Remarks on the Clauser-Horne-Shimony-Holt inequality in relativistic quantum field theory

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We present an investigation of the Clauser-Horne-Shimony-Holt (CHSH) inequality within a relativistic quantum field theory model built up with a pair of free massive scalar fields (φ_A, φ_B) where, as is customary, the indices (A, B) refer to Alice and Bob, respectively. A set of bounded Hermitian operators is introduced by making use of the Weyl operators. A CHSH-type correlator is constructed and evaluated in the Fock vacuum by means of canonical quantization. Although the observed violation of the CHSH inequality turns out to be rather small as compared to Tsirelson's bound of quantum mechanics, the model can be employed for the study of Bell's inequalities in the more physical case of gauge theories such as the Higgs models, for which local Becchi-Rouet-Stora-Tyutin (BRST) invariant operators describing both the massive gauge boson as well as the Higgs particle have been devised. These operators can be naturally exponentiated, leading to BRST invariant type of Weyl operators useful to analyze Bell's inequalities within an invariant BRST environment.

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I. INTRODUCTION

Since their discovery [1–4], Bell's inequalities have much changed the way we look at the quantum world, forcing us to go deeper and deeper in the understanding of quantum mechanics and of the nature of space-time. It is fair to say that, nowadays, the phenomenon of entanglement is a pivotal issue in both theoretical and experimental physics as well as in the creation of new technologies.

This work aims at investigating, within the framework of relativistic quantum field theory, a very popular and extensively studied version of Bell's inequalities, known as the Clauser-Horne-Shimony-Holt (CHSH) inequality [5–8]. Let us briefly recall it, in the form usually presented in quantum mechanics textbooks; see, for example, Refs. [9–11]. One starts by introducing a two spin-1/2 operator

$$\mathcal{C}_{\rm CHSH} = [(\vec{\alpha}\cdot\vec{\sigma}_A + \vec{\alpha}'\cdot\vec{\sigma}_A)\otimes\vec{\beta}\cdot\vec{\sigma}_B + (\vec{\alpha}\cdot\vec{\sigma}_A - \vec{\alpha}'\cdot\vec{\sigma}_A)\otimes\vec{\beta}'\cdot\vec{\sigma}_B], \tag{1}$$

where (A, B) refer to Alice and Bob, $\vec{\sigma}$ are the spin-1/2 Pauli matrices, and $(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}')$ are four arbitrary unit vectors.[1] Because of the properties of the Pauli matrices, one expects that

$$|\mathcal{C}_{\rm CHSH}| \leq 2 \tag{2}$$

for any possible choice of the unit vectors $(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}')$, though it turns out that this inequality is violated by quantum mechanics, due to entanglement. In fact, when evaluating the CHSH correlator in quantum mechanics, i.e., $\langle\psi|\mathcal{C}_{\rm CHSH}|\psi\rangle$, where $|\psi\rangle$ is an entangled state as, for example, the Bell singlet, one gets

$$|\langle\psi|\mathcal{C}_{\rm CHSH}|\psi\rangle| = 2\sqrt{2}, \qquad |\psi\rangle = \frac{|+\rangle_A\otimes|-\rangle_B - |-\rangle_A\otimes|+\rangle_B}{\sqrt{2}}. \tag{3}$$

The bound $2\sqrt{2}$ is known as Tsirelson's bound [12–14], yielding the maximum violation of the CHSH inequality (2). The experiments carried out over the past decades (see Refs. [5–8,15–20], and references therein) have largely confirmed the violation of the CHSH inequality, being in very good agreement with the bound $2\sqrt{2}$.

[1]Notice that, due to $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$, it follows that $(\vec{n}\cdot\vec{\sigma})^2 = 1$ for any unit vector $|\vec{n}| = 1$.

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Concerning now the status of the study of the CHSH inequality within the relativistic quantum field theory framework, the amount of research done so far cannot yet be compared to that of quantum mechanics. We quote here the pioneering work by Refs. [21–25], who have been able to show, by using the techniques of the algebraic quantum field theory, that even free fields lead to a violation of the CHSH inequality. This important result is taken as a strong confirmation of the fact that the phenomenon of entanglement in quantum field theory is believed to be more severe than in quantum mechanics, a property often underlined in the extensive literature on the so-called entanglement entropy, a fundamental quantity in order to quantify the degree of entanglement of a very large class of systems; see Refs. [26–29] for a recent overview on this matter.

It seems, thus, worth to us to pursue the investigation of the CHSH inequality within the realm of relativistic quantum field theory.

The paper is organized as follows. In Sec. II, we present the classical aspects of our field theory model as well as the class of operators eligible in order to construct the CHSH inequality. In Sec. III, we proceed with the canonical quantization and with the evaluation of the correlator of the CHSH operator. Although rather small, we shall be able to already observe a violation of the CHSH inequality, confirming in fact the severity of entanglement in relativistic quantum field theory. In Sec. IV, the violation of the CHSH is analyzed in details. Section V deals with the Becchi-Rouet-Stora-Tyutin (BRST) invariant generalization of the present setup to Higgs gauge theories.

II. THE MODEL: CLASSICAL ASPECTS

As already stated, the model we shall be using is constructed with a pair of free massive real scalar fields $(\varphi_A^i, \varphi_B^i)$, $i = 1, 2, 3$, taken in the adjoint representation of the $SU(2)$ group:

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \varphi_A^i \partial_\mu \varphi_A^i - m_A^2 \varphi_A^i \varphi_A^i) + \frac{1}{2}(\partial^\mu \varphi_B^i \partial_\mu \varphi_B^i - m_B^2 \varphi_B^i \varphi_B^i). \tag{4}$$

Furthermore, we introduce the following bounded operator:

$$\mathcal{U}^a(x, y) = \cos a^i(\hat{\varphi}_A^i(x) + \hat{\varphi}_B^i(y)) = \frac{e^{ia^i(\hat{\varphi}_A^i(x)+\hat{\varphi}_B^i(y))} + e^{-ia^i(\hat{\varphi}_A^i(x)+\hat{\varphi}_B^i(y))}}{2}, \tag{5}$$

where $\{a^i\}$ stands for an arbitrary real vector and where we have introduced the rescaled fields $(\hat{\varphi}_A^i, \hat{\varphi}_B^i)$ in order to deal with dimensionless variables:

$$\hat{\varphi}_A^i = \frac{\hat{\varphi}_A^i}{m_A}, \qquad \hat{\varphi}_B^i = \frac{\hat{\varphi}_B^i}{m_B}. \tag{6}$$

As is apparent from expression (5), the quantity $\mathcal{U}^a(x, y)$ is real and bounded, taking values in the interval $[-1, 1]$. As such, according to Refs. [21–25], it is an eligible operator for the construction of a CHSH inequality which, using the same notations of Refs. [21–25], we write as

$$(A + A')B + (A - A')B', \tag{7}$$

with (A, A', B, B') bounded quantities which take values in the interval $[-1, 1]$. Application of the triangle inequality [12–14] shows that

$$|(A + A')B + (A - A')B'| \leq 2. \tag{8}$$

In terms of the operator $\mathcal{U}$, expression (7) takes the form

$$\mathcal{C}^{aa'bb'}(x, x', y, y') = (\mathcal{U}^a(x, y) + \mathcal{U}^{a'}(x, y))\mathcal{U}^b(x', y') + (\mathcal{U}^a(x, y) - \mathcal{U}^{a'}(x, y))\mathcal{U}^{b'}(x', y'), \tag{9}$$

namely,

$$\begin{aligned}\mathcal{C}^{aa'bb'}&(x, x', y, y') \\ &= [\cos a^i(\hat{\varphi}_A^i(x) + \hat{\varphi}_B^i(y)) + \cos a'^i(\hat{\varphi}_A^i(x) + \hat{\varphi}_B^i(y))] \\ &\quad \times \cos b^i(\hat{\varphi}_A^i(x') + \hat{\varphi}_B^i(y')) + [\cos a^i(\hat{\varphi}_A^i(x) + \hat{\varphi}_B^i(y)) \\ &\quad - \cos a'^i(\hat{\varphi}_A^i(x) + \hat{\varphi}_B^i(y))] \cos b'^i(\hat{\varphi}_A^i(x') + \hat{\varphi}_B^i(y')),\end{aligned} \tag{10}$$

with (a^i, a'^i, b^i, b'^i) being arbitrary vectors. These vectors are akin to the four unit vectors $(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}')$ entering expression (1), though, unlike $(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}')$, (a^i, a'^i, b^i, b'^i) are now not restricted to be unit vectors, due to the fact that expression (5) is already bounded, taking values in the interval $[-1, 1]$. They are independent quantities which, as the four vectors $(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}')$, will be chosen in the most convenient way at the end of the computation. From Eq. (8), we have that, classically,

$$l|\mathcal{C}^{aa'bb'}(x, x', y, y')l| \leq 2. \tag{11}$$

for any choice of the vectors (a^i, a'^i, b^i, b'^i).

