Cobordism and deformation class of the standard model

Juven Wang,[1] Zheyan Wan,[2] and Yi-Zhuang You[3]

[1]*Center of Mathematical Sciences and Applications, Harvard University, Massachusetts 02138, USA*
[2]*Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, China*
[3]*Department of Physics, University of California, San Diego, California 92093, USA*

't Hooft anomalies of quantum field theories (QFTs) with an invertible global symmetry G (including spacetime and internal symmetries) in a dd spacetime are known to be classified by a $d + 1$d cobordism group $\mathrm{TP}_{d+1}(G)$, whose group generator is a $d + 1$d cobordism invariant written as an invertible topological field theory (iTFT) with a partition function $\mathbf{Z}_{d+1}$. It has recently been proposed that the deformation class of QFTs is specified by its symmetry G and an iTFT $\mathbf{Z}_{d+1}$. Seemingly different QFTs of the same deformation class can be deformed each other via quantum phase transitions. In this work, we ask which cobordism class and deformation class control the 4d standard model (SM) of ungauged or gauged $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_q$ group for $q = 1, 2, 3, 6$ with a continuous or discrete baryon minus lepton $(\mathbf{B} - \mathbf{L})$-like symmetry. We show that the answer contains some combination of 5d iTFTs; two $\mathbb{Z}$ classes associated with $(\mathbf{B} - \mathbf{L})^3$ and $(\mathbf{B} - \mathbf{L}) - (\text{gravity})^2$ 4d perturbative local anomalies, a $\mathbb{Z}_{16}$ class Atiyah-Patodi-Singer η invariant, a $\mathbb{Z}_2$ class Stiefel-Whitney $w_2 w_3$ invariant associated with 4d non-perturbative global anomalies, and additional $\mathbb{Z}_3 \times \mathbb{Z}_2$ global anomalies involving higher symmetries whose charged objects are Wilson electric or 't Hooft magnetic line operators. Out of the multiple infinite $\mathbb{Z}$ classes of local anomalies and 24576 classes of global anomalies, we pin down the deformation class of the SM labeled by $(N_f, n_{\nu_R}, \mathrm{p}', q)$, the family number, the total "right-handed sterile" neutrino number, the magnetic monopole datum, and the mod q relation. We show that grand unification such as Georgi-Glashow $su(5)$, Pati-Salam $su(4) \times su(2) \times su(2)$, Barr's flipped $u(5)$, and the familiar or modified $so(n)$ models of $\mathrm{Spin}(n)$ gauge group, e.g., with $n = 10, 18$ can all reside in an appropriate SM deformation class. We show that ultra unification, which replaces some of sterile neutrinos with new exotic gapped/gapless sectors (e.g., topological or conformal field theory) or gravitational sectors with topological origins via cobordism constraints, also resides in an SM deformation class. Neighbor quantum phases near SM or their phase transitions, and neighbor gapless quantum critical regions naturally exhibit beyond SM phenomena.

I. INTRODUCTION AND SUMMARY

Symmetries in quantum field theories (QFTs) in the strict sense mean that the QFT is invariant under certain sets of transformations (i.e., symmetry transformations). The symmetry invariance of QFT may be exhibited in many ways depending on how the QFT is formulated: (1) in the path integral formulation (say in dd for d-dimensional spacetime) the symmetry transformation on quantum fields or spacetime coordinates leaves the path integral $\mathbf{Z}_d$ invariant, (2) in the Hamiltonian approach the symmetry transformation corresponds to a quantum operator $\hat{U}$ commuting with the Hamiltonian $\hat{H}$, i.e., $[\hat{H}, \hat{U}] = 0$. Importantly, the symmetry may only be invariant up to a global complex phase; for example, the symmetry transformation can map the path

integral $\mathbf{Z}_d \mapsto e^{i\Theta} \mathbf{Z}_d$. If $e^{i\Theta}$ cannot be eliminated by dd local counter terms, this $e^{i\Theta}$ manifests a mild violation of symmetry invariance. Such a symmetry is an anomalous symmetry (i.e., symmetry with 't Hooft anomaly [1]). In contrast, any symmetry transformation without any $e^{i\Theta}$ output is an anomaly-free symmetry. Note that $e^{i\Theta}$ is invertible (i.e., $e^{i\Theta} \cdot e^{-i\Theta} = 1$)—the anomaly inflow [2,3] states that the $e^{i\Theta}$ not only characterizes a dd invertible 't Hooft anomaly of a symmetry G, but also characterizes a one-higher dimensional invertible topological field theory (iTFT) with a partition function $\mathbf{Z}_{d+1}$ of the symmetry G. The $\mathbf{Z}_{d+1}$ turns out to be a cobordism invariant in $d + 1$d ([3–8] and references therein). These iTFTs $\mathbf{Z}_{d+1}$ form an Abelian group; the group binary operation is the stacking of iTFTs, the identity element is 1, the inverse is $\mathbf{Z}_{d+1}^{-1}$; they also obey closure, associativity, and commutativity. By [7], this cobordism group is a specific version $\mathrm{TP}_{d+1}(G)$ for smooth manifolds with G-structure.

A. Deformation classes of QFTs

We can use the symmetry to organize the quantum fields into the representations of the symmetry group, and give

powerful selection rules constraining the quantum dynamics. Anomalies are invariant under continuous deformations of parameters of QFTs, including the renormalization group flow [1] ([9,10] and references therein). Given the importance of symmetries and their associated 't Hooft anomalies, Seiberg [11] and others [12] conjectured that any QFTs with the same symmetry G and same anomaly $\mathbf{Z}_{d+1}$ can be deformed to each other by adding degrees of freedom at short distances that preserve the same symmetry and maintain the same anomaly. Namely, the whole system allows a large class of deformations via all symmetric interactions between the original QFT and other new higher-energy sectors of extra QFTs. This organization principle connects a large class of QFTs together within the same data $(G, \mathbf{Z}_{d+1})$ via any symmetric deformation (possibly with discontinuous or continuous quantum phase transitions [13] between different phases). The deformation class of QFTs in dd [11], is indeed related to the cobordism or deformation class of iTFTs in one dimensional higher in $d + 1$d [7].

