Two-point function of a quantum scalar field in the interior region of a Kerr black hole

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Quantum field effects on a classical background spacetime may be obtained from the semiclassical equations of general relativity with the expectation value of the stress-energy tensor of the quantum field as a source. This expectation value can be calculated from Hadamard's elementary two-point function, which in practice is given in terms of sums of products of field modes evaluated at two spacetime points. We derive expressions for the two-point function for a massless scalar field in the Unruh state on a Kerr black hole spacetime. Our main result in this paper is a novel expression valid when the two points lie *inside* the black hole; we also (re)derive, using a new method, the known expression valid when the two points lie outside the black hole. We achieve these expressions by finding relationships between Unruh modes, defined in terms of the retarded Kruskal coordinate, and Eddington modes, defined in terms of the Eddington coordinates. While our starting expression for the two-point function is written in terms of the Unruh modes, we give our final expression in terms of the Eddington modes, which have the computational advantage that they decompose into factors that obey *ordinary* differential equations. In an appendix we also derive expressions for the bare mode contributions to the flux components of the stress-energy tensor for a minimally coupled massless scalar field inside the black hole. Our results thus lay the groundwork for future calculations of quantum effects inside a Kerr black hole.

I. INTRODUCTION

In the semiclassical framework of quantum field theory on a curved spacetime, a gravitational field is treated classically, whereas matter fields on the corresponding (background) spacetime are quantized. In practice, the Einstein field equations of general relativity are sourced by the renormalized expectation value of the stress-energy tensor (RSET) for the matter fields in a certain quantum state. This framework is expected to provide a good approximation to the physics when the scales of the system are above the Planck scales and it has yielded results as important as the emission of quantum (Hawking) radiation by astrophysical black holes (BHs) [1,2].

Within quantum field theory on a curved spacetime, one can define Hadamard's elementary two-point function (HTPF) as the expectation value of the anticommutator

of a (say, scalar) field $\hat{\Phi}$ in a certain quantum state $|\Psi\rangle$: $G^{(1)}_{\Psi}(x, x') = \langle\{\hat{\Phi}(x), \hat{\Phi}(x')\}\rangle_{\Psi}$, where x and x' are spacetime points, and curly brackets denote symmetrization with respect to x and x'.[1] The HTPF is a solution of the homogeneous wave equation satisfied by the field $\hat{\Phi}$ and is an important object for various reasons. First, it is physically relevant in its own right, since it yields the quantum correlations between different points on the background spacetime. Second, when subtracting from it an appropriate, purely geometric, renormalization term (also called the counterterm) [3] and taking the limit $x' \to x$, the renormalized Wick product $\langle\hat{\Phi}^2(x)\rangle^{\Psi}_{\mathrm{ren}}$ is obtained, which is a manifestation of the quantum vacuum fluctuations. Last but not least, by applying a certain differential operator on the HTPF minus the renormalization term [4,5], and then taking the limit $x' \to x$, the RSET is obtained, which is a source in the semiclassical Einstein equations.

In principle, it is possible to define various states for a quantum field on a BH background spacetime. In the case of

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[1]More explicitly, $\{\xi(x), \zeta(x')\} \equiv \xi(x)\zeta(x') + \xi(x')\zeta(x)$, where ξ and ζ are two quantities that depend on the spacetime point.

a spherically symmetric (e.g., Schwarzschild) BH, the most commonly used states are (i) the Boulware state [6,7], which is meant to model the surroundings of a starlike object, since this state is empty at (past and future) null infinity and it diverges on the (past and future) event horizon (EH); (ii) the Unruh state [8], which models an evaporating BH via the emission of Hawking radiation; and (iii) the Hartle-Hawking state [9], which models a BH in equilibrium with its own radiation. When the BH is rotating (Kerr), however, the corresponding Boulware state [10,11] is no longer empty at future null infinity, as it contains the so-called Unruh-Starobinskii radiation (essentially, the quantum version of classical superradiance [12,13]). Also, for bosons, no state in thermal equilibrium with the rotating BH can be constructed, i.e., no Hartle-Hawking-like state exists for bosons in Kerr [11,14].[2] Finally, the Unruh state can be constructed in Kerr ([11] for a scalar field, [15] for a fermion field, and [16] for the electromagnetic field) similarly to the spherically symmetric case.

In this paper we shall focus on the Unruh state in Kerr spacetime, which is the state of relevance for astrophysical BHs.

Our treatment of the Unruh state in Kerr is also significant for the following reason. Often, in spherical symmetry, one first calculates quantities in the Hartle-Hawking state since it allows for the Euclideanization technique, whereby one merely needs to sum over a *discrete* set of field modes thus facilitating the calculation; then, if one wishes to calculate the quantity in another state—such as Unruh—one just calculates the *difference* between that quantity in that other state and in the Hartle-Hawking state (such difference needs no renormalization and so, in principle, it can be carried out relatively easily). In Kerr, however, this method of Hartle-Hawking state subtraction is not applicable, because no Hartle-Hawking-like state exists in Kerr.

Most calculations for quantum fields on a BH spacetime have focused on quantities *outside* the EH of the BH—after all, we are observers located far away from any BH. Expressions for the HTPF with the two points outside the BH are known in both the nonrotating [17,18] and the rotating cases [11,19] and they have been used to calculate Φ^2 and the RSET. While in the nonrotating case the RSET outside the BH has been obtained in various physical settings, in the rotating case the calculation is a lot more technically involved and in fact it has been achieved only recently and in one instance: [20].

In its turn, the investigation of effects of quantum fields inside the EH of a BH may serve to address questions of fundamental conceptual importance. Most notably, the question of whether the inner horizon (IH) of a rotating and/or electrically charged BH is stable under quantum perturbations. Beyond the IH, the Cauchy initial value problem is not well posed and so the Einstein field equations of general relativity cease to be deterministic. Quantum effects have been seen to destroy the regularity of the IH of a nonrotating and electrically charged (Reissner-Nordström, RN) BH [21]. A similar behavior was also found in RN–de Sitter BHs [22] and for quantum perturbations approaching the IH -at least- from the inside of a $2 + 1$-dimensional rotating Bañados, Teitelboim & Zanelli (BTZ) BH [23,24]. In all these cases the HTPF was known for the two points inside the EH and served to calculate the RSET. However, in the most important case of a Kerr BH, an expression for the HTPF with the two points inside the BH was not known until the current work and, consequently, no quantitative investigation of the quantum effects on its IH has yet been carried out.[3]

The main result in this paper is an expression for the HTPF $G_U^{(1)}(x, x') = \langle\{\hat{\Phi}(x), \hat{\Phi}(x')\}\rangle_U$ for a quantum massless scalar field in the Unruh state $|0\rangle_U$ with the two points x and x' located inside the Kerr BH between the EH and the IH. One of the main values of this expression is that it is given in terms of (Eddington) field modes which decompose into factors that obey ordinary differential equations and so are relatively easy to calculate, at least numerically. Thus, our expression for the HTPF is of practical use for potential future calculations of the RSET inside the EH of a Kerr BH in the Unruh state. (We perform a step in this direction in Appendix B, where we derive expressions for the bare mode contribution to the flux components in the BH interior.) Furthermore, once one achieves renormalization in the Unruh state via the HTPF provided in this paper, one can use that as the fiducial state with respect to which to calculate differences and thus easily achieve renormalization in another state. Prior to obtaining this new expression for the HTPF inside a Kerr BH, we derive an expression for the HTPF outside a Kerr BH; although this latter expression was already known, we (re)derive it by employing a new method, which is the one that we subsequently apply inside the BH. Moreover, in order to achieve these expressions for the HTPF, we obtain relationships between the Unruh family of modes (which are defined in terms of the retarded Kruskal coordinate and serve to define the Unruh state) and Eddington families of modes (which are defined in terms of the Eddington coordinates and, as mentioned, decompose by factors). These relationships between families of field modes are useful in their own right in that they may be readily applicable to the calculation of two-point functions other than the HTPF, such as the Wightman function (which is relevant, for example, for the calculation of the transition probability rate of an Unruh-DeWitt quantum particle detector [26]). For the reader who is just interested in the new expression for the HTPF inside the BH, that expression is given in Eq. (6.37) or, equivalently, in Eq. (6.41).

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[2]For fermions, on the other hand, a state in thermal equilibrium can be constructed sufficiently close (specifically, within the so-called speed-of-light surface) to the rotating BH [15].

[3]There have been, however, *qualitative* investigations, see [11,25].

The rest of this paper is organized as follows. Sections II–IV lay the foundations for the subject of the paper: the Unruh HTPF for a scalar field on a Kerr BH interior. In Sec. II we review the Kerr metric and the associated wave equation satisfied by a massless, uncharged scalar field. Section III introduces the various families of field modes which are relevant to this paper. The modes allow us to define the Unruh quantum state in Sec. IV and to derive the (already known) expression for the HTPF outside a Kerr BH in Sec. V [specifically, Eq. (5.29)]. The paper culminates in Sec. VI, where we obtain the (new) expression for the HTPF inside a Kerr BH [specifically, Eq. (6.37) or Eq. (6.41)]. Our main results are briefly discussed in Sec. VII. The paper also has two appendixes: Appendix A addresses the issue of IR regularity (i.e., regularity at small frequencies) of our final expressions for the HTPF, and Appendix B presents a derivation of the bare mode-sum expressions for the flux components of the RSET, based on the HTPF expression derived in this paper for the BH interior.

We use units where $c = G = 1$ (while $\hbar$ is not taken to be equal to 1) and metric signature $(-+++)$.

II. THE KERR METRIC AND THE WAVE EQUATION

A. The Kerr metric and coordinate systems

The Kerr metric is a vacuum solution to the classical Einstein field equations, describing a BH of mass M rotating with angular momentum J. It is given by the line element in Boyer-Lindquist coordinates (t, r, θ, φ),

$$
\begin{aligned}
\mathrm{d}s^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 \\
&+ \left(r^2 + a^2 + \frac{2Mra^2}{\rho^2}\sin^2\theta\right)\sin^2\theta\,\mathrm{d}\varphi^2 \\
&- \frac{4Mra}{\rho^2}\sin^2\theta\,\mathrm{d}\varphi\,\mathrm{d}t,
\end{aligned}
\tag{2.1}
$$

where $a \equiv J/M$ and

$$
\rho^2 \equiv r^2 + a^2 \cos^2\theta,
$$
$$
\Delta \equiv r^2 - 2Mr + a^2.
$$

The horizon radii correspond to the roots of the equation $\Delta = 0$, yielding an EH at

$$
r = r_+ \equiv M + \sqrt{M^2 - a^2}
$$

and an IH at

$$
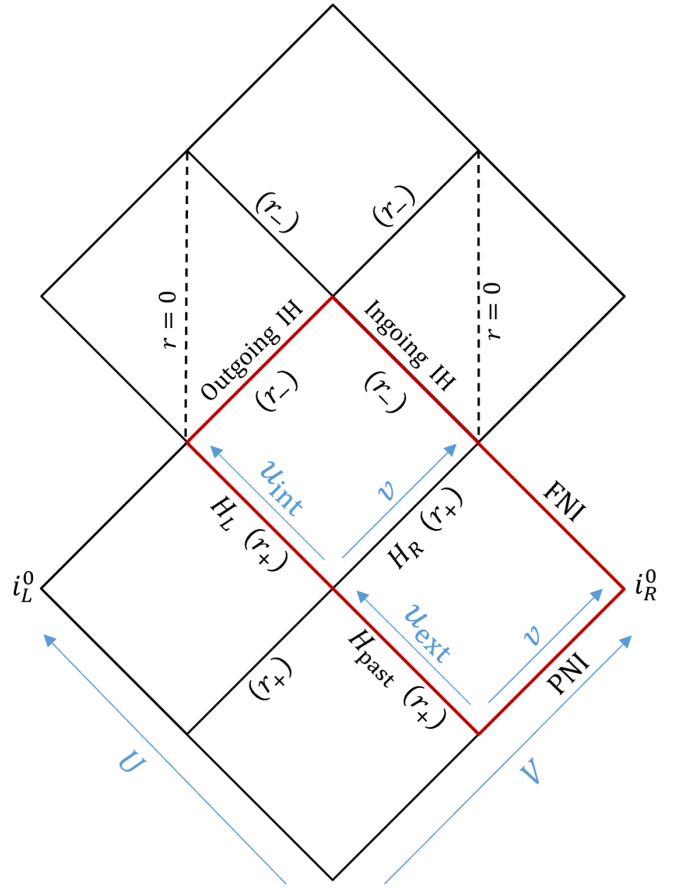r = r_- \equiv M - \sqrt{M^2 - a^2}.
$$

FIG. 1. A portion of the Penrose diagram of the analytically extended subextremal Kerr spacetime, with three systems of coordinates: the Kruskal U and V [Eqs. (2.9) and (2.10)], the outer Eddington u_{ext} and v [Eq. (2.7)], and the inner Eddington u_{int} and v [Eq. (2.8)]. The spacetime regions relevant for this paper are within the red frame, consisting of the BH exterior and interior.

Note the resulting restriction on the BH parameters: $|a|/M \leq 1$. Throughout this paper we shall only treat the subextremal case, corresponding to $|a|/M < 1$, and restrict our attention to the region bounded by $r \geq r_-$. We refer to the region $r > r_+$ (outside the EH) as the external universe or BH "exterior," whereas the region bounded by the horizons $r_- < r < r_+$ is to be referred to as the BH "interior." Note that we might occasionally use the term exterior for $r \geq r_+$ (namely, including $r = r_+$), and likewise the term interior for $r_- \leq r \leq r_+$, depending on the context. See Fig. 1 for (a portion of) the Penrose diagram of the analytically extended subextremal Kerr spacetime.

We shall now briefly discuss the behavior of the standard Boyer-Lindquist coordinates (t, r, θ, φ) for a free-falling observer approaching the EH. As in the case of spherical symmetry (e.g., in the Schwarzschild and RN metrics), the t coordinate diverges at $r = r_+$ for an infalling observer, which motivates the definition of Kruskal coordinates given below in (2.9) and (2.10). However, in Kerr, not only does t

diverge on approaching $r = r_+$ but, unlike in spherical symmetry, also the azimuthal coordinate φ diverges there. In other words, a geodesic approaching the EH undergoes (fictitious) infinite spiraling when presented in the φ coordinate (which is, however, a mere coordinate artifact). One may compute the (constant) angular velocity with which the EH rotates (or, more precisely, the geodesic's limiting value of $d\varphi/dt$ at $r \to r_+$) to be

$$\Omega_+ \equiv \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}. \qquad (2.2)$$

This quantity can be used to construct a coordinate that remains regular on approaching $r \to r_+$, defined by

$$\varphi_+ \equiv \varphi - \Omega_+ t. \qquad (2.3)$$

Similar considerations apply at $r \to r_-$, where we analogously define

$$\Omega_- \equiv \frac{a}{2Mr_-}, \qquad \varphi_- \equiv \varphi - \Omega_- t. \qquad (2.4)$$

For later use, we shall hereby define the tortoise coordinate r_* in Kerr via $dr/dr_* = \Delta/(r^2 + a^2)$. We choose the constant of integration such that[4]

$$r_* = r + \frac{1}{2\kappa_+}\log\left(\frac{|r - r_+|}{r_+ - r_-}\right) - \frac{1}{2\kappa_-}\log\left(\frac{|r - r_-|}{r_+ - r_-}\right), \qquad (2.5)$$

where $\kappa_\pm$ are the two corresponding surface gravity parameters, given by

$$\kappa_\pm \equiv \frac{r_+ - r_-}{2(r_\pm^2 + a^2)}. \qquad (2.6)$$

Note that $r = r_+$ corresponds to $r_* \to -\infty$, while $r = r_-$ (like $r \to \infty$ outside the BH) corresponds to $r_* \to \infty$.

From here we may define the Eddington coordinates,[5] given in the BH exterior by

$$u_{\text{ext}} \equiv t - r_*, \qquad v \equiv t + r_*, \qquad (2.7)$$

and in the BH interior by

$$u_{\text{int}} \equiv r_* - t, \qquad v \equiv r_* + t. \qquad (2.8)$$

The coordinate v is continuous across the EH (and parametrizes it), whereas u_{ext} and u_{int} diverge there. The regularity of the metric at the EH may be seen by

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transforming to a set of Kruskal coordinates, which we shall denote by U and V, given in the BH exterior by

$$U(u_{\text{ext}}) \equiv -\frac{1}{\kappa_+}\exp\left(-\kappa_+ u_{\text{ext}}\right),$$

$$V(v) \equiv \frac{1}{\kappa_+}\exp\left(\kappa_+ v\right), \qquad (2.9)$$

and in the BH interior by

$$U(u_{\text{int}}) \equiv \frac{1}{\kappa_+}\exp\left(\kappa_+ u_{\text{int}}\right),$$

$$V(v) \equiv \frac{1}{\kappa_+}\exp\left(\kappa_+ v\right). \qquad (2.10)$$

Note that both Kruskal coordinates U and V are continuous at the EH: The former vanishes there (from both sides), whereas V, just like v, regularly parametrizes the EH. Furthermore, the metric in the $(U, V, \theta, \varphi_+)$ coordinates is regular and smooth across the EH.

The locus $r = r_+$ marks a four-arms cross in the Penrose diagram in Fig. 1. Out of these four arms, in this paper we are only concerned with the three included in the red frame, being the EH (or "right horizon", denoted H_R), the "past horizon" H_{past} (the white hole horizon), and the "left horizon" H_L. The other arm, the one at the bottom left, will not concern us here as it is located outside the domain of dependence relevant to the Unruh state (namely, the red frame in Fig. 1).

We note that the IH is also a "Cauchy horizon" in the sense that it is the boundary of validity of the Cauchy initial value problem formulated on a spacelike hypersurface extending from i_R^0 to i_L^0, where i_R^0 (i_L^0) is spacelike infinity of the external universe at the right (left) side of Fig. 1. However, in the more physically realistic case of a BH formed by gravitational collapse, which lacks a past horizon as well as the entire left-side external universe, it is only the *ingoing* section of the IH (see Fig. 1) which retains the causal nature of a Cauchy horizon.

In Kruskal coordinates, the past horizon H_{past} is found at $V = 0$ and $U < 0$, the right horizon H_R corresponds to $U = 0$ and $V > 0$, and the left horizon H_L corresponds to $V = 0$ and $U > 0$.

In the BH exterior there are, in addition, two null asymptotic boundaries located at infinity: past null infinity (PNI) is found at $U = -\infty$ and $V > 0$, and future null infinity (FNI) is at $V = \infty$ and $U < 0$. See Fig. 1 for locating all the mentioned null surfaces.

We now make a couple of related observations, relevant to constructing the families of field modes further on in the paper.

Each of the above mentioned asymptotic null surfaces (the three at $r = r_+$ and the two at spacial infinity) can be regularly parametrized by three coordinates—which may be chosen to be two angular coordinates (θ and either φ or φ_+) and one Eddington coordinate—as follows: H_{past} by

$(u_{\text{ext}}, \theta, \varphi_+)$, H_R by (v, θ, φ_+), H_L by $(u_{\text{int}}, \theta, \varphi_+)$, PNI by (v, θ, φ), and FNI by $(u_{\text{ext}}, \theta, \varphi)$. We note that the Eddington coordinate in each of these null surfaces may also be replaced by the corresponding Kruskal coordinate.

We also specify here the *affine* parameters along null geodesics generating each of these asymptotic null surfaces: U along H_{past} and H_L (both with fixed θ and φ_+) and V along H_R (again with fixed θ and φ_+). At PNI and FNI, asymptotic flatness implies that the affine parameters along these surfaces are simply the Eddington coordinates v and u_{ext}, respectively (both with fixed θ and φ).

B. Separation of the wave equation

An uncharged scalar field $\Phi(x)$ of mass m and coupling ξ to curvature obeys the Klein-Gordon (KG) equation,

$$(\Box - m^2 - \xi R)\Phi = 0, \tag{2.11}$$

where $\Box$ is the covariant d'Alembertian associated with the background metric with Ricci scalar R. In the case of a massless field which is minimally coupled ($\xi = 0$) and/or vanishing Ricci scalar (in particular, a vacuum spacetime), this equation becomes

$$\Box \Phi = 0. \tag{2.12}$$

Considering Eq. (2.12) on a Kerr background, one readily obtains the following explicit form:

$$\Box \Phi = \left(\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right) \frac{\partial^2 \Phi}{\partial t^2} + \frac{4aMr}{\Delta} \frac{\partial^2 \Phi}{\partial t \partial \varphi}$$
$$+ \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right) \frac{\partial^2 \Phi}{\partial \varphi^2} - \frac{\partial}{\partial r} \left(\frac{\partial \Phi}{\partial r} \right)$$
$$- \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) = 0. \tag{2.13}$$

We shall refer to this equation as the scalar Teukolsky equation, after the general-spin field case in [28]. Utilizing the axial symmetry and time-translation invariance of the metric and of the master equation, we may decompose the field into modes

$$\Phi_{\omega lm}(x) = \text{const} \cdot \frac{\psi_{\omega lm}(r)}{\sqrt{r^2 + a^2}} e^{-i\omega t} Z_{lm}^{\omega}(\theta, \varphi), \tag{2.14}$$

indexed by the frequency $\omega \in \mathbb{R}$, the azimuthal number $m \in \mathbb{Z}$, and the multipolar number $l \in \mathbb{N}_{\geq |m|}$, where x is a spacetime point and $\psi_{\omega lm}(r)$ is the so-called radial function; the $(r^2 + a^2)^{-1/2}$ factor has been introduced to yield a convenient one-dimensional scatteringlikeequation for $\psi_{\omega lm}(r)$ [see Eq. (2.18) to follow]. The angular functions $Z_{lm}^{\omega}(\theta, \varphi)$ are the "spheroidal harmonics," given by

$$Z_{lm}^{\omega}(\theta, \varphi) = (2\pi)^{-1/2} S_{lm}^{\omega}(\theta) e^{im\varphi}, \tag{2.15}$$

where $S_{lm}^{\omega}(\theta)$ is the spheroidal wave function [29] solving the eigenvalue problem,

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dS_{lm}^{\omega}(\theta)}{d\theta} \right)$$
$$+ \left(a^2\omega^2 \cos^2\theta - \frac{m^2}{\sin^2\theta} + E_{lm}(a\omega) \right) S_{lm}^{\omega}(\theta) = 0, \tag{2.16}$$

with $E_{lm}(a\omega)$ the corresponding eigenvalue, obtained by imposing regularity at $\theta = 0, \pi$. Note that the angular equation is real, and we shall only be concerned here with real angular functions S_{lm}^{ω}.

For a given ω, the functions Z_{lm}^{ω} form a complete basis of orthonormal functions on the two-sphere, fulfilling

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta Z_{lm}^{\omega*}(\theta, \varphi) Z_{l'm'}^{\omega}(\theta, \varphi) = \delta_{ll'} \delta_{mm'}. \tag{2.17}$$

There is no known closed form for the spheroidal harmonics,[6] but in the spherical case (corresponding to $a\omega = 0$) they reduce to the well-known spherical harmonics $Y_{lm}(\theta, \varphi)$, whence the spheroidal wave functions reduce, up to a normalization, to Legendre functions— and the eigenvalue $E_{lm}(a\omega)$ simplifies to $l(l+1)$.

The radial function $\psi_{\omega lm}(r)$ solves the radial equation

$$\frac{d^2\psi_{\omega lm}}{dr_*^2} + V_{\omega lm}(r)\psi_{\omega lm} = 0 \tag{2.18}$$

with the effective potential

$$V_{\omega lm}(r) \equiv \frac{K_{\omega m}^2(r) - \lambda_{lm}(a\omega)\Delta}{(r^2 + a^2)^2} - G^2(r) - \frac{dG(r)}{dr_*}, \tag{2.19}$$

where

$$K_{\omega m}(r) \equiv (r^2 + a^2)\omega - am,$$
$$\lambda_{lm}(a\omega) \equiv E_{lm}(a\omega) - 2am\omega + a^2\omega^2,$$
$$G(r) \equiv \frac{r\Delta}{(r^2 + a^2)^2}. \tag{2.20}$$

From Eq. (2.16) it is evident that flipping the signs of ω and/or m leaves the angular equation invariant. Since the imposed boundary conditions (regularity at the two poles) has no explicit reference to either ω or m, it follows that both the eigenvalue $E_{lm}(a\omega)$ and the angular function S_{lm}^{ω} are invariant (modulo a sign) under such sign flips. For our purposes, we focus on a simultaneous sign flip of m and ω,

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[6]In fact, spheroidal harmonics may be expressed in terms of confluent Heun functions but only with coefficients which are to be determinable numerically.

$$S_{l(-m)}^{-\omega} = (-1)^m S_{lm}^{\omega}, \quad E_{l(-m)}(-a\omega) = E_{lm}(a\omega), \quad (2.21)$$

where we have chosen the sign $(-1)^m$ for S_{lm}^{ω}, so that, for $a\omega = 0$, the spheroidal harmonics agree with the standard sign for the spherical harmonics: $(Z_{l(-m)}^{\omega=0})^* = (Y_l^{-m})^* = (-1)^m Y_l^m = (-1)^m Z_{lm}^{\omega=0}$.

The situation with the radial equation is slightly more delicate. This equation, too, is real, as can be seen in Eqs. (2.18)–(2.20). Therefore, if $\psi_{\omega lm}(r)$ is a solution, its complex conjugate is a solution too. However, we are physically motivated to choose complex boundary conditions to the radial solutions (which would correspond to, e.g., ingoing or upgoing waves)—leading to *complex* radial functions. The radial equation (2.18), too, is invariant under a simultaneous change of signs of ω and m. This implies that if $\psi_{\omega lm}(r)$ solves Eq. (2.18), it will also be a solution of the radial equation with $\omega \mapsto -\omega$ and $m \mapsto -m$, and so will be its complex conjugate. As will become evident in the next section [see Eqs. (3.4)], from the way the boundary conditions for the various modes are defined, flipping the signs of both ω and m will actually take us from the original mode function $\psi_{\omega lm}(r)$ to its complex conjugate.

Examining the effective potential [Eq. (2.19)] in the asymptotic domains of the exterior region, $r_* \to \infty$ (corresponding to $r \to \infty$) and $r_* \to -\infty$ (corresponding to $r \to r_+$), we find

$$V_{\omega lm}^{\text{outside}} \to \begin{cases} \omega^2, & r \to \infty \quad (r_* \to \infty) \\ \omega_+^2, & r \to r_+ \quad (r_* \to -\infty) \end{cases}, \quad (2.22)$$

where we define

$$\omega_+ \equiv \omega - m\Omega_+. \quad (2.23)$$

Thus,[7] the asymptotic behavior of solutions to the radial equation [Eq. (2.18)] outside the BH is generally of the form $e^{\pm i\omega r_*}$ at $r_* \to \infty$ and $e^{\pm i\omega_+ r_*}$ at $r_* \to -\infty$, corresponding to free waves in both these domains.

Similarly, when considering the effective potential in the BH interior, we obtain

$$V_{\omega lm}^{\text{inside}} \to \begin{cases} \omega_-^2, & r \to r_- \quad (r_* \to \infty) \\ \omega_+^2, & r \to r_+ \quad (r_* \to -\infty) \end{cases}, \quad (2.24)$$

where we define

$$\omega_- \equiv \omega - m\Omega_-. \quad (2.25)$$

This is a crucial point for the definition of our modes in Kerr (see Sec. III), and it differs from the spherically symmetric case (similar to the $m = 0$ case here), where the

asymptotic behavior of the effective potential leaves ω^2 in *all* asymptotic domains of the BH exterior and interior.

