Letter

Is first-order relativistic hydrodynamics in a general frame stable and causal for arbitrary interactions?

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We derive a first-order, stable and causal, relativistic hydrodynamic theory from the microscopic kinetic equation using the gradient expansion technique in a general frame. The general frame is introduced from the arbitrary matching conditions for hydrodynamic fields. The interaction is introduced in the relativistic Boltzmann equation through the momentum-dependent relaxation time approximation (MDRTA) with the proposed collision operator that preserves the conservation laws. We demonstrate here for the first time that not only the general frame choice but also the momentum dependence of microscopic interaction rate, captured through MDRTA, is imperative for producing the essential field corrections that give rise to a causal and stable first-order relativistic theory.

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I. INTRODUCTION

The hydrodynamic theory is an effective coarse-grained formulation of the underlying microscopic dynamics at the long-wavelength limit, which has served for decades as an efficient and accessible tool for a vast range of problems in theoretical physics. However convenient, the relativistic extension of the first-order dissipative Navier-Stokes (NS) formalism introduced by Landau-Lifshitz (LL) [1] and Eckart [2] encounters severe issues with instability [3–5] and superluminal signal propagation, which pose serious limitations to the practical application of the theory. Later on, second-order Muller-Israel-Stewart (MIS) theory [6–8] and some of its extended versions [9–13] were introduced to remedy these problems. Recently, a new study was proposed by Bemfica, Disconzi, Noronha, and Kovtun (BDNK) [14–20] for a first-order stable and causal theory by defining the out-of-equilibrium hydrodynamic variables in a general frame other than LL or Eckart through their postulated constitutive relations that include both time and space gradients.

In this work, we derive a first-order theory using a gradient expansion technique in an arbitrary frame where the explicit expressions of the field redefinition coefficients have been estimated from the underlying microscopic dynamics. The homogeneous part of the out-of-equilibrium momentum distribution has been extracted from the hydrodynamic matching conditions. The inhomogeneous part obtained from the Boltzmann equation becomes sensitive to the system interactions through its collision term. The relaxation time approximation (RTA) [21] is proven to be a convenient form for linearization of the collision kernel with a wide range of applications (see Ref. [22] and references therein), and its momentum dependence can be related to the microscopic interaction relevant for the medium under consideration [23]. These two facts provide a strong motivation to use momentum-dependent relaxation time approximation (MDRTA) in the relativistic transport equation to obtain the inhomogeneous part of the solution [24–31]. Here we propose a new collision operator under MDRTA which obeys the fundamental microscopic and macroscopic conservation laws irrespective of the particular momentum dependence of RTA or the matching indices. With this formalism, here we analytically calculate the values of the coefficients in the constitutive relations of hydrodynamic field redefinition from the kinetic theory in a general frame, i.e., for arbitrary matching conditions.

We further analyze the dispersion relation resulting from small perturbations around the hydrostatic equilibrium for this first-order theory to investigate the stability and causality of the system. It is observed that the first-order field correction coefficients responsible for generating causal and stable modes are directly related to the microscopic dynamics of the system. Even in a general frame where the first-order theory is expected to be causal and stable, we find that only nonzero momentum dependence of the relaxation time gives rise to the causal and stable modes. The stability and causality conditions critically depend on the particular momentum dependence of MDRTA. These are the key

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findings of the current work. To the best of our knowledge, for the first time, a correlation between the interaction dynamics and the causality and stability of a relativistic fluid is being reported.

Throughout the paper, we use natural units ($\hbar = c = k_B = 1$) and a flat space-time with a mostly negative metric $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

II. HYDRODYNAMIC FIELD REDEFINITION

The basic idea is to employ the relativistic Boltzmann transport equation to estimate the out-of-equilibrium one-particle distribution function $f(x, p)$ for a general hydrodynamic frame (defined later),

$$p^\mu \partial_\mu f(x, p) = C[f] = -\mathcal{L}[\phi]. \tag{1}$$

Here p is the particle four-momenta, and x denotes the space-time variable, $f = f^{(0)} + f^{(0)}(1 \pm f^{(0)})\phi$ with $f^{(0)}(=[\exp(\frac{p\cdot u}{T} - \frac{\mu}{T}) \mp 1]^{-1}$ for bosons and fermions, respectively) as the equilibrium distribution and ϕ the out-of-equilibrium deviation; $C[f]$ is the collision integral, which corresponds to the two-to-two elastic collisions. It is linearized as $\mathcal{L}[\phi] = \int d\Gamma_{p_1} d\Gamma_{p'} d\Gamma_{p'_1} f^{(0)} f_1^{(0)} (1 \pm f'^{(0)})(1 \pm f_1'^{(0)})\{\phi + \phi_1 - \phi' - \phi'_1\} W(p' p'_1 | p p_1)$, with $d\Gamma_p = \frac{d^3 p}{(2\pi)^3 p^0}$, and W is the transition rate that depends on the cross section of the interactions. In the gradient expansion technique ϕ is expressed as $\phi = \sum_r \phi^{(r)}$, with $\phi^{(r)}$ as the rth-order out-of-equilibrium deviation of the distribution function.