Let us now specify the space-time properties of the regions in which Alice's and Bob's labs are located. The two space-time points (x, x') belong to a space-time region Ω_A in which Alice's lab is located, while (y, y') refer to points of the region Ω_B corresponding to the location of Bob's lab. The two regions (Ω_A, Ω_B) are spacelike separated. Moreover, we consider events within Ω_A which are timelike. The same for those belonging to Ω_B. This means that the measurements performed by Alice and Bob are separated by spacelike intervals, implementing thus the principle of relativistic causality. In summary, we have

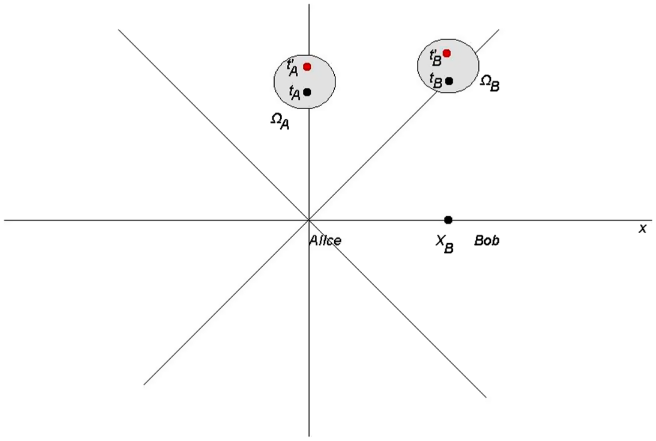

FIG. 1. Location of the labs of Alice and Bob in a two-dimensional space-time diagram.

$$(x-x')^2>0, \qquad (y-y')^2>0, \qquad (x-y)^2<0,$$
$$(x-y')^2<0, \qquad (x'-y)^2<0, \qquad (x'-y')^2<0, \qquad (12)$$

where

$$(x-x')^2=((x^0-x'^0)^2)-(\vec{x}-\vec{x}')^2. \qquad (13)$$

The physical meaning of Eq. (12) can be easily visualized with the help of a two-dimensional (t,x) space-time diagram; see Fig. 1. Alice's lab is located at $x=0$, while Bob's lab at $x=x_B$. Alice performs a first measurement at the time t_A and repeats it at $t'_A>t_A$. On the other hand, Bob does his first measurement at t_B and the second one at $t'_B>t_B$. Moreover, as is apparent from Fig. 1, since the spatial distance between the two labs is greater than the maximum time interval, i.e., $x_B>(t'_B-t_A)$, it follows that Alice and Bob are spacelike separated, according to Eq. (12).

III. CANONICAL QUANTIZATION AND INTRODUCTION OF THE CHSH CORRELATOR BY MEANS OF WEYL OPERATORS

Before facing the quantization of the operator (9), it is useful to shortly recall a few basic properties of the canonical quantization of a free massive scalar field [30]. For such a purpose, the use of a single field φ is enough, the generalization to two free fields being immediate. We start with a free Klein-Gordon field:

$$\mathcal{L}=\frac{1}{2}(\partial^\mu\varphi\partial_\mu\varphi-m^2\varphi^2). \qquad (14)$$

Expanding φ in terms of annihilation and creation operators, we get

$$\varphi(t,\vec{x})=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}(e^{-ikx}a_k+e^{ikx}a_k^\dagger),$$
$$k^0=\omega(k,m)=\sqrt{\vec{k}^2+m^2}, \qquad (15)$$

where

$$[a_k,a_q^\dagger]=(2\pi)^3 2\omega(k,m)\delta^3(\vec{k}-\vec{q}),$$
$$[a_k,a_q]=0, \qquad [a_k^\dagger,a_q^\dagger]=0, \qquad (16)$$

implementing the canonical commutation relations. A quick computation shows that

$$[\varphi(x),\varphi(y)]=i\Delta^m_{\rm PJ}(x-y)=0 \quad \text{for } (x-y)^2<0, \qquad (17)$$

where $\Delta^m_{\rm PJ}(x-y)$ is the Lorentz invariant causal Pauli-Jordan function, encoding the principle of relativistic causality:

$$\Delta^m_{\rm PJ}(x-y)=\frac{1}{i}\int\frac{d^4k}{(2\pi)^3}(\theta(k^0)-\theta(-k^0))\delta(k^2-m^2)e^{-ik(x-y)}, \qquad (18)$$

$$\Delta^m_{\rm PJ}(x-y)=-\Delta^m_{\rm PJ}(y-x), \qquad (\partial_x^2+m^2)\Delta^m_{\rm PJ}(x-y)=0, \qquad (19)$$

$$\Delta^m_{\rm PJ}(x-y)=\left(\frac{\theta(x^0-y^0)-\theta(y^0-x^0)}{2\pi}\right)\times\left(-\delta((x-y)^2)+m\frac{\theta((x-y)^2)J_1(m\sqrt{(x-y)^2})}{2\sqrt{(x-y)^2}}\right), \qquad (20)$$

where J_1 is the Bessel function.

However, as it stands, expression (15) is a too singular object, being in fact an operator-valued distribution in Minkowski space [30]. To give a well-defined meaning to Eq. (15), one introduces the smeared field

$$\varphi(h)=\int d^4x\varphi(x)h(x), \qquad (21)$$

where $h(x)$ is a test function belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^4)$, i.e., to the space of smooth infinitely differentiable functions decreasing as well as their derivatives faster than any power of $(x)\in\mathbb{R}^4$ in any direction. The support of $h(x)$, $supp_h$, is the region in which the test function $h(x)$ is nonvanishing. Introducing the Fourier transform of $h(x)$,[2]

$$\hat{h}(p)=\int d^4xe^{ipx}h(x), \qquad (22)$$

[2]It is well known that the Fourier transform $\hat{h}(p)$ of a test function $h(x)\in\mathcal{S}(\mathbb{R}^4)$ is again a rapidly decreasing function, namely, $\hat{h}(p)\in\mathcal{S}(\mathbb{R}^4)$.

expression (21) becomes

$$\varphi(h) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}(\hat{h}^*(\omega(k,m),\vec{k})a_k + \hat{h}(\omega(k,m),\vec{k})a_k^\dagger) = a_h + a_h^\dagger, \tag{23}$$

where $(a_h, a_h^\dagger)$ stand for

$$a_h = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}\hat{h}^*(\omega(k,m),\vec{k})a_k,$$
$$a_h^\dagger = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}\hat{h}(\omega(k,m),\vec{k})a_k^\dagger. \tag{24}$$