III. FAMILIES OF MODES

It is convenient to decompose the field into sets of modes, each providing a complete set of solutions to Eq. (2.12) on some spacetime region, which are orthonormal with respect to the standard KG scalar product, defined by

$$\langle \psi, \phi \rangle \equiv i \int_\Sigma d\sigma^\mu (\psi^* \phi_{;\mu} - \phi \psi_{;\mu}^*), \quad (3.1)$$

where Σ is the spacelike hypersurface under consideration and $d\sigma^\mu$ is a future directed normal to Σ. Note that the prefactors of the various modes [e.g., the analog of the "const". appearing in Eq. (2.14)] are chosen such that orthonormality with respect to the KG inner product is satisfied.

Three families of modes are of particular importance for our purposes: the outer and inner Eddington modes and the Unruh modes. Each of these three families consists of two distinct sets, a "left-moving" and a "right-moving" one, as specified below. We begin here with a brief general description of the three families of modes, to be followed by a more detailed presentation. The various families of modes are illustrated in Fig. 2.

The "outer Eddington modes" (see Sec. III A) are defined with respect to the Eddington coordinates on the BH exterior, and consist of two sets of modes: the outgoing "up" modes $f_{\omega lm}^{\text{up}}$, which emerge as free waves from H_{past}, and the ingoing "in" modes $f_{\omega lm}^{\text{in}}$, which emerge as free waves from PNI.

The "inner Eddington modes" (see Sec. III B) are similarly defined with respect to the Eddington coordinates on the BH interior and consist of two sets of modes: the outgoing "left" modes $f_{\omega lm}^L$ which emerge as free waves from H_L, and the ingoing "right" modes $f_{\omega lm}^R$ which emerge as free waves from H_R.

The "Unruh modes" (see Sec. III C) are defined on the *entire* combined interior and exterior domain (i.e., the entire red frame in Fig. 1) and consist of two sets of modes: the outgoing "up" modes $g_{\hat{\omega}\hat{l}\hat{m}}^{\text{up}}$ (the indices $\hat{\omega}\,\hat{l}\,\hat{m}$ are to be introduced later on) which emerge from $H_{\text{past}} \cup H_L$ as free waves with respect to the affine parameter there, the Kruskal coordinate U; and the ingoing "in" modes $g_{\omega lm}^{\text{in}}$ which emerge from PNI as free waves with respect to the affine parameter there, the Eddington coordinate v (see the end of Sec. II A).

The Unruh modes are of crucial physical importance for a quantum scalar field evolving on a BH background. See, in particular, the definition of the Unruh vacuum in Sec. IV C. Our motivation for considering the Eddington modes is of a different kind: Remarkably, both outer and inner Eddington modes are entirely decomposable in terms of

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[7]To be more precise, this is a consequence of the fact that the potential is short range: $V_{\omega lm} = \omega^2 + O(1/r^2)$ as $r \to \infty$ and $V_{\omega lm} = \omega_+^2 + O(e^{2\kappa_+ r_*})$ as $r \to r_+$ (corresponding to $r_* \to -\infty$).
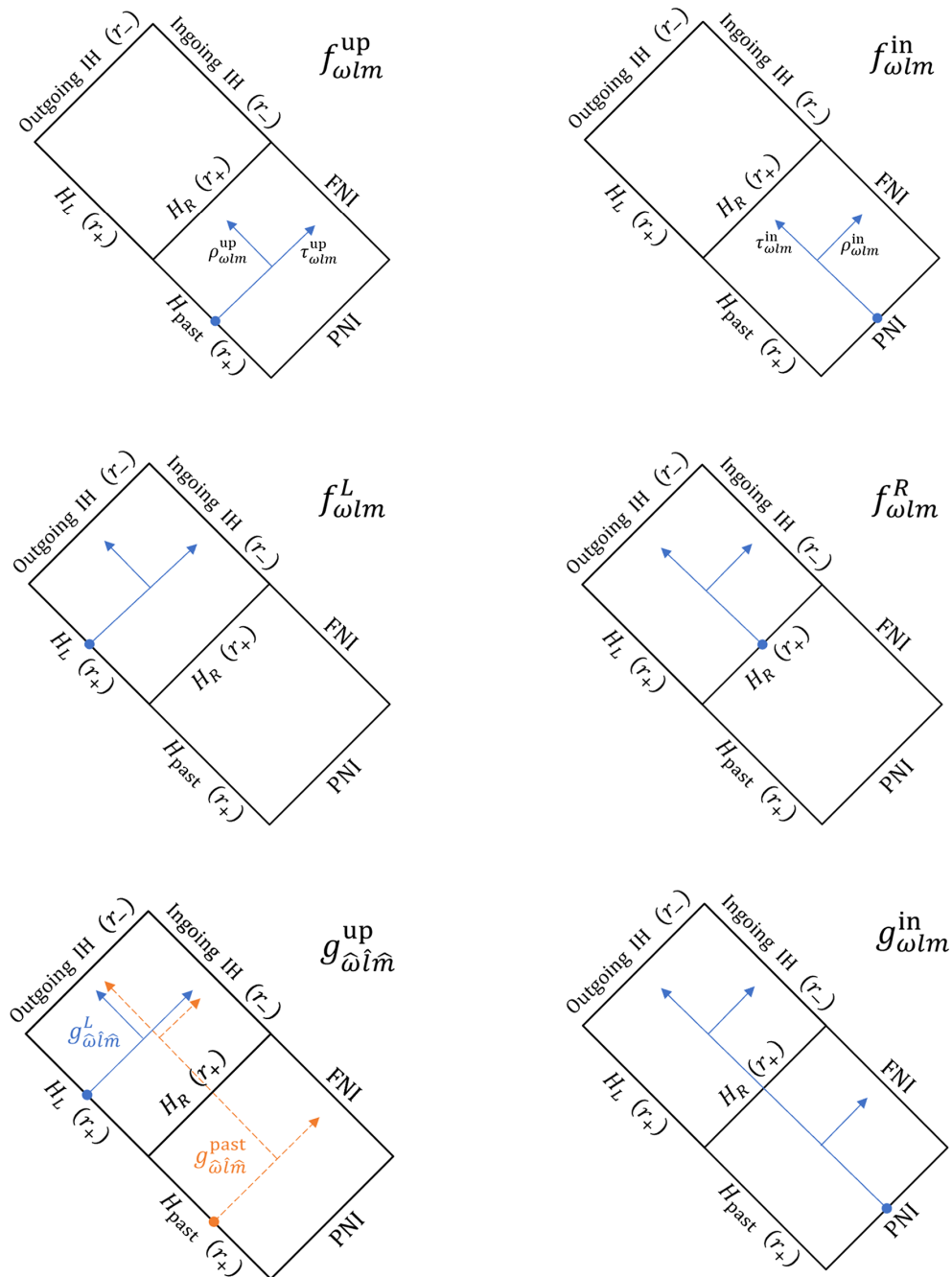
FIG. 2. Penrose diagrams for various field modes on Kerr spacetime: the outer Eddington modes $f_{\omega lm}^{\mathrm{up}}$ (3.8) and $f_{\omega lm}^{\mathrm{in}}$ (3.7); the inner Eddington modes $f_{\omega lm}^{L}$ (3.19) and $f_{\omega lm}^{R}$ (3.18); the Unruh modes $g_{\hat\omega \hat l \hat m}^{\mathrm{up}}$ (3.30) and $g_{\omega lm}^{\mathrm{in}}$ (3.24). All modes start with unit amplitude on whichever hypersurface (PNI, H_{past} or H_L) the corresponding dot lies; part of the wave is reflected and part is transmitted, with the reflection ρ and transmission τ coefficients indicated for $f_{\omega lm}^{\mathrm{in/up}}$. The modes $g_{\hat\omega \hat l \hat m}^{\mathrm{up}}$ are defined equally throughout $H_{\mathrm{past}} \cup H_L$ in terms of U but they may also be constructed as the sum of $g_{\hat\omega \hat l \hat m}^{L}$ [Eq. (3.31), with no support on H_{past}] and $g_{\hat\omega \hat l \hat m}^{\mathrm{past}}$ [Eq. (3.32), with no support on H_L], respectively colored in blue and dashed orange on the diagram for $g_{\hat\omega \hat l \hat m}^{\mathrm{up}}$:

angular, radial, and temporal variables and have a general form analogous to Eq. (2.14). Owing to this decomposition, the Eddington modes are easily handled numerically—which makes them convenient as a mathematical tool for the

decomposition of the Unruh modes—whereas the Unruh modes do not generally admit such decomposition.

Throughout the rest of the paper, whenever we have both ω_+ and ω appearing in the same equation, they should be

understood to be related via the standard relation, $\omega = \omega_+ + m\Omega_+$, or vice versa, via $\omega_+ = \omega - m\Omega_+$. In all of these cases, there will always be a well-defined m value, enabling this transformation between ω and ω_+.

A. Outer Eddington modes

For the sake of separation of the wave equation (2.13) in Kerr, we considered a particular decomposition of the field into modes [Eq. (2.14)], which was given in terms of the spheroidal harmonics $Z_{lm}^\omega(\theta, \varphi) = \frac{1}{\sqrt{2\pi}} S_{lm}^\omega(\theta) e^{im\varphi}$, the temporal dependence $e^{-i\omega t}$, and the radial function $\psi_{\omega lm}(r)$. This decomposition provides the basis for the definition of the Eddington modes, which we shall generally denote by $f_{\omega lm}$. In this subsection, we introduce the outer Eddington modes $f_{\omega lm}^{\text{in}}$ and $f_{\omega lm}^{\text{up}}$ defined exclusively on the BH exterior.

Based on the asymptotic behavior of the effective potential in the BH exterior [as given in Eq. (2.22)] we define two spanning sets of solutions to the radial equation (2.18), $\psi_{\omega lm}^{\text{in}}(r)$ and $\psi_{\omega lm}^{\text{up}}(r)$, uniquely determined by their boundary conditions,

$$\psi_{\omega lm}^{\text{in}}(r) \simeq \begin{cases} \tau_{\omega lm}^{\text{in}} e^{-i\omega_+ r_*}, & r_* \to -\infty \\ e^{-i\omega r_*} + \rho_{\omega lm}^{\text{in}} e^{i\omega r_*}, & r_* \to \infty \end{cases}, \quad (3.2)$$

$$\psi_{\omega lm}^{\text{up}}(r) \simeq \begin{cases} e^{i\omega_+ r_*} + \rho_{\omega lm}^{\text{up}} e^{-i\omega_+ r_*}, & r_* \to -\infty \\ \tau_{\omega lm}^{\text{up}} e^{i\omega r_*}, & r_* \to \infty \end{cases}, \quad (3.3)$$

where $\tau_{\omega lm}^\Lambda$ and $\rho_{\omega lm}^\Lambda$, with Λ denoting either "in" or "up", are the transmission and reflection coefficients, respectively. Determination of these coefficients thus requires numerically solving the one-dimensional scattering of the $\psi_{\omega lm}^\Lambda$ mode off the effective potential (2.19). We use the symbol "$\simeq$" to denote asymptotic equivalence.

When the signs of ω and m are flipped simultaneously, so does the sign of ω_+. This means that the emerging free waves that initiate the in and up modes ($e^{-i\omega r_*}$ or $e^{i\omega_+ r_*}$, respectively) simply undergo complex conjugation under this transformation. We also recall from the previous section that the radial equation is real and is invariant under such a simultaneous sign flip of ω and m. Therefore, the following symmetry relations are satisfied:

$$\psi_{(-\omega)l(-m)}^\Lambda = \psi_{\omega lm}^{\Lambda*} \qquad (3.4)$$

for the radial solutions and

$$\rho_{(-\omega)l(-m)}^\Lambda = \rho_{\omega lm}^{\Lambda*}, \qquad \tau_{(-\omega)l(-m)}^\Lambda = \tau_{\omega lm}^{\Lambda*} \qquad (3.5)$$

for the corresponding reflection and transmission coefficients, where Λ stands here for either "in" or "up".

Having introduced the radial functions $\psi_{\omega lm}^{\text{in}}$ and $\psi_{\omega lm}^{\text{up}}$ and specified their boundary conditions in Eqs. (3.2) and (3.3), the complete orthonormal family of outer Eddington modes are defined in accordance with Eq. (2.14),

$$f_{\omega lm}^{\text{in}}(x) \equiv \frac{1}{\sqrt{4\pi|\omega|(r^2 + a^2)}} Z_{lm}^\omega(\theta, \varphi) e^{-i\omega t}\psi_{\omega lm}^{\text{in}}(r),$$

$$f_{\omega lm}^{\text{up}}(x) \equiv \frac{1}{\sqrt{4\pi|\omega_+|(r^2 + a^2)}}$$
$$\times Z_{lm}^\omega(\theta, \varphi) e^{-i\omega t}\psi_{\omega lm}^{\text{up}}(r), \qquad (3.6)$$

satisfying the boundary conditions [as emerge from Eqs. (3.2) and (3.3)][8]:

$$f_{\omega lm}^{\text{in}}(x) \simeq \frac{1}{\sqrt{8\pi^2|\omega|(r^2 + a^2)}}$$
$$\times S_{lm}^\omega(\theta) \begin{cases} e^{-i\omega v} e^{im\varphi}, & \text{PNI} \\ 0, & H_{\text{past}} \end{cases}, \quad (3.7)$$

$$f_{\omega lm}^{\text{up}}(x) \simeq \frac{1}{\sqrt{8\pi^2|\omega_+|(r^2 + a^2)}}$$
$$\times S_{lm}^\omega(\theta) \begin{cases} 0, & \text{PNI} \\ e^{-i\omega_+ u_{\text{ext}}} e^{im\varphi_+}, & H_{\text{past}} \end{cases}. \quad (3.8)$$

Equation (3.8) makes use of the relation $e^{-i\omega t} e^{im\varphi} = e^{-i\omega_+ t} e^{im\varphi_+}$, also useful later in the paper. Note that, in accordance with the discussion toward the end of Sec. II A, the past asymptotic forms given above [as well as the future asymptotic forms given below in Eqs. (3.14) and (3.15)] are always expressed in terms of the three regular coordinates on each of the asymptotic null surfaces.

For future use, we also find it beneficial to write Eqs. (3.7) and (3.8) in a slightly different manner by absorbing the φ- and φ_+-dependent factors into the angular functions,

$$f_{\omega lm}^{\text{in}}(x) \simeq \frac{1}{\sqrt{4\pi|\omega|(r^2+a^2)}} Z_{lm}^\omega(\theta, \varphi) \begin{cases} e^{-i\omega v}, & \text{PNI} \\ 0, & H_{\text{past}} \end{cases}, \quad (3.9)$$

$$f_{\omega lm}^{\text{up}}(x) \simeq \frac{1}{\sqrt{4\pi|\omega_+|(r^2+a^2)}} Z_{lm}^\omega(\theta, \varphi_+) \begin{cases} 0, & \text{PNI} \\ e^{-i\omega_+ u_{\text{ext}}}, & H_{\text{past}} \end{cases}, \quad (3.10)$$

where

$$Z_{lm}^\omega(\theta, \varphi_+) \equiv \frac{1}{\sqrt{2\pi}} S_{lm}^\omega(\theta) e^{im\varphi_+}. \qquad (3.11)$$

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[8]On the rhs of Eq. (3.8), being nonzero only at H_{past}, we could have replaced r in the prefactor by r_+. However, in Eq. (3.7) this is not the case, as well as in similar equations that follow, being nonzero both at $r = r_+$ and at infinity. We thus choose, for the sake of uniformity, to keep r (rather than r_+) in the prefactor on the rhs of Eq. (3.8), as well as in all similar instances that follow.

Clearly, for any fixed $\omega \in \mathbb{R}$, $Z_{lm}^{\omega}(\theta, \varphi_+)$ is a complete family of functions on the two-sphere, orthonormal in the sense that

$$\int_0^{2\pi} d\varphi_+ \int_0^{\pi} d\theta \sin\theta Z_{lm}^{\omega*}(\theta, \varphi_+) Z_{l'm'}^{\omega}(\theta, \varphi_+) = \delta_{ll'}\delta_{mm'}.$$

$$(3.12)$$

By inspecting Eq. (3.6) along with Eqs. (3.4) and (2.21) we conclude that the exterior Eddington modes are invariant (modulo a sign introduced by the angular functions) under simultaneously flipping the signs of ω and m along with complex conjugation. That is,

$$f_{(-\omega)l(-m)}^{\Lambda} = (-1)^m f_{\omega lm}^{\Lambda*}, \qquad (3.13)$$

where Λ stands for either "in" or "up".

For future use, we also provide the asymptotic behavior of the in and up Eddington modes on the future null hypersurfaces,

$$f_{\omega lm}^{in}(x) \simeq \frac{1}{\sqrt{8\pi^2|\omega|(r^2+a^2)}} S_{lm}^{\omega}(\theta) \begin{cases} \rho_{\omega lm}^{in} e^{-i\omega u_{ext}} e^{im\varphi}, & \text{FNI} \\ \tau_{\omega lm}^{in} e^{-i\omega_+ v} e^{im\varphi_+}, & H_R \end{cases},$$

$$(3.14)$$

$$f_{\omega lm}^{up}(x) \simeq \frac{1}{\sqrt{8\pi^2|\omega_+|(r^2+a^2)}}$$
$$\times S_{lm}^{\omega}(\theta) \begin{cases} \tau_{\omega lm}^{up} e^{-i\omega u_{ext}} e^{im\varphi}, & \text{FNI} \\ \rho_{\omega lm}^{up} e^{-i\omega_+ v} e^{im\varphi_+}, & H_R \end{cases}. \quad (3.15)$$

The in and up outer Eddington modes are illustrated in the top row of Fig. 2. The in mode may be interpreted as a (properly normalized) monochromatic spherical wave propagating *inward* from infinity (PNI) and being partially reflected back to infinity (FNI) and partially transmitted across the horizon (H_R), with the relative coefficients of transmission and reflection being respectively $\tau_{\omega lm}^{in}$ and $\rho_{\omega lm}^{in}$. Similarly, the up mode may be interpreted as a monochromatic spherical wave propagating *upward* from the past horizon (H_{past}) and being partially reflected back to the future horizon (H_R) and partially transmitted to infinity (FNI), with the relative coefficients of transmission and reflection being respectively $\tau_{\omega lm}^{up}$ and $\rho_{\omega lm}^{up}$.

B. Inner Eddington modes

In a similar manner to the definition of the in and up modes outside the BH, we may additionally define two sets of Eddington modes confined to the BH interior. Note that, in this spacetime region, the r_* coordinate serves as a *temporal* coordinate, whereas t has a *spatial* role. This means that a *single* initial condition is required for the radial equation (2.18), which we simply take as a free wave at $r_* \to -\infty$ (corresponding to $r \to r_+$),

$$\psi_{\omega lm}^{int} \simeq e^{-i\omega_+ r_*}, \qquad r \to r_+. \qquad (3.16)$$

With this radial function we now define the right (R) and left (L) sets of orthonormal modes,[9]

$$f_{\omega lm}^R(x) \equiv \frac{1}{\sqrt{4\pi|\omega_+|(r^2+a^2)}} Z_{lm}^{\omega}(\theta, \varphi) e^{-i\omega t} \psi_{\omega lm}^{int}(r),$$

$$f_{\omega lm}^L(x) \equiv \frac{1}{\sqrt{4\pi|\omega_+|(r^2+a^2)}} Z_{lm}^{\omega}(\theta, \varphi) e^{-i\omega t} \psi_{\omega lm}^{int*}(r).$$

$$(3.17)$$

These modes admit the following asymptotic forms at the right and left horizons:

$$f_{\omega lm}^R(x) \simeq \frac{1}{\sqrt{8\pi^2|\omega_+|(r^2+a^2)}} S_{lm}^{\omega}(\theta) e^{im\varphi_+} \begin{cases} 0, & H_L \\ e^{-i\omega_+ v}, & H_R \end{cases},$$

$$(3.18)$$

$$f_{\omega lm}^L(x) \simeq \frac{1}{\sqrt{8\pi^2|\omega_+|(r^2+a^2)}} S_{lm}^{\omega}(\theta) e^{im\varphi_+} \begin{cases} e^{i\omega_+ u_{int}}, & H_L \\ 0, & H_R \end{cases}.$$

$$(3.19)$$

As for the exterior Eddington modes, we shall find it beneficial when constructing the HTPF to present a slightly different form for Eqs. (3.18) and (3.19) by absorbing the φ-and φ_+-dependent factors into angular functions,

$$f_{\omega lm}^R(x) \simeq \frac{1}{\sqrt{4\pi|\omega_+|(r^2+a^2)}} Z_{lm}^{\omega}(\theta, \varphi_+) \begin{cases} 0, & H_L \\ e^{-i\omega_+ v}, & H_R \end{cases},$$

$$(3.20)$$

$$f_{\omega lm}^L(x) \simeq \frac{1}{\sqrt{4\pi|\omega_+|(r^2+a^2)}} Z_{lm}^{\omega}(\theta, \varphi_+) \begin{cases} e^{i\omega_+ u_{int}}, & H_L \\ 0, & H_R \end{cases}.$$

$$(3.21)$$

Analogously to the symmetry in Eq. (3.13) satisfied by the exterior Eddington modes, the interior Eddington modes satisfy

$$f_{(-\omega)l(-m)}^{\Lambda} = (-1)^m f_{\omega lm}^{\Lambda*}, \qquad (3.22)$$

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[9]Note that, in the treatment of the analogous RN case in Ref. [30], the right and left modes are defined with an essentially different temporal dependence [see Eq. (2.16) therein]: the right mode is decomposed with respect to $e^{-i\omega t}$ while the left mode is decomposed with respect to $e^{i\omega t}$, and both share the same radial function. This was possible since, in RN, the wave equation is invariant under $\omega \mapsto -\omega$. In Kerr, however, the latter symmetry does not apply, hence we stick with the canonical decomposition of Eq. (2.14). This difference also leads to some differences between several equations below and their counterparts in Ref. [30].

where Λ stands here for either R or L. The right and left inner Eddington modes are illustrated in the middle row of Fig. 2.

We have readily settled the definition of the inner Eddington family of modes. However, it is interesting to also inspect the form of the radial function $\psi_{\omega lm}^{\text{int}}(r)$ at $r_* \to \infty$ (i.e., on approaching the IH). In correspondence with the asymptotic behavior of the effective potential (2.24), the solution of Eq. (2.18) admits the free asymptotic form

$$\psi_{\omega lm}^{\text{int}} \simeq A_{\omega lm} e^{i\omega_- r_*} + B_{\omega lm} e^{-i\omega_- r_*}, \qquad r \to r_-, \quad (3.23)$$

where $A_{\omega lm}$ and $B_{\omega lm}$ are constant complex coefficients.

C. Unruh modes

The Unruh modes are the basic modes for the field expansion involved in the definition of the Unruh quantum state (see Sec. IV C). There are two distinct sets of Unruh modes and both inhabit the entire union of the BH interior and exterior (namely, the red frame in Fig. 1). We shall occasionally refer to this domain as the "united domain." This domain has two null boundaries in its past: The one on the right is PNI. The other boundary, on the left, is located at $r = r_+$; it is the union of H_{past} and H_L. Recall that, as discussed at the end of Sec. II A, the corresponding affine parameters are Eddington v along the past-right boundary (PNI) and Kruskal U along the past-left boundary ($H_{\text{past}} \cup H_L$). We also note that both these affine parameters span the entire range $(-\infty, \infty)$ along their respective past null boundaries (namely, v along PNI and U along $H_{\text{past}} \cup H_L$).

The two sets of Unruh modes naturally emerge from these two past null boundaries, with positive frequencies in each set defined with respect to the affine parameter along the corresponding boundary. This is unlike the Eddington modes (introduced above), which are always defined asymptotically with respect to the corresponding Eddington coordinates v and u.[10] We now introduce the two sets of Unruh modes, in and up, as outlined above.

1. The in Unruh modes

The in Unruh mode $g_{\omega lm}^{\text{in}}$ originates at PNI as a free wave with respect to the affine parameter there, the Eddington v coordinate (i.e., $\propto e^{-i\omega v}$) and vanishes on both H_{past} and H_L. That is, it is endowed with the following past boundary conditions:

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[10]Since the Unruh modes are introduced here directly for the construction of a quantum state, we only need to define them with positive frequencies. The Eddington modes, however, were introduced to be utilized as a mathematical tool for decomposition. Thus, the latter were defined in Secs. III A and III B for negative frequencies as well (hence the absolute value in their normalization constant).

$$g_{\omega lm}^{\text{in}}(x) \simeq \frac{1}{\sqrt{4\pi\omega(r^2+a^2)}} Z_{lm}^{\omega}(\theta, \varphi) \begin{cases} e^{-i\omega v}, & \text{PNI} \\ 0, & H_{\text{past}} \cup H_L \end{cases}. \quad (3.24)$$

Recall that here ω attains only positive values (see footnote 10). The in Unruh modes are illustrated in the bottom right diagram of Fig. 2.

Evidently, these boundary conditions are regular. Since the d'Alembertian operator in Eq. (2.12) is regular as well, the regularity of the $g_{\omega lm}^{\text{in}}$ modes is guaranteed throughout the (interior of the) united domain.

We now restrict our attention to the BH exterior. Notably, the in Unruh mode $g_{\omega lm}^{\text{in}}$, when constrained to the BH exterior, has the same boundary conditions (on the past asymptotic null surfaces PNI and H_{past}) as the in Eddington mode $f_{\omega lm}^{\text{in}}$: compare Eqs. (3.9) and (3.24). That is,

$$g_{\omega lm}^{\text{in}}|_{\text{PNI}} = f_{\omega lm}^{\text{in}}|_{\text{PNI}},$$
$$g_{\omega lm}^{\text{in}}|_{H_{\text{past}}} = 0 = f_{\omega lm}^{\text{in}}|_{H_{\text{past}}}.$$

Since $g_{\omega lm}^{\text{in}}$ and $f_{\omega lm}^{\text{in}}$ satisfy the same wave equation (2.13), it follows that these two quantities are identical not only on the initial null hypersurfaces H_{past} and PNI but at every spacetime point in the BH exterior,

$$g_{\omega lm}^{\text{in}}(x) = f_{\omega lm}^{\text{in}}(x), \qquad r \geq r_+. \quad (3.25)$$

In order to find the behavior of $g_{\omega lm}^{\text{in}}$ in the BH interior, we carry it using Eq. (3.25) to H_R, where it fulfills

$$g_{\omega lm}^{\text{in}}|_{H_R} = f_{\omega lm}^{\text{in}}|_{H_R} = \sqrt{\frac{|\omega_+|}{\omega}} \tau_{\omega lm}^{\text{in}} f_{\omega lm}^R|_{H_R} \quad (3.26)$$

[for the last equality, compare Eq. (3.14) with Eq. (3.18)]. Again, since $g_{\omega lm}^{\text{in}}$ and $\sqrt{\frac{|\omega_+|}{\omega}} \tau_{\omega lm}^{\text{in}} f_{\omega lm}^R$ coincide on H_R and on H_L as well [since they both vanish on the latter hypersurface, see Eq. (3.24) and Eq. (3.18)], these solutions are identical everywhere in the BH interior,

$$g_{\omega lm}^{\text{in}}(x) = \sqrt{\frac{|\omega_+|}{\omega}} \tau_{\omega lm}^{\text{in}} f_{\omega lm}^R(x), \qquad r_- \leq r \leq r_+. \quad (3.27)$$

Equations (3.25) and (3.27) demonstrate a useful property of the in Unruh modes, namely, that we may match each in Unruh mode, at any given neighborhood, with a particular Eddington mode: with an in mode at $r \geq r_+$ and with a right mode (up to a specified multiplicative constant) at $r_- \leq r \leq r_+$. In particular, this means that, throughout the united domain, $g_{\omega lm}^{\text{in}}$ decomposes into radial, angular, and temporal terms, as may also be anticipated from its initial conditions given in Eq. (3.24) (because $e^{-i\omega v}$ decomposes naturally into a temporal factor $e^{-i\omega t}$ times a function of r, and this separable form of the t dependence

is preserved as $g^{\text{in}}_{\omega l m}$ evolves according to the t-independent wave equation). This situation changes when considering the up Unruh modes, as we shall do next.