In general, $\phi^{(r)}$ can be expressed as a linear combination of rth-order field gradients with appropriate tensor coefficients [32]:

$$\phi^{(r)} = \sum_l A_l^{(r)} X^{(r)l} + \sum_m B_m^{(r)\mu} Y_\mu^{(r)m} + \sum_n C_n^{(r)\mu\nu} Z_{\mu\nu}^{(r)n}, \tag{2}$$

where $X^{(r)l}, Y_\mu^{(r)m}$, and $Z_{\mu\nu}^{(r)n}$ are the rth-order scalar, vector, and rank-2 tensor gradient corrections of the l, m, and nth kind, respectively. Here, $A_l^{(r)}, B_m^{(r)\mu}$, and $C_n^{(r)\mu\nu}$ are the unknown coefficient functions of space-time, particle momentum, and the ratio of rest mass to temperature $z = m/T$. We expand the coefficients in a polynomial basis to extract their values as $A_l^{(r)} = \sum_{s=0}^{\infty} A_l^{r,s}(z, x) P_s^{(0)}$, $B_m^{(r)\mu} = \sum_{s=0}^{\infty} B_m^{r,s}(z,x) P_s^{(1)} \tilde{p}^{\langle\mu\rangle}$, $C_n^{(r)\mu\nu} = \sum_{s=0}^{\infty} C_n^{r,s}(z, x) \times P_s^{(2)} \tilde{p}^{\langle\mu} \tilde{p}^{\nu\rangle}$. Inspired by [33] and being convenient for the current analysis, we employ an orthogonal polynomial basis which is partially orthogonal in the scalar sector. For our case the first two polynomials, $P_0^{(0)} = 1, P_1^{(0)} = \tilde{E}_p$, are not orthogonal, but all other higher polynomials are chosen to be orthogonal to these two as well as to each other, and monic (in $P_s^{(n)}$ the coefficient of maximum power of $\tilde{E}_p$, i.e., $\tilde{E}_p{}^s$ is 1). Concisely, they are given by

$$P_0^{(0)} = 1, \quad P_1^{(0)} = \tilde{E}_p, \quad P_0^{(1)} = 1, \quad P_0^{(2)} = 1, \tag{3}$$

$$\int dF_p (\tilde{E}_p/\tau_R)(\Delta_{\mu\nu} p^\mu p^\nu)^n P_s^{(n)} P_r^{(n)} \sim \delta_{s,r}, \tag{4}$$

with τ_R the relaxation time of a single particle distribution function that will be introduced later with more details. The notations we use are as follows: $dF_p = d\Gamma_p f^{(0)}(1 \pm f^{(0)})$, $\tilde{p}^\mu = p^\mu/T$, $\tilde{\mu} = \mu/T$, $\tilde{E}_p = u_\mu p^\mu/T$, $\tilde{p}_{\langle\mu\rangle} = \Delta_{\mu\nu}\tilde{p}^\nu$, and $\tilde{p}_{\langle\mu}\tilde{p}_{\nu\rangle} = \Delta_{\mu\nu}^{\alpha\beta}\tilde{p}_\alpha\tilde{p}_\beta$, with T, μ, and u^μ the temperature, chemical potential, and fluid four-velocity of the system at equilibrium and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$.

It is observed that, by virtue of the collision integral properties $\mathcal{L}[1] = 0$ and $\mathcal{L}[p^\mu] = 0$ that follow from the particle number and energy-momentum conservation, respectively, the coefficients $A_l^{r,0}, A_l^{r,1}$, and $B_m^{r,0}$ cannot be determined from the transport equation (1), and hence they are called the coefficients of the homogeneous solution. The rest of the coefficients, $A_l^{r,s}, B_m^{r,s}$, and $C_n^{r,s}$, can be estimated from the transport equation, and they are called inhomogeneous or interaction solutions. We take the recourse of the matching conditions, which are constraints that set the thermodynamic fields (such as temperature, chemical potential, etc.) to their equilibrium values even in the presence of dissipation, to extract the coefficients of the homogeneous part of the distribution function. Each such matching condition produces one out of an infinite number of possible "hydrodynamic frames" [19]. From the requirement of setting two scalars and one vector homogeneous coefficients from these constraints, we use the following three matching conditions,

$$\int dF_p \tilde{E}_p^i \phi = 0, \quad \int dF_p \tilde{E}_p^j \phi = 0, \quad \int dF_p \tilde{E}_p^k \tilde{p}^{\langle\mu\rangle} \phi = 0, \tag{5}$$

where $i \neq j$, i, j, k are non-negative integers. We identify the set of matching indices (1,2,1) and (1,2,0) to represent the LL and Eckart frames, respectively. Substituting Eq. (2) in Eq. (5), we find the homogeneous part in terms of the interaction part $\phi_{\mathrm{int}}^{(r)} = \sum_{s=2}^{\infty} P_s^{(0)} \sum_l A_l^{r,s} X^{(r)l} + \tilde{p}^{\langle\mu\rangle} \sum_{s=1}^{\infty} P_s^{(1)} \sum_m B_m^{r,s} Y_\mu^{(r)m} + \tilde{p}^{\langle\mu}\tilde{p}^{\mu\rangle} \sum_{s=0}^{\infty} P_s^{(2)} \sum_n C_n^{r,s} Z_{\mu\nu}^{(r)n}$ and the matching indices. Using this prescription, the entire out-of-equilibrium distribution function for any order becomes