One sees, thus, that the smearing procedure has turned the too singular object $\varphi(x)$ [Eq. (15)] into an operator [Eq. (23)] acting on the Hilbert space of the system. When rewritten in terms of the operators $(a_h, a_{h'}^\dagger)$, the canonical commutation relations (16) take the form

$$[a_h, a_{h'}^\dagger] = \langle h|h'\rangle_m, \tag{25}$$

where $\langle h|h'\rangle_m$ is the Lorentz invariant scalar product between the test functions h and h'; i.e.,

$$\langle h|h'\rangle_m = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}\hat{h}^*(\omega(k,m),\vec{k})\hat{h}'(\omega(k,m),\vec{k}) = \int \frac{d^4\vec{k}}{(2\pi)^4}(2\pi\theta(k^0)\delta(k^2-m^2))\hat{h}^*(k)\hat{h}(k). \tag{26}$$

We indicate by a subscript m the mass that appears in the scalar product (26), in view of the fact that our model [Eq. (4)] contains more than one mass. The scalar product (26) can be rewritten in configuration space. Taking the Fourier transform, one has

$$\langle h|h'\rangle_m = \int d^4x d^4x' h(x)\mathcal{D}(x-x')h(x'), \tag{27}$$

where $\mathcal{D}^m(x-x')$ is the so-called Wightman function

$$\mathcal{D}^m(x-x') = \langle 0|\varphi(x)\varphi(x')|0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}e^{-ik(x-x')}, \quad k^0 = \omega(k,m), \tag{28}$$

which can be decomposed as

$$\mathcal{D}^m(x-x') = \frac{i}{2}\Delta_{\rm PJ}^m(x-x') + H^m(x-x'), \tag{29}$$

where $\Delta_{\rm PJ}^m(x-x')$ is the Pauli-Jordan function and $H^m(x-x') = H^m(x'-x)$ is the real symmetric quantity [31]

$$H^m(x-x') = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}(e^{-ik(x-x')} + e^{ik(x-x')}), \quad k^0 = \omega(k,m). \tag{30}$$

The commutation relation (17) can be expressed in terms of smeared fields as

$$[\varphi(h), \varphi(h')] = i\Delta_{\rm PJ}^m(h,h'), \tag{31}$$

where h and h' are test functions and

$$\Delta_{\rm PJ}^m(h,h') = \int d^4x d^4x' h(x)\Delta_{\rm PJ}^m(x-x')h'(x'). \tag{32}$$

Therefore, the causality condition in terms of smeared fields becomes

$$[\varphi(h), \varphi(h')] = 0, \tag{33}$$

if $supp_h$ and $supp_{h'}$ are spacelike.

A. A few words on the test functions

As a concrete example of test functions, we might consider the class of test functions that have compact support, known as *bump functions*. A good example of a bump function is the function

$$f_{\rm bump}(x) = \begin{cases} \mathcal{C}e^{-\frac{1}{\alpha^2 - m^2|x|^2}} & \text{if } \alpha^2 \geq m^2|x|^2, \\ 0 & \text{if } \alpha^2 < m^2|x|^2, \end{cases} \tag{34}$$

where α is a real number, $\mathcal{C}$ is a normalization factor, and $|x|^2 = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$ is the Euclidean distance from the origin. The function (37) is a smooth function, infinitely differentiable, with compact support. It is nonvanishing only within the region $\alpha^2 \geq m^2|x|^2$. Bump functions such as that in Eq. (37) have many interesting properties; see Ref. [31]. Since $f_{\rm bump}(x) = f_{\rm bump}(-x)$, its Fourier transform

$$\hat{f}_{\rm bump}(p) = \int d^4x e^{ipx} f_{\rm bump}(x) \tag{35}$$

is a real symmetric function:

$$\hat{f}_{\rm bump}(p)^* = \hat{f}_{\rm bump}(p), \quad \hat{f}_{\rm bump}(p) = \hat{f}_{\rm bump}(-p). \tag{36}$$

It turns out that $\hat{f}_{\rm bump}(p)$ has no compact support. However, it is a smooth, infinitely differentiable function, exhibiting an exponential decay for large $|p|$. As such, both $f_{\rm bump}(x)$ and its Fourier transform $\hat{f}_{\rm bump}(p)$ belong to the Schwartz space $\mathcal{S}(\mathbb{R}^4)$. Another important property of the bump functions is that their derivatives are still bump functions. For example,

$$f'_{\rm bump}(x) = \begin{cases} \mathcal{C}' \frac{\partial(e^{-\frac{1}{\beta^2 - m^2|x|^2}})}{\partial x^0} & \text{if } \beta^2 \geq m^2|x|^2, \\ 0 & \text{if } \beta^2 < m^2|x|^2, \end{cases} \quad (37)$$

is an antisymmetric bump function: $f'_{\rm bump}(x) = -f'_{\rm bump}(-x)$. As a consequence, its Fourier transform reads

$$\hat{f}'_{\rm bump}(p) = -ip^0\mathcal{C}' \int_{\beta \geq m|x|} d^4x e^{ipx} e^{-\frac{1}{\beta^2 - m^2|x|^2}}. \quad (38)$$

We see thus that $\hat{f}'_{\rm bump}(p)$ is a purely imaginary function which is antisymmetric:

$$\hat{f}'_{\rm bump}(p)^* = -\hat{f}'_{\rm bump}(p), \qquad \hat{f}'_{\rm bump}(p) = -\hat{f}'_{\rm bump}(-p). \quad (39)$$

In particular, the normalization constants $(\mathcal{C}, \mathcal{C}')$ as well as the properties (36) and (39) can be used to define a pair of test functions (f, f') fulfilling the following properties (see also Refs. [21–25]):

$$\begin{aligned} f(x) &= f(-x), \qquad f'(x) = -f'(-x), \\ \|f\|_m^2 &= \langle f|f\rangle_m = m^2, \qquad \|f'\|_m^2 = m^2, \\ \langle f|f'\rangle_m &= \text{purely imaginary} \\ &= \frac{i}{2}\int d^4x d^4x' f(x)\Delta_{\rm PJ}^m(x-x')f'(x'). \end{aligned} \quad (40)$$

The appearance of the mass parameter m^2 in the normalization of the test functions [Eq. (40)] is due to our conventions for the engineering dimensions of the quantities $(\hat{\varphi}_A^i, \hat{\varphi}_B^i)$ [Eq. (6)], which will be kept dimensionless throughout. Moreover, as we shall see in the next section, the final dependence of the quantum CHSH correlator from the parameters (m_A^2, m_B^2) will enable us to discuss the zero mass limit.

Finally, we recall the Cauchy-Schwarz inequality for the scalar product

$$l|\langle f|f'\rangle_m l|^2 \leq \|f\|_m^2 \|f'\|_m^2 = m^4. \quad (41)$$

B. Weyl operators

Let us now recall a few features of the so-called Weyl operators, which will be the building blocks for the construction of the CHSH operator [Eqs. (9) and (10)] at the quantum level. The Weyl operators are bounded unitary operators built out by exponentiating the smeared field, namely,

$$\mathcal{A}_h = e^{i\hat{\varphi}(h)}, \quad (42)$$

where $\hat{\varphi}(h) = \varphi(h)/m$ is the dimensionless smeared field defined in Eqs. (21) and (23). Making use of the following relation:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}, \quad (43)$$

valid for two operators (A, B) commuting with $[A, B]$, one immediately checks that the Weyl operators give rise to the following algebraic structure:

$$\mathcal{A}_h \mathcal{A}'_h = e^{-\frac{1}{2}[\hat{\varphi}(h), \hat{\varphi}(h')]}\mathcal{A}_{(h+h')} = e^{-\frac{i}{2m^2}\Delta_{\rm PJ}^m(h,h')}\mathcal{A}_{(h+h')}, \quad (44)$$

where $\Delta_{\rm PJ}^m(h, h')$ is the causal Pauli-Jordan function [Eq. (48)]. Also, using the canonical commutation relations written in the form (25), for the vacuum expectation value of $\mathcal{A}_h$, one gets

$$\langle 0|\mathcal{A}_h|0\rangle = e^{-\frac{1}{2m^2}\|h\|_m^2}. \quad (45)$$

As already underlined, the vacuum state $|0\rangle$ is the Fock vacuum: $a_k|0\rangle = 0$, for all modes k.