However, it is widely believed that there is no global symmetry in quantum gravity (QG) ([14] and references therein). Skeptical readers may hold prejudice against this deformation class labeled by $(G, \mathbf{Z}_{d+1})$, possibly not universal nor useful in QG. But several reasons below suggest its importance even in QG. First, there could still be various QFTs with global symmetries localized on a brane or obtained via compactification from QG. Second, McNamara-Vafa [15] suggests that if a dd effective theory is obtained via the compactification from a quantum gravity in a total spacetime Dd, and if the compact $(D - d)$d manifold is in a cobordism class $\mathbf{Z}_{D-d}$, it either suggests the theory belongs to the swampland in QG, or suggests the $\mathbf{Z}_{D-d}$ must be appropriately trivialized to zero in the full suitably interpreted QG cobordism group $\Omega_{D-d}^{\mathrm{QG}} = 0$ (e.g., the trivialization of cobordism class via introducing new defects to cancel each other's cobordism class). This original dd theory can be connected by a finite-energy $(d - 1)$d domain wall to nothing on the other side. So we may say that Ref. [15] justifies the use of deformation class of quantum gravity. Third, although there is no global symmetry in QG, there can be either explicitly broken global symmetry or dynamically gauged symmetry (i.e., global symmetry is dynamically gauged as gauge symmetry, which we shall call Weyl's gauge principle [16]) in QG. Therefore, for all the reasons above, it is promising to explore the idea of cobordism and deformation class further, in the context of both QFT [11] and QG [15].

In this work, we shall focus on a specific question; What is the deformation class of the 4d standard model (SM) [17–20]? Before attempting to solve this, we shall sharpen the question further. We need to first specify the spacetime-internal symmetries of SM, typically written as $G \equiv [(G_{\mathrm{spacetime}} \times G_{\mathrm{internal}})/N_{\mathrm{shared}}] \equiv G_{\mathrm{spacetime}} \times_{N_{\mathrm{shared}}} G_{\mathrm{internal}}$. The N_{shared} is the shared common normal subgroup symmetry between $G_{\mathrm{spacetime}}$ and G_{internal}, e.g., N_{shared} can be the fermion parity symmetry $\mathbb{Z}_2^F$, which acts on fermions by $\psi \mapsto -\psi$. The Lie algebra of the

internal symmetry of SM is $su(3) \times su(2) \times u(1)$, but the global structure Lie group G_{SM_q} has four possible versions [21,22], $G_{\mathrm{SM}_q} \equiv [\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_{\tilde{Y}}]/\mathbb{Z}_q$, with $q = 1, 2, 3, 6$. We also include the baryon minus lepton $(\mathbf{B} - \mathbf{L})$ vector symmetry respected by the SM gauge structure and Yukawa-Higgs terms. We consider both a continuous $\mathrm{U}(1)_{\mathbf{B}-\mathbf{L}}$ or $\mathrm{U}(1)_X$, or a discrete order-4 finite Abelian $\mathbb{Z}_{4,X}$ group; there $X \equiv 5(\mathbf{B} - \mathbf{L}) - \frac{2}{3}\tilde{Y}$ is Wilczek-Zee's chiral symmetry [23] with the electroweak hypercharge $\tilde{Y}$. We decide not to include extra discrete charge conjugation, parity, or time-reversal (C, P, or T) symmetries as they are already individually broken at the SM energy scale. The SM is a Lorentz invariant QFT that obeys the spacetime rotation and boost symmetry $G_{\mathrm{spacetime}} =$ Spin group [Spin(3,1) in Lorentz signature and Spin(4) in Euclidean signature]. Gathering the above data, we learn that two physically pertinent versions of spacetime-internal symmetries of SM are

$$G_{\mathrm{SM}_q}^{\mathrm{U}(1)} \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{U}(1) \times G_{\mathrm{SM}_q} \quad \text{and}$$

$$G_{\mathrm{SM}_q}^{\mathbb{Z}_{4,X}} \equiv \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\mathrm{SM}_q}. \tag{1}$$

Both versions of the symmetry (1) are compatible with the representation (rep) of the three families of 15 Weyl fermions of quarks and leptons (below written as left-handed 2-component Weyl spinors of the spacetime Spin group) confirmed by experiments, possibly with or without the 16th Weyl fermion (the right-handed neutrino sterile to SM gauge force),

$$\bar{d}_R \oplus l_L \oplus q_L \oplus \bar{u}_R \oplus \bar{e}_R \oplus n_{\nu_{j,R}} \bar{\nu}_{j,R} \sim (\bar{\mathbf{3}}, \mathbf{1})_2$$

$$\oplus (\mathbf{1}, \mathbf{2})_{-3} \oplus (\mathbf{3}, \mathbf{2})_1 \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \mathbf{1})_6$$

$$\oplus n_{\nu_{j,R}}(\mathbf{1}, \mathbf{1})_0 \text{ for each family of } su(3) \times su(2) \times u(1)_{\tilde{Y}}. \tag{2}$$

We use $n_{\nu_{e,R}}, n_{\nu_{\mu,R}}, n_{\nu_{\tau,R}} \in \{0, 1\}$ to label either the absence or presence of the 16th Weyl fermion for each of the three families of electron e, muon μ, or tauon τ types. In general, we can consider N_f families (typically $N_f = 3$) and a total sterile neutrino number $n_{\nu_R} \equiv \sum_j n_{\nu_{j,R}}$ that can be equal, smaller, or larger than 3.