$$\begin{aligned}\phi^{(r)} = \phi_{\mathrm{int}}^{(r)} &- \tilde{E}_p \left[\frac{I_j}{\mathcal{D}_{i,j}^{1,0}} \int dF_p \tilde{E}_p^i \phi_{\mathrm{int}}^{(r)} + (i \leftrightarrow j)\right] \\ &- \left[\frac{I_{j+1}}{\mathcal{D}_{i,j}^{0,1}} \int dF_p \tilde{E}_p^i \phi_{\mathrm{int}}^{(r)} + (i \leftrightarrow j)\right] \\ &- \frac{\tilde{p}_{\langle\nu\rangle}}{J_k} \int dF_p \tilde{E}_p^k \tilde{p}^{\langle\nu\rangle} \phi_{\mathrm{int}}^{(r)}.\end{aligned} \tag{6}$$

Here we use the shorthand notation $\mathcal{D}_{i,j}^{m,n} = I_{i+m} I_{j+n} - I_{i+n} I_{j+m}$ with the properties $\mathcal{D}_{i,j}^{m,n} = -\mathcal{D}_{i,j}^{n,m}$ and

$\mathcal{D}^{m,n}_{i,j} = -\mathcal{D}^{m,n}_{j,i}$. The moment integrals are defined as $I_n = \int dF_p \tilde{E}^n_p$, $\Delta^{\mu\nu} J_n = \int dF_p \tilde{p}^{\langle\mu\rangle} \tilde{p}^{\langle\nu\rangle} \tilde{E}^n_p$. Equation (6) provides the out-of-equilibrium parts of the two most general hydrodynamic field variables, namely, the particle four-flow (N^μ) and the energy-momentum tensor ($T^{\mu\nu}$), respectively, for the rth order of gradient correction as

$$\delta N^{(r)\mu} = \int dF_p p^\mu \phi^{(r)}, \quad \delta T^{(r)\mu\nu} = \int dF_p p^\mu p^\nu \phi^{(r)}. \tag{7}$$

Utilizing Eq. (7), the nonequilibrium correction to the particle number density ($\delta n^{(r)} = u^\mu \delta N^{(r)}_\mu$), the energy density ($\delta\epsilon^{(r)} = u^\mu u^\nu \delta T^{(r)}_{\mu\nu}$), pressure ($\delta P^{(r)} = -\frac{1}{3}\Delta^{\mu\nu}\delta T^{(r)}_{\mu\nu}$), energy flux or momentum density ($W^{(r)\alpha} = \Delta^\alpha_\mu u_\nu \delta T^{(r)\mu\nu}$), and the particle flux ($V^{(r)\alpha} = \Delta^\alpha_\mu \delta N^{(r)\mu}$) can be estimated order by order as

$$\delta n^{(r)} = \int dF_p (p^\mu u_\mu) \phi^{(r)}_{\rm int} + a^{(r)} \frac{\partial n_0}{\partial\tilde{\mu}} + u^\mu b^{(r)}_\mu T \frac{\partial n_0}{\partial T}, \tag{8}$$

$$\delta\epsilon^{(r)} = \int dF_p (p^\mu u_\mu)^2 \phi^{(r)}_{\rm int} + a^{(r)} \frac{\partial \epsilon_0}{\partial\tilde{\mu}} + u^\mu b^{(r)}_\mu T \frac{\partial \epsilon_0}{\partial T}, \tag{9}$$

$$\delta P^{(r)} = \frac{1}{3}\int dF_p \{(p^\mu u_\mu)^2 - m^2\} \phi^{(r)}_{\rm int} + a^{(r)} \frac{\partial P_0}{\partial\tilde{\mu}} + u^\mu b^{(r)}_\mu T \frac{\partial P_0}{\partial T}, \tag{10}$$

$$W^{(r)\mu} = \int dF_p p^{\langle\mu\rangle} (p^\mu u_\mu) \phi^{(r)}_{\rm int} - (\epsilon_0 + P_0)\Delta^{\mu\nu} b^{(r)}_\nu, \tag{11}$$

$$V^{(r)\mu} = \int dF_p p^{\langle\mu\rangle} \phi^{(r)}_{\rm int} - n_0 \Delta^{\mu\nu} b^{(r)}_\nu. \tag{12}$$

Here, n_0, ϵ_0, and P_0 are the equilibrium values of particle number density, energy density, and pressure, respectively, and $a^{(r)}$ and $b^{(r)\mu}$ are the dimensionless momentum-independent quantities given by $a^{(r)} = \frac{I_{i+1}}{\mathcal{D}^{0,1}_{i,j}} \int dF_p \tilde{E}^j_p \phi^{(r)}_{\rm int} + (i \leftrightarrow j)$, $u^\mu b^{(r)}_\mu = \frac{I_i}{\mathcal{D}^{1,0}_{i,j}} \int dF_p \tilde{E}^j_p \phi^{(r)}_{\rm int} + (i \leftrightarrow j)$, $\Delta^{\mu\nu} b^{(r)}_\nu = -\frac{1}{J_k} \int dF_p \tilde{E}^k_p \times \tilde{p}^{\langle\mu\rangle} \phi^{(r)}_{\rm int}$, which define the homogeneous part of ϕ as $\phi^{(r)}_{\rm h} = a^{(r)} + b^{(r)\mu}\tilde{p}_\mu$. Adding up the field corrections for all orders, the most general expressions for N^μ and $T^{\mu\nu}$ are given by

$$N^\mu = (n_0 + \delta n) u^\mu + V^\mu, \tag{13}$$

$$T^{\mu\nu} = (\epsilon_0 + \delta\epsilon) u^\mu u^\nu - (P_0 + \delta P)\Delta^{\mu\nu} + (W^\mu u^\nu + W^\nu u^\mu) + \pi^{\mu\nu}, \tag{14}$$

with $\pi^{\mu\nu}$ the shear stress tensor.