C. Construction of the CHSH quantum operator

At the classical level, the model we are considering is characterized by the Lagrangian density (4). We have two free fields $(\varphi_A^i, \varphi_B^i)$, for Alice and Bob, respectively. Each field satisfies the Klein-Gordon equation and can be expanded in term of annihilation and creation operators [Eq. (15)], namely,

$$\begin{aligned} \varphi_A^i(t, \vec{x}) &= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega(k, m_A)}(e^{-ikx}a_k^i + e^{ikx}a_k^{i\dagger}), \\ & k^0 = \omega(k, m_A), \\ \varphi_B^i(t, \vec{x}) &= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega(k, m_B)}(e^{-ikx}b_k^i + e^{ikx}b_k^{i\dagger}), \\ & k^0 = \omega(k, m_B), \end{aligned} \quad (46)$$

where the only nonvanishing commutators among the annihilation and creation operators are

$$\begin{aligned} [a_k^i, a_q^{j\dagger}] &= (2\pi)^3 2\omega(k, m_A)\delta^3(\vec{k} - \vec{q})\delta^{ij}, \\ [b_k^i, b_q^{j\dagger}] &= (2\pi)^3 2\omega(k, m_B)\delta^3(\vec{k} - \vec{q})\delta^{ij}. \end{aligned} \quad (47)$$

To have well-defined operators in the Fock-Hilbert space, these fields are smeared with test functions, as described in Eq. (21), resulting in $(\varphi_A^i(h), \varphi_B^i(h))$. It is thus straightforward to evaluate the following commutation relations for the smeared fields:

$$\begin{aligned} [\varphi_A^i(h), \varphi_A^j(h')] &= i\delta^{ij}\Delta_{\rm PJ}^{m_A}(h, h'), \\ [\varphi_B^i(\tilde{h}), \varphi_B^j(\tilde{h}')] &= i\delta^{ij}\Delta_{\rm PJ}^{m_B}(\tilde{h}, \tilde{h}'), \\ [\varphi_A^i(h), \varphi_B^j(\tilde{h})] &= 0, \end{aligned} \quad (48)$$

valid for any pair of test functions (h, h') and $(\tilde{h}, \tilde{h}')$. The presence of the Pauli-Jordan function in expressions (48) implements the relativistic causality in the model. In fact, if $supp_h$ and $supp_{h'}$ are spacelike as well as those of $(\tilde{h}, \tilde{h}')$, then the commutator of the corresponding smeared fields vanishes. The Fock vacuum of the model is defined as being the state $|0\rangle$ such that

$$a^i_k|0\rangle = 0, \quad b^i_k|0\rangle = 0, \tag{49}$$

for any $i = 1, 2, 3$ and any momentum k.

We are now ready to write down the quantum version of the CHSH operator [Eqs. (9) and (10)]. We first introduce the smeared operator

$$\mathcal{U}^{ab}_{ff'gg'} = \cos a^i(\hat{\varphi}^i_A(f) + \hat{\varphi}^i_B(g)) \cos b^i(\hat{\varphi}^i_A(f') + \hat{\varphi}^i_B(g')) = \left[\frac{e^{ia^i(\hat{\varphi}^i_A(f)+\hat{\varphi}^i_B(g))} + e^{-ia^i(\hat{\varphi}^i_A(f)+\hat{\varphi}^i_B(g))}}{2}\right] \times \left[\frac{e^{ib^i(\hat{\varphi}^i_A(f')+\hat{\varphi}^i_B(g'))} + e^{-ib^i(\hat{\varphi}^i_A(f')+\hat{\varphi}^i_B(g'))}}{2}\right], \tag{50}$$

where (f, f') and (g, g') are test functions belonging, respectively, to Alice's and Bob;s space-time regions, Ω_A and Ω_B, respectively; see Fig. 1. More precisely, the supports of (f, f') are spacelike with respect to those of (g, g'):

$$(\mathrm{supp}_{(f,f')}) \text{ spacelike with respect to } (\mathrm{supp}_{(g,g')}). \tag{51}$$

Furthermore, we introduce the Hermitian operator

$$\hat{\mathcal{U}}^{ab}_{ff'gg'} = (\hat{\mathcal{U}}^{ab}_{ff'gg'})^\dagger = \frac{1}{2}(\mathcal{U}^{ab}_{ff'gg'} + (\mathcal{U}^{ab}_{ff'gg'})^\dagger). \tag{52}$$

Finally, for the quantum version of the CHSH operator [Eqs. (9) and (10)], we write

$$\mathcal{C}^{aa'bb'}_{(ff'gg')} = \hat{\mathcal{U}}^{ab}_{ff'gg'} + \hat{\mathcal{U}}^{a'b}_{ff'gg'} + \hat{\mathcal{U}}^{ab'}_{ff'gg'} - \hat{\mathcal{U}}^{a'b'}_{ff'gg'}. \tag{53}$$

In the following, we shall compute the vacuum correlator

$$\langle 0|\mathcal{C}^{aa'bb'}_{(ff'gg')}|0\rangle \tag{54}$$

by means of the algebraic properties of the Weyl operators. According to Refs. [21–25], we shall speak of a violation of the CHSH classical inequality [Eq. (11)] if

$$l|\langle 0|\mathcal{C}^{aa'bb'}_{(ff'gg')}|0\rangle l| > 2 \tag{55}$$

for some suitable choice of (a^i, a'^i, b^i, b'^i).

Expression (54) can be calculated in closed form using Eqs. (43)–(45) and the fact that the vacuum is annihilated by a^i_h and b^j_h; see Eq. (49). The outcome of our result reads

$$\langle \mathcal{C}^{aa'bb'}_{(ff'gg')}\rangle = e^{-\frac{1}{2}\left[\vec{a}\cdot\vec{a}\left(\frac{\|f\|^2_{m_A}}{m_A^2}+\frac{\|g\|^2_{m_B}}{m_B^2}\right)+\vec{b}\cdot\vec{b}\left(\frac{\|f'\|^2_{m_A}}{m_A^2}+\frac{\|g'\|^2_{m_B}}{m_B^2}\right)\right]} \times \cos\left(\frac{\vec{a}\cdot\vec{b}}{2}(\omega_A+\omega_B)\right)\cosh(\vec{a}\cdot\vec{b}(\tilde{\omega}_A+\tilde{\omega}_B)) + (a\to a') + (b\to b') - (a\to a', b\to b'), \tag{56}$$

where

$$\omega_A = \frac{1}{m_A^2}\Delta^{m_A}_{\mathrm{PJ}}(f, f'), \quad \omega_B = \frac{1}{m_B^2}\Delta^{m_B}_{\mathrm{PJ}}(g, g'), \quad \tilde{\omega}_A = \frac{1}{m_A^2}\mathrm{Re}\langle f|f'\rangle_{m_A}, \quad \tilde{\omega}_B = \frac{1}{m_B^2}\mathrm{Re}\langle g|g'\rangle_{m_B}. \tag{57}$$

Instead of explicitly writing all terms in Eq. (56), we have simply indicated that the other terms are obtained from the first one by replacing the vectors $\vec{a} = (a^1, a^2, a^3)$ and $\vec{b} = (b^1, b^2, b^3)$ as denoted by the arrows. The scalar product between the vectors in Eq. (56) is the tridimensional Euclidian scalar product, i.e., $\vec{a}\cdot\vec{b} = \sum_{i=1}^3 a^i b^i$.

IV. ANALYSIS OF THE VIOLATION OF THE CHSH INEQUALITY

Having evaluated the CHSH correlator [Eq. (56)], we can face now the issue of the violation of the CHSH inequality. To a first look, one might have the impression that Eq. (56) contains a lot of free parameters, so that it would be relatively simple to find a violation of the CHSH inequality. But things are not that easy, the main reason being the presence of the exponentials which decay very fast. As a consequence, the allowed space of parameters turns out to be quite small.