2. The up Unruh modes

The up Unruh modes are solutions to Eq. (2.13) emerging from $H_{\text{past}} \cup H_L$ with positive frequency, which we denote by $\hat{\omega}$ (to distinguish it from the Killing frequency ω). That is, the up Unruh modes originate from (the ingoing arms of) $r = r_+$ as $\propto e^{-i\hat{\omega}U}$, with the Kruskal U as defined in Eqs. (2.9) and (2.10).

The desired orthonormal set of up modes is conveniently defined by specifying the initial value of each of the modes at the initial null hypersurface $H_{\text{past}} \cup H_L$—as a function of the three regular coordinates U, θ, φ_+ which span it (see the end of Sec. II A). This initial value setup for the up modes is complemented by requiring these modes to vanish at PNI, see Eq. (3.29) below. In order for the up modes to provide (when combined with the in modes defined in the previous subsection) a complete KG-orthonormal set of solutions to the wave equation, the aforementioned three-parameter set of up-modes initial functions has to be in itself a complete KG-orthonormal set of functions of U, θ, and φ_+ on $H_{\text{past}} \cup H_L$. We already chose the modes' initial U dependence at $H_{\text{past}} \cup H_L$ to be $\propto e^{-i\hat{\omega}U}$, so all that is left is to specify a complete orthonormal set of functions of the remaining coordinates θ and φ_+ on the two-sphere. This set can be chosen quite arbitrarily (and, in principle, it could also depend on $\hat{\omega}$). It should depend on two discrete parameters (reflecting the dimensionality of the two-sphere), which we here schematically denote by $\hat{l}, \hat{m}$; hence we may generally denote such a set of "initial" angular functions as $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+)$.[11] We choose it to be orthonormal in the usual sense,

$$\int_0^{2\pi} d\varphi_+ \int_0^{\pi} d\theta \sin\theta \hat{Z}^{\hat{\omega}*}_{\hat{l}\hat{m}}(\theta, \varphi_+) \hat{Z}^{\hat{\omega}}_{\hat{l}'\hat{m}'}(\theta, \varphi_+) = \delta_{\hat{l}\hat{l}'}\delta_{\hat{m}\hat{m}'}.$$

(3.28)

The set of up modes is then generally defined via its initial conditions at $H_{\text{past}} \cup H_L$ and PNI by

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[11]As mentioned, the Unruh modes introduced here are to be utilized in Sec. IV C for construction of the Unruh quantum state. Generally speaking, what determines a quantum state is the *frequency* (and thereby the implied choice of *positive-frequency* modes), which was here chosen to be the parameter $\hat{\omega}$ appearing in $e^{-i\hat{\omega}U}$. Then, the remaining choice of angular functions $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+)$ for the up Unruh modes' initial conditions does not affect the resultant quantum state. In particular, the final mode-sum structure of the Unruh-state HTPF, as appears in Eq. (5.4) or (6.37), does not depend at all on the choice of $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}$— we show this explicitly in Secs. V B and VI B.

$$g^{\text{up}}_{\hat{\omega}\hat{l}\hat{m}}(x) \simeq \frac{1}{\sqrt{4\pi\hat{\omega}(r^2 + a^2)}} \hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+) \begin{cases} 0, & \text{PNI} \\ e^{-i\hat{\omega}U}, & H_{\text{past}} \cup H_L \end{cases}.$$

(3.29)

Recall that $\hat{\omega}$ attains only positive values (see footnote 10). The up Unruh modes are illustrated at the bottom left diagram of Fig. 2.

We shall choose our arbitrary angular functions $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+)$ to be the simplest complete orthonormal set of angular functions on the two-sphere—namely, the conventional spherical harmonics,

$$Y_{\hat{l}\hat{m}}(\theta, \varphi_+) = \sqrt{\frac{(2\hat{l}+1)}{4\pi}\frac{(\hat{l}-\hat{m})!}{(\hat{l}+\hat{m})!}}P^{\hat{m}}_{\hat{l}}(\cos\theta)e^{i\hat{m}\varphi_+},$$

where $P^{\hat{m}}_{\hat{l}}$ are the associated Legendre polynomials.

Thus, we define $g^{\text{up}}_{\hat{\omega}\hat{l}\hat{m}}$ as a solution to Eq. (2.13) with the initial conditions

$$g^{\text{up}}_{\hat{\omega}\hat{l}\hat{m}}(x) \simeq \frac{1}{\sqrt{4\pi\hat{\omega}(r^2 + a^2)}} Y_{\hat{l}\hat{m}}(\theta, \varphi_+) \begin{cases} 0, & \text{PNI} \\ e^{-i\hat{\omega}U}, & H_{\text{past}} \cup H_L \end{cases}.$$

(3.30)

Conveniently, any given up mode may be written as a sum of two other solutions to Eq. (2.13), denoted $g^{\text{past}}_{\hat{\omega}\hat{l}\hat{m}}$ and $g^L_{\hat{\omega}\hat{l}\hat{m}}$, which are endowed with the following initial conditions:

$$g^L_{\hat{\omega}\hat{l}\hat{m}}(x) \simeq \frac{1}{\sqrt{4\pi\hat{\omega}(r^2 + a^2)}} Y_{\hat{l}\hat{m}}(\theta, \varphi_+) \begin{cases} 0, & \text{PNI} \\ 0, & H_{\text{past}}, \\ e^{-i\hat{\omega}U}, & H_L \end{cases}$$

(3.31)

$$g^{\text{past}}_{\hat{\omega}\hat{l}\hat{m}}(x) \simeq \frac{1}{\sqrt{4\pi\hat{\omega}(r^2 + a^2)}} Y_{\hat{l}\hat{m}}(\theta, \varphi_+) \begin{cases} 0, & \text{PNI} \\ e^{-i\hat{\omega}U}, & H_{\text{past}} \\ 0, & H_L \end{cases}.$$

(3.32)

That is, while the initial support of $g^{\text{up}}_{\hat{\omega}\hat{l}\hat{m}}(x)$ is on the entire null surface $H_{\text{past}} \cup H_L$, the function $g^L_{\hat{\omega}\hat{l}\hat{m}}(x)$ has its initial support on H_L alone, whereas $g^{\text{past}}_{\hat{\omega}\hat{l}\hat{m}}(x)$ is initially supported on H_{past} only.

One might be slightly confused about our choice of spherical harmonics as the angular functions here, because it is customary to use the *spheroidal* harmonics for a Kerr BH. But the only reason for this common use of spheroidal harmonics in Kerr is to achieve angular separability of the wave equation, as in Eq. (2.14). Recall, however, that since the spheroidal harmonics explicitly depend on ω, this angular separability can only be achieved when the

temporal dependence is precisely of the form $e^{-i\omega t}$ (multiplying some function of r, θ, and φ). Quite unluckily, an up Unruh mode admits a more intricate temporal dependence, which does not fit any single Killing frequency ω. Instead, each such mode is a superposition of Eddington modes with potentially all possible ω values; this may be seen already at the initial hypersurface $H_{\text{past}} \cup H_L$, where an up Unruh mode is $\propto e^{-i\hat{\omega}U}$, recalling that U cannot be expressed as a sum of t- and r-dependent pieces, unlike the Eddington coordinates. Since angular separability cannot be achieved anyway in this case, there is no advantage in using the spheroidal harmonics for the up Unruh modes—and we choose the much simpler (and frequency-independent) spherical harmonics instead.

We point out, however, that despite our choice of spherical harmonics for the definition of the up Unruh modes, our final mode-sum expressions (in Eddington modes) for the HTPF are actually given in terms of spheroidal harmonics (as one would naturally expect for Kerr)—as may explicitly be seen in Eqs. (5.4) and (6.37) [along with Eqs. (3.6) and (3.17)]. In fact, these final mode-sum expressions for the Unruh-state HTPF are entirely independent of the choice of angular functions $\hat{Z}_{\hat{l}\hat{m}}^{\hat{\omega}}(\theta, \varphi_+)$ at this stage of constructing the up Unruh modes, see footnote 11 above.

It is clear from the discussion above that, unlike what we did for the in Unruh modes [see Eqs. (3.25) and (3.27)], it is not possible to match a specific up Unruh mode with a single Eddington mode (neither in the exterior nor in the interior of the BH). This reflects our inability to reduce the partial differential equation (PDE) (2.13) to an ordinary differential equation (ODE) [such as (2.18)] for the up Unruh modes. Therefore, in order to allow a convenient numerical implementation involving the solution of ODEs rather than PDEs, we shall later Fourier decompose the $g_{\hat{\omega}\hat{l}\hat{m}}^{\text{up}}(x)$ modes in terms of the (separable) Eddington modes.

D. Wronskian relations

The absence of a first derivative in the radial equation (2.18) leads to r_* independence of the Wronskian of any pair of solutions. This Wronskian conservation yields well-known relations involving the exterior reflection and transmission coefficients $\rho_{\omega lm}$ and $\tau_{\omega lm}$ [defined via Eqs. (3.2) and (3.3)], as well as relations involving the interior near-IH coefficients $A_{\omega lm}$ and $B_{\omega lm}$ [defined via Eq. (3.23)].

1. Relations involving $\rho_{\omega lm}$ and $\tau_{\omega lm}$

Using the Wronskian conservation on pairs of solutions chosen from $\psi_{\omega lm}^{\text{in}}$, $\psi_{\omega lm}^{\text{up}}$, $\psi_{\omega lm}^{\text{in}*}$, and $\psi_{\omega lm}^{\text{up}*}$ [in particular, equating their Wronskian at $r_* \to -\infty$ with their Wronskian at $r_* \to \infty$, using the asymptotic forms given in Eqs. (3.2) and (3.3)], yields the following constraints on the reflection and transmission coefficients:

$$|\rho_{\omega lm}^{\text{in}}|^2 + \frac{\omega_+}{\omega}|\tau_{\omega lm}^{\text{in}}|^2 = 1,$$

$$|\rho_{\omega lm}^{\text{up}}|^2 + \frac{\omega}{\omega_+}|\tau_{\omega lm}^{\text{up}}|^2 = 1,$$

$$\tau_{\omega lm}^{\text{up}} = \frac{\omega_+}{\omega}\tau_{\omega lm}^{\text{in}},$$

$$\frac{\rho_{\omega lm}^{\text{up}*}}{\rho_{\omega lm}^{\text{in}}} = -\frac{\tau_{\omega lm}^{\text{up}*}}{\tau_{\omega lm}^{\text{up}}}. \tag{3.33}$$

The last equation yields, in particular, $|\rho_{\omega lm}^{\text{up}}| = |\rho_{\omega lm}^{\text{in}}|$.

Notably, from the first (second) constraint, modes with $\omega\omega_+ < 0$ have $|\rho_{\omega lm}^{\text{in}}|^2 > 1$ ($|\rho_{\omega lm}^{\text{up}}|^2 > 1$). That is, the reflected in (up) wave has, at FNI (H_R), an amplitude greater than it originally had at PNI (H_{past}). This is the classical phenomenon of "superradiance" [12,13].[12]

2. Relations involving $A_{\omega lm}$ and $B_{\omega lm}$

Similarly, Wronskian conservation of the interior radial function $\psi_{\omega lm}^{\text{int}}$ [see Eq. (3.16)] and its conjugate $\psi_{\omega lm}^{\text{int}*}$ relates the internal scattering coefficients $A_{\omega lm}$ and $B_{\omega lm}$ [defined through Eq. (3.23)] as follows:

$$|B_{\omega lm}|^2 - |A_{\omega lm}|^2 = \frac{\omega_+}{\omega_-}. \tag{3.34}$$

IV. QUANTUM STATES IN A KERR SPACETIME

All topics outlined in the paper so far were basically purely classical. We shall now promote our scalar field from a classical field Φ to a quantum field operator $\hat{\Phi}$. We first provide a brief review of its decomposition via annihilation and creation operators and then introduce various quantum states in Kerr spacetime.

A. Generic construction of quantum states

Consider a space of generic positive-frequency mode solutions Φ_i with respect to some temporal coordinate (clearly, in curved spacetime, this choice is not unique). These solutions fulfill the following orthonormality relations with respect to the KG inner product,

$$\langle\Phi_i,\Phi_j\rangle = \delta_{ij}, \quad \langle\Phi_i^*,\Phi_j^*\rangle = -\delta_{ij}, \quad \langle\Phi_i,\Phi_j^*\rangle = 0, \tag{4.1}$$

so that the union of the set Φ_i (for all i) and the set Φ_j^* (for all j) is a complete family of orthonormal solutions to the KG equation (2.12). We may now expand the field in terms of this basis of solutions via creation ($\hat{a}_i^\dagger$) and annihilation ($\hat{a}_i$) operators as follows:

---

[12]Note that in RN we have $\omega = \omega_+$, which leads to the simple analogous relation $|\rho_{\omega l}|^2 + |\tau_{\omega l}|^2 = 1$, implying that there exist *no* superradiant modes.

$$\hat{\Phi}(x) = \sum_i (\hat{a}_i \Phi_i(x) + \hat{a}_i^\dagger \Phi_i^*(x)),$$

where the following commutation relations are imposed:

$$[\hat{a}_i, \hat{a}_j^\dagger] = \hbar \delta_{ij} \hat{1}, \qquad [\hat{a}_i, \hat{a}_j] = 0, \qquad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0.$$

The vacuum with respect to this family is a state $|0\rangle$ such that

$$\hat{a}_i |0\rangle = 0, \quad \text{for all } i.$$

Acting on the vacuum state with the creation operators $\hat{a}_i^\dagger$ yields the one-particle states

$$|1_i\rangle = \hbar^{-1/2} \hat{a}_i^\dagger |0\rangle,$$

and from here one may construct the entire many-particle Fock space.

B. The Boulware and (lack of) Hartle-Hawking quantum states

The outlined decomposition scheme is utilized in the construction of quantum states for a scalar field. As a concrete example, consider the field decomposition in the Kerr BH exterior in terms of the outer Eddington modes [Eq. (3.6)]

$$\hat{\Phi}(x) = \sum_{l=0}^\infty \sum_{m=-l}^l \int_0^\infty d\omega (\hat{b}_{\omega lm}^{\text{in}} f_{\omega lm}^{\text{in}}(x) + \hat{b}_{\omega lm}^{\text{in}\dagger} f_{\omega lm}^{\text{in}*}(x))$$
$$+ \sum_{l=0}^\infty \sum_{m=-l}^l \int_0^\infty d\omega_+ (\hat{b}_{\omega lm}^{\text{up}} f_{\omega lm}^{\text{up}}(x) + \hat{b}_{\omega lm}^{\text{up}\dagger} f_{\omega lm}^{\text{up}*}(x)),$$

$$(4.2)$$

for some operator coefficients $\hat{b}_{\omega lm}^{\text{in}}$ and $\hat{b}_{\omega lm}^{\text{up}}$. Note that the up modes are defined with respect to the positive frequency ω_+ rather than ω [this is a direct result of the asymptotic behavior of the effective potential (2.22)]. Therefore, the corresponding integration is over positive ω_+. The decomposition in Eq. (4.2) serves to define the so-called (past) Boulware state $|0\rangle_B$ (see Refs. [6,7] for the Schwarzschild case and Refs. [10,11] for Kerr) via

$$\hat{b}_{\omega lm}^{\text{in}} |0\rangle_B = 0, \quad \text{for all } \omega > 0;$$
$$\hat{b}_{\omega lm}^{\text{up}} |0\rangle_B = 0, \quad \text{for all } \omega_+ > 0.$$

In nonrotating BHs (i.e., Schwarzschild and RN), this state is irregular on both H_{past} and H_R and is empty on both PNI and FNI (and so it is said to model a cold star); in the rotating case, it continues to be empty on PNI but it contains quantum superradiance at FNI (the Unruh-Starobinskii effect).

The focus of this paper is another state: the Unruh state, which we define in the next subsection. Before turning to the Unruh state, however, we wish to give the following remark on another, third state. In nonrotating BHs, one may consider the Hartle-Hawking (HH) state [9,31], which corresponds to a BH in thermal equilibrium, coupled to an infinite bath of radiation. Although not too realistic, the HH state provides (in the nonrotating case) relative simplicity due to its time-reversal and time-translational invariance, and so historically it was used to make some progress in the study of the RSET. However, a state analogous to HH is ill-defined in Kerr (see Ref. [14], as well as Refs. [11,19]). This may be intuitively understood from the existence of superradiant modes (see Sec. III D), for which waves are reflected back to infinity with increased amplitude, conflicting with the feasibility of a state of thermal equilibrium. We shall thus consider only the (highly physically relevant) Unruh state from now on.

C. The Unruh quantum state

The Unruh state [8] is widely recognized as a physically realistic vacuum quantum state, describing an evaporating BH (and thus, by definition, is not time-reversal invariant). The Unruh state is constructed to resemble the quantum state arising at late times for a BH formed by gravitational collapse. It is defined by taking positive frequencies with respect to the affine parameters along both initial null hypersurfaces (see the formulation of the Unruh modes in Sec. III C). That is, positive frequencies are defined with respect to v (the affine coordinate on PNI) for incoming modes and with respect to U (the affine coordinate on H_{past} and H_L) for outgoing modes.

For a straightforward definition of the Unruh vacuum state, we decompose the metric in terms of the Unruh modes [Eqs. (3.24) and (3.30)],

$$\hat{\Phi}(x) = \sum_{l=0}^\infty \sum_{m=-l}^l \int_0^\infty d\omega (\hat{a}_{\omega lm}^{\text{in}} g_{\omega lm}^{\text{in}}(x) + \hat{a}_{\omega lm}^{\text{in}\dagger} g_{\omega lm}^{\text{in}*}(x))$$
$$+ \sum_{\hat{l}=0}^\infty \sum_{\hat{m}=-\hat{l}}^{\hat{l}} \int_0^\infty d\hat{\omega} (\hat{a}_{\hat{\omega}\hat{l}\hat{m}}^{\text{up}} g_{\hat{\omega}\hat{l}\hat{m}}^{\text{up}}(x) + \hat{a}_{\hat{\omega}\hat{l}\hat{m}}^{\text{up}\dagger} g_{\hat{\omega}\hat{l}\hat{m}}^{\text{up}*}(x)),$$

$$(4.3)$$

where $\hat{a}_{\omega lm}^{\text{in}}$ and $\hat{a}_{\omega lm}^{\text{in}\dagger}$ ($\hat{a}_{\hat{\omega}\hat{l}\hat{m}}^{\text{up}}$ and $\hat{a}_{\hat{\omega}\hat{l}\hat{m}}^{\text{up}\dagger}$) are the creation and annihilation operators corresponding to the in (up) Unruh modes, and the set of quantum numbers $\hat{\omega}\hat{l}\hat{m}$ classifying the up Unruh modes are as discussed in Sec. III C 2.

The Unruh state $|0\rangle_U$ is then defined as the vacuum state with respect to the Unruh decomposition (4.3), namely, it is the state annihilated by all Unruh-modes annihilation operators,

$$\hat{a}_{\omega lm}^{\mathrm{in}}|0\rangle_U = 0, \quad \text{for all } \omega > 0;$$

$$\hat{a}_{\hat{\omega}\hat{l}\hat{m}}^{\mathrm{up}}|0\rangle_U = 0, \quad \text{for all } \hat{\omega} > 0.$$

The Unruh state involves no incoming flux at PNI and an outgoing flux of thermal radiation at FNI, in correspondence with Hawking radiation of an evaporating BH. The corresponding RSET is expected to be regular across the interior of the united domain (since the united domain is the future domain of dependence of the two Unruh-state initial null hypersurfaces, PNI and $H_{\mathrm{past}} \cup H_L$). In particular, this expectation for regularity applies at H_R, but not at $H_{\mathrm{past}} \cup H_L$.

V. CONSTRUCTING THE UNRUH-STATE HTPF IN THE EXTERIOR OF A KERR BH

From the decomposition of the field $\hat{\Phi}$ in terms of Unruh modes [Eq. (4.3)], applying the commutation relations $[\hat{a}_I^\Lambda, \hat{a}_{I'}^{\Lambda'\dagger}] = \hbar \delta_{II'} \delta_{\Lambda\Lambda'} \hat{1}$ (where I denotes the set of all three quantum numbers and Λ is either "up" or "in"), we obtain the mode-sum expression of the Unruh-state HTPF in terms of the Unruh modes $g_{\omega lm}^{\mathrm{in}}$ and $g_{\hat{\omega}\hat{l}\hat{m}}^{\mathrm{up}}$ [defined in Eqs. (3.24) and (3.30)],

$$
\begin{aligned}
G_U^{(1)}(x,x') &\equiv \langle\{\hat{\Phi}(x),\hat{\Phi}(x')\}\rangle_U \\
&= \langle\hat{\Phi}(x)\hat{\Phi}(x') + \hat{\Phi}(x')\hat{\Phi}(x)\rangle_U \\
&= G_U^{\mathrm{in}}(x,x') + G_U^{\mathrm{up}}(x,x'),
\end{aligned}
\tag{5.1}
$$

where x and x' are spacetime points and we denote

$$
G_U^{\mathrm{in}}(x,x') \equiv \hbar \sum_{l,m} \int_0^\infty d\omega \{g_{\omega lm}^{\mathrm{in}}(x), g_{\omega lm}^{\mathrm{in}*}(x')\}, \tag{5.2}
$$

$$
G_U^{\mathrm{up}}(x,x') \equiv \hbar \sum_{\hat{l},\hat{m}} \int_0^\infty d\hat{\omega} \{g_{\hat{\omega}\hat{l}\hat{m}}^{\mathrm{up}}(x), g_{\hat{\omega}\hat{l}\hat{m}}^{\mathrm{up}*}(x')\}, \tag{5.3}
$$

and, recall, $\{\xi(x), \zeta(x')\} \equiv \xi(x)\zeta(x') + \xi(x')\zeta(x)$.

Throughout the paper, we shall use the shorthand notation $\sum_{l,m} \equiv \sum_{l=0}^\infty \sum_{m=-l}^l$ (and likewise for $\sum_{\hat{l},\hat{m}}$). Remarkably, it has been found [see Eq. (3.22) in Ref. [19], as well as Eq. (3.18c) in Ref. [11],[13] with the analogous Schwarzschild case in Refs. [17,18]], that the mode-sum expression in Eq. (5.1) may be decomposed in terms of the more manageable outer Eddington modes (defined and discussed in Sec. III A), yielding the expression

$$
\begin{aligned}
G_U^{(1)}(x,x') = \hbar \sum_{l,m} &\Bigg[\int_0^\infty d\omega \{f_{\omega lm}^{\mathrm{in}}(x), f_{\omega lm}^{\mathrm{in}*}(x')\} \\
&+ \int_0^\infty d\omega_+ \coth\left(\frac{\pi\omega_+}{\kappa_+}\right) \{f_{\omega lm}^{\mathrm{up}}(x), f_{\omega lm}^{\mathrm{up}*}(x')\} \Bigg].
\end{aligned}
\tag{5.4}
$$

In what follows in this section, we shall present the derivation of Eq. (5.4) in the Kerr BH exterior via a procedure different from the methods previously used in the mentioned references. This procedure is the same as that which we use later on to derive the HTPF in the Kerr interior and is an extension to Kerr of the procedure used in Ref. [30] for the interior of a RN BH. In doing so, we recover the known result [Eq. (5.4)]. Proceeding in this way will allow us to demonstrate our method on a Kerr BH background in a simpler case (i.e., outside the BH) before delving into the BH interior and to handle various issues special to Kerr that arise already in the BH exterior.

A. Mode decomposition of the exterior Unruh HTPF

Concentrating on the BH exterior, we wish to express Eqs. (5.2) and (5.3) in terms of the exterior Eddington modes, $f_{\omega lm}^{\mathrm{in}}$ and $f_{\omega lm}^{\mathrm{up}}$, defined in Sec. III A.

In various stages of the computation to be carried out, it turns out to be very useful to define a new version of $f_{\omega lm}^{\mathrm{up}}$, which carries an index ω_+ rather than ω. We shall use the notation $f_{\omega_+ lm}^{\mathrm{up}(+)}$ as the ω_+-indexed version of $f_{\omega lm}^{\mathrm{up}}$. That is, for a certain set ωlm, we define

$$
f_{\omega_+ lm}^{\mathrm{up}(+)} \equiv f_{\omega(\omega_+,m)lm}^{\mathrm{up}}, \tag{5.5}
$$

where ω on the rhs is related to ω_+ and m on the lhs by the standard relation, $\omega(\omega_+, m) = \omega_+ + m\Omega_+$. All relations and equations from the previous sections that include $f_{\omega lm}^{\mathrm{up}}$ may now be carried to this section with the simple replacement $f_{\omega lm}^{\mathrm{up}} \mapsto f_{\omega_+ lm}^{\mathrm{up}(+)}$.[14] In what follows, we shall use the object $f_{\omega_+ lm}^{\mathrm{up}(+)}$ (rather than $f_{\omega lm}^{\mathrm{up}}$) as a tool until we reach the final expression [Eq. (5.28)], which will then be reexpressed in terms of the usual $f_{\omega lm}^{\mathrm{up}}$.

We begin with the in contribution. We may readily use the equality presented in Eq. (3.25) between $g_{\omega lm}^{\mathrm{in}}$ and $f_{\omega lm}^{\mathrm{in}}$ in the BH exterior to express Eq. (5.2) as the mode sum

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[13]Note that Ref. [11] considers a slightly different two-point function (TPF) from our $G_U^{(1)}(x,x')$, namely, the *nonsymmetrized* TPF, usually called the Wightman function $G^+(x,x') = \langle\hat{\Phi}(x)\hat{\Phi}(x')\rangle$ rather than the Hadamard two-point function $G^{(1)}(x,x') = \langle\{\hat{\Phi}(x),\hat{\Phi}(x')\}\rangle$. In particular, it is trivial to derive the symmetrized TPF from the nonsymmetrized one provided in Ref. [11], thereby obtaining Eq. (5.4).

[14]A clarification regarding our notation may be in order here (particularly related to the interchangeability of $f_{\omega lm}^{\mathrm{up}}$ and $f_{\omega_+ lm}^{\mathrm{up}(+)}$): Once the object $f_{\omega_+ lm}^{\mathrm{up}(+)}$ has been defined here [in Eq. (5.5)], our notational rules allow us to write equations of the form, e.g., $f_{\omega_+ lm}^{\mathrm{up}(+)} = f_{\omega lm}^{\mathrm{up}}$ (just to give a simple illustrative example). The exact meaning of an equality of this type has been clarified in Sec. III, and we repeat it here for clarity: Whenever a part of a given equation depends on ω and another part depends on ω_+, the latter is to be viewed as given by $\omega - m\Omega_+$. (Or, if one prefers, the other way around: ω may be viewed to be given by $\omega_+ + m\Omega_+$.)

$$G_U^{\rm in}(x, x') = \hbar \sum_{l,m} \int_0^\infty d\omega \{f_{\omega lm}^{\rm in}(x), f_{\omega lm}^{\rm in*}(x')\}. \quad (5.6)$$

Likewise, the up counterpart $G_U^{\rm up}(x, x')$ requires establishing a relation between the up Unruh modes $g_{\hat{\omega}\hat{l}\hat{m}}^{\rm up}$ and the exterior Eddington modes. However, as discussed at the end of Sec. III C 2, that task is a more complicated one—compared to the in Unruh modes, but also compared to the spherical symmetry counterpart.