III. FIRST-ORDER THEORY WITH MDRTA

Up to now, the discussion has been completely general, and the results are applicable for any order in the gradient expansion. To provide the explicit expression for the distribution function from Eq. (6), one needs to estimate the interaction part of the distribution function for a specific order. For this purpose, we employ here the MDRTA for solving the relativistic transport equation (1) as a dynamical model study. The idea is to replace $\mathcal{L}[\phi]$ in Eq. (1) with the Anderson-Witting-type relaxation kernel, but now we generalize the relaxation time to be momentum dependent. For this purpose, we propose here a collision operator under MDRTA in Eq. (1) as the following:

$$\begin{aligned}\mathcal{L}_{\rm MDRTA}[\phi] = \frac{(p\cdot u)}{\tau_R} f^{(0)}(1 \pm f^{(0)}) \\ \times \Bigg[\phi - \frac{\langle \frac{\tilde{E}_p}{\tau_R}\tilde{E}^2_p\rangle\langle\frac{\tilde{E}_p}{\tau_R}\phi\rangle - \langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\rangle\langle\frac{\tilde{E}_p}{\tau_R}\phi\tilde{E}_p\rangle}{\langle\frac{\tilde{E}_p}{\tau_R}\rangle\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}^2_p\rangle - \langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\rangle^2} \\ - \tilde{E}_p \frac{\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\rangle\langle\frac{\tilde{E}_p}{\tau_R}\phi\rangle - \langle\frac{\tilde{E}_p}{\tau_R}\rangle\langle\frac{\tilde{E}_p}{\tau_R}\phi\tilde{E}_p\rangle}{\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}_p\rangle^2 - \langle\frac{\tilde{E}_p}{\tau_R}\rangle\langle\frac{\tilde{E}_p}{\tau_R}\tilde{E}^2_p\rangle} \\ - \tilde{p}_{\langle\nu\rangle} \frac{\langle\frac{\tilde{E}_p}{\tau_R}\phi\tilde{p}^{\langle\nu\rangle}\rangle}{\frac{1}{3}\langle\frac{\tilde{E}_p}{\tau_R}\tilde{p}^{\langle\mu\rangle}\tilde{p}_{\langle\mu\rangle}\rangle}\Bigg],\end{aligned} \tag{15}$$

with $\langle\cdots\rangle = \int dF_p(\cdots)$. Equation (15) readily gives $\mathcal{L}_{\rm MDRTA}[\phi] = 0$ if $\phi = a + b(p\cdot u) + c^\mu p_{\langle\mu\rangle}$ with a, b, c^μ being arbitrary momentum-independent coefficients. It satisfies the self-adjoint property as well, $\int d\Gamma_p \psi \mathcal{L}_{\rm MDRTA}[\phi] = \int d\Gamma_p \phi \mathcal{L}_{\rm MDRTA}[\psi]$. These two combined give the summation invariant property $\int d\Gamma_p \psi \mathcal{L}_{\rm MDRTA}[\phi] = 0$ for $\psi = a + b(p\cdot u) + c^\mu p_{\langle\mu\rangle}$ which immediately results in the conservation laws $\partial_\mu N^\mu = 0$ and $\partial_\mu T^{\mu\nu} = 0$ microscopically. These conservation laws do not need to be estimated order by order and are treated nonperturbatively. The preservation of particle number and energy-momentum conservation in $\mathcal{L}_{\rm MDRTA}[\phi]$ is irrespective of the frame indices or particular momentum dependence of τ_R. Equation (15) resembles the novel relaxation time collision operator introduced in [26] apart from the fact that it uses the polynomial basis given in Eqs. (3) and (4). The advantage of using this basis is that the polynomials associated with the homogeneous part of the solution are in the form of simple exponents, which reduces the computational complexity significantly.

In the current analysis, the momentum dependence of τ_R is expressed as a power law of $\tilde{E}_p$ in the comoving frame, with τ^0_R the momentum-independent part; the parameter Λ specifies the power of the scaled energy.

