral anomaly in a generalized helicity conservation equation has been shown in [13]. The framework is entirely different from ours as seen in the difference between explicit anomaly equations. In Ref. [13], the anomaly appears to be induced as a noninertial effect. But we have followed throughout a systematic Hamiltonian approach from first principles, starting from a semiclassical Poisson algebra with Berry phase corrections that induces a generalized (anomalous) fluid algebra. Subsequently, the Hamiltonian equations of motion yield the generalized continuity and Euler equations leading to the anomaly. Furthermore, current algebra and Schwinger terms do not play any role in Ref. [13].

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