We shall now introduce a notation to be used occasionally throughout the rest of the paper. In determining a quantum state, the frequency has a special role over the other quantum numbers. Let us then denote collectively all other quantum numbers by J (or sometimes $\hat{J}$). Here, more specifically, $J \equiv (l, m)$ and $\hat{J} \equiv (\hat{l}, \hat{m})$ (where the latter indices were introduced in Sec. III C 2). Then, we shall also use the shorthand notation $\sum_J \equiv \sum_{l,m} = \sum_{l=0}^\infty \sum_{m=-l}^l$, and likewise for $\sum_{\hat{J}}$.

To relate the up Unruh modes $g_{\hat{\omega}\hat{J}}^{\rm up}$ and the Eddington modes on the BH exterior, we turn our attention to the relevant asymptotic surfaces. Recall that on the past horizon, we have [see Eqs. (3.30) and (3.10)]

$$g_{\hat{\omega}\hat{J}}^{\rm up}\big|_{H_{\rm past}} = \frac{1}{\sqrt{4\pi\hat{\omega}(r_+^2 + a^2)}} Y_{\hat{J}}(\theta, \varphi_+) e^{-i\hat{\omega}U(u_{\rm ext})} \quad (5.7)$$

and

$$f_{\omega_+ J}^{{\rm up}(+)}\big|_{H_{\rm past}} = \frac{1}{\sqrt{4\pi|\omega_+|(r_+^2 + a^2)}} Z_J^\omega(\theta, \varphi_+) e^{-i\omega_+ u_{\rm ext}}. \quad (5.8)$$

As was already spelled out in Eq. (3.12), the set of spheroidal harmonics $Z_J^\omega(\theta, \varphi_+)$ is a complete orthonormal family on the two-sphere for any fixed ω. For the analysis below, it is important to note that this set also forms a complete orthonormal family for any fixed ω_+. It is orthonormal in the sense

$$\int_0^{2\pi} d\varphi_+ \int_0^\pi d\theta \sin\theta [Z_{lm}^{\omega(\omega_+, m)}(\theta, \varphi_+)]^*$$
$$\times Z_{l'm'}^{\omega(\omega_+, m')}(\theta, \varphi_+) = \delta_{ll'}\delta_{mm'}, \quad (5.9)$$

where, as usual, $\omega(\omega_+, m) \equiv \omega_+ + m\Omega_+$. This is so because $\int_0^{2\pi} e^{i(m'-m)\varphi_+} d\varphi_+ = 2\pi\delta_{mm'}$, hence the θ integral in Eq. (5.9) needs only to be carried out for $m' = m$—in which case the equality of ω_+ (in the two Z functions) is fully equivalent to the equality of ω. A fairly similar argument may be used to show that the completeness of the family $Z_J^\omega(\theta, \varphi_+)$ with fixed ω also implies its completeness with fixed ω_+.

Now, in order to decompose $g_{\hat{\omega}\hat{J}}^{\rm up}$ in terms of $f_{\omega_+ J}^{{\rm up}(+)}$ on $H_{\rm past}$, we introduce two sets of coefficients, $\alpha_{\hat{\omega}\omega_+}$ and $C_{\hat{J}J}^{\omega_+}$, aimed at handling the frequential and angular factors,

respectively. The Fourier coefficients $\alpha_{\hat{\omega}\omega_+}$ are given by the inverse Fourier transform

$$\alpha_{\hat{\omega}\omega_+} \equiv \int_{-\infty}^\infty du_{\rm ext} e^{-i\hat{\omega}U(u_{\rm ext})} e^{i\omega_+ u_{\rm ext}}. \quad (5.10)$$

This integral may be evaluated as described in Ref. [30],[15] yielding

$$\alpha_{\hat{\omega}\omega_+} = \frac{1}{\kappa_+}\left(\frac{\hat{\omega}}{\kappa_+}\right)^{i\omega_+/\kappa_+} e^{\pi\omega_+/2\kappa_+}\Gamma\left(-i\frac{\omega_+}{\kappa_+}\right). \quad (5.11)$$

Similarly, $C_{\hat{J}J}^{\omega_+}$, the coefficients translating between the two bases of orthonormal functions on the two-sphere, Z_J^ω and $Y_{\hat{J}}$, are defined by

$$C_{\hat{J}J}^{\omega_+} \equiv \int_0^{2\pi} d\varphi_+ \int_0^\pi d\theta \sin\theta [Z_J^{\omega(\omega_+, m)}(\theta, \varphi_+)]^* Y_{\hat{J}}(\theta, \varphi_+) \quad (5.12)$$

[recall that in the square brackets the information about the value of m in $\omega(\omega_+, m)$ is encoded in J].

The $\alpha_{\hat{\omega}\omega_+}$ and $C_{\hat{J}J}^{\omega_+}$ coefficients, as defined in Eqs. (5.10) and (5.12), allow a translation from the Unruh to the Eddington frequential and angular factors via the relations

$$e^{-i\hat{\omega}U(u_{\rm ext})} = \frac{1}{2\pi}\int_{-\infty}^\infty d\omega_+ \alpha_{\hat{\omega}\omega_+} e^{-i\omega_+ u_{\rm ext}} \quad (5.13)$$

and

$$Y_{\hat{J}}(\theta, \varphi_+) = \sum_J C_{\hat{J}J}^{\omega_+} Z_J^{\omega(\omega_+, m)}(\theta, \varphi_+), \quad (5.14)$$

where, recall, in the last equality the sum over J is carried out with fixed ω_+. The first of these relations is just the inversion of the (inverse) Fourier transform in Eq. (5.10). To derive the second relation, we use the (fixed-ω_+) completeness of the spheroidal harmonics to decompose the spherical harmonics as

$$Y_{\hat{J}}(\theta, \varphi_+) = \sum_J P_{\hat{J}J}^{\omega_+} Z_J^{\omega(\omega_+, m)}(\theta, \varphi_+), \quad (5.15)$$

where $P_{\hat{J}J}^{\omega_+}$ denote the coefficients of the decomposition. Substituting this decomposition into the rhs of Eq. (5.12), and recalling the spheroidal harmonics orthonormality, one readily sees that $C_{\hat{J}J}^{\omega_+} = P_{\hat{J}J}^{\omega_+}$, hence Eq. (5.15) reduces to the desired relation (5.14). In a similar manner (this time employing the orthonormality and completeness of the *spherical* harmonics), one can decompose the spheroidal harmonics and show that

---

[15]See Eq. (3.3) therein and apply the notation change $\tilde{\omega} \mapsto \omega_+, \omega \mapsto \hat{\omega}$.

$$Z_J^{\omega(\omega_+, m)}(\theta, \varphi_+) = \sum_{\hat{\jmath}} C_{\hat{\jmath}J}^{\omega_+ *} Y_{\hat{\jmath}}(\theta, \varphi_+). \quad (5.16)$$

For future use, we also note that the following relation holds:

$$\sum_{\hat{\jmath}} C_{\hat{\jmath}J}^{\omega_+} C_{\hat{\jmath}J'}^{\omega_+ *} = \delta_{JJ'}. \quad (5.17)$$

To see this, we substitute Eq. (5.14) into the rhs of Eq. (5.16) (with index renaming $J \mapsto J'$ in the latter) and obtain

$$Z_{J'}^{\omega(\omega_+, m')}(\theta, \varphi_+) = \sum_J \left[\sum_{\hat{\jmath}} C_{\hat{\jmath}J}^{\omega_+} C_{\hat{\jmath}J'}^{\omega_+ *} \right] Z_J^{\omega(\omega_+, m)}(\theta, \varphi_+). \quad (5.18)$$

Recalling the orthonormality of the spheroidal harmonics, the term in square brackets must be the identity matrix with components $\delta_{JJ'}$, yielding Eq. (5.17).

Substituting Eqs. (5.13) and (5.14) in Eq. (5.7) and comparing to Eq. (5.8), the decomposition of $g_{\hat{\omega}\hat{\jmath}}^{\rm up}$ in terms of $f_{\omega_+ J}^{\rm up(+)}$ on the null hypersurface $H_{\rm past}$ may now be written as follows:

$$\sqrt{\hat{\omega}} g_{\hat{\omega}\hat{\jmath}}^{\rm up}\Big|_{H_{\rm past}} = \frac{1}{2\pi} \sum_J \int_{-\infty}^{\infty} d\omega_+ \sqrt{|\omega_+|} \alpha_{\hat{\omega}\omega_+} C_{\hat{\jmath}J}^{\omega_+} f_{\omega_+ J}^{\rm up(+)}\Big|_{H_{\rm past}}. \quad (5.19)$$

In addition, recall that both $g_{\hat{\omega}\hat{\jmath}}^{\rm up}$ and $f_{\omega_+ J}^{\rm up(+)}$ vanish on PNI: see Eqs. (3.10) and (3.30). Thus,

$$\sqrt{\hat{\omega}} g_{\hat{\omega}\hat{\jmath}}^{\rm up}\Big|_{\rm PNI} = 0 = \frac{1}{2\pi} \sum_J \int_{-\infty}^{\infty} d\omega_+ \sqrt{|\omega_+|} \alpha_{\hat{\omega}\omega_+} C_{\hat{\jmath}J}^{\omega_+} f_{\omega_+ J}^{\rm up(+)}\Big|_{\rm PNI}. \quad (5.20)$$

Since $g_{\hat{\omega}\hat{\jmath}}^{\rm up}$ and $f_{\omega_+ J}^{\rm up(+)}$ satisfy the same wave equation (2.13), it follows that these two quantities are related as prescribed in Eqs. (5.19) and (5.20), not only on the initial null hypersurfaces $H_{\rm past}$ and PNI but throughout the BH exterior. That is,

$$\sqrt{\hat{\omega}} g_{\hat{\omega}\hat{\jmath}}^{\rm up}(x) = \frac{1}{2\pi} \sum_J \int_{-\infty}^{\infty} d\omega_+ \sqrt{|\omega_+|} \alpha_{\hat{\omega}\omega_+} C_{\hat{\jmath}J}^{\omega_+} f_{\omega_+ J}^{\rm up(+)}(x),$$
$$r \geq r_+. \quad (5.21)$$

Next, the HTPF up mode contribution $G_U^{\rm up}(x, x')$ in terms of Eddington modes is achieved by substituting Eq. (5.21) in Eq. (5.3),

$$G_U^{\rm up}(x, x') = \frac{\hbar}{4\pi^2} \sum_{\hat{\jmath}} \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \sum_J \int_{-\infty}^{\infty} d\omega_+ \sqrt{|\omega_+|} \alpha_{\hat{\omega}\omega_+} C_{\hat{\jmath}J}^{\omega_+}$$
$$\times \sum_{J'} \int_{-\infty}^{\infty} d\omega'_+ \sqrt{|\omega'_+|} \alpha_{\hat{\omega}\omega'_+}^* C_{\hat{\jmath}J'}^{\omega'_+ *}$$
$$\times \{f_{\omega_+ J}^{\rm up(+)}(x), f_{\omega'_+ J'}^{\rm up(+)*}(x')\}. \quad (5.22)$$

We conveniently rearrange Eq. (5.22) as follows[16]:

$$G_U^{\rm up}(x, x') = \frac{\hbar}{4\pi^2} \sum_J \int_{-\infty}^{\infty} d\omega_+ \sqrt{|\omega_+|}$$
$$\times \sum_{J'} \int_{-\infty}^{\infty} d\omega'_+ \sqrt{|\omega'_+|} \{f_{\omega_+ J}^{\rm up(+)}(x), f_{\omega'_+ J'}^{\rm up(+)*}(x')\}$$
$$\times \sum_{\hat{\jmath}} C_{\hat{\jmath}J}^{\omega_+} C_{\hat{\jmath}J'}^{\omega'_+ *} \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \alpha_{\hat{\omega}\omega_+} \alpha_{\hat{\omega}\omega'_+}^*. \quad (5.23)$$

We can now perform the $\hat{\omega}$ integral appearing in the above equation, using Eq. (5.11)[17]:

$$\int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \alpha_{\hat{\omega}\omega_+} \alpha_{\hat{\omega}\omega'_+}^* = \frac{4\pi^2}{\omega_+} \frac{1}{1 - e^{-2\pi\omega_+/\kappa_+}} \delta(\omega_+ - \omega'_+). \quad (5.24)$$

With this identity, Eq. (5.23) reduces, after performing the trivial integration of the δ function over ω'_+, to

$$G_U^{\rm up}(x, x') = \hbar \sum_J \int_{-\infty}^{\infty} d\omega_+ \frac{|\omega_+|}{\omega_+} \frac{1}{1 - e^{-2\pi\omega_+/\kappa_+}}$$
$$\times \sum_{J'} \{f_{\omega_+ J}^{\rm up(+)}(x), f_{\omega_+ J'}^{\rm up(+)*}(x')\} \sum_{\hat{\jmath}} C_{\hat{\jmath}J}^{\omega_+} C_{\hat{\jmath}J'}^{\omega_+ *}.$$

Next we use Eq. (5.17) to obtain

$$G_U^{\rm up}(x, x') = \hbar \sum_J \int_{-\infty}^{\infty} d\omega_+ {\rm sign}(\omega_+) \frac{1}{1 - e^{-2\pi\omega_+/\kappa_+}}$$
$$\times \{f_{\omega_+ J}^{\rm up(+)}(x), f_{\omega_+ J}^{\rm up(+)*}(x')\}. \quad (5.25)$$

Finally, we would like to "fold" the ω_+ integral in this equation so that only modes with $\omega_+ > 0$ show up. To this

---

[16]This rearrangement involves interchanges of the summation over $\hat{\jmath}$ and the integration over $\hat{\omega}$ with all subsequent operations (summations and integrations). We do not attempt to rigorously justify this manipulation (or similar ones that appear later on). Nevertheless, after implementing this rearrangement, we do recover the correct, well-known result quoted in Eq. (5.4) above. This may be considered a justification for the manipulations entailed.

[17]To obtain this integral, one may rewrite it as $\frac{1}{\kappa_+} e^{(\omega_+ + \omega'_+)\pi/2\kappa_+} \Gamma(-i\omega_+/\kappa_+) \Gamma(i\omega'_+/\kappa_+) \int_{-\infty}^{\infty} ds\, e^{is(\omega_+ - \omega'_+)}$, where $s \equiv \frac{1}{\kappa_+} \ln(\hat{\omega}/\kappa_+)$, and use the relation $|\Gamma(i\omega_+/\kappa_+)|^2 = \pi(\kappa_+/\omega_+)/\sinh(\pi\omega_+/\kappa_+)$.

end, we note that since the mapping $(\omega, m) \mapsto (-\omega, -m)$ is equivalent to $(\omega_+, m) \mapsto (-\omega_+, -m)$, we may rewrite Eq. (3.13) (with Λ taken as "up") as

$$f^{\mathrm{up}(+)}_{(-\omega_+)l(-m)} = (-1)^m f^{\mathrm{up}(+)*}_{\omega_+ lm}. \qquad (5.26)$$

We now recall that $\sum_J \equiv \sum_{l=0}^{\infty} \sum_{m=-l}^{l}$ and concentrate on the summation over m. For each given l we have

$$\sum_{m=-l}^{l} \{f^{\mathrm{up}(+)}_{(-\omega_+)lm}(x), f^{\mathrm{up}(+)*}_{(-\omega_+)lm}(x')\}$$

$$= \sum_{m=-l}^{l} \{f^{\mathrm{up}(+)}_{\omega_+ l(-m)}(x), f^{\mathrm{up}(+)*}_{\omega_+ l(-m)}(x')\}$$

$$= \sum_{m=-l}^{l} \{f^{\mathrm{up}(+)}_{\omega_+ lm}(x), f^{\mathrm{up}(+)*}_{\omega_+ lm}(x')\},$$

where the first equality follows from Eq. (5.26), and the last equality simply involves a renaming of the summation index $m \mapsto -m$. Using the group index J, this may be expressed as

$$\sum_J \{f^{\mathrm{up}(+)}_{(-\omega_+)J}(x), f^{\mathrm{up}(+)*}_{(-\omega_+)J}(x')\} = \sum_J \{f^{\mathrm{up}(+)}_{\omega_+ J}(x), f^{\mathrm{up}(+)*}_{\omega_+ J}(x')\}.$$

Equation (5.25) can thus be rewritten as

$$G^{\mathrm{up}}_U(x, x') = \hbar \sum_J \int_0^\infty \mathrm{d}\omega_+ \left[\frac{1}{1 - e^{-2\pi\omega_+/\kappa_+}} - \frac{1}{1 - e^{2\pi\omega_+/\kappa_+}} \right]$$

$$\times \{f^{\mathrm{up}(+)}_{\omega_+ J}(x), f^{\mathrm{up}(+)*}_{\omega_+ J}(x')\}.$$

As one can easily see, the term in square brackets is $\coth(\pi\omega_+/\kappa_+)$. Our final result for G^{up}_U is, therefore,

$$G^{\mathrm{up}}_U(x, x') = \hbar \sum_J \int_0^\infty \mathrm{d}\omega_+ \coth\left(\frac{\pi\omega_+}{\kappa_+}\right)$$

$$\times \{f^{\mathrm{up}(+)}_{\omega_+ J}(x), f^{\mathrm{up}(+)*}_{\omega_+ J}(x')\}. \qquad (5.27)$$

As prescribed in Eq. (5.1), we may now put together the up and in mode contributions, as given in Eqs. (5.6) and (5.27), to yield $G^{(1)}_U(x, x')$,

$$G^{(1)}_U(x, x') = \hbar \sum_{l,m} \Bigg[\int_0^\infty \mathrm{d}\omega \{f^{\mathrm{in}}_{\omega lm}(x), f^{\mathrm{in}*}_{\omega lm}(x')\}$$

$$+ \int_0^\infty \mathrm{d}\omega_+ \coth\left(\frac{\pi\omega_+}{\kappa_+}\right)$$

$$\times \{f^{\mathrm{up}(+)}_{\omega_+ lm}(x), f^{\mathrm{up}(+)*}_{\omega_+ lm}(x')\} \Bigg]. \qquad (5.28)$$

Finally, we may retrieve the standard ω-indexed notation, replacing $f^{\mathrm{up}(+)}_{\omega_+ lm}$ by $f^{\mathrm{up}}_{\omega lm}$ (since, as mentioned above,

they represent the same object). Then, the mode-sum expression of the Unruh-state HTPF outside a Kerr BH, in terms of Eddington modes, is as previously quoted [in Eq. (5.4)],

$$G^{(1)}_U(x, x') = \hbar \sum_{l,m} \Bigg[\int_0^\infty \mathrm{d}\omega \{f^{\mathrm{in}}_{\omega lm}(x), f^{\mathrm{in}*}_{\omega lm}(x')\}$$

$$+ \int_0^\infty \mathrm{d}\omega_+ \coth\left(\frac{\pi\omega_+}{\kappa_+}\right) \{f^{\mathrm{up}}_{\omega lm}(x), f^{\mathrm{up}*}_{\omega lm}(x')\} \Bigg].$$

$$(5.29)$$

It may be shown (however, outside the scope of this paper) that at small ω_+, the up radial function $\psi^{\mathrm{up}}_{\omega lm}$ behaves (to leading order) as ω_+, which ensures regularity of the $\int_0^\infty \mathrm{d}\omega_+$ integral at small ω_+. (Regularity of the $\int_0^\infty \mathrm{d}\omega$ integral at small ω is similarly ensured.)

B. Invariance to the choice of the angular functions $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+)$

Finally, we comment on the invariance of our final result (5.29) in the exterior of a Kerr BH with respect to the choice of the angular functions $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+)$, used for prescribing the up Unruh-modes initial data at $H_{\mathrm{past}} \cup H_L$. As discussed in Sec. III C 2, $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+)$ can be any set of angular functions which is orthonormal and complete on the two-sphere (and, in particular, it may depend on the mode's Kruskal frequency $\hat{\omega}$). However, for the sake of concreteness (as well as simplicity and brevity), in the analysis above we made the specific choice $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+) = Y_{\hat{l}\hat{m}}(\theta, \varphi_+)$. Here we shall briefly consider how the analysis would proceed and ultimately what would be the final resultant mode structure of the HTPF, if one chose to work with generic angular functions $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}$ rather than the specific functions $Y_{\hat{l}\hat{m}}$.

Let us examine the consequence of replacing $Y_{\hat{l}\hat{m}}$ everywhere by $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}$ (and likewise $Y_{\hat{j}}$ by $\hat{Z}^{\hat{\omega}}_{\hat{j}}$), starting at Eq. (5.7). The C coefficients, relating the spheroidal harmonics with the up Unruh-modes initial angular functions (which now depend on $\hat{\omega}$), accordingly acquire an extra index $\hat{\omega}$. That is, Eq. (5.12) is replaced by

$$C^{\hat{\omega}\omega_+}_{\hat{j}J} \equiv \int_0^{2\pi} \mathrm{d}\varphi_+ \int_0^\pi \mathrm{d}\theta \sin\theta [Z^{\omega(\omega_+, m)}_J(\theta, \varphi_+)]^* \hat{Z}^{\hat{\omega}}_{\hat{j}}(\theta, \varphi_+),$$

$$(5.30)$$

and every instance of $C^{\omega_+}_{\hat{j}J}$ is replaced by $C^{\hat{\omega}\omega_+}_{\hat{j}J}$. Under these replacements, all equations up to Eq. (5.22) (inclusive) hold in their new analogous form. For later use, we quote, in particular, the new form of Eq. (5.17), simply adding the index $\hat{\omega}$ to both coefficients,

$$\sum_{\hat{j}} C_{\hat{j}J}^{\hat{\omega}\omega_+} C_{\hat{j}J'}^{\hat{\omega}\omega_+*} = \delta_{JJ'}. \tag{5.31}$$

Things become slightly more delicate when arriving at (the $Y_{\hat{j}} \mapsto \hat{Z}_{\hat{j}}^{\hat{\omega}}$ counterpart of) Eq. (5.23): Here, the term $\sum_{\hat{j}} C_{\hat{j}J}^{\omega_+} C_{\hat{j}J'}^{\omega'_+*}$ should of course be replaced by $\sum_{\hat{j}} C_{\hat{j}J}^{\hat{\omega}\omega_+} C_{\hat{j}J'}^{\hat{\omega}\omega'_+*}$. Naively, one might be concerned that since this term exhibits explicit dependence on $\hat{\omega}$, it would be necessary to keep it inside the $\hat{\omega}$ integral [i.e., the integral at the very end of the rhs of Eq. (5.23), which is then evaluated in Eq. (5.24)]. It is therefore crucial to note that (as we shall shortly justify)

$$\sum_{\hat{j}} C_{\hat{j}J}^{\hat{\omega}\omega_+} C_{\hat{j}J'}^{\hat{\omega}\omega'_+*} = \tilde{C}_{JJ'}^{\omega_+\omega'_+*}, \tag{5.32}$$

where $\tilde{C}_{JJ'}^{\omega_+\omega'_+}$ is defined by

$$\tilde{C}_{JJ'}^{\omega_+\omega'_+} \equiv \int_0^{2\pi} d\varphi_+ \int_0^{\pi} d\theta \sin\theta [Z_{J'}^{\omega(\omega'_+,m')}(\theta,\varphi_+)]^* \\ \times Z_J^{\omega(\omega_+,m)}(\theta,\varphi_+), \tag{5.33}$$

which by construction is independent of $\hat{\omega}$.[18]

Note that taking $\omega'_+ = \omega_+$ on the rhs of Eq. (5.33) reduces to the lhs of the orthonormality relation (3.12). Hence

$$\tilde{C}_{JJ'}^{\omega_+\omega_+} = \delta_{JJ'}, \tag{5.34}$$

and Eq. (5.32) thus reduces appropriately to Eq. (5.31).

In order to establish Eq. (5.32) we first note [using an argument similar to the one employed above for the justification of Eq. (5.14)] that $\tilde{C}_{JJ'}^{\omega_+\omega'_+}$ are actually the coefficients relating the two sets of spheroidal harmonics $Z_J^{\omega(\omega_+,m)}$ and $Z_{J'}^{\omega(\omega'_+,m')}$, via[19]

$$Z_J^{\omega(\omega_+,m)}(\theta,\varphi_+) = \sum_{J'} \tilde{C}_{JJ'}^{\omega_+\omega'_+} Z_{J'}^{\omega(\omega'_+,m')}(\theta,\varphi_+). \tag{5.35}$$

Then, Eq. (5.32) naturally follows from the completeness and orthonormality of each of the three involved families of angular functions, namely, $Z_J^{\omega(\omega_+,m)}$, $Z_{J'}^{\omega(\omega'_+,m')}$, and $\hat{Z}_{\hat{j}}^{\hat{\omega}}$, by a slight generalization of the argument described right after Eq. (5.17).[20]

Now, owing to Eq. (5.32), we are allowed to place the term $\sum_{\hat{j}} C_{\hat{j}J}^{\hat{\omega}\omega_+} C_{\hat{j}J'}^{\hat{\omega}\omega'_+*}$ out of the $\hat{\omega}$ integral, just as in Eq. (5.23). From this point on, the analysis proceeds in a completely analogous manner [recalling Eq. (5.31)] to the previous subsection, and the final result (5.29) is again obtained, this time using the generic angular functions $\hat{Z}_{\hat{l}\hat{m}}^{\hat{\omega}}$ rather than the spherical harmonics $Y_{\hat{l}\hat{m}}$.

To avoid confusion, we also emphasize that, in this final expression for the mode structure of the Unruh-state HTPF, the modes that appear are the Eddington modes $f_{\omega lm}^{in}$ and $f_{\omega lm}^{up}$, which are of course separable in terms of *spheroidal harmonics* [this is regardless of the nature of the angular functions $\hat{Z}_{\hat{l}\hat{m}}^{\hat{\omega}}(\theta,\varphi_+)$ that were chosen earlier in the process].

VI. CONSTRUCTING THE UNRUH-STATE HTPF IN THE INTERIOR OF A KERR BH

In this section we shall finally construct the mode-sum expression for the Unruh-state HTPF *inside* a Kerr BH in terms of Eddington modes. We shall follow here an analogous procedure to the one carried out in Ref. [30] in the RN case, while noting that the presence of rotation induces some essential differences. Basically, it is the procedure demonstrated already in Sec. V for the exterior of a Kerr BH, although there are some notable technical differences. In the first subsection we shall carry out the actual derivation of the expression for the HTPF and in the second one we shall prove that the expression is invariant with respect to the choice of the initial angular functions.

---

[18]As one can easily see by performing the integration over φ_+, $\tilde{C}_{JJ'}^{\omega_+\omega'_+}$ vanishes for any $m' \neq m$; but this specific property is not needed here.

[19]Explicitly, the derivation is as follows: From the (fixed-ω_+) completeness of the spheroidal harmonics, one can write [in analogy to Eq. (5.15)]

$$Z_J^{\omega(\omega_+,m)}(\theta,\varphi_+) = \sum_{J'} P_{JJ'}^{\omega_+\omega'_+} Z_{J'}^{\omega(\omega'_+,m')}(\theta,\varphi_+),$$

where $P_{JJ'}^{\omega_+\omega'_+}$ are the coefficients of the decomposition. Substituting this decomposition into Eq. (5.33), and using the orthonormality of the spheroidal harmonics, one obtains $P_{JJ'}^{\omega_+\omega'_+} = \tilde{C}_{JJ'}^{\omega_+\omega'_+}$.

[20]More precisely, one would need to combine the $Y_{\hat{j}} \mapsto \hat{Z}_{\hat{j}}^{\hat{\omega}}$ counterparts of Eqs. (5.14) and (5.16), taking $J \mapsto J'$ and $\omega_+ \mapsto \omega'_+$ in the former, to yield the (slightly generalized) counterpart of Eq. (5.18),

$$Z_J^{\omega(\omega_+,m)}(\theta,\varphi_+) = \sum_{J'} \left[\sum_{\hat{j}} C_{\hat{j}J}^{\hat{\omega}\omega_+*} C_{\hat{j}J'}^{\hat{\omega}\omega'_+} \right] Z_{J'}^{\omega(\omega'_+,m')}(\theta,\varphi_+).$$

Recalling the coefficients of the decomposition in Eq. (5.35) are unique, one obtains

$$\sum_{\hat{j}} C_{\hat{j}J}^{\hat{\omega}\omega_+*} C_{\hat{j}J'}^{\hat{\omega}\omega'_+} = \tilde{C}_{JJ'}^{\omega_+\omega'_+}.$$

The desired result, Eq. (5.32), is then achieved by complex conjugation.