In Sec. II, we consider the 4d SM with internal symmetry G_{SM_q} ungauged as a global symmetry group. Both versions of spacetime-internal symmetries (1) are treated as global symmetries, so we study the global symmetries and their 't Hooft anomalies of the ungauged Standard Model at $d = 4$. However, 't Hooft anomaly of the ungauged theory is the obstruction of gauging. Some 't Hooft anomalies of global symmetries directly descend to the dynamical gauge anomalies of gauge symmetries, after gauging G_{SM_q}. So we can also use the cobordism group $\mathrm{TP}_5(G)$ to check the anomaly cancellation consistency of the gauge theory.

In Sec. III, the internal symmetry G_{SM_q} is thereafter dynamically gauged and treated as a gauge group in the gauged standard model with dynamical Yang-Mills gauge fields. We will see that the spacetime-internal symmetry

Eq. (1) has to be modified to a new G' by quotienting out the gauge group $[G_{SM_q}]$ and including extra generalized higher symmetry [24] (which is also an internal global symmetry but whose charged objects are extended 1d Wilson-'t Hooft line operators). This $TP_5(G')$ instead will be used to constrain the quantum dynamics of the gauge theory.

II. STANDARD MODEL WITH INTERNAL SYMMETRY UNGAUGED AND 'T HOOFT ANOMALIES

We aim to determine the deformation class $(G, \mathbf{Z}_5)$ of ungauged 4d SM with spacetime-internal symmetries G given in (1). References [8,12,25–29] consider systematic anomaly classifications include several SM-like or grand unification (GUT)-like groups. The relevant cobordism groups that classify the 4d anomaly and 5d iTFT $\mathbf{Z}_5$ are [28–32]

$$TP_5(G_{SM_q}^{U(1)}) = TP_5(Spin \times_{\mathbb{Z}_2^F} U(1) \times G_{SM_q})$$
$$= TP_5(Spin^c \times G_{SM_q}) = \mathbb{Z}^{11}, \qquad (3)$$

$$TP_5(G_{SM_q}^{\mathbb{Z}_{4,X}}) = TP_5(Spin \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{SM_q})$$
$$= \begin{cases} \mathbb{Z}^5 \times \mathbb{Z}_2 \times \mathbb{Z}_4^2 \times \mathbb{Z}_{16}, & q = 1, 3 \\ \mathbb{Z}^5 \times \mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_{16}, & q = 2, 6. \end{cases} \qquad (4)$$

Here are some remarks:

a. We turn on all possible background fields (which are also called classical nondynamical fields, or symmetry twists [6], that are not summed over in the path integral) that couple to the charged objects of QFT. Then we can perform all symmetry transformations of QFT by changing the values of background fields to detect all invertible $\mathbf{Z}_5 = e^{i\Theta}$.

b. The symmetry G consists of internal symmetry and spacetime symmetry. Internal symmetry transformations act on charged objects that form a representation (rep) of group $G_{internal}$. We couple the 0d charged particle to a 1d background gauge field A. Spacetime symmetry transformations act on the spacetime coordinates or metrics by spacetime diffeomorphisms. The anomalies are detected by turning on (i) internal symmetry's background gauge fields, (ii) spacetime symmetry's background fields via diffeomorphism or via placing the theory on G-structured manifolds, or (iii) mixed internal-spacetime background fields. These anomalies are respectively called (i) gauge anomaly, (ii) gravitational anomaly, or (iii) mixed gauge-gravity or gauge-diffeomorphism anomaly. These adjectives, gauge or gravity, characterize the nature of the background fields of those anomalies.

c. Cobordism group $TP_{d+1}(G)$ not only contains all finite subgroup $\mathbb{Z}_n$ (the torsion part) of the standard bordism group Ω_{d+1}^G, but also contains the integer $\mathbb{Z}$ classes descended from those $\mathbb{Z}$ (the free part) of the bordism group Ω_{d+2}^G.

d. The dd perturbative local anomalies [33–35] correspond to the torsion free $\mathbb{Z}$ classes of $TP_{d+1}(G)$. They

are detected by infinitesimal gauge/diffeomorphism transformations and captured by Feynman diagram calculations. The dd nonperturbative global anomalies [36–38] correspond to the torsion $\mathbb{Z}_n$ classes of $TP_{d+1}(G)$. They are detected via large gauge/diffeomorphism transformations that cannot be continuously deformed from the identity. These adjectives, perturbative or not, characterize the local or global nature of those anomalies.

Now we can obtain the deformation class $\mathbf{Z}_5$ of the ungauged SM by checking which 't Hooft anomaly persists:

1. For SM with a $U(1)_{\mathbf{B-L}}$ vector symmetry, Eq. (3) shows $\mathbb{Z}^{11}$ classes of local anomalies captured by triangle Feynman diagrams. Given the SM chiral fermion content (2), all of local anomalies cancel to zero [39] except two $\mathbb{Z}$ classes [30]; the cubic pure gauge $U(1)_{\mathbf{B-L}}^3$ and mixed gauge-gravity $U(1)_{\mathbf{B-L}} -$ (gravity)^2 anomalies. Their 5d iTFT evaluated on a 5d manifold M^5 is

$$\mathbf{Z}_5^{U(1)} \equiv \exp\left(i(-N_f + n_{\nu_R})\left(\left(\int_{M^5} A c_1^2\right) + \frac{1}{48} CS_3^{T(PD(c_1))}\right)\right), \qquad (5)$$