To solve Eq. (1), we adopt a perturbative expansion introduced in [20]. By decomposing the space-time derivative, the left-hand side of Eq. (1) gives rise to a number of time and space derivatives over the fundamental thermodynamic quantities T, μ, and u^μ. In popular perturbation approaches like the Chapman-Enskog method, the time derivatives are replaced by the spatial ones in order to make the left-hand side of Eq. (1) orthogonal to zero modes (homogeneous solutions). By virtue of the collision operator $\mathcal{L}_{\rm MDRTA}$ given in Eq. (15), the right-hand side of Eq. (1) now retains only the interaction part of ϕ. It singularly excludes the zero modes of the linearized collision operator; i.e., any functions proportional to 1 and p^μ are not present from the momentum basis of the unknown coefficients in Eq. (2). Because of this fact, the left-hand side of Eq. (1) does not necessarily need to be orthogonal to zero modes in order to extract the remaining nonzero mode coefficients, which are themselves orthogonal to zero modes as well as each other. Hence, the covariant time derivatives appearing on the left-hand side of Eq. (1) do not need to be exchanged by the spatial gradients. Employing this method, the inhomogeneous or interaction part of the first-order out-of-equilibrium distribution function turns out to be

$$\frac{\phi^{(1)}_{\rm int}}{\tau_R^0} = -\tilde{E}_p^{\Lambda-1}\Bigg[\tilde{E}_p^2\frac{DT}{T} + \tilde{E}_p D\tilde{\mu} + \left(\frac{\tilde{E}_p^2}{3} - \frac{z^2}{3}\right)(\partial\cdot u) + \tilde{E}_p\tilde{p}^{\langle\mu\rangle}\left(\frac{\nabla_\mu T}{T} - Du_\mu\right) + \tilde{p}^{\langle\mu\rangle}\nabla_\mu\tilde{\mu} - \tilde{p}^{\langle\mu}\tilde{p}^{\nu\rangle}\sigma_{\mu\nu}\Bigg], \tag{16}$$

where $\sigma_{\mu\nu} = \nabla_{\langle\mu}u_{\nu\rangle}$, $D = u^\mu\partial_\mu$, and $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$ are symmetric traceless shear tensor, temporal, and spatial counterparts of the total space-time derivative, respectively. Next, we use Eq. (16) in Eq. (6) to construct $\phi^{(1)}$ in order to calculate the first-order field correction coefficients. From Eqs. (8)–(12), the first-order thermodynamic field corrections in a general frame and with arbitrary interactions are given by

$$\delta n^{(1)}, \delta\epsilon^{(1)}, \delta P^{(1)} = \nu_1, \varepsilon_1, \pi_1\frac{DT}{T} + \nu_2, \varepsilon_2, \pi_2(\partial\cdot u) + \nu_3, \varepsilon_3, \pi_3 D\tilde{\mu}, \tag{17}$$

$$W^{(1)\mu}, V^{(1)\mu} = \theta_1, \gamma_1\left[\frac{\nabla^\mu T}{T} - Du^\mu\right] + \theta_3, \gamma_3\nabla^\mu\tilde{\mu}. \tag{18}$$

The explicit expressions of the field correction coefficients turn out to be elaborate and complicated functions of the frame indices i, j, k and the parameter Λ of MDRTA. These field corrections, along with $\pi^{(1)\mu\nu} = 2\eta\sigma^{\mu\nu}$ (η is the shear viscosity), constitute the first-order out-of-equilibrium N^μ and $T^{\mu\nu}$ from Eqs. (13) and (14), respectively.

Here we end up with 14 field correction coefficients ($\nu_{1,2,3}, \epsilon_{1,2,3}, \pi_{1,2,3}, \theta_{1,3}, \gamma_{1,3}$, and η). It was shown in [17,19] that not all coefficients are invariant under the first-order field redefinition (due to the arbitrariness in the definition of temperature, fluid four-velocity, and chemical potential for the out-of equilibrium case). We check that our coefficients satisfy the combinations $f_i = \pi_i - \varepsilon_i(\frac{\partial P_0}{\partial\epsilon_0})_{n_0} - \nu_i(\frac{\partial P_0}{\partial n_0})_{\epsilon_0}$ and $l_i = \gamma_i - \frac{n_0}{\epsilon_0+P_0}\theta_i$ to be frame invariant (i.e., independent of the indices i, j, k), which further reduce to the physical transport coefficients, bulk viscosity $\zeta = -f_2 + (\frac{\partial P_0}{\partial\epsilon_0})_{n_0}f_1 + \frac{1}{T}(\frac{\partial P_0}{\partial n_0})_{\epsilon_0}f_3$, and charge conductivity $k_n = l_3 - \frac{n_0 T}{(\epsilon_0+P_0)}l_1$. The detailed expressions of ζ and k_n with MDRTA are given in [29]. The corrections further reveal that the LL and Eckart limits of the scalar indices ($i=1$, $j=2$ or vice versa) give $\delta n^{(1)} = 0, \delta\epsilon^{(1)} = 0$ (such that ζ is entirety taken up by the pressure correction), where for the vector index, the LL limit ($k=1$) gives $W^{(1)\mu} = 0$ and the Eckart limit ($k=0$) gives $V^{(1)\mu} = 0$. Most significantly, we find that for the momentum-independent relaxation time (i.e., for $\Lambda = 0$), all the correction coefficients associated with the first-order time derivatives ($\nu_1, \nu_3, \varepsilon_1, \varepsilon_3, \pi_1, \pi_3, \theta_1, \gamma_1$) in Eqs. (17) and (18) identically vanish for all hydrodynamic frame conditions (irrespective of i, j, k values), which will be shown later to have crucial implications on the causality and stability of the theory.