A. Mode decomposition of the interior Unruh HTPF

In the analysis to follow, just like in the exterior counterpart presented in Sec. V, we shall use the ω_+-indexed versions (that is, objects carrying an index ω_+ rather than ω) of the Eddington modes $f_{\omega lm}^R$, $f_{\omega lm}^L$, and $f_{\omega lm}^{up}$. The ω_+ version of $f_{\omega lm}^{up}$ has been introduced in Eq. (5.5), and for the interior modes we define in a similar manner

$$f_{\omega_+ lm}^{\Lambda(+)} \equiv f_{\omega(\omega_+,m)lm}^{\Lambda},$$

where Λ is either R or L, and $\omega(\omega_+,m) \equiv \omega_+ + m\Omega_+$. In addition, we shall use the notation

$$\rho_{\omega_+ lm}^{up(+)} \equiv \rho_{\omega(\omega_+,m)lm}^{up}$$

as the ω_+-indexed version of $\rho_{\omega lm}^{up}$. All relations and equations containing $f_{\omega lm}^R$, $f_{\omega lm}^L$, $f_{\omega lm}^{up}$, and $\rho_{\omega lm}^{up}$ are to be carried from previous sections to this section along with a simple replacement of the ω-indexed objects with their ω_+-indexed counterparts, as defined above (see also footnote 14).

We begin with Eqs. (5.1)–(5.3), which are valid in the interior as well as the exterior of the BH, aiming for a mode-sum decomposition of both the in and up contributions in terms of the interior Eddington modes. Starting with $G_U^{in}(x,x')$, we may readily use the relation between the in Unruh modes and the R Eddington modes, as given in Eq. (3.27), which holds throughout the BH interior. Then, Eq. (5.2) may be written as

$$G_U^{in}(x,x') = \hbar \sum_{l,m} \int_0^\infty d\omega \frac{|\omega_+|}{\omega} |\tau_{\omega lm}^{in}|^2 \{f_{\omega_+ lm}^{R(+)}(x), f_{\omega_+ lm}^{R(+)*}(x')\}.$$

(6.1)

The contribution from the up Unruh modes is, as expected (see discussion in Sec. III C), less straightforward to decompose in terms of Eddington modes. We find it convenient to start by writing $G_U^{up}(x,x')$ in Eq. (5.3) in terms of the two $g_{\hat\omega \hat l \hat m}^{up}$ components: $g_{\hat\omega \hat l \hat m}^{past}$ and $g_{\hat\omega \hat l \hat m}^{L}$ [see Eqs. (3.31) and (3.32)]. Recalling that

$$g_{\hat\omega \hat l \hat m}^{up}(x) = g_{\hat\omega \hat l \hat m}^{past}(x) + g_{\hat\omega \hat l \hat m}^{L}(x),$$

which is valid throughout the united domain, and substituting this relation into Eq. (5.3), we readily obtain

$$G_U^{up}(x,x') = \hbar \sum_{\hat j} \int_0^\infty d\hat\omega [\{g_{\hat\omega \hat j}^{past}(x), g_{\hat\omega \hat j}^{past*}(x')\}$$
$$+ \{g_{\hat\omega \hat j}^{L}(x), g_{\hat\omega \hat j}^{L*}(x')\} + \{g_{\hat\omega \hat j}^{past}(x), g_{\hat\omega \hat j}^{L*}(x')\}$$
$$+ \{g_{\hat\omega \hat j}^{L}(x), g_{\hat\omega \hat j}^{past*}(x')\}],$$

(6.2)

where the integration is over the Kruskal frequency $\hat\omega$ and we use the notation previously introduced, $\hat J = (\hat l, \hat m)$.

Aiming for a decomposition of both $g_{\hat\omega \hat j}^{past}$ and $g_{\hat\omega \hat j}^{L}$ in terms of Eddington modes, we shall follow the same reasoning as in Sec. V, where we defined coefficients relating the frequential and angular factors of the Unruh and Eddington modes under consideration, constrained to the relevant asymptotic null surfaces. The angular coefficients $C_{\hat j J}^{\omega_+}$, defined in Eq. (5.12), will be utilized in the BH interior exactly as they were in the BH exterior. However, as we shall see, adjusting the various frequential factors will require defining an additional set of Fourier coefficients, along with the ones already defined in the BH exterior. For future use, we rename the $\alpha_{\hat\omega\omega_+}$ coefficients, as defined in Eq. (5.10), by $\alpha_{\hat\omega\omega_+}^{past}$ (adding a superscript "past"). That is,

$$\alpha_{\hat\omega\omega_+}^{past} \equiv \int_{-\infty}^\infty du_{ext} e^{-i\hat\omega U(u_{ext})} e^{i\omega_+ u_{ext}},$$

(6.3)

and it is explicitly given by [see Eq. (5.11)]

$$\alpha_{\hat\omega\omega_+}^{past} = \frac{1}{\kappa_+} \left(\frac{\hat\omega}{\kappa_+}\right)^{i\omega_+/\kappa_+} e^{\pi\omega_+/2\kappa_+} \Gamma\left(-i\frac{\omega_+}{\kappa_+}\right).$$

(6.4)

We shall begin with $g_{\hat\omega \hat j}^{past}$, emerging from H_{past} and vanishing on H_L and PNI, hence identical to $g_{\hat\omega \hat j}^{up}$ when restricted to the BH exterior [compare Eqs. (3.30) and (3.32)]. This allows us to relate to the analysis carried out in Sec. V and replace $g_{\hat\omega \hat j}^{up}$ by $g_{\hat\omega \hat j}^{past}$ in the lhs of Eq. (5.21). Explicitly, this relates $g_{\hat\omega \hat j}^{past}$ and $f_{\omega_+ J}^{up(+)}$ throughout the BH exterior as follows:

$$\sqrt{\hat\omega} g_{\hat\omega \hat j}^{past} = \frac{1}{2\pi} \sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|} \alpha_{\hat\omega\omega_+}^{past} C_{\hat j J}^{\omega_+} f_{\omega_+ J}^{up(+)},$$

$$r \geq r_+.$$

Aiming at the BH interior, we carry the above relation over to the common boundary of the BH exterior and interior—namely, the hypersurface H_R (the EH). This yields

$$\sqrt{\hat\omega} g_{\hat\omega \hat j}^{past}\big|_{H_R} = \frac{1}{2\pi} \sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|} \alpha_{\hat\omega\omega_+}^{past} C_{\hat j J}^{\omega_+} f_{\omega_+ J}^{up(+)}\big|_{H_R}.$$

(6.5)

Now, we wish to reexpress this in terms of the interior Eddington modes instead of the exterior up modes. By comparing Eq. (3.15) with Eq. (3.18), we register their relation on H_R,

$$f_{\omega_+ J}^{up(+)}\big|_{H_R} = \rho_{\omega_+ J}^{up(+)} f_{\omega_+ J}^{R(+)}\big|_{H_R}.$$

(6.6)

Substituting this into Eq. (6.5) we find

$$
\sqrt{\hat{\omega}} g_{\hat{\omega}\hat{j}}^{\text{past}}\big|_{H_R} = \frac{1}{2\pi} \sum_J \int_{-\infty}^{\infty} d\omega_+ \sqrt{|\omega_+|} \alpha_{\hat{\omega}\omega_+}^{\text{past}} C_{\hat{j}J}^{\omega_+} \rho_{\omega_+J}^{\text{up}(+)} f_{\omega_+J}^{R(+)}\big|_{H_R}. \tag{6.7}
$$

In addition, both $g_{\hat{\omega}\hat{j}}^{\text{past}}$ and $f_{\omega_+J}^{R(+)}$ vanish on H_L [see Eqs. (3.32) and (3.18)], implying that this relation actually holds throughout the BH interior,

$$
\sqrt{\hat{\omega}} g_{\hat{\omega}\hat{j}}^{\text{past}}(x) = \frac{1}{2\pi} \sum_J \int_{-\infty}^{\infty} d\omega_+ \sqrt{|\omega_+|} \alpha_{\hat{\omega}\omega_+}^{\text{past}} C_{\hat{j}J}^{\omega_+} \rho_{\omega_+J}^{\text{up}(+)} f_{\omega_+J}^{R(+)}(x), \quad r_- \le r \le r_+. \tag{6.8}
$$

We now proceed similarly with $g_{\hat{\omega}\hat{j}}^L$, whose form on H_L is [see Eq. (3.31)]

$$
g_{\hat{\omega}\hat{j}}^L(x)\big|_{H_L} = \frac{Y_{\hat{j}}(\theta, \varphi_+)}{\sqrt{4\pi\hat{\omega}(r_+^2 + a^2)}} e^{-i\hat{\omega}U(u_{\text{int}})},
$$

aiming to relate it to the left Eddington mode, whose form on H_L is [see Eq. (3.21)]

$$
f_{\omega_+J}^{L(+)}(x)\big|_{H_L} = \frac{Z_J^{\omega}(\theta, \varphi_+)}{\sqrt{4\pi|\omega_+|(r_+^2 + a^2)}} e^{i\omega_+ u_{\text{int}}}.
$$

This resembles the case of decomposing $g_{\hat{\omega}\hat{j}}^{\text{up}}$ in terms of $f_{\omega_+J}^{\text{up}(+)}$ on H_{past}, as carried out in Sec. V, with a modification in the frequential factors: here we have $e^{-i\hat{\omega}U(u_{\text{int}})}$ and $e^{i\omega_+ u_{\text{int}}}$ as the Unruh and Eddington frequential factors, respectively, instead of $e^{-i\hat{\omega}U(u_{\text{ext}})}$ and $e^{-i\omega_+ u_{\text{ext}}}$, respectively.

We thus define new Fourier coefficients relating $e^{-i\hat{\omega}U(u_{\text{int}})}$ and $e^{i\omega_+ u_{\text{int}}}$, which we shall denote by $\alpha_{\hat{\omega}\omega_+}^L$, given by the inverse Fourier transform

$$
\alpha_{\hat{\omega}\omega_+}^L \equiv \int_{-\infty}^{\infty} du_{\text{int}} e^{-i\hat{\omega}U(u_{\text{int}})} e^{-i\omega_+ u_{\text{int}}}. \tag{6.9}
$$

This integral may be found by inspecting Eq. (6.9) alongside Eq. (6.3), changing the integration variable from u_{int} to $-u_{\text{ext}}$ [which also implies $U(u_{\text{int}}) \mapsto -U(u_{\text{ext}})$, see Eqs. (2.9) and (2.10)], which results in $\alpha_{\hat{\omega}\omega_+}^L = \alpha_{\hat{\omega}(-\omega_+)}^{\text{past}*}$. Applying this relation to Eq. (6.4) then yields[21]

$$
\alpha_{\hat{\omega}\omega_+}^L = \frac{1}{\kappa_+} \left(\frac{\hat{\omega}}{\kappa_+}\right)^{i\omega_+/\kappa_+} e^{-\pi\omega_+/2\kappa_+} \Gamma\left(-i\frac{\omega_+}{\kappa_+}\right). \tag{6.10}
$$

Comparing Eq. (6.10) with Eq. (6.4), we find $\alpha_{\hat{\omega}\omega_+}^L$ in terms of $\alpha_{\hat{\omega}\omega_+}^{\text{past}}$ [22],

$$
\alpha_{\hat{\omega}\omega_+}^L = \alpha_{\hat{\omega}\omega_+}^{\text{past}} e^{-\pi\omega_+/\kappa_+}. \tag{6.11}
$$

---

[21]This expression is also given in Eq. (3.5) in Ref. [30], however, there are some notational differences in this case that need to be bridged: Basically, the notation in Ref. [30] is related to ours as prescribed in footnote 15, namely, $\tilde{\omega} \mapsto \omega_+$, $\omega \mapsto \hat{\omega}$. However, on the rhs of Eq. (6.10) our ω_+ should actually be mapped to $-\tilde{\omega}$ (this extra change of sign has to do with the issue discussed earlier in footnote 9).

[22]One might be concerned about the notable difference between our Eq. (6.11), describing the relation between $\alpha_{\hat{\omega}\omega_+}^L$ and $\alpha_{\hat{\omega}\omega_+}^{\text{past}}$, and the rightmost side of Eq. (3.5) in Ref. [30] (which states that $\alpha_{\omega\tilde{\omega}}^L = \alpha_{\omega\tilde{\omega}}^{\text{past}*}$). To reconcile this difference, note that when transforming our $\alpha_{\hat{\omega}\omega_+}^{\text{past}}$ and $\alpha_{\hat{\omega}\omega_+}^L$ to the notation of Ref. [30], the former becomes $\alpha_{\omega\tilde{\omega}}^{\text{past}}$, but $\alpha_{\hat{\omega}\omega_+}^L$ is translated to $\alpha_{\omega(-\tilde{\omega})}^L$ (see footnotes 15 and, 21 as well as 9). Indeed, considering the relation between $\alpha_{\omega(-\tilde{\omega})}^L$ and $\alpha_{\omega\tilde{\omega}}^{\text{past}}$ therein would yield the analog of Eq. (6.11) (namely, $\alpha_{\omega(-\tilde{\omega})}^L = \alpha_{\omega\tilde{\omega}}^{\text{past}} e^{-\pi\tilde{\omega}/\kappa_+}$).

The two H_L-projected functions $g^L_{\hat\omega\hat\jmath}(x)|_{H_L}$ and $f^{L(+)}_{\omega_+ J}(x)|_{H_L}$ may now be related using the conversion coefficients $\alpha^L_{\hat\omega\omega_+}$ and $C^{\omega_+}_{\hat\jmath J}$ [in full analogy with Eq. (5.21), replacing $\alpha_{\hat\omega\omega_+} = \alpha^{\text{past}}_{\hat\omega\omega_+}$ by $\alpha^L_{\hat\omega\omega_+}$],

$$\sqrt{\hat\omega}\, g^L_{\hat\omega\hat\jmath}\big|_{H_L} = \frac{1}{2\pi}\sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\, \alpha^L_{\hat\omega\omega_+} C^{\omega_+}_{\hat\jmath J} f^{L(+)}_{\omega_+ J}\big|_{H_L}. \tag{6.12}$$

Since both the left Unruh and Eddington modes vanish on H_R, this relation between $g^L_{\hat\omega\hat\jmath}$ and $f^{L(+)}_{\omega_+ J}$ actually holds throughout the BH interior,

$$\sqrt{\hat\omega}\, g^L_{\hat\omega\hat\jmath}(x) = \frac{1}{2\pi}\sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\, \alpha^L_{\hat\omega\omega_+} C^{\omega_+}_{\hat\jmath J} f^{L(+)}_{\omega_+ J}(x), \qquad r_- \le r \le r_+. \tag{6.13}$$

Now, substituting Eqs. (6.8) and (6.13) into Eq. (6.2), $G^{\text{up}}_U(x,x')$ may be written as

$$G^{\text{up}}_U(x,x') = \hbar(I_{RR} + I_{LL} + I_{RL} + I_{LR}), \tag{6.14}$$

where

$$I_{RR} \equiv$$
$$\sum_{\hat\jmath} \int_0^\infty \frac{d\hat\omega}{4\pi^2\hat\omega} \sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\, \alpha^{\text{past}}_{\hat\omega\omega_+} C^{\omega_+}_{\hat\jmath J} \rho^{\text{up}(+)}_{\omega_+ J} \sum_{J'} \int_{-\infty}^\infty d\omega'_+ \sqrt{|\omega'_+|}\, \alpha^{\text{past}*}_{\hat\omega\omega'_+} C^{\omega'_+ *}_{\hat\jmath J'} \rho^{\text{up}(+)*}_{\omega'_+ J'} \{f^{R(+)}_{\omega_+ J}(x), f^{R(+)*}_{\omega'_+ J'}(x')\}, \tag{6.15}$$

$$I_{LL} \equiv \sum_{\hat\jmath} \int_0^\infty \frac{d\hat\omega}{4\pi^2\hat\omega} \sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\, \alpha^L_{\hat\omega\omega_+} C^{\omega_+}_{\hat\jmath J} \sum_{J'} \int_{-\infty}^\infty d\omega'_+ \sqrt{|\omega'_+|}\, \alpha^{L*}_{\hat\omega\omega'_+} C^{\omega'_+ *}_{\hat\jmath J'} \{f^{L(+)}_{\omega_+ J}(x), f^{L(+)*}_{\omega'_+ J'}(x')\}, \tag{6.16}$$

$$I_{RL} \equiv \sum_{\hat\jmath} \int_0^\infty \frac{d\hat\omega}{4\pi^2\hat\omega} \sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\, \alpha^{\text{past}}_{\hat\omega\omega_+} C^{\omega_+}_{\hat\jmath J} \rho^{\text{up}(+)}_{\omega_+ J} \sum_{J'} \int_{-\infty}^\infty d\omega'_+ \sqrt{|\omega'_+|}\, \alpha^{L*}_{\hat\omega\omega'_+} C^{\omega'_+ *}_{\hat\jmath J'} \{f^{R(+)}_{\omega_+ J}(x), f^{L(+)*}_{\omega'_+ J'}(x')\}, \tag{6.17}$$

$$I_{LR} \equiv \sum_{\hat\jmath} \int_0^\infty \frac{d\hat\omega}{4\pi^2\hat\omega} \sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\, \alpha^L_{\hat\omega\omega_+} C^{\omega_+}_{\hat\jmath J} \sum_{J'} \int_{-\infty}^\infty d\omega'_+ \sqrt{|\omega'_+|}\, \alpha^{\text{past}*}_{\hat\omega\omega'_+} C^{\omega'_+ *}_{\hat\jmath J'} \rho^{\text{up}(+)*}_{\omega'_+ J'} \{f^{L(+)}_{\omega_+ J}(x), f^{R(+)*}_{\omega'_+ J'}(x')\}. \tag{6.18}$$

We rearrange Eqs. (6.15)–(6.18) into a form similar to that of Eq. (5.23),

$$I_{RR} = \frac{1}{4\pi^2}\sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\sum_{J'} \int_{-\infty}^\infty d\omega'_+ \sqrt{|\omega'_+|}\, \rho^{\text{up}(+)}_{\omega_+ J} \rho^{\text{up}(+)*}_{\omega'_+ J'} \cdot \{f^{R(+)}_{\omega_+ J}(x), f^{R(+)*}_{\omega'_+ J'}(x')\}\sum_{\hat\jmath} C^{\omega_+}_{\hat\jmath J} C^{\omega'_+ *}_{\hat\jmath J'} \int_0^\infty \frac{d\hat\omega}{\hat\omega}\, \alpha^{\text{past}}_{\hat\omega\omega_+} \alpha^{\text{past}*}_{\hat\omega\omega'_+},$$
$$\tag{6.19}$$

$$I_{LL} = \frac{1}{4\pi^2}\sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\sum_{J'} \int_{-\infty}^\infty d\omega'_+ \sqrt{|\omega'_+|} \cdot \{f^{L(+)}_{\omega_+ J}(x), f^{L(+)*}_{\omega'_+ J'}(x')\}\sum_{\hat\jmath} C^{\omega_+}_{\hat\jmath J} C^{\omega'_+ *}_{\hat\jmath J'} \int_0^\infty \frac{d\hat\omega}{\hat\omega}\, \alpha^L_{\hat\omega\omega_+} \alpha^{L*}_{\hat\omega\omega'_+}, \tag{6.20}$$

$$I_{RL} = \frac{1}{4\pi^2}\sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\sum_{J'} \int_{-\infty}^\infty d\omega'_+ \sqrt{|\omega'_+|}\, \rho^{\text{up}(+)}_{\omega_+ J} \cdot \{f^{R(+)}_{\omega_+ J}(x), f^{L(+)*}_{\omega'_+ J'}(x')\}\sum_{\hat\jmath} C^{\omega_+}_{\hat\jmath J} C^{\omega'_+ *}_{\hat\jmath J'} \int_0^\infty \frac{d\hat\omega}{\hat\omega}\, \alpha^{\text{past}}_{\hat\omega\omega_+} \alpha^{L*}_{\hat\omega\omega'_+}, \tag{6.21}$$

$$I_{LR} = \frac{1}{4\pi^2}\sum_J \int_{-\infty}^\infty d\omega_+ \sqrt{|\omega_+|}\sum_{J'} \int_{-\infty}^\infty d\omega'_+ \sqrt{|\omega'_+|}\, \rho^{\text{up}(+)*}_{\omega'_+ J'} \cdot \{f^{L(+)}_{\omega_+ J}(x), f^{R(+)*}_{\omega'_+ J'}(x')\}\sum_{\hat\jmath} C^{\omega_+}_{\hat\jmath J} C^{\omega'_+ *}_{\hat\jmath J'} \int_0^\infty \frac{d\hat\omega}{\hat\omega}\, \alpha^L_{\hat\omega\omega_+} \alpha^{\text{past}*}_{\hat\omega\omega'_+}. \tag{6.22}$$

At this point it becomes clear (after renaming the indices $\omega_+ \leftrightarrow \omega'_+$ and $J \leftrightarrow J'$ in the last equation) that $I_{LR} = I^*_{RL}$.

In order to proceed as we did in the BH exterior, we need to perform the integrals on the rightmost side of each equality, of the form $\int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \alpha_{\hat{\omega}\omega_+}^{\Lambda_1} \alpha_{\hat{\omega}\omega'_+}^{\Lambda_2*}$ with $\Lambda_{1,2}$ either "past" or L. The result of the integral $\int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \alpha_{\hat{\omega}\omega_+}^{\text{past}} \alpha_{\hat{\omega}\omega'_+}^{\text{past}*}$ has already been given in Eq. (5.24). Next, we use the relation between $\alpha_{\hat{\omega}\omega_+}^{L}$ and $\alpha_{\hat{\omega}\omega_+}^{\text{past}}$ given in Eq. (6.11), along with the previously mentioned integral [Eq. (5.24)], to find

$$\int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \alpha_{\hat{\omega}\omega_+}^{L} \alpha_{\hat{\omega}\omega'_+}^{L*} = e^{-\pi(\omega_+ + \omega'_+)/\kappa_+} \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \alpha_{\hat{\omega}\omega_+}^{\text{past}} \alpha_{\hat{\omega}\omega'_+}^{\text{past}*} = \frac{4\pi^2}{\omega_+} \frac{1}{e^{2\pi\omega_+/\kappa_+} - 1} \delta(\omega_+ - \omega'_+) \tag{6.23}$$

and

$$\int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \alpha_{\hat{\omega}\omega_+}^{\text{past}} \alpha_{\hat{\omega}\omega'_+}^{L*} = e^{-\pi\omega'_+/\kappa_+} \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}} \alpha_{\hat{\omega}\omega_+}^{\text{past}} \alpha_{\hat{\omega}\omega'_+}^{\text{past}*} = \frac{4\pi^2}{\omega_+} \frac{1}{e^{\pi\omega_+/\kappa_+} - e^{-\pi\omega_+/\kappa_+}} \delta(\omega_+ - \omega'_+). \tag{6.24}$$

Performing a computation very similar to the one carried out in Eq. (5.25) in the BH exterior, substituting the integrals in Eqs. (5.24), (6.23), and (6.24) into Eqs. (6.19), (6.20), and (6.21), respectively, and making use of Eq. (5.17), we obtain, after translating the exponential factors into corresponding hyperbolic-geometric functions,

$$I_{RR} = \frac{1}{2} \sum_J \int_{-\infty}^\infty d\omega_+ \frac{|\omega_+|}{\omega_+} \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) + 1 \right] [|\rho_{\omega_+ J}^{\text{up}(+)}|^2 \{f_{\omega_+ J}^{R(+)}(x), f_{\omega_+ J}^{R(+)*}(x')\}], \tag{6.25}$$

$$I_{LL} = \frac{1}{2} \sum_J \int_{-\infty}^\infty d\omega_+ \frac{|\omega_+|}{\omega_+} \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) - 1 \right] [\{f_{\omega_+ J}^{L(+)}(x), f_{\omega_+ J}^{L(+)*}(x')\}], \tag{6.26}$$

and

$$I_{RL} = I_{LR}^* = \frac{1}{2} \sum_J \int_{-\infty}^\infty d\omega_+ \frac{|\omega_+|}{\omega_+} \left[\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right) \right] [\rho_{\omega_+ J}^{\text{up}(+)} \{f_{\omega_+ J}^{R(+)}(x), f_{\omega_+ J}^{L(+)*}(x')\}], \tag{6.27}$$

where $\operatorname{cosech} \equiv 1/\sinh$.