depending on the family number N_f and total sterile neutrino number n_{ν_R}. Typically, for $N_f = 3$, we have $-N_f + n_{\nu_R} \equiv -N_f + \sum_j n_{\nu_{j,R}} = -3 + n_{\nu_{e,R}} + n_{\nu_{\mu,R}} + n_{\nu_{\tau,R}} + \dots$. Let us explain the components in (5). The A is the $Spin \times_{\mathbb{Z}_2^F} U(1)_{\mathbf{B-L}} \equiv Spin^c$ gauge field, while $2A$ is the ordinary Abelian gauge field, whose first Chern class $c_1 = \frac{d(2A)}{2\pi}$ satisfies a spin-charge relation $c_1 = w_2(TM)$ mod 2 [40] with the second Stiefel-Whitney class $w_2(TM)$ of spacetime tangent bundle TM of the base manifold M. The gravitational Chern-Simons 3-form is $CS_3^{TM^3} \equiv \frac{1}{4\pi}\int_{M^3=\partial M^4} Tr(\omega d\omega + \frac{2}{3}\omega^3) = \frac{1}{4\pi}\int_{M^4} Tr(R(\omega) \wedge R(\omega))$ where M^3 is evaluated as a boundary ∂M^4, while ω is the 1-connection of tangent bundle TM and $R(\omega)$ is the Riemann curvature 2-form of ω. In Eq. (5), we take $M^3 = PD(c_1)$ to be a 3-manifold Poincaré dual (PD) to the degree-2 first Chern class c_1 on the 5d M^5. On a closed oriented M^4, we further have $\frac{1}{4\pi}\int_{M^4} Tr(R(\omega) \wedge R(\omega)) = 2\pi \int_{M^4} p_1(TM) = 2\pi \cdot 3\sigma$, where p_1 is the first Pontryagin class of TM and the σ is the M^4's signature. Overall, the 5d term (5) descends from a 6d anomaly polynomial on M^6 at the $U(1)$-valued $\theta = 2\pi$, $\exp\{i\theta(-N_f + n_{\nu_R}) \times [(\int_{M^6} \frac{1}{2} c_1^3) + \frac{1}{16}\sigma(PD(c_1))]\}$. Since $U(1)_Y$ is fully anomaly free, our above result is applicable to SM with a $U(1)_{X \equiv 5(\mathbf{B-L}) - \frac{2}{3}\tilde{Y}}$ chiral symmetry [23]; we still obtain the same deformation class (5) as $\mathbf{Z}_5^{U(1)}$ when replacing $U(1)_{\mathbf{B-L}}$ to $U(1)_X$.

2. For SM with a $\mathbb{Z}_{4,X}$ chiral symmetry, given the fermion content (2), all $\mathbb{Z}^5$ classes of local anomalies

in (4) cancel to zero. All extra $\mathbb{Z}_2$ and $\mathbb{Z}_4$ classes of global anomalies in (4) also cancel, but the $\mathbb{Z}_{16}$ mixed gauge-gravity global anomaly can be nonzero [30,32], whose corresponding 5d iTFT is

$$\mathbf{Z}_5^{\mathbb{Z}_{4,X}} \equiv \exp\left(\mathrm{i}(-N_f + n_{\nu_R})\frac{2\pi\mathrm{i}}{16}\eta_{4\mathrm{d}}(\mathrm{PD}(A_{\mathbb{Z}_{2,X}}))\right). \quad (6)$$

The background gauge field $A_{\mathbb{Z}_{2,X}}$ is the generator of the cohomology group $H^1(\mathrm{B}\mathbb{Z}_{2,X}, \mathbb{Z}_2)$ with $\mathbb{Z}_{2,X} \equiv \mathbb{Z}_{4,X}/\mathbb{Z}_2^F$ of $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$-structure manifold M. The classifying space $\mathrm{B}^n\mathrm{A} \equiv K(\mathrm{A}, n)$ is the Eilenberg-MacLane space for an Abelian group A. Hereafter we use the standard convention, all cohomology classes are pulled back to the manifold M along the maps given in the definition of cobordism groups, thus $H^1(\mathrm{B}\mathbb{Z}_{2,X}, \mathbb{Z}_2)$ can be pulled back to $H^1(M, \mathbb{Z}_2)$. Here a 5d $\mathbb{Z}_{16}$ class Atiyah-Patodi-Singer (APS [41]) eta invariant $\eta_{5\mathrm{d}} = \eta_{4\mathrm{d}}(\mathrm{PD}(A_{\mathbb{Z}_{2,X}}))$ is written as the 4d eta invariant $\eta_{4\mathrm{d}} \in \mathbb{Z}_{16}$ on the Poincaré dual (PD) submanifold of $A_{\mathbb{Z}_{2,X}}$ (i.e., 4d Pin^+-manifold), based on a Smith homomorphism for the bordism group $\Omega_5^{\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4} \cong \Omega_4^{\mathrm{Pin}^+} = \mathbb{Z}_{16}$ [42–45]. The $\mathbb{Z}_{16}$ class 4d eta invariant $\eta_{4\mathrm{d}}$ is the topological invariant of the interacting fermionic topological superconductor of condensed matter in three spatial dimensions with an antiunitary time-reversal symmetry $\mathbb{Z}_4^T$ such that the time-reversal symmetry generator T squares to the fermion parity operator, namely $T^2 = (-1)^F$ (see a review [46,47]). In contrast, in SM, we have the unitary $(\mathbf{B} - \mathbf{L})$-like symmetry $\mathbb{Z}_{4,X}$ whose generator X squares to $X^2 = (-1)^F$.