IV. STABILITY AND CAUSALITY ANALYSIS

Here we investigate the causality and stability of the theory by linearizing the conservation equations for small perturbations of fluid variables around the hydrostatic equilibrium in the local rest frame, $\epsilon(t,x) = \epsilon_0 + \delta\epsilon(t,x)$, $n = n_0 + \delta n(t,x)$, $P(t,x) = P_0 + \delta P(t,x)$, $u^\mu(t,x) = (1,\vec{0}) + \delta u^\mu(t,x)$. In linear approximation, δu^μ has only spatial components to retain the normalization condition. For convenience, these fluctuations are further expressed in their plane wave solutions via a Fourier transformation $\delta\psi(t,x) \to e^{i(\omega t - kx)}\delta\psi(\omega,k)$, with wave 4-vector $k^\mu = (\omega,k,0,0)$. The resulting dispersion relation for the transverse or shear channel is

$$(i\omega)^2 + i\omega\frac{(\epsilon_0 + P_0)}{\theta} + \frac{\eta}{\theta}k^2 = 0, \tag{19}$$

where we define $\theta = -\theta_1$. At the small k limit, the obtained modes are $\omega_1^T = i\frac{\eta}{(\epsilon_0+P_0)}k^2 + \mathcal{O}(k^4)$ and $\omega_2^T = i\frac{(\epsilon_0+P_0)}{\theta} + \mathcal{O}(k^2)$. Both the modes are nonpropagating, where ω_1^T is a hydrodynamic mode (vanishes at $k=0$) and ω_2^T is a nonhydro mode. Note that ω_1^T is the conventional shear mode of NS theory. At small k, the stability is guaranteed if $\theta > 0$, because in that case the imaginary part of ω_2^T is positive definite and gives rise to exponentially decaying perturbations. At large k, the modes turn out to be $\omega_{1,2}^T = \pm\sqrt{\eta/\theta}k + i\frac{(\epsilon_0+P_0)}{2\theta} + \mathcal{O}(\frac{1}{k})$. These are propagating modes where causality holds for $\theta > \eta$, which also guarantees the stability condition. Here, θ_1 plays a crucial role in stability and causality of the shear channel. From Eq. (18) the explicit expression of θ_1 turns out to be

$$\theta_1 = -\tau_R^0 T^2\left(J_{\Lambda+1} + \frac{\epsilon_0 + P_0}{T^2}\frac{J_{k+\Lambda}}{J_k}\right). \tag{20}$$

We can see that $\theta_1 = 0$ for both $k = 1$ (LL frame) with any interaction or $\Lambda = 0$ (momentum-independent RTA) for any general frame. This will give rise to superluminal velocities in the shear channel. In Fig. 1, the left panel shows $\theta(= -\theta_1)$ scaled by τ_R^0 as a function of Λ for different vector matching indices k. Here, θ is always positive for $\Lambda > 0$. The right panel shows the group velocity $v_g = \sqrt{\eta/\theta}$ which obeys causality for $\Lambda > 1$, where $\eta = \tau_R^0 T^2 K_{\Lambda-1}/2$ with $\Delta^{\alpha\beta\mu\nu} K_n = \int dF_p \tilde{p}^{\langle\mu} \tilde{p}^{\nu\rangle} \tilde{p}^{\langle\alpha} \tilde{p}^{\beta\rangle} \tilde{E}_p^n$. We also see that larger values of k and Λ reduce group velocity. So, even in a general frame, the choice of Λ crucially decides the stability and causality of the shear channel. Throughout the numerical analysis, the parameters have been set to $T = 300$ MeV, $m = 300$ MeV.

For the longitudinal or sound mode, the dispersion relation turns out to be a sixth-order polynomial,

$$(i\omega)^6 A_6 + (i\omega)^5 A_5 + (i\omega)^4 A_4 + (i\omega)^3 A_3 + (i\omega)^2 A_2 + (i\omega) A_1 + A_0 = 0, \tag{21}$$

with $A_4 = A_4^0 + A_4^2 k^2$, $A_3 = A_3^0 + A_3^2 k^2$, $A_2 = A_2^2 k^2 + A_2^4 k^4$, $A_1 = A_1^2 k^2 + A_1^4 k^4$, $A_0 = A_0^4 k^4 + A_0^6 k^6$. Equation (21) agrees with the result obtained in [34], where the coefficient A's are functions of $\nu_{1,2,3}, \epsilon_{1,2,3}, \pi_{1,2,3}, \theta_{1,3}, \gamma_{1,3}$ defined earlier (the detailed analysis will be reported elsewhere). Equation (21) cannot be solved analytically, and hence we present results for the $k \to 0$ limit. At this limit, Eq. (21) gives three hydrodynamic modes as $\omega_6^L = i\hat{h}^2 \frac{k_n T}{(\epsilon_0 + P_0)} k^2$ and $\omega_{4,5}^L = \pm c_s k + i\frac{\Gamma_s}{2} k^2 + \mathcal{O}(k^3)$, with scaled enthalpy per particle $\hat{h} = (\epsilon_0 + P_0)/n_0 T$, velocity of sound squared $c_s^2 = (\frac{\partial P_0}{\partial \epsilon_0})_{n_0} + \frac{1}{\hat{h}} \frac{1}{T} (\frac{\partial P_0}{\partial n_0})_{\epsilon_0}$, and sound attenuation coefficients $\Gamma_s = [\frac{4}{3}\eta + \zeta + \frac{k_n T}{c_s^2} (\frac{1}{T} \frac{\partial P_0}{\partial n_0})_{\epsilon_0}^2]/(\epsilon_0 + P_0)$. Here, ω_6^L and $\omega_{4,5}^L$ are the conventional heat diffusion and sound modes of the NS theory, respectively.