Next, we would like to fold these three integrals through $\omega_+ = 0$, just as we did in Sec. V for the BH exterior. To this end, we first note that in all three equations (6.25)–(6.27), the rhs is of the general form

$$I \equiv \frac{1}{2} \sum_J \int_{-\infty}^\infty d\omega_+ \frac{|\omega_+|}{\omega_+} H(\omega_+) F_{\omega_+ J}(x, x') = \frac{1}{2} \sum_{l,m} \int_{-\infty}^\infty d\omega_+ \operatorname{sign}(\omega_+) H(\omega_+) F_{\omega_+ lm}(x, x'), \tag{6.28}$$

where $H(\omega_+)$ and $F_{\omega_+ lm}(x, x')$, respectively, stand for the first and second terms in square brackets in each of these three equations. Furthermore, we once again note that since $(\omega, m) \mapsto (-\omega, -m)$ is identical to $(\omega_+, m) \mapsto (-\omega_+, -m)$, we may rewrite Eq. (3.22) as

$$f_{(-\omega_+)l(-m)}^{\Lambda(+)} = (-1)^m f_{\omega_+ lm}^{\Lambda(+)*} \tag{6.29}$$

(with Λ either R or L) and the invariance relation of ρ^{up} included in Eq. (3.5) as

$$\rho_{(-\omega_+)l(-m)}^{\text{up}(+)} = \rho_{\omega_+ lm}^{\text{up}(+)*}. \tag{6.30}$$

Equations (6.29) and (6.30) may now be used to show that the function $F_{\omega_+ lm}(x, x')$ in all three cases (6.25)–(6.27) satisfies

$$F_{(-\omega_+)lm}(x, x') = F_{\omega_+ l(-m)}^*(x, x').$$

Then, summing over l and m (recalling $\sum_{l,m} = \sum_{l=0}^\infty \sum_{m=-l}^l$), we obtain

$$\sum_{l,m} F_{(-\omega_+)lm}(x,x') = \sum_{l,m} F^*_{\omega_+lm}(x,x') = \sum_J F^*_{\omega_+J}(x,x').$$

Therefore, by folding the ω_+ integral in Eq. (6.28), we obtain the following explicit form:

$$I = \frac{1}{2}\sum_J \int_0^\infty d\omega_+ [H(\omega_+)F_{\omega_+J}(x,x') - H(-\omega_+)F^*_{\omega_+J}(x,x')]. \tag{6.31}$$

Note that the function $H(\omega_+)$ potentially has both a part that is an even function of ω_+ and a part that is an odd function. Then, from Eq. (6.31) it is clear that the even part of $H(\omega_+)$ leaves out the imaginary part of $F_{\omega_+J}(x,x')$, while the odd part of $H(\omega_+)$ leaves out the real part of $F_{\omega_+J}(x,x')$. Applying this general folding structure to Eqs. (6.25)–(6.27) and noticing that $F_{\omega_+J}(x,x')$ is actually real for I_{RR} and I_{LL} [Eqs. (6.25) and (6.26), while for Eq. (6.27) this is not the case], we obtain the three desired folded integrals,

$$I_{RR} = \sum_J \int_0^\infty d\omega_+ \coth\left(\frac{\pi\omega_+}{\kappa_+}\right) |\rho_{\omega_+J}^{up(+)}|^2 \\ \times \{f_{\omega_+J}^{R(+)}(x), f_{\omega_+J}^{R(+)*}(x')\}, \tag{6.32}$$

$$I_{LL} = \sum_J \int_0^\infty d\omega_+ \coth\left(\frac{\pi\omega_+}{\kappa_+}\right) \{f_{\omega_+J}^{L(+)}(x), f_{\omega_+J}^{L(+)*}(x')\}, \tag{6.33}$$

and

$$I_{RL} = I_{LR} = \sum_J \int_0^\infty d\omega_+ \operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right) \\ \times \Re\left(\rho_{\omega_+J}^{up(+)} \{f_{\omega_+J}^{R(+)}(x), f_{\omega_+J}^{L(+)*}(x')\}\right). \tag{6.34}$$

Notably, all four individual contributions to $G_U^{up}(x,x')$, namely, I_{RR}, I_{LL}, and $I_{RL} = I_{LR}$, are real. Combining now Eqs. (6.32)–(6.34), we obtain the up contribution,

$$G_U^{up}(x,x') = \hbar\sum_{l,m} \int_0^\infty d\omega_+ \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right)(\{f_{\omega_+lm}^{L(+)}(x), f_{\omega_+lm}^{L(+)*}(x')\} + |\rho_{\omega_+lm}^{up(+)}|^2 \{f_{\omega_+lm}^{R(+)}(x), f_{\omega_+lm}^{R(+)*}(x')\}) \\ + 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho_{\omega_+lm}^{up(+)}\{f_{\omega_+lm}^{R(+)}(x), f_{\omega_+lm}^{L(+)*}(x')\})\right]. \tag{6.35}$$

Finally, combining the in contribution [Eq. (6.1)] with the up contribution [Eq. (6.35)], we obtain the full HTPF in the BH interior,

$$G_U^{(1)}(x,x') = \hbar\sum_{l,m} \int_0^\infty d\omega_+ \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right)(\{f_{\omega_+lm}^{L(+)}(x), f_{\omega_+lm}^{L(+)*}(x')\} + |\rho_{\omega_+lm}^{up(+)}|^2 \{f_{\omega_+lm}^{R(+)}(x), f_{\omega_+lm}^{R(+)*}(x')\}) \\ + 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho_{\omega_+lm}^{up(+)}\{f_{\omega_+lm}^{R(+)}(x), f_{\omega_+lm}^{L(+)*}(x')\})\right] \\ + \hbar\sum_{l,m} \int_0^\infty d\omega \frac{|\omega_+|}{\omega} |\tau_{\omega lm}^{in}|^2 \{f_{\omega_+lm}^{R(+)}(x), f_{\omega_+lm}^{R(+)*}(x')\}, \tag{6.36}$$

and, translating back to the standard ω-indexing notation, we reach our final result,

$$G_U^{(1)}(x,x') = \hbar\sum_{l,m} \int_0^\infty d\omega_+ \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right)(\{f_{\omega lm}^{L}(x), f_{\omega lm}^{L*}(x')\} + |\rho_{\omega lm}^{up}|^2 \{f_{\omega lm}^{R}(x), f_{\omega lm}^{R*}(x')\}) \\ + 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho_{\omega lm}^{up}\{f_{\omega lm}^{R}(x), f_{\omega lm}^{L*}(x')\})\right] + \hbar\sum_{l,m} \int_0^\infty d\omega \frac{|\omega_+|}{\omega} |\tau_{\omega lm}^{in}|^2 \{f_{\omega lm}^{R}(x), f_{\omega lm}^{R*}(x')\}. \tag{6.37}$$

One may be concerned about the apparent singularity of the mode-sum at $\omega_+ \to 0$ (and, similarly, at $\omega \to 0$). In Appendix A, we address this issue and show that the integrands in Eq. (6.37) are in fact entirely regular at both limits.

B. Invariance to the choice of the angular functions $\hat{Z}_{\hat{l}\hat{m}}^{\hat{\omega}}(\theta, \varphi_+)$

In Sec. V B we established that in the construction of the HTPF outside the BH, the final result remains unchanged if in the definition of the up Unruh modes (described in

Sec. III C 2) one replaces the spherical harmonics $Y_{\hat{l}\hat{m}}(\theta, \varphi_+)$ by any other complete orthonormal set of angular functions $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+)$ [fulfilling Eq. (3.28)]. As one can easily verify, all the arguments and considerations made there are equally valid for the interior.

More specifically, in order to carry out this generalization in the construction of the interior HTPF, throughout the analysis in the present section one simply has to replace everywhere $Y_{\hat{l}\hat{m}}$ by $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}$ and $C^{\omega_+}_{\hat{j}J}$ by $C^{\hat{\omega}\omega_+}_{\hat{j}J}$ (and similarly replace $Y_{\hat{j}}$ by $\hat{Z}^{\hat{\omega}}_{\hat{j}}$). Then, recalling Eq. (5.32), in (the $Y_{\hat{j}} \mapsto \hat{Z}^{\hat{\omega}}_{\hat{j}}$ counterparts of) each of the four equations (6.19)–(6.22), the factor $\sum_{\hat{j}} C^{\hat{\omega}\omega_+}_{\hat{j}J} C^{\hat{\omega}\omega'_+ *}_{\hat{j}J'}$ may be justifiably kept out of the $\hat{\omega}$ integral. From that point on, making use of Eq. (5.31), the analysis proceeds with no further modifications.

We conclude that the result in Eq. (6.37) for the mode-sum expression of the Unruh HTPF inside the BH is invariant with respect to the choice of the initial angular functions $\hat{Z}^{\hat{\omega}}_{\hat{l}\hat{m}}(\theta, \varphi_+)$ in the up Unruh modes, as anticipated.

C. Alternative forms of the final result

We propose here alternative forms of the final result for the HTPF given in Eq. (6.37), which may prove to be useful in future applications. In particular, we shall provide an expression in which the integral over ω_+ (arising from the up contribution) is replaced by an integral over ω. To this end, we shall proceed as follows:

We begin by introducing the functions $\tilde{f}^R_{\omega lm}$ and $\tilde{f}^L_{\omega lm}$ related to the standard interior Eddington modes $f^R_{\omega lm}$ and $f^L_{\omega lm}$ [see Eq. (3.17)] by eliminating the normalization factor, that is,

$$\tilde{f}^R_{\omega lm} \equiv \sqrt{4\pi|\omega_+|} f^R_{\omega lm} = \frac{1}{\sqrt{r^2 + a^2}} Z^\omega_{lm}(\theta, \varphi) e^{-i\omega t} \psi^{int}_{\omega lm},$$

$$\tilde{f}^L_{\omega lm} \equiv \sqrt{4\pi|\omega_+|} f^L_{\omega lm} = \frac{1}{\sqrt{r^2 + a^2}} Z^\omega_{lm}(\theta, \varphi) e^{-i\omega t} \psi^{int*}_{\omega lm}.$$

$$(6.38)$$

Rewriting Eq. (6.37) in terms of $\tilde{f}^R_{\omega lm}$ and $\tilde{f}^L_{\omega lm}$, we obtain

$$G^{(1)}_U(x, x') = \hbar \sum_{l,m} \int_0^\infty \frac{d\omega_+}{4\pi\omega_+} \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) \left(\{\tilde{f}^L_{\omega lm}(x), \tilde{f}^{L*}_{\omega lm}(x')\} + |\rho^{up}_{\omega lm}|^2 \{\tilde{f}^R_{\omega lm}(x), \tilde{f}^{R*}_{\omega lm}(x')\} \right) \right.$$

$$\left. + 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right) \Re(\rho^{up}_{\omega lm} \{\tilde{f}^R_{\omega lm}(x), \tilde{f}^{L*}_{\omega lm}(x')\}) \right] + \hbar \sum_{l,m} \int_0^\infty \frac{d\omega}{4\pi\omega} |\tau^{in}_{\omega lm}|^2 \{\tilde{f}^R_{\omega lm}(x), \tilde{f}^{R*}_{\omega lm}(x')\}, \qquad (6.39)$$

where we note that the first integral is over *positive* ω_+ only, crucially allowing us to replace $|\omega_+|$ by ω_+ there.

Recall that there are two parts making up $G^{(1)}_U(x, x')$: the $\int_0^\infty d\omega_+$ integral arising from the up modes contribution $G^{up}_U(x, x')$ and the $\int_0^\infty d\omega$ integral corresponding to $G^{in}_U(x, x')$. We now concentrate on the former.

Inspecting the form of the integrand of $G^{up}_U(x, x')$ as written in the first part of Eq. (6.39) (namely, the integrand of $\int_0^\infty d\omega_+$ there), we find that it is invariant under the simultaneous sign changes $m \mapsto -m$ and $\omega \mapsto -\omega$. To see this, apply the symmetries given in Eqs. (3.22) and (3.5) along with the odd nature of the coth and cosech functions [recalling that $(\omega, m) \mapsto (-\omega, -m)$ also implies $\omega_+ \mapsto -\omega_+$].

We shall now establish the following statement: Given a function $E_m(\omega)$ with the property

$$E_{(-m)}(-\omega) = E_m(\omega),$$

we may formally write

$$\sum_{m=-l}^l \int_0^\infty d\omega_+ E_m(\omega) = \sum_{m=-l}^l \int_0^\infty d\omega E_m(\omega). \qquad (6.40)$$

That is, one may replace $\sum_{l,m} \int_0^\infty d\omega_+$ (in the case of a symmetric integrand as described) by $\sum_{l,m} \int_0^\infty d\omega$.

To see this, we may denote

$$I_m \equiv \int_0^\infty d\omega_+ E_m(\omega),$$

and then express I_m and I_{-m} as follows:

$$I_{\pm m} = \int_{\pm m\Omega_+}^0 d\omega E_{\pm m}(\omega) + \int_0^\infty d\omega E_{\pm m}(\omega).$$

We now concentrate on the finite-domain integration term of I_{-m},

$$\int_{-m\Omega_+}^0 d\omega E_{-m}(\omega) = -\int_{m\Omega_+}^0 d\omega E_{-m}(-\omega)$$

$$= -\int_{m\Omega_+}^0 d\omega E_m(\omega),$$

where we have changed variables from ω to $-\omega$ and then used the symmetry of $E_m(\omega)$. We readily see that this term exactly cancels the finite-domain integration term in the

corresponding expression for I_m. Summing over m in pairs of $\pm m$ (and noting that the $m = 0$ term does not contain such a finite-domain integration piece), we are thus left with the desired relation (6.40).

$$G_U^{(1)}(x, x') = \hbar \sum_{l,m} \int_0^\infty \frac{d\omega}{4\pi\omega_+} \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) (\{\tilde{f}_{\omega lm}^L(x), \tilde{f}_{\omega lm}^{L*}(x')\} + |\rho_{\omega lm}^{up}|^2 \{\tilde{f}_{\omega lm}^R(x), \tilde{f}_{\omega lm}^{R*}(x')\}) \right.$$

$$\left. + 2 \, \text{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right) \Re(\rho_{\omega lm}^{up} \{\tilde{f}_{\omega lm}^R(x), \tilde{f}_{\omega lm}^{L*}(x')\}) + \frac{\omega_+}{\omega} |\tau_{\omega lm}^{in}|^2 \{\tilde{f}_{\omega lm}^R(x), \tilde{f}_{\omega lm}^{R*}(x')\} \right]. \quad (6.41)$$

The integrand in this equation is regular at both $\omega \to 0$ and $\omega_+ \to 0$, as mentioned after Eq. (6.37).

One can also modify the form of Eq. (6.41) [or likewise Eq. (6.37)] by various applications of the Wronskian relations, given in Eq. (3.33). In particular, one may replace $\frac{\omega_+}{\omega} |\tau_{\omega lm}^{in}|^2$ by $1 - |\rho_{\omega lm}^{up}|^2$, thereby entirely eliminating τ from the final expression.

In Appendix B we harness Eq. (6.41) in order to construct the bare mode-sum expressions for the Unruh fluxes $\langle T_{uu} \rangle^U$ and $\langle T_{vv} \rangle^U$ (where $T_{\mu\nu}$ is the stress-energy tensor) for a minimally coupled massless scalar quantum field, starting at a general r value in the BH interior and then taking it to the horizons $r \to r_\pm$. The resulting expressions will serve as a basis for future research.

VII. DISCUSSION

In this paper, we have derived expressions for Hadamard's elementary two-point function $G^{(1)}(x, x')$ for the Unruh state inside a Kerr BH in the domain between the EH and the IH (that is, with both points located at $r_- < r < r_+$; note that the Unruh state is not defined beyond the IH, because the latter is a Cauchy horizon).

In order to introduce and facilitate our approach, we first rederived the (already known) expression for the HTPF in the BH exterior (namely, $r > r_+$), with the result given in Eq. (5.29). This expression consists of an (nonthermalized) in contribution as well as a thermalized [$\propto \coth(\pi\omega_+/\kappa_+)$] up contribution. A similar (though more intricate) approach is then applied to the BH interior.

The main result of the paper (the HTPF in the BH interior) appears in its most useful form in Eq. (6.41) [also in slightly different forms in Eqs. (6.37) and (6.39)]. This expression may naturally be divided into three terms: (i) a term proportional to $|\tau_{\omega lm}^{in}|^2$, which emerges from the in modes (see the bottom right panel of Fig. 2). These modes arise from PNI, penetrate the potential barrier at $r > r_+$, and then cross the EH and propagate in the BH interior (this term is evidently the counterpart of the in contribution in the BH exterior, but here only the transmitted part remains, hence the $|\tau_{\omega lm}^{in}|^2$ factor), (ii) a term proportional to $\coth(\pi\omega_+/\kappa_+)$ arising from the up modes (see the bottom left panel of Fig. 2), composed of a left contribution

Following the discussion above, this property may now be applied to the integral $\int_0^\infty d\omega_+$ in Eq. (6.39), replacing it by an integral $\int_0^\infty d\omega$. Then, the entire $G_U^{(1)}(x, x')$ may be written in terms of an integral over ω,

originating at H_L, and a $\propto |\rho_{\omega lm}^{up}|^2$ past contribution arising from H_{past}, then being outside the BH and crossing the EH [this may be viewed as the interior counterpart of the thermalized piece $\propto \coth(\pi\omega_+/\kappa_+)$ in the BH exterior], and (iii) a term proportional to cosech $(\pi\omega_+/\kappa_+)$ coming from the up modes and involving interference between left and reflected past modes, thus not having a counterpart in the BH exterior (where the up modes consist only of a past contribution).

The expression (6.41) for the HTPF in the BH interior includes factors (such as $1/\omega$, $1/\omega_+$, as well as hyperbolic functions that diverge at $\omega_+ \to 0$) which may raise concerns regarding potential IR divergencies. In Appendix A we analyze both the $\omega \to 0$ and $\omega_+ \to 0$ limits of $G^{(1)}(x, x')$ and show that both these limits are regular, that is, no IR divergence is present.

Our result (6.41) is expressed in terms of the computationally favorable Eddington modes (whose determination only requires solving ODEs), and thus lays the ground for an array of useful applications, such as the computation of $\langle \Phi^2 \rangle$ or $\langle T_{\alpha\beta} \rangle$ in the physically motivated Unruh state in the interior of a Kerr BH. Of special interest are the flux components of the RSET, $\langle T_{uu} \rangle$ and $\langle T_{vv} \rangle$, where u and v denote the standard Eddington coordinates in the BH interior. In Appendix B, we use our expression for the HTPF to develop explicit expressions for the "bare" mode contributions to these flux components inside a Kerr BH, at a general r value between the two horizons. These general r expressions, given in Eqs. (B34)–(B38), provide the basis for a future paper of ours [32], in which we compute $\langle T_{uu} \rangle$ and $\langle T_{vv} \rangle$ between the EH and the IH of a spinning BH. Then, we take the (physically interesting) IH limit, leading to Eqs. (B49) and (B50). These limiting expressions are readily applied in our subsequent paper [33], in which we compute the Unruh-state fluxes $\langle T_{uu} \rangle$ and $\langle T_{vv} \rangle$ at the IH vicinity of a Kerr BH, hence shedding light on the remarkable open question regarding the fate of the IH inside a spinning BH under quantum perturbations.

The Kerr geometry admits the well-known superradiance phenomenon: all in modes in the "superradiance band" $0 < \omega < m\Omega_+$ (for $m > 0$) are amplified when scattered off the BH, that is, they satisfy $|\rho_{\omega lm}^{in}| > 1$. Although this is

basically a purely classical phenomenon, it also has some interesting implications at the quantum level. One such remarkable implication is the fact that the semiclassical quantum outflux emitted from a Kerr BH to infinity (in the Unruh state) does *not* vanish at the extremal limit, on which we shall focus in the remainder of this section: There remains a nonvanishing contribution (known as the "Unruh-Starobinskii radiation") coming from the super-radiant modes. It is interesting to note that a closely analogous superradiance-related phenomenon also occurs *inside* the BH (again in the extremal limit), as may be seen in Eq. (B52) at the end of Appendix B. As pointed out there, the integrand in that equation is negative throughout (off the pole), which implies $\langle T^-_{uu}\rangle^U_{\rm ren} - \langle T^-_{vv}\rangle^U_{\rm ren} < 0$.[23] This last inequality guarantees that at least one of the fluxes at the IH (and possibly both of them) does not vanish. [This situation may be confronted with the extremal RN limit, in which both $\langle T^-_{uu}\rangle$ and $\langle T^-_{vv}\rangle$ vanish [34].[24]) This non-vanishing flux (or possibly fluxes) at the IH in the extremal limit is a clear signature of the superradiance phenomenon inside the BH.

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$$G^{\rm up}_{\omega lm}(x,x') = \coth\left(\frac{\pi\omega_+}{\kappa_+}\right)(\{f^L_{\omega lm}(x), f^{L*}_{\omega lm}(x')\} + \{f^R_{\omega lm}(x), f^{R*}_{\omega lm}(x')\}|\rho^{\rm up}_{\omega lm}|^2)$$

$$+ 2\,{\rm cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho^{\rm up}_{\omega lm}\{f^R_{\omega lm}(x), f^{L*}_{\omega lm}(x')\}), \qquad (A2)$$

where $x = (t, r, \theta, \varphi)$ and $x' = (t', r', \theta', \varphi')$. We denote $\delta t \equiv t - t'$ and $\delta\varphi \equiv \varphi - \varphi'$.

Equation (A2) is composed of the inner Eddington modes $f^R_{\omega lm}(x)$ and $f^L_{\omega lm}(x)$, given in Eq. (3.17). These mode functions involve in their definition a factor $|\omega_+|^{-1/2}$.

APPENDIX A: THE HTPF INTEGRANDS AT SMALL FREQUENCIES

Equation (6.37) expresses the HTPF as a sum of an up part, involving an integral over positive ω_+, and an in part, which involves an integral over positive ω. The expressions appearing in Eq. (6.37) for these two integrals may raise concerns about possible divergences at two specific frequencies: The up piece includes terms proportional to $\coth(\pi\omega_+/\kappa_+)$ or $\,{\rm cosech}(\pi\omega_+/\kappa_+)$, both diverging at $\omega_+ \to 0$; the in piece includes a $1/\omega$ factor, which diverges at $\omega \to 0$. In addition, both the up and in pieces include products of $f^R_{\omega lm}$ and/or $f^L_{\omega lm}$ functions, each entailing a factor of $1/\sqrt{|\omega_+|}$ in its definition [see Eq. (3.17)], which also contribute to a potential divergence at $\omega_+ \to 0$. Our goal in this appendix is to analyze these potential divergences and to show that no divergence actually occurs in neither $\omega_+ \to 0$ nor the $\omega \to 0$ limit. We shall show this separately for the up and in pieces.

1. The up integrand

We begin with the up part of the HTPF, given by

$$G^{\rm up}_U(x,x') = \hbar\sum_{l,m}\int_0^\infty d\omega_+ G^{\rm up}_{\omega lm}(x,x'), \qquad (A1)$$

with the individual mode contribution

In order to explicitly reveal this divergent factor, we shall here rewrite $f^R_{\omega lm}(x)$ and $f^L_{\omega lm}(x)$ in the form

$$f^\Lambda_{\omega lm}(x) = \frac{S^\omega_{lm}(\theta)}{\sqrt{8\pi^2|\omega_+|(r^2+a^2)}}\bar{f}^\Lambda_{\omega lm}(x) \qquad (A3)$$

(for Λ either R or L), where we denote

$$\bar{f}^\Lambda_{\omega lm}(x) \equiv e^{im\varphi}e^{-i\omega t}\psi^\Lambda_{\omega lm}(r). \qquad (A4)$$

As before, $\psi^\Lambda_{\omega lm}$ is the radial function and we have $\psi^R_{\omega lm} = \psi^{L*}_{\omega lm} \equiv \psi^{\rm int}_{\omega lm}$. At this stage it becomes clear that there is a potential divergence in $G^{\rm up}_{\omega lm}$ that goes like $1/\omega^2_+$, due to the $|\omega_+|$ factor appearing in the radical in the denominator of Eq. (A3), combined with the $\coth(\pi\omega_+/\kappa_+)$ or $\sinh^{-1}(\pi\omega_+/\kappa_+)$ factors in Eq. (A2).

In order to facilitate the analysis of this potential divergence at $\omega_+ \to 0$, we next write $\{f^{\Lambda_1}_{\omega lm}(x), f^{\Lambda_2*}_{\omega lm}(x')\}$ (with Λ_1, Λ_2 either R or L) as

$$\{f^{\Lambda_1}_{\omega lm}(x), f^{\Lambda_2*}_{\omega lm}(x')\} = \frac{S^\omega_{lm}(\theta)S^\omega_{lm}(\theta')}{8\pi^2|\omega_+|\sqrt{(r^2+a^2)(r'^2+a^2)}}$$
$$\times \{\bar{f}^{\Lambda_1}_{\omega lm}(x), \bar{f}^{\Lambda_2*}_{\omega lm}(x')\}. \qquad (A5)$$

Then $\{\bar{f}^{\Lambda_1}(x), \bar{f}^{\Lambda_2*}(x')\}$ is given by

$$\{\bar{f}^{\Lambda_1}_{\omega lm}(x), \bar{f}^{\Lambda_2*}_{\omega lm}(x')\} = e^{-iK}\psi^{\Lambda_1}_{\omega lm}(r)\psi^{\Lambda_2*}_{\omega lm}(r')$$
$$+ e^{iK}\psi^{\Lambda_2*}_{\omega lm}(r)\psi^{\Lambda_1}_{\omega lm}(r'), \qquad (A6)$$

where $K \equiv \omega\delta t - m\delta\varphi$.

In what follows, we suppress the superscript "int" for brevity (that is, $\psi^{\text{int}}_{\omega lm}$ is to be denoted by $\psi_{\omega lm}$). Explicitly, the three relevant cases for $\{\bar{f}^{\Lambda_1}_{\omega lm}(x), \bar{f}^{\Lambda_2*}_{\omega lm}(x')\}$ are

$$\{\bar{f}^L_{\omega lm}(x), \bar{f}^{L*}_{\omega lm}(x')\} = 2\Re[e^{-iK}\psi^*_{\omega lm}(r)\psi_{\omega lm}(r')], \qquad (A7)$$

$$\{\bar{f}^R_{\omega lm}(x), \bar{f}^{R*}_{\omega lm}(x')\} = 2\Re[e^{-iK}\psi_{\omega lm}(r)\psi^*_{\omega lm}(r')], \qquad (A8)$$

$$\{\bar{f}^R_{\omega lm}(x), \bar{f}^{L*}_{\omega lm}(x')\} = 2\cos(K)\psi_{\omega lm}(r)\psi_{\omega lm}(r'). \qquad (A9)$$

Recalling the small-ω_+ expansions $\coth(\pi\omega_+/\kappa_+) = \kappa_+/\pi\omega_+ + O(\omega_+)$ and $\operatorname{cosech}(\pi\omega_+/\kappa_+) = \kappa_+/\pi\omega_+ + O(\omega_+)$, along with Eqs. (A7)–(A9) for the various $\{\bar{f}^{\Lambda_1}_{\omega lm}(x), \bar{f}^{\Lambda_2*}_{\omega lm}(x')\}$ factors, we obtain

$$G^{\text{up}}_{\omega lm}(x, x') = \frac{S^\omega_{lm}(\theta')S^\omega_{lm}(\theta)}{8\pi^2\sqrt{(r^2+a^2)(r'^2+a^2)}}$$
$$\times \frac{\kappa_+}{\pi}[G^{\text{up}(A)}_{\omega lm}(r, r') + G^{\text{up}(B)}_{\omega lm}(r, r')$$
$$+ G^{\text{up}(C)}_{\omega lm}(r, r')] + O(\omega^0_+), \qquad (A10)$$

where

$$G^{\text{up}(A)}_{\omega lm}(r, r') \equiv \frac{2}{\omega^2_+}\Re[e^{iK}\psi_{\omega lm}(r)\psi^*_{\omega lm}(r')], \qquad (A11)$$

$$G^{\text{up}(B)}_{\omega lm}(r, r') \equiv \frac{2}{\omega^2_+}|\rho^{\text{up}}_{\omega lm}|^2\Re[e^{-iK}\psi_{\omega lm}(r)\psi^*_{\omega lm}(r')], \quad (A12)$$

$$G^{\text{up}(C)}_{\omega lm}(r, r') \equiv \frac{4}{\omega^2_+}\cos(K)\Re[\rho^{\text{up}}_{\omega lm}\psi_{\omega lm}(r)\psi_{\omega lm}(r')]. \quad (A13)$$

We next resort to the small-ω_+ expansion of $\psi_{\omega lm}(r)$ and $\rho^{\text{up}}_{\omega lm}$. For $\psi_{\omega lm}(r)$, the analysis in Sec. A 3 below [see, in particular, Eqs. (A37) and (A47)] implies that

$$\psi_{\omega lm}(r) = \psi^{(0)}_{lm}(r) + \psi^{(1)}_{lm}(r)\omega_+ + o(\omega_+), \qquad (A14)$$

where $\psi^{(0)}_{lm}(r)$ is a *real* function, and $o(\omega_+)$ denotes terms whose decay rate at $\omega_+ \to 0$ is faster than ω_+. For $\rho^{\text{up}}_{\omega lm}$, it can be shown[25] that

$$\rho^{\text{up}}_{\omega lm} = -1 + \rho^{(1)}_{lm}\omega_+ + o(\omega_+). \qquad (A15)$$

Naturally, the expansion coefficients $\psi^{(0)}_{lm}(r)$, $\psi^{(1)}_{lm}(r)$, and $\rho^{(1)}_{lm}$ are independent of ω (or ω_+), as their lower indices (being solely lm) also indicate.