3. Thanks to the Lie group embedding $(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times G_{\mathrm{SM}_6}) \subset \mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(10) \equiv G^{\mathrm{Spin}(10)}$ [28], we can embed the 16-Weyl-fermion SM_6 (i.e., $n_{\nu_{e,R}}, n_{\nu_{\mu,R}}, n_{\nu_{\tau,R}} = 1$) with rep **16** into a Spin(10) chiral gauge theory, the $so(10)$ GUT. We can again ask the deformation class $(G^{\mathrm{Spin}(10)}, \mathbf{Z}_5^{\mathrm{Spin}(10)})$ of the ungauged 4d model. Its cobordism group [8,12,28] is a $\mathrm{p} \in \mathbb{Z}_2$ class for $\mathrm{Spin}(n)$ with $n \geq 7$,

$$\mathrm{TP}_5(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathrm{Spin}(n \geq 7)) = \mathbb{Z}_2, \quad (7)$$

$$\mathbf{Z}_5^{\mathrm{Spin}(n)} \equiv \exp\left(\mathrm{i}\,\pi\,\mathrm{p}\int_{M^5} w_2 w_3\right), \quad (8)$$

which is $\exp(\mathrm{i}\pi\mathrm{p}\int_{M^5} w_2(TM)w_3(TM)) = \exp(\mathrm{i}\pi\mathrm{p}\int_{M^5} w_2(V_{\mathrm{SO}(n)})w_3(V_{\mathrm{SO}(n)}))$. Here $w_j(TM)$ or $w_j(V_{\mathrm{SO}(n)})$ is the jth Stiefel-Whitney class of the spacetime tangent bundle TM, or the associated vector bundle of the principal gauge bundle of $\mathrm{SO}(n) = \mathrm{Spin}(n)/\mathbb{Z}_2^F$. The only 5d iTFT represents a $\mathrm{p} \in \mathbb{Z}_2$ class 4d mixed gauge-gravity global anomaly. The conventional $so(10)$ GUT [48] has no 4d anomaly thus resides in a $\mathrm{p} = 0$ trivial deformation class $\mathbf{Z}_5^{\mathrm{Spin}(10)} = 1$ [12,38,49].

Ref. [50,51] construct a modified $so(10)$ GUT with an extra discrete torsion class of 4d–5d Wess-Zumino-Witten term from the GUT-Higgs field. The modified $so(10)$ GUT has a $w_2 w_3$ anomaly, thus it resides in a $\mathrm{p} = 1$ deformation class. Since $\mathrm{Spin}(18) \supset \mathrm{Spin}(10)$ contains the same $w_2 w_3$ deformation class, one can work out a similar story for a Spin(18) chiral gauge theory or $so(18)$ GUT [52]. In fact, within either $\mathrm{p} \in \mathbb{Z}_2$ deformation class, Ref. [50,51] show that the SM can quantum-phase transition to the GUT-like models of $su(5)$ [53], $su(4) \times su(2) \times su(2)$ [54], or flipped $u(5)$ [55] models. Especially for $\mathrm{p} = 1$ deformation class, Ref. [50,51] suggest a gapless deconfined quantum critical region (analogous to [56]) with beyond-SM phenomena near the SM vacuum by tuning the GUT-Higgs potential.

In summary, the deformation class $(G, \mathbf{Z}_5)$ of ungauged SM with $\mathrm{U}(1)_{\mathbf{B-L}}$ or $\mathrm{U}(1)_X$ is $(G_{\mathrm{SM}_q}^{\mathrm{U}(1)}, \mathbf{Z}_5^{\mathrm{U}(1)})$, and that of ungauged SM with $\mathbf{Z}_5^{\mathbb{Z}_{4,X}}$ is $(G_{\mathrm{SM}_q}^{\mathbb{Z}_{4,X}}, \mathbf{Z}_5^{\mathbb{Z}_{4,X}})$, for any $q = 1, 2, 3, 6$. For the ungauged SM embeddable into a conventional or modified $so(n)$ GUT, we obtain an extra deformation class data $(G^{\mathrm{Spin}(n)}, \mathbf{Z}_5^{\mathrm{Spin}(n)})$.

III. STANDARD MODEL WITH INTERNAL SYMMETRY GAUGED AND WITH HIGHER SYMMETRIES

Next we determine the deformation class of G_{SM_q}-dynamically gauged SM. Before solving that problem, let us recall the relationship between the deformation classes of the ungauged SM (Sec. II) and the gauged SM:

a. The vanishing 't Hooft anomalies (thus not presented in the deformation class $\mathbf{Z}_5$) for the symmetry that we will gauge in the ungauged SM indeed one-to-one correspond to the dynamical gauge-diffeomorphism anomaly cancellation conditions for the gauged SM.

b. If the gauged SM has only G_{SM_q} gauged, while keeping the global symmetry $\mathrm{U}(1)_{\mathbf{B-L}}$, $\mathrm{U}(1)_X$, or $\mathbb{Z}_{4,X}$ ungauged, then this SM deformation class still includes Eqs. (5), (6), and (8).

c. If the gauged SM has not only G_{SM_q} but also $\mathbf{B} - \mathbf{L}$ or X dynamically gauged, then the 5d iTFT also has to be dynamically gauged to a new theory, which is either an noninvertible gapped TFT or a gapless theory. In this scenario, we could no longer study the 4d SM on its own. The 4d SM and 5d theories are strongly correlated and coupled with each other through $\mathbf{B} - \mathbf{L}$ or X gauge forces mediating long-range entanglements.

d. In a gauge theory, the internal symmetry also includes higher n symmetries as generalized global symmetries [24] that act on nd charged objects (1d lines, 2d surfaces, etc.) by coupling the nd objects to $n + 1$d background gauge fields. We denote the higher n-symmetry group G as $G_{[n]}$. The $n + 1$d gauge field lives in a higher classifying space $\mathrm{B}^n G$ (namely, the