The remaining nonhydro modes are given by

$$(i\omega^L)^3 A_6 + (i\omega^L)^2 A_5 + (i\omega^L) A_4^0 + A_3^0 = 0. \tag{22}$$

Using Routh-Hurwitz criteria, we find the following conditions for stability of the nonhydro modes,

$$A_6 > 0, \qquad A_5 > 0, \qquad A_3^0 > 0, \tag{23}$$

$$B_2 = (A_4^0 A_5 - A_3^0 A_6)/A_5 > 0. \tag{24}$$

Among these coefficients, $A_3^0 = n_0(\epsilon_0 + P_0)$ is always positive. The remaining coefficients are given by

$$A_6 = \frac{\theta_1}{(\epsilon_0 + P_0)} \hat{h} c_s^2 (\nu_1 \epsilon_3 - \nu_3 \epsilon_1), \tag{25}$$

$$A_5 = \hat{h} c_s^2 (\nu_3 \epsilon_1 - \nu_1 \epsilon_3) - \theta_1 \left[(\nu_1 f + \nu_3 c) + \frac{1}{\hat{h} T} (\epsilon_1 g + \epsilon_3 d) \right], \tag{26}$$

$$A_4^0 = (\epsilon_0 + P_0)(\nu_1 f + \nu_3 c) + n_0 (\epsilon_1 c + \epsilon_3 d - \theta_1), \tag{27}$$

with $c = J_0 I_3/(I_2^2 - I_1 I_3)$, $d = -J_1 I_2/(I_2^2 - I_1 I_3)$, $f = -J_0 I_2/(I_2^2 - I_1 I_3)$, $g = J_1 I_1/(I_2^2 - I_1 I_3)$. The concerned field correction coefficients are given by

$$\epsilon_1 = \tau_R^0 \left[\frac{\partial \epsilon_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda+1,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial \epsilon_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda+1,0}}{\mathcal{D}_{i,j}^{1,0}} - T^2 I_{\Lambda+3} \right], \tag{28}$$

$$\epsilon_3 = \tau_R^0 \left[\frac{\partial \epsilon_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial \epsilon_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda,0}}{\mathcal{D}_{i,j}^{1,0}} - T^2 I_{\Lambda+2} \right], \tag{29}$$

$$\nu_1 = \tau_R^0 \left[\frac{\partial n_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda+1,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial n_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda+1,0}}{\mathcal{D}_{i,j}^{1,0}} - T I_{\Lambda+2} \right], \tag{30}$$

$$\nu_3 = \tau_R^0 \left[\frac{\partial n_0}{\partial \tilde{\mu}} \frac{\mathcal{D}_{i,j}^{\Lambda,1}}{\mathcal{D}_{i,j}^{0,1}} + T \frac{\partial n_0}{\partial T} \frac{\mathcal{D}_{i,j}^{\Lambda,0}}{\mathcal{D}_{i,j}^{1,0}} - T I_{\Lambda+1} \right]. \tag{31}$$

The coefficients ν_1, ϵ_1 vanish both for $i = 1$, $j = 2$ (LL + Eckart) $\forall\, \Lambda$ and also at $\Lambda = 0$ for all frame choices. Note that ν_3, ϵ_3 obey the same but also vanish for $\Lambda = 1$ at all frames. The coefficients make A_5 and A_4^0 vanish for $\Lambda = 0$ and A_6 vanish for both $\Lambda = 0$ and 1 at any frame. From Fig. 2 we can see that A_6 becomes positive for $\Lambda > 1$, but it becomes negative for the region $\Lambda = 0$ to 1 excluding the end points. This is shown in the inset of Fig. 2, where we can see that in this region higher values of the frame indices make the situation worse with larger negative values of A_6, resulting in more increased instability. This is the

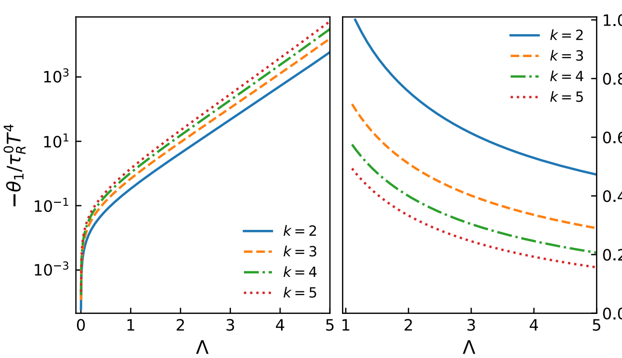

FIG. 1. θ_1 and v_g as a function of Λ in general frames.

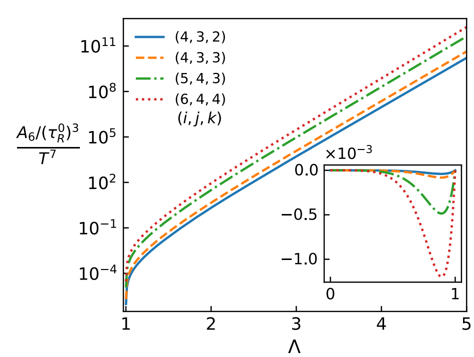

FIG. 2. A_6 as a function of Λ in general frames.