Before analyzing the best choice for the parameters (a^i, a'^i, b^i, b'^i), we fix the norms of the test functions (f, f') and (g, g'), with supports in the regions of Alice's lab Ω_A and Bob's lab Ω_B, respectively, according to

$$\|f\|^2_{m_A} = \|f'\|^2_{m_A} = m_A^2, \qquad \|g\|^2_{m_B} = \|g'\|^2_{m_B} = m_B^2. \tag{58}$$

It is worth remarking here that, as expected, the choice of the norm of the test functions does not play much role in expression (56). These norms are easily seen to be reabsorbable into the vectors $(\vec{a}, \vec{b}, \vec{a}', \vec{b}')$, which are arbitrary. Therefore, the choice of working with normalized test functions [Eq. (58)] does not change the final output.

Concerning now the scalar products $\langle f|f'\rangle$ and $\langle g|g'\rangle$, we have followed the same prescription adopted in the original work [22] and have taken $(\langle f|f'\rangle, \langle g|g'\rangle)$ purely imaginary,[3] as described in Eqs. (40), namely,

$$\langle f|f'\rangle = \text{purely imaginary} = \frac{i}{2}\Delta_{\rm PJ}^{m_A}(f,f'),$$
$$\langle g|g'\rangle = \text{purely imaginary} = \frac{i}{2}\Delta_{\rm PJ}^{m_B}(g,g'). \tag{59}$$

Because of Eqs. (58) and (59), the CHSH correlator gets simplified:

$$\begin{aligned}\langle \mathcal{C}^{aa'bb'}_{(ff'gg')}\rangle &= e^{-\vec{a}\cdot\vec{a}-\vec{b}\cdot\vec{b}}\cos\left(\frac{\vec{a}\cdot\vec{b}}{2}(\omega_A+\omega_B)\right)\\ &+ e^{-\vec{a}'\cdot\vec{a}'-\vec{b}\cdot\vec{b}}\cos\left(\frac{\vec{a}'\cdot\vec{b}}{2}(\omega_A+\omega_B)\right)\\ &+ e^{-\vec{a}\cdot\vec{a}-\vec{b}'\cdot\vec{b}'}\cos\left(\frac{\vec{a}\cdot\vec{b}'}{2}(\omega_A+\omega_B)\right)\\ &- e^{-\vec{a}'\cdot\vec{a}'-\vec{b}'\cdot\vec{b}'}\cos\left(\frac{\vec{a}'\cdot\vec{b}'}{2}(\omega_A+\omega_B)\right).\end{aligned} \tag{60}$$

In addition to the vectors (a^i, a'^i, b^i, b'^i), expression (60) contains the quantity

$$\omega_A+\omega_B = \frac{1}{m_A^2}\Delta_{\rm PJ}^{m_A}(f,f') + \frac{1}{m_B^2}\Delta_{\rm PJ}^{m_B}(g,g'), \tag{61}$$

which is the smearing of the Pauli-Jordan function. Because of the choice of the scalar products $\langle f|f'\rangle$ and $\langle g|g'\rangle$ done in Eq. (59), from the Cauchy-Schwarz inequality it follows

$$\Delta_{\rm PJ}^{m_A}(f,f') = \frac{2}{i}\langle f|f'\rangle_{m_A}, \tag{62}$$

which implies that

$$|\Delta_{\rm PJ}^{m_A}(f,f')| \le 2\|f\|_{m_A}\|f'\|_{m_A} = 2m_A^2. \tag{63}$$

The same holds for $\Delta_{\rm PJ}^{m_B}(g,g')$. Thus, taking into account the even character of the cosine, it is convenient to parametrize $(\omega_A+\omega_B)$ by introducing the quantity σ defined as

$$\omega_A+\omega_B = 4\sigma, \qquad 0\le\sigma\le 1. \tag{64}$$

Therefore, we have

$$\begin{aligned}\langle \mathcal{C}^{aa'bb'}_{(ff'gg')}\rangle &= e^{-a^2-b^2}\cos(2\vec{a}\cdot\vec{b}\sigma) + e^{-a'^2-b^2}\cos(2\vec{a}'\cdot\vec{b}\sigma)\\ &+ e^{-a^2-b'^2}\cos(2\vec{a}\cdot\vec{b}'\sigma)\\ &- e^{-a'^2-b'^2}\cos(2\vec{a}'\cdot\vec{b}'\sigma),\end{aligned} \tag{65}$$

where $a=|\vec{a}|$, $b=|\vec{b}|$, $a'=|\vec{a}'|$, and $b'=|\vec{b}'|$. Before focusing on expression (65), let us devote a little discussion to the parameter σ, which has a deep physical meaning. This is the task of the next subsection.

A. The meaning of the parameter σ

As we have seen [Eq. (64)], the parameter σ is directly related to the smearing of the Pauli-Jordan functions $(\Delta_{\rm PJ}^{m_A}(f,f'), \Delta_{\rm PJ}^{m_B}(g,g'))$. As such, σ encodes all information about the relativistic causality of our model. It is important, thus, to have a more precise idea of its behavior and of its explicit relation with Alice's and Bob's spacetime configurations, as depicted in Fig. 1. To that end, it suffices to pick up Alice's factor ω_A [Eq. (57)] and proceed by smearing it with two narrowed Gaussians in order to be able to work out analytic expressions which will provide a more transparent understanding of σ. Accordingly, for the pair of Gaussian test functions (f,f') around $\vec{x}=0$, we write

$$f(x) = \sqrt{\frac{8\pi^2}{0,41411\ldots}}m_A\partial_t e^{-m_A^2(t-t_A)^2}e^{-m_A^2r^2},$$
$$f'(x) = \sqrt{\frac{8\pi^2}{0,18301\ldots}}m_A^2 e^{-m_A^2(t-t'_A)^2}e^{-m_A^2r^2}, \tag{66}$$

where $r=|\vec{x}|$ and where the numerical factors take into account the normalization of (f,f'). The two temporal coordinates t_A and t'_A in Eq. (66) can be thought of as the instants in which Alice performs her measurements; see Fig. 1. Moving to momentum space, one gets[4]

$$\hat{f}(p) = -\sqrt{\frac{8\pi^2}{0,41411\ldots}}\frac{ip^0 e^{-\frac{p^2}{4m^2}-\frac{p_0^2}{4m^2}+ip^0t_A}}{m\sqrt{2}},$$
$$\hat{f}'(p) = \sqrt{\frac{8\pi^2}{0,18301\ldots}}\frac{e^{-\frac{p^2}{4m^2}-\frac{p_0^2}{4m^2}+ip^0t'_A}}{\sqrt{2}}. \tag{67}$$

[3]We notice that, since at least one of the labs is not located at the origin of the coordinate system, one pair of test functions will not have the odd and even symmetries with respect to $x=0$, as assumed in Eqs. (40). However, due to the translation invariance of the Wightman function and, consequently, of the scalar product, a pair of test functions can still satisfy Eqs. (40) if they are odd and even with respect to a certain point of the space-time which, in the present case, is the location of Bob's lab; see Fig. 1.

[4]Strictly speaking, f and f' being two Gaussians, they do not display properties (40); that is, $\langle f|f'\rangle$ has a real part as well as an imaginary part. Moreover, one easily verifies that the real part gets smaller and smaller as $m_A|t_A-t'_A|$ becomes small, so that $\langle f|f'\rangle$ fulfills, in practice, property (40).