To facilitate the analysis below, we now rewrite Eqs. (A14) and (A15) by absorbing their $o(\omega_+)$ parts into the first-order coefficients $\psi^{(1)}_{lm}(r)$ and $\rho^{(1)}_{lm}$. As a result, these first-order coefficients now become ω dependent, and correspondingly we denote them by $\psi^{(1)}_{\omega lm}(r)$ and $\rho^{(1)}_{\omega lm}$. [Nevertheless, this dependence on ω will not cause any complication: The only relevant fact is that both $\psi^{(1)}_{\omega lm}(r)$ and $\rho^{(1)}_{\omega lm}$ remain finite as $\omega_+ \to 0$.] Thus, we rewrite Eqs. (A14) and (A15) as follows:

$$\psi_{\omega lm}(r) = \psi^{(0)}_{lm}(r) + \psi^{(1)}_{\omega lm}(r)\omega_+, \qquad (A16)$$

$$\rho^{\text{up}}_{\omega lm} = -1 + \rho^{(1)}_{\omega lm}\omega_+. \qquad (A17)$$

The last equation also implies

$$|\rho^{\text{up}}_{\omega lm}|^2 = 1 - 2\omega_+\Re(\rho^{(1)}_{\omega lm}) + O(\omega^2_+). \qquad (A18)$$

We now plug the expansions (A16)–(A18) of $\psi_{\omega lm}$, $\rho^{\text{up}}_{\omega lm}$, and $|\rho^{\text{up}}_{\omega lm}|^2$ into Eqs. (A11)–(A13). The three quantities $G^{\text{up}(A)}_{\omega lm}$, $G^{\text{up}(B)}_{\omega lm}$, and $G^{\text{up}(C)}_{\omega lm}$ then split accordingly into terms multiplying ω^{-2}_+ and ω^{-1}_+ [plus an $O(\omega^0_+)$ term], and they all take the form

$$G^{\text{up}(X)}_{\omega lm}(r, r') = \frac{1}{\omega^2_+}G^{\text{up}(X_{-2})}_{\omega lm}(r, r')$$
$$+ \frac{1}{\omega_+}G^{\text{up}(X_{-1})}_{\omega lm}(r, r') + O(\omega^0_+), \quad (A19)$$

where X stands here for either A, B, or C. The computation of the 2×3 coefficients $G^{\text{up}(X_{-2})}_{\omega lm}$ and $G^{\text{up}(X_{-1})}_{\omega lm}$ is straightforward [and uses the fact that $\psi^{(0)}_{lm}(r)$ is real]. For $X = A$ the two coefficients are

$$G^{\text{up}(A_{-2})}_{\omega lm}(r, r') = 2\cos(K)\psi^{(0)}_{lm}(r)\psi^{(0)}_{lm}(r') \qquad (A20)$$

and

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[25]We analytically derived this small-ω_+ expansion of $\rho^{\text{up}}_{\omega lm}$, both for $m \neq 0$ and $m = 0$ (in which case $\omega_+ \to 0$ means $\omega \to 0$). We also verified this small-ω_+ expansion numerically. We do not provide the analytical derivation here, as this issue (being solely related to wave scattering *outside* the BH) is beyond the scope of the present paper.

$$G_{\omega lm}^{\mathrm{up}(A_{-1})}(r,r') = 2\psi_{lm}^{(0)}(r)\Re[e^{-iK}\psi_{\omega lm}^{(1)}(r')]$$
$$+ 2\psi_{lm}^{(0)}(r')\Re[e^{iK}\psi_{\omega lm}^{(1)}(r)], \qquad (A21)$$

for $X = B$ the coefficients are

$$G_{\omega lm}^{\mathrm{up}(B_{-2})}(r,r') = 2\cos(K)\psi_{lm}^{(0)}(r)\psi_{lm}^{(0)}(r') \qquad (A22)$$

and

$$G_{\omega lm}^{\mathrm{up}(B_{-1})}(r,r') = 2\psi_{lm}^{(0)}(r)\Re[e^{iK}\psi_{\omega lm}^{(1)}(r')]$$
$$+ 2\psi_{lm}^{(0)}(r')\Re[e^{-iK}\psi_{\omega lm}^{(1)}(r)]$$
$$- 4\cos(K)\Re[\rho_{\omega lm}^{(1)}]\psi_{lm}^{(0)}(r)\psi_{lm}^{(0)}(r'), \qquad (A23)$$

and for $X = C$,

$$G_{\omega lm}^{\mathrm{up}(C_{-2})}(r,r') = -4\cos(K)\psi_{lm}^{(0)}(r)\psi_{lm}^{(0)}(r') \qquad (A24)$$

and

$$G_{\omega lm}^{\mathrm{up}(C_{-1})}(r,r') = -4\cos(K)(\psi_{lm}^{(0)}(r)\Re[\psi_{\omega lm}^{(1)}(r')]$$
$$+ \psi_{lm}^{(0)}(r')\Re[\psi_{\omega lm}^{(1)}(r)])$$
$$+ 4\cos(K)\Re[\rho_{\omega lm}^{(1)}]\psi_{lm}^{(0)}(r)\psi_{lm}^{(0)}(r'). \qquad (A25)$$

Concentrating first on the three $\propto 1/\omega_+^2$ coefficients, namely, $G_{\omega lm}^{\mathrm{up}(X_{-2})}$, notice that

$$G_{\omega lm}^{\mathrm{up}(A_{-2})} + G_{\omega lm}^{\mathrm{up}(B_{-2})} = -G_{\omega lm}^{\mathrm{up}(C_{-2})}, \qquad (A26)$$

so they cancel out in $G_{\omega lm}^{\mathrm{up}}$. Turning next to the three $\propto 1/\omega_+$ coefficients $G_{\omega lm}^{\mathrm{up}(X_{-1})}$, we see the same structure here again,

$$G_{\omega lm}^{\mathrm{up}(A_{-1})} + G_{\omega lm}^{\mathrm{up}(B_{-1})} = -G_{\omega lm}^{\mathrm{up}(C_{-1})}, \qquad (A27)$$

so that part is canceled out as well. Substituting this back into Eq. (A10), we are left with

$$G_{\omega lm}^{\mathrm{up}}(x,x') = O(\omega_+^0) \qquad (A28)$$

at $\omega_+ \to 0$; that is, the potential divergence of the individual up mode contributions at $\omega_+ = 0$ is gone.

Finally we consider the behavior of $G_{\omega lm}^{\mathrm{up}}(x,x')$ at $\omega \to 0$. Both $\psi_{\omega lm}$ and $\rho_{\omega lm}^{\mathrm{up}}$ are regular at that limit.[26] For $m \neq 0$ (in which case $\omega \to 0$ implies that ω_+ stays remote from zero), no divergence can occur in $G_{\omega lm}^{\mathrm{up}}$ at

$\omega \to 0$. In the special case $m = 0$, taking the limit $\omega \to 0$ also implies $\omega_+ \to 0$ (which in turn implies there are potentially divergent terms in the above expression for $G_{\omega lm}^{\mathrm{up}}$); nevertheless, it was already shown above that the overall expression for $G_{\omega lm}^{\mathrm{up}}$ is actually regular at $\omega_+ \to 0$.

We therefore conclude that $G_{\omega lm}^{\mathrm{up}}(x,x')$ is regular at both limits $\omega_+ \to 0$ and $\omega \to 0$.

2. The in integrand

We now consider the in mode contribution, which is much simpler and is given by

$$G_U^{\mathrm{in}}(x,x') = \hbar \sum_{l,m} \int_0^\infty d\omega\, G_{\omega lm}^{\mathrm{in}}(x,x'), \qquad (A29)$$

with the integrand

$$G_{\omega lm}^{\mathrm{in}}(x,x') = \frac{|\omega_+|}{\omega}|\tau_{\omega lm}^{\mathrm{in}}|^2 \{f_{\omega lm}^R(x), f_{\omega lm}^{R*}(x')\}. \qquad (A30)$$

The Wronskian relations in Eq. (3.33) yield

$$\frac{\omega_+}{\omega}|\tau_{\omega lm}^{\mathrm{in}}|^2 = 1 - |\rho_{\omega lm}^{\mathrm{in}}|^2,$$

and therefore (recalling $|\rho_{\omega lm}^{\mathrm{in}}| = |\rho_{\omega lm}^{\mathrm{up}}|$) also

$$\frac{|\omega_+|}{\omega}|\tau_{\omega lm}^{\mathrm{in}}|^2 = \mathrm{sign}(\omega_+)(1 - |\rho_{\omega lm}^{\mathrm{up}}|^2).$$

Plugging this relation along with Eq. (3.33) into Eq. (A30), we obtain

$$G_{\omega lm}^{\mathrm{in}}(x,x') = \frac{1}{\omega_+}\left(1 - |\rho_{\omega lm}^{\mathrm{up}}|^2\right)\frac{S_{lm}^\omega(\theta')S_{lm}^\omega(\theta)}{8\pi^2\sqrt{(r^2+a^2)(r'^2+a^2)}}$$
$$\times \{\bar{f}_{\omega lm}^R(x), \bar{f}_{\omega lm}^{R*}(x')\}, \qquad (A31)$$

which highlights the potential divergence at $\omega_+ \to 0$. However, Eq. (A18) reads

$$1 - |\rho_{\omega lm}^{\mathrm{up}}|^2 = 2\omega_+\Re(\rho_1) + O(\omega_+^2), \qquad (A32)$$

and the ω_+ factor on the right-hand side cancels out the $1/\omega_+$ factor in Eq. (A31). Also, $\bar{f}_{\omega lm}^R$ is regular at $\omega_+ \to 0$, as directly follows from Eqs. (A4) and (A14). We are therefore left with

$$G_{\omega lm}^{\mathrm{in}}(x,x') = O(\omega_+^0) \qquad (A33)$$

at the limit $\omega_+ \to 0$.

The form of Eq. (A31) also guarantees that no irregularity occurs at $\omega \to 0$ either.[27]

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[26]For $\psi_{\omega lm}$ in the case $m \neq 0$, this regularity naturally follows from the definition of $\psi_{\omega lm}$ based on boundary conditions specified at the EH in terms of ω_+ rather than ω. For $\rho_{\omega lm}^{\mathrm{up}}$ in the case $m \neq 0$, we analytically computed $\rho_{\omega lm}^{\mathrm{up}}$ at $\omega \to 0$ and found it to be finite and well defined (and it satisfies $|\rho_{(\omega=0)lm}^{\mathrm{up}}| = 1$), but again, this analysis of $\rho_{\omega lm}^{\mathrm{up}}$ is beyond the scope of this paper. We also numerically verified smoothness of $\rho_{\omega lm}^{\mathrm{up}}$ at the limit $\omega \to 0$. In the other case $m = 0$, the limit $\omega \to 0$ coincides with the limit $\omega_+ \to 0$, for which regularity of $\psi_{\omega lm}$ and $\rho_{\omega lm}^{\mathrm{up}}$ has already been established above [see Eqs. (A14) and (A15), and also footnote 25].

[27]As before, we use the fact that $\rho_{\omega lm}^{\mathrm{up}}$ and $\psi_{\omega lm}$ (and hence also $\bar{f}_{\omega lm}^R$) are regular at $\omega \to 0$. We also recall that, in the special case $m = 0$, for which the $1/\omega_+$ factor in Eq. (A31) diverges as $\omega \to 0$ (because now this limit also implies $\omega_+ \to 0$), this potential divergence is already handled in the above analysis, which showed that $G_{\omega lm}^{\mathrm{in}}$ is actually regular at $\omega_+ \to 0$.

3. $\psi_{\omega lm}^{\text{int}}$ at small ω_+

The function $\psi_{\omega lm}(r)$ [which, recall, denotes $\psi_{\omega lm}^{\text{int}}(r)$ in this appendix] satisfies the radial equation

$$\psi_{\omega lm, r_* r_*} = -V_{\omega lm}(r)\psi_{\omega lm}, \qquad (A34)$$

with an effective potential $V_{\omega lm}(r)$ given explicitly in Eqs. (2.19) and (2.20). The initial condition for this ODE is specified at the EH (corresponding to $r_* \to -\infty$) by

$$\psi_{\omega lm} \simeq e^{-i\omega_+ r_*} \equiv \psi_{\omega lm}^{(\text{init})} (r_* \to -\infty). \qquad (A35)$$

(Recall, the symbol $\simeq$ denotes equality at the relevant asymptotic boundary, namely, $r_* \to -\infty$ in the present case.)

Our goal is to analyze the behavior of $\psi_{\omega lm}(r)$ at small ω_+. To this end, we first write the asymptotic behavior of the initial condition (A35) at small ω_+,

$$\psi_{\omega lm}^{(\text{init})} = 1 - i\omega_+ r_* - \omega_+^2 \frac{r_*^2}{2} + \cdots. \qquad (A36)$$

The effective potential $V_{\omega lm}(r)$, too, can be decomposed in a power series in ω_+ around $\omega_+ = 0$ (see below). We are therefore motivated to adopt the following ansatz for the form of $\psi_{\omega lm}(r)$ at small ω_+, as a power series in ω_+:

$$\psi_{\omega lm}(r) = \psi_{lm}^{(0)}(r) + \psi_{lm}^{(1)}(r)\omega_+ + \psi_{lm}^{(2)}(r)\omega_+^2 + \cdots. \qquad (A37)$$

Each coefficient $\psi_{lm}^{(n)}(r)$ in this expansion should satisfy its own ODE (with its own initial condition at the EH), as we shall now describe.

To find the specific ODE that each term $\psi_{lm}^{(n)}(r)$ satisfies, we have to expand the potential $V_{\omega lm}(r)$ in powers of ω_+. Since $V_{\omega lm}$ depends on the angular eigenvalue $\lambda_{\omega lm}$, we first expand this eigenvalue,[28]

$$\lambda_{\omega lm} = \lambda_{lm}^{(0)} + \lambda_{lm}^{(1)} a\omega_+ + \lambda_{lm}^{(2)} (a\omega_+)^2 + \cdots. \qquad (A38)$$

Then we can expand the effective potential in the same manner,

$$V_{\omega lm}(r) = V_{lm}^{(0)}(r) + V_{lm}^{(1)}(r)\omega_+ + V_{lm}^{(2)}(r)\omega_+^2 + \cdots. \qquad (A39)$$

The leading-order coefficient is given by

$$V_{lm}^{(0)}(r) = (m\Omega_+)^2 \left(\frac{r^2 - r_+^2}{r^2 + a^2} \right)^2 - G^2$$

$$- \frac{\Delta}{r^2 + a^2} \frac{dG}{dr} - \frac{\lambda_{lm}^{(0)} \Delta}{(r^2 + a^2)^2}, \qquad (A40)$$

where, recall, $\Omega_+ = a/(r_+^2 + a^2)$ and $G = r\Delta/(r^2 + a^2)^2$. The first-order coefficient is

$$V_{lm}^{(1)}(r) = 2m\Omega_+ \frac{r^2 - r_+^2}{r^2 + a^2} - \frac{a\lambda_{lm}^{(1)} \Delta}{(r^2 + a^2)^2}, \qquad (A41)$$

and the second-order coefficient is

$$V_{lm}^{(2)}(r) = 1 - \frac{a^2 \lambda_{lm}^{(2)} \Delta}{(r^2 + a^2)^2}. \qquad (A42)$$

[In fact, all higher-order coefficients are of the same simple form: $V_{lm}^{(n>2)} = -a^n \lambda_{lm}^{(n)} \Delta / (r^2 + a^2)^2$.]

It is important to recall that the potential $V_{\omega lm}(r)$ is real—and so are all its expansion coefficients $V_{lm}^{(n)}$. In addition, note that both coefficients $V_{lm}^{(0)}$ and $V_{lm}^{(1)}$ vanish at the EH like $\propto \Delta \propto r - r_+$—hence, they both decay exponentially with r_* at the EH limit $r_* \to -\infty$. In fact, at the EH we have $V_{\omega lm} = \omega_+^2$.[29]

Inserting the ansatz (A37) for $\psi_{\omega lm}$ into the radial equation (A34) with the expanded form (A39) of $V_{\omega lm}$ and grouping powers of ω_+, we obtain the following hierarchy of ODEs:

$$\psi_{lm, r_* r_*}^{(0)} + V_{lm}^{(0)} \psi_{lm}^{(0)} = 0,$$
$$\psi_{lm, r_* r_*}^{(1)} + V_{lm}^{(0)} \psi_{lm}^{(1)} = -V_{lm}^{(1)} \psi_{lm}^{(0)},$$
$$\psi_{lm, r_* r_*}^{(2)} + V_{lm}^{(0)} \psi_{lm}^{(2)} = -V_{lm}^{(1)} \psi_{lm}^{(1)} - V_{lm}^{(2)} \psi_{lm}^{(0)},$$
$$\cdots. \qquad (A43)$$

Note that $\psi_{lm}^{(0)}(r)$ satisfies a homogeneous ODE, but all other functions $\psi_{lm}^{(n)}(r)$ satisfy inhomogeneous ones (having $V_{lm}^{(0)}$ as their potential and a source term involving other coefficients in the expansion of $V_{\omega lm}$).

The initial conditions for these ODEs are to be specified at the EH limit, just like those of the original function $\psi_{\omega lm}(r)$. They are naturally obtained by the Taylor expansion (A36) of the original initial data $\psi_{\omega lm}^{(\text{init})}$,

$$\psi_{lm}^{(0)} \simeq 1 \ (r_* \to -\infty), \qquad (A44)$$

$$\psi_{lm}^{(1)} \simeq -ir_* \ (r_* \to -\infty), \qquad (A45)$$

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[28]For $m \neq 0$ it is trivial, as the formulation of the angular eigenvalue problem is insensitive to the $\omega_+ \to 0$ limit. For $m = 0$, it has been shown [35] that such a power-series expansion exists.

[29]Correspondingly $V_{lm}^{(2)}(r) = 1$ at the EH, as one can also see from Eq. (A42).

$$\psi_{lm}^{(2)} \simeq -\frac{r_*^2}{2} \quad (r_* \to -\infty), \tag{A46}$$

etc.

Of particular importance to our analysis is the leading-order function $\psi_{lm}^{(0)}(r)$. It satisfies a real ODE (as $V_{lm}^{(0)}$ is real), with real initial conditions. It therefore follows that $\psi_{lm}^{(0)}(r)$ is a real function,

$$\psi_{lm}^{(0)}(r) \in \mathbb{R}. \tag{A47}$$

Although not necessary for the regularity analysis carried out in this appendix, it may be interesting to consider the properties of $\psi_{lm}^{(1)}$ as well. It still satisfies a real ODE (because both its potential $V_{lm}^{(0)}$ and its source term $-V_{lm}^{(1)}\psi_{lm}^{(0)}$ are real). However, its initial condition at the EH limit ($\simeq -ir_*$) is imaginary. Therefore, $\psi_{lm}^{(1)}(r)$ is not real. It is not purely imaginary either, because it is fed by a real source term $-V_{lm}^{(1)}\psi_{lm}^{(0)}$. All higher-order terms $\psi_{lm}^{(n>1)}$ are expected to be complex too.

APPENDIX B: THE UNRUH-STATE BARE FLUX EXPRESSIONS INSIDE THE BH

This appendix is dedicated to developing the mode-sum expressions for the Unruh-state RSET components T_{uu} and T_{vv} [where hereafter $u = u_{\rm int}$ and v are the interior Eddington coordinates given in Eq. (2.8)]. We first construct this mode-sum expression at a general point $r_- < r < r_+$ in the BH interior, then we concentrate on the horizon limits $r \to r_-$ and $r \to r_+$.

We focus here on T_{uu} and T_{vv} because these two components are especially meaningful for the semiclassical study of backreaction on BH interiors (in particular, at the IH vicinity). At the horizons, these components play the role of energy fluxes,[30] and we shall thus refer to them as the "flux components" or, in short, the "fluxes." In addition, these components reveal notable simplicity at the IH limit, as we shall briefly note later on.

1. Bare fluxes at general r

Before we begin with the construction, we note that the components of a tensor such as $T_{\alpha\beta}$ clearly depend on the underlying coordinate system. Here we shall particularly be interested in three coordinate systems, which only differ from each other by the choice of the azimuthal coordinate. We collectively denote these three coordinate systems as

$(u, v, \theta, \tilde{\varphi})$, where $\tilde{\varphi}$ stands for either φ, φ_+, or φ_-. Recall that φ is the original Boyer-Lindquist azimuthal coordinate, while φ_+ and φ_- are the two modified azimuthal coordinates constructed to be regular, respectively, at the EH and IH, and they are given by $\varphi \equiv \varphi_\pm - \Omega_\pm t$ (see Sec. II A). Thus, we may generally define $\tilde{\varphi}$ as

$$\tilde{\varphi} \equiv \varphi - \tilde{\Omega}t, \tag{B1}$$

where the constant $\tilde{\Omega}$ is either zero, Ω_+, or Ω_-, for φ, φ_+, and φ_- respectively.[31]

We shall restrict our attention here to a minimally coupled massless scalar field [i.e., $m = \xi = 0$ in Eq. (2.11)]. Then, at the classical level, the stress-energy tensor $T_{\alpha\beta}$ of this field may be expressed as

$$T_{\alpha\beta} = \bar{T}_{\alpha\beta} - (1/2)g_{\alpha\beta}\bar{T}_\mu^\mu, \tag{B2}$$

where $\bar{T}_{\alpha\beta}$ (the "trace-reversed" stress-energy tensor) is given in terms of the first-order scalar field derivatives by

$$\bar{T}_{\alpha\beta} = \Phi_{,\alpha}\Phi_{,\beta}. \tag{B3}$$

For the analysis below, it will be useful to reexpress $\bar{T}_{\alpha\beta}$ as a second-order differential operator acting on a certain quantity bilinear in Φ [this form will later allow us to conveniently express the quantum expectation value of $T_{\alpha\beta}$ in terms of a differential operator acting on the quantity $G_U^{(1)}(x, x')$ that is already available to us]. To this end, we reexpress $\bar{T}_{\alpha\beta}$ (still at the classical level) as

$$\bar{T}_{\alpha\beta}(x) = \lim_{x' \to x}[\Phi(x)\Phi(x')]_{,\alpha\beta'}. \tag{B4}$$

The symbol $_{,\alpha\beta'}$ denotes differentiation with respect to x^α and x'^β, where $\partial/\partial x^\alpha$ acts on functions of the spacetime point x, while $\partial/\partial x'^\beta$ acts on functions of the spacetime point x'. We then further rewrite it in the form

$$\bar{T}_{\alpha\beta}(x) = \frac{1}{2}\lim_{x' \to x}[\Phi(x)\Phi(x') + \Phi(x')\Phi(x)]_{,\alpha\beta'} \tag{B5}$$

(which, although trivial, sets the stage for the quantum treatment that will now follow).

Transitioning from the classical- to the quantum-field context, we want to compute the expectation value of $\bar{T}_{\alpha\beta}(x)$, for our minimally coupled quantum field $\hat{\Phi}$, in the Unruh state.[32] Applying the $\langle \ldots \rangle_U$ expectation value

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[30]Note that the Eddington coordinates u and v are spacelike at $r_- < r < r_+$ but they become asymptotically null at $r \to r_-$ and $r \to r_+$. (To be more precise, we can look at the corresponding Kruskal coordinates, which are found to be spacelike between the horizons and null at the horizons. These properties are then carried over to the corresponding Eddington coordinates.)

[31]We point out that, specifically, the choice of the azimuthal coordinate $\tilde{\varphi}$ does affect the values of the flux components T_{uu} and T_{vv}.

[32]Note that in the quantum context, both $T_{\alpha\beta}$ and $\bar{T}_{\alpha\beta}$ are treated as quantum operators. To avoid notational complications (especially for $\bar{T}_{\alpha\beta}$), we do not add any special symbol (e.g., an overhat) to make this quantum nature explicitly visible. Nevertheless, in the equations below, the expectation value symbol $\langle \ldots \rangle^U$ will always reveal the quantum-operator nature of $T_{\alpha\beta}$ and $\bar{T}_{\alpha\beta}$.

operation to the two sides of Eq. (B5) (which are viewed now as quantum operators) and recalling that $G_U^{(1)}(x, x') = \langle \hat{\Phi}(x)\hat{\Phi}(x') + \hat{\Phi}(x')\hat{\Phi}(x) \rangle_U$ [see Eq. (5.1)], we obtain the following formal expression for $\langle \bar{T}_{\alpha\beta} \rangle_{\text{bare}}^U$:

$$\langle \bar{T}_{\alpha\beta} \rangle_{\text{bare}}^U(x) = \frac{1}{2} \lim_{x' \to x} [G_U^{(1)}(x, x')_{,\alpha\beta'}]. \qquad (B6)$$

This is complemented by the quantum version of Eq. (B2), namely,

$$\langle T_{\alpha\beta} \rangle_{\text{bare}}^U = \langle \bar{T}_{\alpha\beta} \rangle_{\text{bare}}^U - (1/2)g_{\alpha\beta}\langle \bar{T}_{\mu}^{\mu} \rangle_{\text{bare}}^U. \qquad (B7)$$

To avoid confusion we emphasize again that the split appearing on the right-hand side of Eqs. (B4)–(B6) is *not*

aimed for regularization [recall we deal in Eqs. (B4) and (B5) with the *classical* expressions and in Eq. (B6) with the *bare* quantum expression]: The only purpose of this split is to allow differentiation with respect to x and x' separately—in order to eventually express the RSET in terms of the already-known function $G_U^{(1)}(x, x')$.