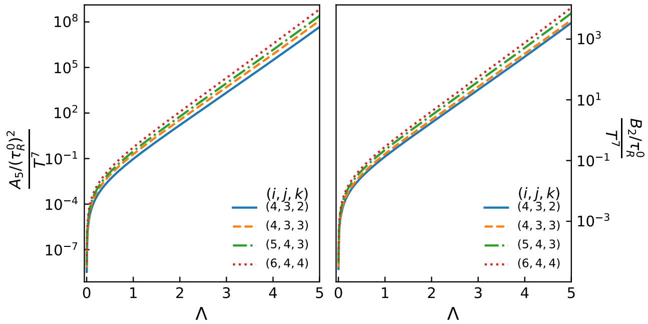

FIG. 3. A_5 and B_2 as a function of Λ in general frames.

essence of the current work. In short, we conclude that a general frame and the nature of underlying interactions are both crucial for the stability and causality of a first-order theory. Figure 3 shows the dependency of A_5 and B_2 on Λ for different frames, which turns out to be positive for general frames and $\Lambda > 0$.

V. CONCLUSION

In this work, a first-order, relativistic stable, and causal hydrodynamic theory has been derived in a general frame from the Boltzmann transport equation, where the system interactions are introduced via the microscopic particle momenta captured through τ_R and an appropriate collision operator $\mathcal{L}_{\mathrm{MDRTA}}$. We have shown that in order to hold stability and causality at first-order theories, besides a general frame, the system interactions need to be carefully taken into account. The conventional momentum-independent RTA leads to acausality by diverging the shear modes even in a general frame. The momentum dependence employed through MDRTA is shown to subtly control the stability and causality of the theory in a general frame.

We believe that this correlation between system dynamics (microscopic interactions) and relativistic hydrodynamics (macroscopic frame variables), along with the precise estimation of causality and stability conditions, makes the current work an acceptable first-order hydrodynamic theory, ready for practical applications.

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[1] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Elsevier, Amsterdam, 1987).
[2] C. Eckart, Phys. Rev. **58**, 919 (1940).
[3] W. A. Hiscock and L. Lindblom, Ann. Phys. (N.Y.) **151**, 466 (1983).
[4] W. A. Hiscock and L. Lindblom, Phys. Rev. D **31**, 725 (1985).
[5] W. A. Hiscock and L. Lindblom, Phys. Rev. D **35**, 3723 (1987).
[6] I. Muller, Z. Phys. **198**, 329 (1967).
[7] W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976).
[8] W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[9] A. Muronga, Phys. Rev. Lett. **88**, 062302 (2002).
[10] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev. D **85**, 114047 (2012).
[11] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, J. High Energy Phys. 04 (2008) 100.
[12] A. Jaiswal, Phys. Rev. C **87**, 051901 (2013); **88**, 021903 (2013).
[13] S. Grozdanov and N. Kaplis, Phys. Rev. D **93**, 066012 (2016).
[14] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. D **98**, 104064 (2018).
[15] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. D **100**, 104020 (2019).
[16] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. X **12**, 021044 (2022).
[17] P. Kovtun, J. High Energy Phys. 10 (2019) 034.
[18] R. E. Hoult and P. Kovtun, J. High Energy Phys. 06 (2020) 067.
[19] R. E. Hoult and P. Kovtun, arXiv:2112.14042.
[20] G. S. Rocha, G. S. Denicol, and J. Noronha, arXiv: 2205.00078.
[21] J. Anderson and H. Witting, Physica (Utrecht) **74**, 466 (1974).
[22] W. Florkowski, M. P. Heller, and M. Spalinski, Rep. Prog. Phys. **81**, 046001 (2018).
[23] K. Dusling, G. D. Moore, and D. Teaney, Phys. Rev. C **81**, 034907 (2010).
[24] D. Teaney and L. Yan, Phys. Rev. C **89**, 014901 (2014).
[25] A. Kurkela and U. A. Wiedemann, Eur. Phys. J. C **79**, 776 (2019).
[26] G. S. Rocha, G. S. Denicol, and J. Noronha, Phys. Rev. Lett. **127**, 042301 (2021).
[27] G. S. Rocha and G. S. Denicol, Phys. Rev. D **104**, 096016 (2021).
[28] S. Mitra, Phys. Rev. C **103**, 014905 (2021).
[29] S. Mitra, Phys. Rev. C **105**, 014902 (2022).
[30] S. Mitra, Phys. Rev. C **105**, 054910 (2022).
[31] D. Dash, S. Bhadury, S. Jaiswal, and A. Jaiswal, Phys. Lett. B **831**, 137202 (2022).
[32] S. R. De Groot, W. A. Van Leeuwen, and C. G. Van Weert, *Relativistic Kinetic Theory, Principles and Applications* (North-Holland, Amsterdam, 1980).
[33] A. Dobado, F. J. Llanes-Estrada, and J. M. Torres-Rincon, Phys. Lett. B **702**, 43 (2011).
[34] F. Taghinavaz, J. High Energy Phys. 08 (2020) 119.