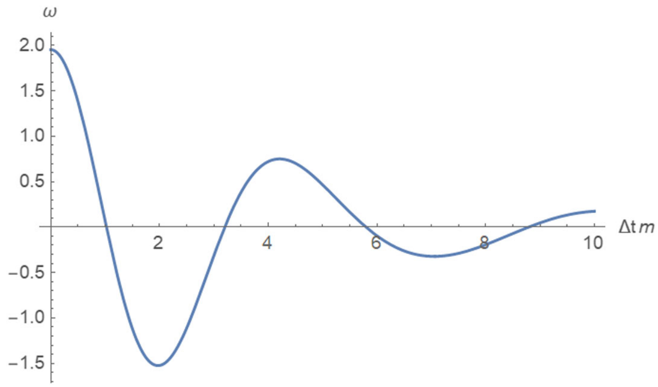

FIG. 2. ω_A as a function of the variable $(\Delta t m_A)$.

Evaluating ω_A, one finds

$$\omega_A = \frac{2}{\sqrt{0,41411\ldots}\sqrt{0,18301\ldots}} \times \int_0^\infty du u^2 \cos\left(m_A(t_A - t'_A)\sqrt{1+u^2}\right) e^{-\frac{1+2u^2}{2}}. \quad (68)$$

The behavior of ω_A as a function of $m_A \Delta t \equiv m_A |t_A - t'_A|$ is shown in Fig. 2. Essentially, ω_A shows an exponential decay modulated by a periodic function. This exponential decay in the variable $(m_A \Delta t)$ is in full agreement with one of the main results of Refs. [21–25]; see, in particular, Corollary 4.2 in Ref. [21]. It means that the violation of the CHSH inequality decreases exponentially with the magnitude of the masses of the particles and with the size of the time intervals involved. For lighter particles and short time intervals, i.e., when $\sigma \approx 1$, we shall in fact be able to show that the violation of the CHSH inequality of our model is the biggest one.

B. The violation of the CHSH inequality

Let us now dive into the analysis of the CHSH correlator (65), which is recognized to be a bounded quantity. To analyze $\langle \mathcal{C}^{aa'bb'}_{(ff'gg')} \rangle$, we consider the space of parameter as being

$$(a, a', b, b', \alpha, \beta, \gamma, \delta, \sigma),$$

where

$$\begin{aligned} \vec{a} \cdot \vec{b} &= ab\cos\alpha, \\ \vec{a}' \cdot \vec{b} &= a'b\cos\beta, \\ \vec{a} \cdot \vec{b}' &= ab'\cos\gamma, \\ \vec{a}' \cdot \vec{b}' &= a'b'\cos\delta. \end{aligned} \quad (69)$$

Regarding the parameters (a, a', b, b'), which correspond to the norms of the vectors (a^i, a'^i, b^i, b'^i), one sees that, due to the exponential decay of expression (65), they cannot take large values; otherwise, the whole correlator will be exponentially suppressed, becoming too small in order to detect a violation of the CHSH inequality. For the two parameters (a, b), the best values seem to be $a = b = 0,001$. In Figs. 3 and 4, one finds the behavior of $\langle \mathcal{C}^{aa'bb'}_{(ff'gg')} \rangle$ as a function of the remaining parameters a' and b' and for the choices $\sigma = 0,85$ and $\sigma = 1$, respectively, in a configuration in which all vectors (a^i, a'^i, b^i, b'^i) are parallel; that is, $\alpha = \beta = \gamma = \delta = 0$.

It turns out that (see cyan surface and the blue curve in Figs. 3 and 4) $\langle \mathcal{C}^{aa'bb'}_{(ff'gg')} \rangle$ violates the CHSH inequality in both cases, although the violation is rather small, its maximum value being located in the interval [2.029, 2.03]. According to Ref. [21] and as discussed before, the violation of the CHSH inequality reaches its optimal value for $\sigma = 1$, corresponding to light particles and short time intervals.

Let us mention that, although in our analysis we have employed many different configurations for the vectors $(\vec{a}, \vec{b}, \vec{a}', \vec{b}')$, the correlator $\langle \mathcal{C}^{aa'bb'}_{(ff'gg')} \rangle$ turns out to be

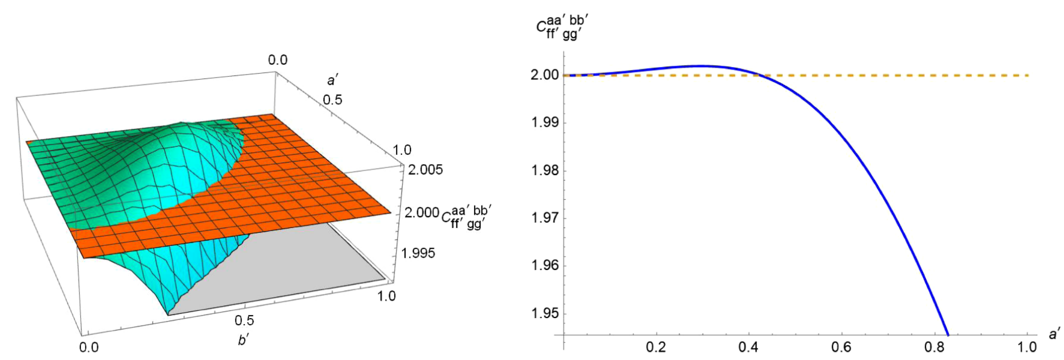

FIG. 3. CHSH correlator for $\sigma = 0,85$. Behavior of the CHSH correlator $\langle \mathcal{C}^{aa'bb'}_{(ff'gg')} \rangle$ (cyan surface) for $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b}' = \vec{a}' \cdot \vec{b} = 0$, $\vec{a}' \cdot \vec{b}' = a'b'$, $a = b = 0,001$, and $\sigma = 0,85$. To observe the violation more easily, we have also plotted the plane $z = 2$, corresponding to the orange surface. The blue line in the right-hand side shows the behavior of $\langle \mathcal{C}^{aa'bb'}_{(ff'gg')} \rangle$ for $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b}' = \vec{a}' \cdot \vec{b} = 0$, $\vec{a}' \cdot \vec{b}' = a'b'$, $a = b = 0,001$, $\sigma = 0,85$, and $b' = 0,7$.

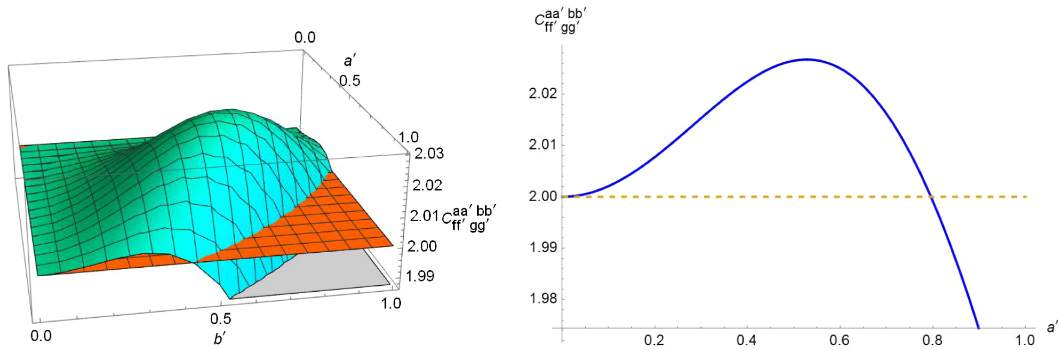

FIG. 4. CHSH correlator for $\sigma = 1$. The CHSH correlator $\langle \mathcal{C}^{aa'bb'}_{(ff'gg')} \rangle$ (cyan surface) for $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{b}' = \vec{a}'\cdot\vec{b} = 0$, $\vec{a}'\cdot\vec{b}' = a'b'$, $a = b = 0, 001$, and $\sigma = 1$. To observe the violations more easily, we have also plotted the plane $z = 2$ (orange surface). The blue line in the right-hand side shows the behavior of $\langle \mathcal{C}^{aa'bb'}_{(ff'gg')} \rangle$ for $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{b}' = \vec{a}'\cdot\vec{b} = 0$, $\vec{a}'\cdot\vec{b}' = a'b'$, $a = b = 0, 001$, $\sigma = 1$, and $b' = 0, 7$.

sensible only to the relative orientation of $\vec{a}'$ and $\vec{b}'$. For instance, if $\vec{a}'$ and $\vec{b}'$ are perpendicular, i.e., $\delta = \pi/2$, the violation no longer occurs, whereas parallel, $\delta = 0$, or antiparallel, $\delta = \pi$, configurations lead to a violation.