Eilenberg-MacLane space) of the group $G_{[n]}$. Once G_{SM_q} is gauged, kinematically, we may obtain extra 1-form electric and magnetic symmetries, $G^e_{[1]}$ and $G^m_{[1]}$, which are determined from the center of gauge group $Z(G_{\mathrm{SM}_q})$, and the Pontryagin dual of the homotopy

group $\pi_1(G_{\mathrm{SM}_q})^\vee$. Moreover the 0d SM gauge charge fermionic particle (2) explicitly breaks the $G^e_{[1]}$ from $Z(G_{\mathrm{SM}_q})$ to a subgroup [51,57–59] that acts trivially on all particle representations,

	$Z(G_{\mathrm{SM}_q})$	$\pi_1(G_{\mathrm{SM}_q})^\vee$	1-form e sym $G^e_{[1]}$	1-form m sym $G^m_{[1]}$
$G_{\mathrm{SM}_q} \equiv \frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)_{\tilde{Y}}}{\mathbb{Z}_q}$	$\mathbb{Z}_{6/q}\times\mathrm{U}(1)$	$\mathrm{U}(1)$	$\mathbb{Z}^e_{6/q,[1]}$	$\mathrm{U}(1)^m_{[1]}$

$$(9)$$

Thus, we have to modify (1), by removing G_{SM_q} and including $G^e_{[1]}\times G^m_{[1]}$, to

$$G'^{\mathrm{U}(1)}_{\mathrm{SM}_q} \equiv \mathrm{Spin}\times_{\mathbb{Z}^F_2}\mathrm{U}(1)\times\mathbb{Z}^e_{6/q,[1]}\times\mathrm{U}(1)^m_{[1]} \quad \text{and} \quad G'^{\mathbb{Z}_{4,X}}_{\mathrm{SM}_q} \equiv \mathrm{Spin}\times_{\mathbb{Z}^F_2}\mathbb{Z}_{4,X}\times\mathbb{Z}^e_{6/q,[1]}\times\mathrm{U}(1)^m_{[1]}. \quad (10)$$

We denote $q'\equiv 6/q$, where $q=1,2,3,6$ maps to $q'=6,3,2,1$, and define $q'\equiv 2^{n_2}\cdot 3^{n_3}$ where $q'=1,2,3,6$ corresponds to $(n_2,n_3)=(0,0),(1,0),(0,1),(1,1)$. Below we examine what are additional new terms in the deformation class of SM due to 't Hooft anomalies of higher symmetries of 1d line operators. All possible 4d invertible anomalies are classified by the 5d cobordism groups TP_5 of (10) [29],

$$\mathrm{TP}_5(G'^{\mathrm{U}(1)}_{\mathrm{SM}_q}) = \mathrm{TP}_5(\mathrm{Spin}\times_{\mathbb{Z}^F_2}\mathrm{U}(1)\times\mathrm{B}\mathbb{Z}_{q'}\times\mathrm{BU}(1)) = \mathrm{TP}_5(\mathrm{Spin}^c\times\mathrm{B}\mathbb{Z}_{q'}\times\mathrm{BU}(1)) = \mathbb{Z}^2\times\mathbb{Z}_{q'},$$

$$\text{generated by } Ac_1^2, \quad \frac{1}{48}\mathrm{CS}_3^{T(\mathrm{PD}(c_1))}, \quad \text{and} \quad B^e_{\mathbb{Z}_{q',[1]}}(\mathrm{d}B^m_{\mathrm{U}(1),[1]} \bmod q'), \quad (11)$$

$$\mathrm{TP}_5(G'^{\mathbb{Z}_{4,X}}_{\mathrm{SM}_q}) = \mathrm{TP}_5(\mathrm{Spin}\times_{\mathbb{Z}^F_2}\mathbb{Z}_{4,X}\times\mathrm{B}\mathbb{Z}_{q'}\times\mathrm{BU}(1)) = \mathbb{Z}_{16}\times(\mathbb{Z}_4)^{n_2}\times\mathbb{Z}_{q'},$$

$$\text{generated by } \eta_{4d}(\mathrm{PD}(A_{\mathbb{Z}_{2,X}})), \quad \eta'_{1d}(\mathrm{PD}(\mathcal{P}(B^e_{\mathbb{Z}_{2,[1]}}))), \quad \text{and} \quad B^e_{\mathbb{Z}_{q',[1]}}(\mathrm{d}B^m_{\mathrm{U}(1),[1]} \bmod q'). \quad (12)$$

Below we first explain the 5d iTFTs of (11) and (12), and then determine the deformation class of the gauged SM:

1. The $\mathbb{Z}^2$ and $\mathbb{Z}_{16}$ classes give the same 5d iTFTs already appeared in Sec. II. So the deformation class of ungauged SM in Eqs. (5) and (6) still remains in that of gauged SM.

2. $B^e_{\mathbb{Z}_{q',[1]}}$ and $B^m_{\mathrm{U}(1),[1]}$ are the background fields of 1-form electric and magnetic symmetries respectively, namely $\mathbb{Z}^e_{q',[1]}$ and $\mathrm{U}(1)^m_{[1]}$ in (9). Again, all cohomology classes are pulled back to the manifold M. Since the maps $M\to\mathrm{B}^2\mathbb{Z}_{q'}$ and $M\to\mathrm{B}^2\mathrm{U}(1)$ are given in the definitions of the cobordism groups, the cohomology classes $B^e_{\mathbb{Z}_{q',[1]}}\in H^2(\mathrm{B}^2\mathbb{Z}_{q'},\mathbb{Z}_{q'})$ and $B^m_{\mathrm{U}(1),[1]}\in H^2(\mathrm{B}^2\mathrm{U}(1),\mathrm{U}(1))$ are pulled back to $H^2(M,\mathbb{Z}_{q'})$ and $H^2(M,\mathrm{U}(1))$, respectively.