Finally, as already underlined, the violation of the CHSH inequality increases as $\sigma \to 1$.

V. A BRST INVARIANT FORMULATION OF THE CHSH INEQUALITY IN GAUGE THEORIES: THE EXAMPLE OF THE $U(1)$ HIGGS MODEL

This section is devoted to outline a BRST invariant setup for the study of the CHSH inequality in gauge theories, taking as an explicit example the renormalizable $U(1)$ Higgs model whose action, including the gauge fixing, is given by

$$S_{\rm Higgs} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}(A) F^{\mu\nu}(A) + (D_\mu\varphi)^*(D^\mu\varphi) - \frac{\lambda}{2}\left(\varphi^*\varphi - \frac{v^2}{2}\right)^2 + b\partial_\mu A^\mu + \bar{c}\partial^2 c \right], \tag{70}$$

where

$$\varphi = \frac{1}{\sqrt{2}}(v + h + i\rho) \tag{71}$$

is a complex scalar field whose components h and ρ denote the Higgs and the Goldstone fields, respectively. The massive parameter v is the vacuum expectation value of φ, i.e., $\langle\varphi\rangle = \frac{v}{\sqrt{2}}$, implementing the Higgs mechanism. The field b is known as the Nakanishi-Lautrup field, needed to impose the gauge condition which, in the present case, has been chosen to be the transverse Landau gauge

$$\partial A = 0. \tag{72}$$

Also, the fields $(\bar{c}, c)$ are the Faddeev-Popov ghosts.

The action $S_{\rm Higgs}$ enjoys an exact BRST invariance:

$$sS_{\rm Higgs} = 0, \qquad s^2 = 0, \tag{73}$$

where s is the nilpotent BRST operator, whose action on the fields $(A_\mu, h, \rho, b, \bar{c}, c)$ is specified by

$$\begin{aligned} sA_\mu &= -\partial_\mu c, \\ sh &= -ec\rho, \\ s\rho &= ec(v+h), \\ sc &= 0, \\ s\bar{c} &= b, \\ sb &= 0. \end{aligned} \tag{74}$$

The interest in the $U(1)$ Higgs model is due to a set of articles [32–43] where a fully BRST invariant description of the massive gauge boson has been worked out.

More precisely, it has been shown that the following dimension-three vector operator

$$V_\mu = \frac{1}{2}(-\rho\partial_\mu h + (v+h)\partial_\mu\rho + eA_\mu(v^2 + h^2 + 2vh + \rho^2)), \tag{75}$$

displays the following properties (see Refs. [35–40]):

(i) V_μ is BRST invariant, belonging to the local cohomology [44] of the operator s:

$$sV_\mu = 0, \qquad V_\mu \neq sQ_\mu, \tag{76}$$

for some local field polynomial Q_μ.

(ii) V_μ turns out to be the conserved Noether current corresponding to the global $U(1)$ invariance of the action (70), namely,

$$\begin{aligned} \delta h &= -\omega\rho, \qquad \delta\rho = \omega(v+h), \\ \delta(A_\mu, b, \bar{c}, c) &= 0, \qquad \delta S_{\rm Higgs} = 0, \end{aligned} \tag{77}$$

where ω is a constant parameter. Thus,

$$\partial^{\mu} V_{\mu} = \text{equations of motion.} \tag{78}$$

From this property, it follows that the anomalous dimension of V_{μ} vanishes to all orders in perturbation theory.

(iii) The transverse component of the two-point function $\langle V_{\mu}(p) V_{\nu}(-p) \rangle^T$,

$$\langle V_{\mu}(p) V_{\nu}(-p) \rangle^T = \mathcal{P}_{\mu\sigma} \langle V^{\sigma}(p) V_{\nu}(-p) \rangle,$$
$$\mathcal{P}_{\mu\sigma} = \left(g_{\mu\sigma} - \frac{p_{\mu} p_{\sigma}}{p^2} \right), \tag{79}$$

has the same pole mass of the elementary two-point function $\langle A_{\mu}(p) A_{\nu}(-p) \rangle$, a key property which extends to all orders of perturbation theory, due to a set of Ward identities.

(iv) The longitudinal component of $\langle V_{\mu}(p) V_{\nu}(-p) \rangle^L$,

$$\langle V_{\mu}(p) V_{\nu}(-p) \rangle^L = \mathcal{L}_{\mu\sigma} \langle V^{\sigma}(p) V_{\nu}(-p) \rangle,$$
$$\mathcal{L}_{\mu\sigma} = \left(\frac{p_{\mu} p_{\sigma}}{p^2} \right), \tag{80}$$

has only tree-level contributions to all orders. Moreover, the tree-level term is momentum independent, so that $\langle V_{\mu}(p) V_{\nu}(-p) \rangle^L$ does not correspond to any propagating mode.

(v) The two-point transverse function $\langle V_{\mu}(p) V_{\nu}(-p) \rangle^T$ exhibits a Källén-Lehmann spectral representation with positive definite spectral density.

All these nontrivial properties enable us to employ the operator V_{μ} to achieve a fully gauge-invariant description of the massive gauge boson in the $U(1)$ Higgs model. It worth underlining that the whole set of properties listed above generalize to the non-Abelian $SU(2)$ case with a single scalar field in the fundamental representation; see Refs. [35–40].

Being BRST invariant, the operator V_{μ} leads to a natural construction of BRST invariant bounded Weyl-type operators, i.e.,

$$\mathcal{A}_V = e^{i \hat{V}(f)} = e^{i \int_{\Omega} d^4x f_{\mu}(x) \hat{V}(x)^{\mu}}, \tag{81}$$

where $\hat{V}_{\mu}$ stands for the dimensionless quantity

$$\hat{V}_{\mu}(x) = \frac{1}{e v^3} V_{\mu}(x) \tag{82}$$

and where $\{f_{\mu}(x)\}$ are a set of smooth functions with compact support, introduced in order to localize the operator $\mathcal{A}_V$ in the desired region of the space-time Ω.

It is helpful to remind here that, in the case of a gauge field $A_{\mu}(x)$, the smearing procedure is done by means of a set $\{f_{\mu}(x)\}$ of test functions carrying a Lorentz index (see Ref. [45]):

$$A(f) = \int d^4x A^{\mu}(x) f_{\mu}(x), \tag{83}$$

where $\{f_{\mu}(x)\}$ are required to transform in such a way to leave expression (83) Lorentz invariant.

As is apparent, the operator $\mathcal{A}_V$ displays the important property of being BRST invariant, providing thus a way to construct suitable CHSH operators in order to investigate the violation of the CHSH inequality in Higgs models within an explicit BRST invariant environment. From the computational side, the operator $\mathcal{A}_V$ can be evaluated order by order in a loop expansion, much like the usual way we deal with the perturbative treatment of the Wilson loop $\mathcal{W}_{\gamma} = e^{i \int_{\gamma} dx^{\mu} A_{\mu}}$.

Let us end this section by giving a short account of what we are currently doing on the $U(1)$ Higgs model, whose detailed analysis will be reported in a forthcoming work [46].