3. $\eta'_{1d}(\mathrm{PD}(\mathcal{P}(B^e_{\mathbb{Z}_{2,[1]}})))$; η'_{1d} is a generator of the cobordism group $\mathrm{TP}_1(\mathrm{Spin}\times_{\mathbb{Z}^F_2}\mathbb{Z}^X_4)=\mathbb{Z}_4$. The Pontryagin square $\mathcal{P}(B^e_{\mathbb{Z}_{2,[1]}})\equiv B^e_{\mathbb{Z}_{2,[1]}}\smile B^e_{\mathbb{Z}_{2,[1]}}+B^e_{\mathbb{Z}_{2,[1]}}\smile_1\delta B^e_{\mathbb{Z}_{2,[1]}}$ is a cohomology class in $H^4(M,\mathbb{Z}_4)$, while $B^e_{\mathbb{Z}_{2,[1]}}$ is a cohomology class in $H^2(M,\mathbb{Z}_2)$, the $\smile_j$ is the jth cup product between cochains (we make the 0th cup product $\smile$ implicit intentionally throughout the article), and δ is a coboundary operator. Since M is $\mathbb{Z}$ orientable and hence $\mathbb{Z}_4$ orientable, the Poincaré dual $\mathrm{PD}(\mathcal{P}(B^e_{\mathbb{Z}_{2,[1]}}))$ is represented by a 1d $\mathrm{Spin}\times_{\mathbb{Z}_2}\mathbb{Z}_4$ submanifold of M; thus $\eta'_{1d}(\mathrm{PD}(\mathcal{P}(B^e_{\mathbb{Z}_{2,[1]}})))$ is

well defined. Whether the gauged SM has this mixed anomaly between the chiral $\mathbb{Z}_{4,X}$ and the $\mathbb{Z}^e_{q',[1]}$ symmetries (only at $q'=2,6$ thus $q=3,1$) can be computed via the techniques in [60–63].

Reference ([62]) finds that two adjoint Weyl fermions of SU(2) gauge theory with $\mathrm{Spin}\times_{\mathbb{Z}^F_2}\mathbb{Z}_8\times\mathbb{Z}^e_{2,[1]}$ symmetry has an anomaly index $1\in\mathbb{Z}_4$ class of $A'\mathcal{P}(B^e_{\mathbb{Z}_{2,[1]}})$ due to fractional SO(3) instantons, where $A'\in H^1(\mathrm{B}(\frac{\mathbb{Z}_8}{\mathbb{Z}_2}),\mathbb{Z}_4)$ is pulled back to $H^1(M,\mathbb{Z}_4)$. In comparison, we consider that each family of SM has four SU(2) fundamental Weyl fermions as four SU(2) doublets **2** gauged under $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge theory for G_{SM_q} at $q=1,3$ with $\mathrm{Spin}\times_{\mathbb{Z}^F_2}\mathbb{Z}_{4,X}\times\mathbb{Z}^e_{2,[1]}$ symmetry. But this theory does not carry an anomaly index in the $\mathbb{Z}_4$ class of $\eta'_{1d}(\mathrm{PD}(\mathcal{P}(B^e_{\mathbb{Z}_{2,[1]}})))$, because the $\mathrm{U}(1)_X$ symmetry is not broken down to $\mathbb{Z}_{4,X}$ due to SU(2) instantons. So any fractional U(2) instantons cannot induce any further mixed anomaly between the $\mathbb{Z}_{4,X}$-$\mathbb{Z}^e_{q',[1]}$ symmetry.

4. $B^e_{\mathbb{Z}_{q',[1]}}(\mathrm{d}B^m_{\mathrm{U}(1),[1]} \bmod q')$: This 5d term represents a mixed anomaly between the electric 1-form $\mathbb{Z}_{q'}$ symmetry and the magnetic 1-form U(1) symmetry that we will detect in the 4d SM. To define this 5d BdB'-like term mathematically precisely, we first define a generator $\tilde{\tau}_3\equiv\mathrm{d}B^m_{\mathrm{U}(1),[1]}\in H^3(\mathrm{B}^2\mathrm{U}(1),\mathbb{Z})$

represented by the identity map $B^2U(1) = B^3\mathbb{Z} \to B^3\mathbb{Z} = B^2U(1)$. The $\tilde{\tau}_3$ is pulled back to $H^3(M, \mathbb{Z})$ along the map $M \to B^2U(1)$ given in the definition of the cobordism group. Here we regard $B_{U(1),[1]}^m$ as the generator of $H^2(B^2U(1), U(1)) = \mathbb{Z}$. Also, we regard d as the boundary map in the long exact sequence $\cdots \to H^2(-, U(1)) \xrightarrow{\text{d}} H^3(-, \mathbb{Z}) \to \cdots$ induced from the short exact sequence $1 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1$. Note that d is not the coboundary map δ which sends a cochain to a coboundary. The factorization $\mathbb{Z}_{q'} = (\mathbb{Z}_3)^{n_3} \times (\mathbb{Z}_2)^{n_2}$ implies $B_{\mathbb{Z}_{q'},[1]}^e (\mathrm{d}B_{U(1),[1]}^m \bmod q') = -2n_3 B_{\mathbb{Z}_3,[1]}^e \times (\mathrm{d}B_{U(1),[1]}^m \bmod 3) + 3n_2 B_{\mathbb{Z}_2,[1]}^e (\mathrm{d}B_{U(1),[1]}^m \bmod 2)$. We check that, for G_{SM_q} with Weyl fermion content in (2), there is indeed a 4d mixed higher anomaly of 1-form symmetries between $\mathbb{Z}_{6/q,[1]}^e$ and $U(1)_{[1]}^m$. For any family, the gauged SM contains the anomaly index $\mathrm{p}' = 1 \in \mathbb{Z}_{q'}$ of the 5d iTFT. However, if the $U(1)_{\tilde{Y}}$ magnetic monopole exists, kinematically it can break $U(1)_{[1]}^m$ down to a discrete subgroup or to none (thus $\mathrm{p}' = 0$) at the higher-energy monopole scale; nonetheless $U(1)_{[1]}^m$ can still reemerge at the lower-energy SM scale. In short, the value $\mathrm{p}' \in \mathbb{Z}_{q'}$ depends on both the $U(1)_{\tilde{Y}}$ magnetic monopole and the energy scale.

5. For a discrete $\mathbf{B} - \mathbf{L}$ vector or X chiral symmetry, we have so far focused on $(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X} \times \ldots)$ for the SM but we could have other alternatives, e.g., $(\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2m} \times \ldots)$ for $\mathrm{m} = 2, 3, 4, \ldots$ for a larger discrete $\mathbb{Z}_{2m,X}$ symmetry [43,44]. We could also include the discrete $\mathbf{B} + \mathbf{L}$ vector symmetry; the $U(1)_{\mathbf{B+L}}$ is explicitly broken down to $\mathbb{Z}_{2N_f, \mathbf{B+L}}$ by the Adler-Bell-Jackiw anomaly [33,34] due to SU(2) instantons [64], or to $\mathbb{Z}_{2N_f\#, \mathbf{B+L}}$ with an extra q-dependent factor denoted as # due to fractional instantons [58]. But we have excluded $\mathbf{B} + \mathbf{L}$ in our work because it is not a good symmetry for the $su(5)$ or other GUTs. We leave other scenarios for future directions [65].

6. In summary, out of the multiple $\mathbb{Z}$ classes of local anomalies and $\mathbb{Z}_n$ classes of global anomalies [counted as a total finite group of order $2 \cdot 4^2 \cdot 16 \cdot 2 \cdot 4 \cdot 6 = 2^{13} \cdot 3 = 24576$ from (3), (4), (7), (11), and (12)], we determine the deformation class of ungauged or gauged SM, labeled by the family number N_f, total sterile neutrino number n_{ν_R}, and magnetic monopole datum p', the mod $q \equiv 6/q'$ of G_{SM_q}, written as a 5d iTFT,

$$\mathbf{Z}_5 \equiv \exp\left(\mathrm{i}(-N_f + n_{\nu_R})(\ldots) \right.$$
$$\left. + \frac{2\pi \mathrm{i} \mathrm{p}'}{q'} \int_{M^5} B_{\mathbb{Z}_{q'},[1]}^e (\mathrm{d}B_{U(1),[1]}^m \bmod q') \right). \quad (13)$$

Here the $(\ldots)$ are the $\mathbb{Z}^2$ or $\mathbb{Z}_{16}$ class iTFT already detected in Eq. (5) and (6). For the ungauged SM, if we embed the SM into the modified $so(10)$ GUT [50,51], we could add Eq. (8) into (13). But again, once the $\mathbf{B} - \mathbf{L}$, X, or Spin(10) are dynamically gauged, the 4d SM and the 5d becoming non-invertible theories are correlated via mediating $\mathbf{B} - \mathbf{L}$, X, or Spin(10) gauge forces.

The last term of (13) with p'/q' dependence imposes a dynamical constraint on the 4d quantum gauge theory of the gauged SM; the high-energy theory or low-energy fate of this gauged SM must match the same 't Hooft anomaly. For example, the $B^e \mathrm{d}B^m$ term suggests that either $\mathbb{Z}_{q',[1]}^e$ or $U(1)_{[1]}^m$ or both are spontaneously broken [24], or there is no symmetric gapped trivial vacuum. The $\mathbb{Z}_{16}$ class 4d anomaly of (6) and (13) imposes another dynamical constraint in 4d; there is no symmetric gapped phase for the odd classes in $\mathbb{Z}_{16}$ but there exists symmetric gapped phases (namely gapped without $\mathbb{Z}_{4,X}$ chiral symmetry breaking) with low-energy TQFTs for the even classes in $\mathbb{Z}_{16}$ [43,66–68].

7. Ultra unification (UU) [30–32] replaces some of "right-handed sterile neutrinos" (sterile only to G_{SM_q} gauge force, but not sterile to X or certain GUT forces) by new exotic gapped/gapless sectors (e.g., topological or conformal field theory) or by gravitational sectors with topological origins, via nonperturbative global anomaly cancellation and co-bordism constraints. Namely $(-N_f + n_{\nu_R} + \ldots) = 0$ in $\mathbb{Z}_{16}$ for $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{4,X}$ symmetry, where $\ldots$ is the anomaly index of the new gapped/gapless sectors from 4d or 5d. It can be generalized to $\mathrm{Spin} \times_{\mathbb{Z}_2^F} \mathbb{Z}_{2^m,X}$ symmetry with the $\mathbb{Z}_{2^{m+2}}$ global anomaly constraint for $m \geq 2$ [43,44]. If we remove the new gapped/gapless sectors out of UU, the UU also resides in an SM deformation class. There is a rich interplay between SM, GUT, and UU, where topological quantum phase transitions occur between the 15 vs 16 Weyl fermion models. The new sectors [30–32,50,51] (gapped or gapless, including quantum phases or phase transitions, and quantum critical regions), neighbor to the SM vacuum in the quantum phase diagram landscape and in the deformation class, naturally exhibit beyond SM phenomena and thus may provide dark matter candidates. Gapless quantum criticality also challenges the spacetime concept near the highly-entangled critical regions, which may affect the energy density or pressure of spacetime, thus may relate to dark energy. The deformation between neutrinos and exotic gapped/gapless sectors may shed new light on the quantum version of the equivalence principle on the inertia mass (of the quantum gauge theory origin) and gravitational mass (of the gravitational origin).

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