As we have learned from the pioneering work [21–25], free fields are already able to produce a violation of the CHSH inequality. Therefore, as a first step, we are looking at the purely quadratic part of the Higgs action S_{Higgs} [Eq. (70)], namely,

$$S^{\text{quad}}_{\text{Higgs}} = \int d^4x \bigg[-\frac{1}{4} F_{\mu\nu}(A) F^{\mu\nu}(A) + \frac{m^2}{2} A_{\mu} A^{\mu} + \frac{1}{2} \partial_{\mu} h \partial^{\mu} h - \frac{m_h^2}{2} h^2 + \frac{1}{2} \partial_{\mu} \rho \partial^{\mu} \rho + m A_{\mu} \partial^{\mu} \rho + b \partial_{\mu} A^{\mu} - \bar{c} \partial^2 c \bigg], \tag{84}$$

where $m^2 = e^2 v^2$ and $m_h^2 = \lambda v^2$ are the masses of the gauge vector boson and of the Higgs field h, respectively.

Even at the quadratic level, the action $S^{\text{quad}}_{\text{Higgs}}$ exhibits an exact BRST invariance, corresponding to the linear part of the transformations of Eqs. (74), i.e.,

$$s_0 S^{\text{quad}}_{\text{Higgs}} = 0, \qquad s_0 s_0 = 0, \tag{85}$$

where

$$\begin{aligned} s_0 A_{\mu} &= -\partial_{\mu} c, \\ s_0 h &= 0, \\ s_0 \rho &= e v c, \\ s_0 c &= 0, \\ s_0 \bar{c} &= b, \\ s_0 b &= 0. \end{aligned} \tag{86}$$

At the same order, the vector operator V_μ [Eq. (75)] becomes

$$V_\mu^{\rm lin} = \frac{1}{2} v(\partial_\mu \rho + e v A_\mu), \tag{87}$$

with

$$s_0 V_\mu^{\rm lin} = 0. \tag{88}$$

One easily recognizes that $V_\mu^{\rm lin}$ coincides precisely with the physical part of the gauge boson field, displaying the content of the Higgs mechanism: The Goldstone mode is eaten by the gauge field, which becomes massive.

We underline that the quadratic action in Eq. (84) can be canonically quantized by following the well-known Kugo-Ojima procedure [47–50] for the construction of the physical Fock space with positive norm states through the extensive use of the cohomology of the BRST charge. We can therefore repeat the same analysis done in the previous sections and built an s_0-invariant CHSH correlator by means of the s_0-invariant Weyl operator

$$\mathcal{A}_{V^{\rm lin}} = e^{i\hat{V}^{\rm lin}(f)}, \qquad s_0 \mathcal{A}_{V^{\rm lin}} = 0, \tag{89}$$

allowing thus to investigate the possible violation of the CHSH inequality in the $U(1)$ Higgs system already at the quadratic level [46].

We mention here that a BRST invariant operator O can be introduced also for the Higgs field h [35–40]:

$$O(x) = \frac{1}{2}(2vh(x) + h^2(x) + \rho^2(x)), \qquad sO = 0. \tag{90}$$

The operator O shares many of the properties of the operator V_μ at the quantum level [35–40] and can be employed in order to have a BRST invariant description of the Higgs field h. As in the case of V_μ, BRST invariant Weyl operators can be introduced by exponentiating O:

$$\mathcal{A}_O = e^{i\hat{O}(g)} = e^{i\int_\Omega d^4x g(x)\hat{O}(x)}, \tag{91}$$

where $\hat{O}$ denotes the dimensionless quantity

$$\hat{O}(x) = \frac{1}{v^2} O(x). \tag{92}$$

Therefore, even in the quadratic approximation, it will be possible to investigate the violation of the CHSH inequality by using Weyl operators of the Higgs type [Eq. (92)].

VI. CONCLUSION

In this work, we have analyzed the violation of the CHSH inequality in a relativistic quantum field theory model. Following the pioneering work of Refs. [21–25], we started with a pair of free massive real scalar fields $(\varphi_A^i, \varphi_B^i)$, $i = 1, 2, 3$, taken in the adjoint representation of the $SU(2)$ group [Eq. (4)]. These fields have been employed to introduce a CHSH-type operator $\mathcal{C}_{(ff'gg')}^{aa'bb'}$ [Eqs. (50), (52), and (53)], obtained by means of Hermitian combinations of Weyl operators. Making use of the canonical quantization, the correlation function $\langle \mathcal{C}_{(ff'gg')}^{aa'bb'} \rangle$ of the above-mentioned operator has been evaluated in closed form [Eq. (56)], allowing us to already detect a violation of the CHSH inequality in the free case; see Figs. 3 and 4.

Although the reported violation turns out to be rather small as compared to Tsirelson's bound, we believe that the present work might be helpful for the investigation of more physical models.

In particular, as discussed in Sec. V, we have paid attention to devise a BRST invariant framework for the study of the violation of the CHSH inequality in the case of gauge theories, taking as an explicit example the $U(1)$ Higgs model [46]. We highlight that the setup presented in Sec. V generalizes as well to the case of the non-Abelian $SU(2)$ model [35–40], a feature which might lead to a quantum field theory investigation of the CHSH inequality in electroweak theory, a subject of great phenomenological and experimental interest; see Ref. [51] and references therein.

A second topic which we are starting to look at is the possibility of obtaining a formulation of the CHSH inequality by means of direct use of the Feynman path integral. This would enable us to treat interacting field theories through the usual dictionary of the Feynman diagrams. Even if the task might seem to not present much difficulty, it requires one, nevertheless, to face the challenging issue of the renowned lack of causality of the Feynman propagator $\Delta_F(x-y)$ [52], namely,

$$\Delta_F(x-y) = \begin{cases} \frac{1}{4\pi}\delta((x-y)^2) - \frac{m}{8\pi\sqrt{(x-y)^2}} H_1^{(2)}(m\sqrt{(x-y)^2}), & (x-y)^2 \geq 0, \\ \frac{im}{4\pi^2\sqrt{(-(x-y)^2)}} K_1(m\sqrt{-(x-y)^2}), & (x-y)^2 < 0, \end{cases} \tag{93}$$

where $H_1^{(2)}$ is the Hankel function, while K_1 is the modified Bessel function. Expression (93) shows the lack of causality of $\Delta_F(x-y)$: It receives nonvanishing contributions from the spacelike region $(x-y)^2 < 0$. This feature requires a deeper understanding of the relationship between entanglement and Feynman propagator; see, for instance, the discussion in Ref. [53]. This is certainly a topic worth to be investigated, due to the large number of applications of the Feynman path integral in quantum field theory.

Overall, it seems fair to state that the study of the Bell-CHSH inequalities in quantum field theory can be considered at its very beginning. Many issues remain to be investigated. Our results, obtained by employing a rather simple free field model, suggest that the entanglement properties of the vacuum state of a relativistic quantum field theory should play a fundamental role, as corroborated by the general Reeh-Schlieder theorem [30]; see also the detailed Sec. III, Eq. (3.22), in Ref. [26].

Certainly, the inclusion of interaction terms in the starting action represents a major step toward a complete understanding of the Bell-CHSH inequalities in quantum field theory. In particular, the treatment of non-Abelian theories looks of particular interest, due to the asymptotic freedom as well as to genuine nonperturbative effects ranging from the existence of the Gribov copies to the inclusion of instanton and monopole contributions. We hope to report soon on these very attractive topics.

Finally, we add to this short list the establishment of a quantum field theory version of Bell's inequality [1], which we reproduce below in its original form [1]:

$$|E(a,b) - E(a,c)| \leq 1 + E(b,c), \tag{94}$$

where $E(a,b)$ stands for the expectation value of the product of Alice's and Bob's measurements [1]. From the work of Refs. [21–25], we have learned how to formulate the CHSH inequality in relativistic quantum field theory, though we are unaware of a similar formulation for the original Bell inequality. This would be a nice achievement, in view of the pivotal role played by this inequality in the physics of entanglement.

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Correction: The names Schwarz and Schwartz were interchanged in the sentence below Eq. (21), two sentences below Eq. (36), and in the sentence above Eq. (41) and have been set right.