To proceed, we write the Unruh-state HTPF $G_U^{(1)}(x, x')$ (for points inside the BH) given in Eq. (6.41) as the mode-sum

$$G_U^{(1)}(x, x') = \hbar \sum_{l,m} \int_0^\infty G_{\omega lm}(x, x')d\omega, \qquad (B8)$$

where

$$G_{\omega lm}(x, x') = \frac{1}{4\pi\omega_+} \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) (\{\tilde{f}_{\omega lm}^L(x), \tilde{f}_{\omega lm}^{L*}(x')\} + |\rho_{\omega lm}^{\text{up}}|^2 \{\tilde{f}_{\omega lm}^R(x), \tilde{f}_{\omega lm}^{R*}(x')\}) \right.$$
$$\left. + 2\text{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right) \Re(\rho_{\omega lm}^{\text{up}} \{\tilde{f}_{\omega lm}^R(x), \tilde{f}_{\omega lm}^{L*}(x')\}) + \frac{\omega_+}{\omega} |\tau_{\omega lm}^{\text{in}}|^2 \{\tilde{f}_{\omega lm}^R(x), \tilde{f}_{\omega lm}^{R*}(x')\} \right]. \qquad (B9)$$

Correspondingly we may then rewrite Eq. (B6) as the mode sum

$$\langle \bar{T}_{\alpha\beta} \rangle_{\text{bare}}^U(x) = \sum_{l,m} \int_0^\infty \bar{T}_{\alpha\beta(\omega lm)}d\omega, \qquad (B10)$$

where the integrand is defined as

$$\bar{T}_{\alpha\beta(\omega lm)} \equiv \frac{\hbar}{2} \lim_{x' \to x} [G_{\omega lm}(x, x')_{,\alpha\beta'}]. \qquad (B11)$$

From this point on we shall concentrate on the two flux components, taking $\alpha\beta$ to be yy, with y hereafter denoting either u or v. The computation of $\bar{T}_{yy(\omega lm)}$ then involves two simple stages: (i) differentiating $G_{\omega lm}(x, x')$ with respect to

y and y', and then (ii) taking the coincidence limit $x' \to x$. The rhs of Eq. (B9) consists of several terms of the form

$$\{F_1(x), F_2(x')\} = F_1(x)F_2(x') + F_1(x')F_2(x).$$

Applying these stages (i) and (ii) to the term $F_1(x)F_2(x')$ simply yields $F_{1,y}F_{2,y}$ (evaluated at the point x), and applying it to the other term $F_1(x')F_2(x)$ yields exactly the same result; that is,

$$\lim_{x' \to x} \{F_1(x), F_2(x')\}_{,yy'} = 2F_{1,y}F_{2,y}$$

(evaluated at the point x as mentioned above). Implementing this in Eq. (B11), we obtain

$$\bar{T}_{yy(\omega lm)} = \frac{\hbar}{4\pi\omega_+} \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) (\tilde{f}_{\omega lm,y}^L \tilde{f}_{\omega lm,y}^{L*} + |\rho_{\omega lm}^{\text{up}}|^2 \tilde{f}_{\omega lm,y}^R \tilde{f}_{\omega lm,y}^{R*}) \right.$$
$$\left. + 2\text{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right) \Re(\rho_{\omega lm}^{\text{up}} \tilde{f}_{\omega lm,y}^R \tilde{f}_{\omega lm,y}^{L*}) + \frac{\omega_+}{\omega} |\tau_{\omega lm}^{\text{in}}|^2 \tilde{f}_{\omega lm,y}^R \tilde{f}_{\omega lm,y}^{R*} \right]. \qquad (B12)$$

The functions $\tilde{f}_{\omega lm}^\Lambda$ (with Λ denoting either R or L) were defined in Eq. (6.38). Recalling that

$$Z_{lm}^\omega(\theta, \varphi) = \frac{1}{\sqrt{2\pi}} S_{lm}^\omega(\theta)e^{im\varphi},$$

we may rewrite these functions in the more explicit form,

$$\tilde{f}_{\omega lm}^R = \frac{1}{\sqrt{2\pi(r^2 + a^2)}} S_{lm}^\omega(\theta)e^{im\varphi}e^{-i\omega t}\psi_{\omega lm}^{\text{int}},$$

$$\tilde{f}_{\omega lm}^L = \frac{1}{\sqrt{2\pi(r^2 + a^2)}} S_{lm}^\omega(\theta)e^{im\varphi}e^{-i\omega t}\psi_{\omega lm}^{\text{int*}}.$$

We need to differentiate these functions with respect to y with *fixed $\tilde{\varphi}$* (rather than fixed φ). To this end, we define a general frequency parameter $\tilde{\omega}$ of the form

$$\tilde{\omega} \equiv \omega - m\tilde{\Omega} \qquad (B13)$$

(the tilde "~" in $\tilde{\omega}$ links this frequency to the choice of the azimuthal coordinate $\tilde{\varphi}$). Noting that

$$e^{-i\omega t}e^{im\varphi} = e^{-i\tilde{\omega}t}e^{im\tilde{\varphi}}, \qquad (B14)$$

we may now reexpress $\tilde{f}^{\Lambda}_{\omega lm}$ as

$$\tilde{f}^{\Lambda}_{\omega lm}(x) = \frac{1}{\sqrt{8\pi}}S^{\omega}_{lm}(\theta)e^{im\tilde{\varphi}}\hat{f}^{\Lambda}_{\omega lm}(t,r), \qquad (B15)$$

where we define

$$\hat{f}^{R}_{\omega lm}(t,r) \equiv \frac{2}{\sqrt{r^2+a^2}}e^{-i\tilde{\omega}t}\psi^{\text{int}}_{\omega lm}(r),$$

$$\hat{f}^{L}_{\omega lm}(t,r) \equiv \frac{2}{\sqrt{r^2+a^2}}e^{-i\tilde{\omega}t}\psi^{\text{int}*}_{\omega lm}(r). \qquad (B16)$$

(The variables t and r should be viewed here as functions of the coordinates u, v.) Substituting Eq. (B15) in Eq. (B12) [recalling that $S^{\omega}_{lm}(\theta)$ is real], we may now entirely factor out the angular dependence,

$$\bar{T}_{yy(\omega lm)} = \hbar\frac{[S^{\omega}_{lm}(\theta)]^2}{32\pi^2\omega_+}\bar{\mathcal{T}}_{yy(\omega lm)}, \qquad (B17)$$

where

$$\bar{\mathcal{T}}_{yy(\omega lm)} \equiv \coth\left(\frac{\pi\omega_+}{\kappa_+}\right)\left(\hat{f}^{L}_{\omega lm,y}\hat{f}^{L*}_{\omega lm,y} + |\rho^{\text{up}}_{\omega lm}|^2\hat{f}^{R}_{\omega lm,y}\hat{f}^{R*}_{\omega lm,y}\right)$$

$$+ 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re\left(\rho^{\text{up}}_{\omega lm}\hat{f}^{R}_{\omega lm,y}\hat{f}^{L*}_{\omega lm,y}\right)$$

$$+ \frac{\omega_+}{\omega}|\tau^{\text{in}}_{\omega lm}|^2\hat{f}^{R}_{\omega lm,y}\hat{f}^{R*}_{\omega lm,y}. \qquad (B18)$$

To further process this expression, we next specify the four combinations entailed in $\hat{f}^{\Lambda}_{\omega lm,y}$ (corresponding to $\Lambda = L, R$ and $y = u, v$). Using the relations

$$t = \frac{v-u}{2}, \qquad r_* = \frac{u+v}{2},$$

as well as $dr/dr_* = \Delta/(r^2+a^2)$, we find

$$\hat{f}^{R}_{\omega lm,u} = \frac{e^{-i\tilde{\omega}t}}{\sqrt{r^2+a^2}}[(i\tilde{\omega}\psi^{\text{int}}_{\omega lm} + \psi^{\text{int}}_{\omega lm,r_*}) - \mathcal{H}(r)\psi^{\text{int}}_{\omega lm}\Delta], \qquad (B19)$$

$$\hat{f}^{R}_{\omega lm,v} = \frac{e^{-i\tilde{\omega}t}}{\sqrt{r^2+a^2}}[(-i\tilde{\omega}\psi^{\text{int}}_{\omega lm} + \psi^{\text{int}}_{\omega lm,r_*}) - \mathcal{H}(r)\psi^{\text{int}}_{\omega lm}\Delta], \qquad (B20)$$

$$\hat{f}^{L}_{\omega lm,u} = \frac{e^{-i\tilde{\omega}t}}{\sqrt{r^2+a^2}}[(i\tilde{\omega}\psi^{\text{int}*}_{\omega lm} + \psi^{\text{int}*}_{\omega lm,r_*}) - \mathcal{H}(r)\psi^{\text{int}*}_{\omega lm}\Delta], \qquad (B21)$$

$$\hat{f}^{L}_{\omega lm,v} = \frac{e^{-i\tilde{\omega}t}}{\sqrt{r^2+a^2}}[(-i\tilde{\omega}\psi^{\text{int}*}_{\omega lm} + \psi^{\text{int}*}_{\omega lm,r_*}) - \mathcal{H}(r)\psi^{\text{int}*}_{\omega lm}\Delta], \qquad (B22)$$

where

$$\mathcal{H}(r) \equiv \frac{r}{(r^2+a^2)^2}.$$

Notice that $\hat{f}^{\Lambda}_{\omega lm,v}$ and $\hat{f}^{\Lambda}_{\omega lm,u}$ are related by the transformation $\tilde{\omega} \mapsto -\tilde{\omega}, t \mapsto -t$.[33] Also, $\hat{f}^{L}_{\omega lm,y}$ and $\hat{f}^{R}_{\omega lm,y}$ are related by the transformation $\tilde{\omega} \mapsto -\tilde{\omega}$ combined with overall complex conjugation. These relations will be useful below.

We now combine these derivatives to form the various bilinear combinations of the form $\hat{f}^{\Lambda_1}_{\omega lm,y}\hat{f}^{\Lambda_2*}_{\omega lm,y}$ appearing in Eq. (B18), namely, the three combinations $\Lambda_1\Lambda_2 = (LL, RR, RL)$. One immediately notices that the factors $e^{-i\tilde{\omega}t}$ (and indeed the entire dependence on t) cancel out in all these combinations. It is convenient to express each of these contributions in the form

$$\hat{f}^{\Lambda_1}_{\omega lm,y}\hat{f}^{\Lambda_2*}_{\omega lm,y} = \frac{1}{r^2+a^2}\left[\mathcal{A}^{\Lambda_1\Lambda_2}_{\omega lm(y)} - 2\mathcal{H}(r)\mathcal{B}^{\Lambda_1\Lambda_2}_{\omega lm(y)}\Delta\right.$$

$$\left.+ \mathcal{H}^2(r)\mathcal{C}^{\Lambda_1\Lambda_2}_{\omega lm(y)}\Delta^2\right]. \qquad (B23)$$

By a direct substitution of Eqs. (B19)–(B22), recalling the Wronskian relation

$$\psi^{\text{int}}_{\omega lm}\psi^{\text{int}*}_{\omega lm,r_*} - \psi^{\text{int}*}_{\omega lm}\psi^{\text{int}}_{\omega lm,r_*} = 2i\omega_+,$$

we obtain the following expressions for the $\mathcal{A}$ coefficients:

$$\mathcal{A}^{RR}_{\omega lm(v)} = \mathcal{A}^{LL}_{\omega lm(u)} = |\psi^{\text{int}}_{\omega lm,r_*}|^2 + \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 + 2\tilde{\omega}\omega_+, \qquad (B24)$$

$$\mathcal{A}^{LL}_{\omega lm(v)} = \mathcal{A}^{RR}_{\omega lm(u)} = |\psi^{\text{int}}_{\omega lm,r_*}|^2 + \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 - 2\tilde{\omega}\omega_+, \qquad (B25)$$

$$\mathcal{A}^{RL}_{\omega lm(u)} = \mathcal{A}^{RL}_{\omega lm(v)} = (\psi^{\text{int}}_{\omega lm,r_*})^2 + \tilde{\omega}^2(\psi^{\text{int}}_{\omega lm})^2. \qquad (B26)$$

For the $\mathcal{B}$ and $\mathcal{C}$ coefficients we will omit the (y) subscript, as they attain the same value for both $y = v$ and $y = u$. We obtain

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[33]In $\tilde{\omega} \mapsto -\tilde{\omega}$ we refer to any *explicit* occurrence of $\tilde{\omega}$—we do not touch the indices ωlm [as an illustration, see how the described transformation relates Eqs. (B19) and (B20) or (B21) and (B22)].

$$\mathcal{B}^{RR}_{\omega lm} = \mathcal{B}^{LL}_{\omega lm} = \Re(\psi^{\text{int}}_{\omega lm}\psi^{\text{int}*}_{\omega lm, r_*}), \qquad \mathcal{B}^{RL}_{\omega lm} = \psi^{\text{int}}_{\omega lm}\psi^{\text{int}}_{\omega lm, r_*},$$
$$(B27)$$

$$\mathcal{C}^{LL}_{\omega lm} = \mathcal{C}^{RR}_{\omega lm} = |\psi^{\text{int}}_{\omega lm}|^2, \qquad \mathcal{C}^{RL}_{\omega lm} = (\psi^{\text{int}}_{\omega lm})^2. \quad (B28)$$

Note also that none of the $\mathcal{B}$, $\mathcal{C}$ coefficients depend explicitly on $\tilde{\omega}$ (but the $\mathcal{A}$ coefficients do).

We have mentioned above simple rules for the transformations $R \leftrightarrow L$ and $u \leftrightarrow v$ in the expressions for $\hat{f}^{\Lambda}_{\omega lm, y}$. We can use them to derive corresponding rules for (overall) interchanges $R \leftrightarrow L$ and/or $u \leftrightarrow v$ in $\hat{f}^{\Lambda_1}_{\omega lm, y}\hat{f}^{\Lambda_2*}_{\omega lm, y}$. It follows that both transformations $RR \leftrightarrow LL$ and $u \leftrightarrow v$ amount to changing $\tilde{\omega} \mapsto -\tilde{\omega}$[34] (and therefore the combined transformation $RR \leftrightarrow LL, u \leftrightarrow v$ leaves the expression unchanged). One can easily verify that the above expressions (B24)–(B28) indeed satisfy these simple exchange rules.

The next stage would be to substitute Eq. (B23) in Eq. (B18) for $\bar{\mathcal{T}}_{yy(\omega lm)}$. It is again convenient to rewrite the latter (just as we did in the former) explicitly in powers of Δ, so we write

$$\bar{\mathcal{T}}_{yy(\omega lm)} = \frac{1}{r^2 + a^2}[\bar{\mathcal{T}}^{\mathcal{A}}_{yy(\omega lm)} - 2\mathcal{H}(r)\bar{\mathcal{T}}^{\mathcal{B}}_{(\omega lm)}\Delta + \mathcal{H}^2(r)\bar{\mathcal{T}}^{\mathcal{C}}_{(\omega lm)}\Delta^2]. \quad (B29)$$

We find $\bar{\mathcal{T}}^{\mathcal{A}}_{uu(\omega lm)}$ to be given by

$$\bar{\mathcal{T}}^{\mathcal{A}}_{uu(\omega lm)} = \coth\left(\frac{\pi\omega_+}{\kappa_+}\right)[|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 + 2\tilde{\omega}\omega_+$$
$$+ |\rho^{\text{up}}_{\omega lm}|^2(|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 - 2\tilde{\omega}\omega_+)]$$
$$+ 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho^{\text{up}}_{\omega lm}[(\psi^{\text{int}}_{\omega lm, r_*})^2$$
$$+ \tilde{\omega}^2(\psi^{\text{int}}_{\omega lm})^2]) + \frac{\omega_+}{\omega}|\tau^{\text{in}}_{\omega lm}|^2(|\psi^{\text{int}}_{\omega lm, r_*}|^2$$
$$+ \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 - 2\tilde{\omega}\omega_+). \quad (B30)$$

The vv counterpart, $\bar{\mathcal{T}}^{\mathcal{A}}_{vv(\omega lm)}$, is obtained by taking $\tilde{\omega} \mapsto -\tilde{\omega}$ in the above expression,

$$\bar{\mathcal{T}}^{\mathcal{A}}_{vv(\omega lm)} = \coth\left(\frac{\pi\omega_+}{\kappa_+}\right)[|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 - 2\tilde{\omega}\omega_+$$
$$+ |\rho^{\text{up}}_{\omega lm}|^2(|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 + 2\tilde{\omega}\omega_+)]$$
$$+ 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho^{\text{up}}_{\omega lm}[(\psi^{\text{int}}_{\omega lm, r_*})^2$$
$$+ \tilde{\omega}^2(\psi^{\text{int}}_{\omega lm})^2]) + \frac{\omega_+}{\omega}|\tau^{\text{in}}_{\omega lm}|^2(|\psi^{\text{int}}_{\omega lm, r_*}|^2$$
$$+ \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 + 2\tilde{\omega}\omega_+). \quad (B31)$$

Finally, the $\mathcal{B}$ and $\mathcal{C}$ coefficients are given by

$$\bar{\mathcal{T}}^{\mathcal{B}}_{(\omega lm)} = \coth\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re\left(\psi^{\text{int}}_{\omega lm}\psi^{\text{int}*}_{\omega lm, r_*}\right)\left(1 + |\rho^{\text{up}}_{\omega lm}|^2\right)$$
$$+ 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho^{\text{up}}_{\omega lm}\psi^{\text{int}}_{\omega lm}\psi^{\text{int}}_{\omega lm, r_*})$$
$$+ \frac{\omega_+}{\omega}|\tau^{\text{in}}_{\omega lm}|^2\Re(\psi^{\text{int}}_{\omega lm}\psi^{\text{int}*}_{\omega lm, r_*}), \quad (B32)$$

$$\bar{\mathcal{T}}^{\mathcal{C}}_{(\omega lm)} = \coth\left(\frac{\pi\omega_+}{\kappa_+}\right)|\psi^{\text{int}}_{\omega lm}|^2(1 + |\rho^{\text{up}}_{\omega lm}|^2)$$
$$+ 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho^{\text{up}}_{\omega lm}(\psi^{\text{int}}_{\omega lm})^2)$$
$$+ \frac{\omega_+}{\omega}|\tau^{\text{in}}_{\omega lm}|^2|\psi^{\text{int}}_{\omega lm}|^2. \quad (B33)$$

Note that the $\mathcal{B}$ and $\mathcal{C}$ contributions are the same for uu and vv—and the same applies to $\bar{T}^{\mathcal{B},\mathcal{C}}_{(\omega lm)}$ defined below.

Finally, we substitute the expression (B29) for $\bar{\mathcal{T}}_{yy(\omega lm)}$ into Eq. (B17) for $\bar{T}_{yy(\omega lm)}$. Again, we rewrite the latter in powers of Δ,

$$\bar{T}_{yy(\omega lm)} = \bar{T}^{\mathcal{A}}_{yy(\omega lm)} + \bar{T}^{\mathcal{B}}_{(\omega lm)}\Delta + \bar{T}^{\mathcal{C}}_{(\omega lm)}\Delta^2. \quad (B34)$$

The coefficients $\bar{T}^{\mathcal{A}}_{yy(\omega lm)}$, $\bar{T}^{\mathcal{B}}_{(\omega lm)}$, and $\bar{T}^{\mathcal{C}}_{(\omega lm)}$ are then given by

$$\bar{T}^{\mathcal{A}}_{uu(\omega lm)} = \hbar \frac{[S^{\omega}_{lm}(\theta)]^2}{32\pi^2\omega_+(r^2 + a^2)}\left(\coth\left(\frac{\pi\omega_+}{\kappa_+}\right)[|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 + 2\tilde{\omega}\omega_+\right.$$
$$+ |\rho^{\text{up}}_{\omega lm}|^2(|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 - 2\tilde{\omega}\omega_+)]$$
$$+ 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho^{\text{up}}_{\omega lm}[(\psi^{\text{int}}_{\omega lm, r_*})^2 + \tilde{\omega}^2(\psi^{\text{int}}_{\omega lm})^2])$$
$$\left.+ (1 - |\rho^{\text{up}}_{\omega lm}|^2)(|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2|\psi^{\text{int}}_{\omega lm}|^2 - 2\tilde{\omega}\omega_+)\right), \quad (B35)$$

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[34]To this end, one should recall that (i) $\hat{f}^{\Lambda_1}_{\omega lm, y}\hat{f}^{\Lambda_2*}_{\omega lm, y}$ is independent of t, and (ii) all these $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ coefficients for LL or RR are real.

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$$\bar{T}^{\mathcal{A}}_{vv(\omega lm)} = \hbar \frac{[S^{\omega}_{lm}(\theta)]^2}{32\pi^2 \omega_+ (r^2 + a^2)} \left(\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) [|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2 |\psi^{\text{int}}_{\omega lm}|^2 - 2\tilde{\omega}\omega_+ + |\rho^{\text{up}}_{\omega lm}|^2 (|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2 |\psi^{\text{int}}_{\omega lm}|^2 + 2\tilde{\omega}\omega_+)] \right.$$

$$\left. + 2 \operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right) \Re(\rho^{\text{up}}_{\omega lm}[(\psi^{\text{int}}_{\omega lm, r_*})^2 + \tilde{\omega}^2 (\psi^{\text{int}}_{\omega lm})^2]) + (1 - |\rho^{\text{up}}_{\omega lm}|^2)(|\psi^{\text{int}}_{\omega lm, r_*}|^2 + \tilde{\omega}^2 |\psi^{\text{int}}_{\omega lm}|^2 + 2\tilde{\omega}\omega_+) \right), \quad \text{(B36)}$$

$$\bar{T}^{\mathcal{B}}_{(\omega lm)} = -\hbar \frac{[S^{\omega}_{lm}(\theta)]^2 r}{16\pi^2 \omega_+ (r^2 + a^2)^3} \left(\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) \Re(\psi^{\text{int}}_{\omega lm}\psi^{\text{int*}}_{\omega lm, r_*})(1 + |\rho^{\text{up}}_{\omega lm}|^2) \right.$$

$$\left. + 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right) \Re(\rho^{\text{up}}_{\omega lm}\psi^{\text{int}}_{\omega lm}\psi^{\text{int}}_{\omega lm, r_*}) + (1 - |\rho^{\text{up}}_{\omega lm}|^2)\Re(\psi^{\text{int}}_{\omega lm}\psi^{\text{int*}}_{\omega lm, r_*}) \right), \quad \text{(B37)}$$

$$\bar{T}^{\mathcal{C}}_{(\omega lm)} = \hbar \frac{[S^{\omega}_{lm}(\theta)]^2 r^2}{32\pi^2 \omega_+ (r^2 + a^2)^5}$$

$$\times \left(\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) |\psi^{\text{int}}_{\omega lm}|^2 (1 + |\rho^{\text{up}}_{\omega lm}|^2) + 2\operatorname{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right) \Re(\rho^{\text{up}}_{\omega lm}(\psi^{\text{int}}_{\omega lm})^2) + (1 - |\rho^{\text{up}}_{\omega lm}|^2)|\psi^{\text{int}}_{\omega lm}|^2 \right), \quad \text{(B38)}$$

where we made the substitution $(\omega_+/\omega)|\tau^{\text{in}}_{\omega lm}|^2 = 1 - |\rho^{\text{up}}_{\omega lm}|^2$ [see Wronskian relations, Eq. (3.33)] to eliminate τ from the final results.

This provides the desired mode-sum expression for $\langle \bar{T}_{yy} \rangle^U_{\text{bare}}(x)$, see Eq. (B10). The translation from the trace-reversed to the original bare RSET then proceeds according to Eq. (B7) (although, this last stage requires also the RSET trace mode sum, which we have not addressed here).

We worked here in coordinates $(u, v, \theta, \tilde{\varphi})$, with an azimuthal coordinate $\tilde{\varphi}$ whose general form is given in Eq. (B1). Evidently, dependence on the choice of azimuthal coordinate $\tilde{\varphi}$ only appears (through $\tilde{\omega}$) in $\bar{T}^{\mathcal{A}}_{yy(\omega lm)}$, the part that does not vanish at the horizons (in particular, for $\tilde{\varphi} = \varphi$, the parameter $\tilde{\omega}$ is replaced by ω; and for $\tilde{\varphi} = \varphi_{\pm}$, the parameter $\tilde{\omega}$ is replaced by $\omega_{\pm}$, respectively).

2. The difference $T_{uu} - T_{vv}$

Note that, as mentioned, the terms $\bar{T}^{\mathcal{B}}_{(\omega lm)}$ and $\bar{T}^{\mathcal{C}}_{(\omega lm)}$ are shared by the uu and vv components. In addition, the difference between $\bar{T}^{\mathcal{A}}_{uu(\omega lm)}$ and $\bar{T}^{\mathcal{A}}_{vv(\omega lm)}$ is only in the sign of the three $\propto \tilde{\omega}$ terms. The difference between $\langle \bar{T}_{uu} \rangle^U_{\text{bare}}$ and $\langle \bar{T}_{vv} \rangle^U_{\text{bare}}$ [which also equals the difference between $\langle T_{uu} \rangle^U_{\text{bare}}$ and $\langle T_{vv} \rangle^U_{\text{bare}}$, since $g_{uu} = g_{vv}$ in coordinates $(u, v, \theta, \tilde{\varphi})$], therefore has a rather simple form,

$$\langle T_{uu} \rangle^U_{\text{bare}} - \langle T_{vv} \rangle^U_{\text{bare}}$$

$$= \hbar \sum_{l,m} \int_0^\infty d\omega \frac{[S^{\omega}_{lm}(\theta)]^2}{8\pi^2(r^2 + a^2)} \tilde{\omega}$$

$$\times \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) - 1 \right] (1 - |\rho^{\text{up}}_{\omega lm}|^2). \quad \text{(B39)}$$

Next we consider the renormalized version of Eq. (B39). Performing a coordinate transformation from $(u, v, \theta, \tilde{\varphi})$ to $(t, r_*, \theta, \tilde{\varphi})$ and then to $(t, r_*, \theta, \varphi)$ yields

$$(T_{uu} - T_{vv})_{(u,v,\theta,\tilde{\varphi})} = (-T_{r_*t})_{(t,r_*,\theta,\tilde{\varphi})}$$

$$= -(\tilde{\Omega}T_{\varphi r_*} + T_{r_*t})_{(t,r_*,\theta,\varphi)}. \quad \text{(B40)}$$

From Eq. (3.30) in Ref. [19], we see that the counterterms $T^{\text{div}}_{r_*t}$ and $T^{\text{div}}_{\varphi r_*}$ vanish. Thus, the renormalized difference $\langle T_{uu} \rangle^U_{\text{ren}} - \langle T_{vv} \rangle^U_{\text{ren}}$ is equal to the bare difference $\langle T_{uu} \rangle^U_{\text{bare}} - \langle T_{vv} \rangle^U_{\text{bare}}$, given on the rhs of Eq. (B39) [in coordinates $(u, v, \theta, \tilde{\varphi})$],

$$\langle T_{uu} \rangle^U_{\text{ren}} - \langle T_{vv} \rangle^U_{\text{ren}}$$

$$= \hbar \sum_{l,m} \int_0^\infty d\omega \frac{[S^{\omega}_{lm}(\theta)]^2}{8\pi^2(r^2 + a^2)} \tilde{\omega}$$

$$\times \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) - 1 \right] (1 - |\rho^{\text{up}}_{\omega lm}|^2). \quad \text{(B41)}$$

Evidently, $(r^2 + a^2)(\langle T_{uu} \rangle^U_{\text{ren}} - \langle T_{vv} \rangle^U_{\text{ren}})$ is independent of r (reflecting energy-momentum conservation).

The mode-sum expression for the Hawking outflux (per solid angle) may then be obtained from Eq. (B41) by choosing the Boyer-Lindquist azimuthal coordinate $\tilde{\varphi} = \varphi$ (that is, taking $\tilde{\omega} = \omega$) and multiplying by $(r^2 + a^2)$. This yields the expression

$$(r^2 + a^2)(\langle T_{uu} \rangle^U_{\text{ren}} - \langle T_{vv} \rangle^U_{\text{ren}})$$

$$= \hbar \sum_{l,m} \int_0^\infty d\omega \frac{[S^{\omega}_{lm}(\theta)]^2}{8\pi^2} \omega \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) - 1 \right] (1 - |\rho^{\text{up}}_{\omega lm}|^2).$$

$$\text{(B42)}$$

This is a well-known result [see, e.g., Eq. (5.5) in Ref. [11][35]].

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[35]Equation (5.5) in Ref. [11] gives a quantity denoted by $K_{U^-}(\theta)$, which coincides with $-(r^2 + a^2)(\langle T_{uu} \rangle^U_{\text{ren}} - \langle T_{vv} \rangle^U_{\text{ren}})$. For comparison with Eq. (B42), note that in Ref. [11] $\tilde{\omega}$ denotes ω_+ and $B^-_{\omega lm}$ is our $\tau^{\text{up}}_{\omega lm}$, and use the Wronskian relation relating $|\tau^{\text{up}}_{\omega lm}|$ with $|\rho^{\text{up}}_{\omega lm}|$ [see Eq. (3.33)].

3. Bare fluxes at the horizons

We are particularly interested in the behavior of the fluxes at the EH and IH of the BH. To this end, we take the limits $r \to r_\pm$ of the general r expressions for the fluxes $\langle \bar{T}_{yy} \rangle_{\mathrm{bare}}^U$ (where y is either u or v). Since Δ vanishes at $r = r_\pm$, $\bar{T}_{yy(\omega lm)}^{\mathcal{A}}$ is the only piece that contributes at the horizons [see Eq. (B34)].

Hereafter, a superscript $\pm$ denotes $r \to r_\pm$ along with the coordinate system in use—$(u, v, \theta, \varphi_\pm)$ at $r_\pm$, respectively (recalling that the regular azimuthal coordinate at $r_\pm$ is $\varphi_\pm$). Note that since the coordinate system (at both horizons) is chosen such that, in particular, $g_{uu} = g_{vv} = 0$ there, each flux component coincides with its trace-reversed counterpart at the horizons [see Eq. (B7)].[36]

We hereby introduce the summation/integration operator,

$$\hat{\sum}_\pm (\ldots) \equiv \hbar \int_0^\infty \sum_{l=0}^\infty \sum_{m=-l}^l \frac{[S_{lm}^\omega(\theta)]^2}{8\pi^2(r_\pm^2 + a^2)} (\ldots) \mathrm{d}\omega. \quad \text{(B43)}$$

4. The event horizon

At the EH, as prescribed in Eq. (3.16), the radial function $\psi_{\omega lm}^{\mathrm{int}}$ behaves as $e^{-i\omega_+ r_*}$. Also, in Eqs. (B36) and (B35) we now substitute $\tilde{\omega} = \omega_+$ (corresponding to the choice $\tilde{\Omega} = \Omega_+$ and hence $\tilde{\varphi} = \varphi_+$). This leads to a remarkable simplification, because now both combinations $(\psi_{\omega lm,r_*}^{\mathrm{int}})^2 + \tilde{\omega}^2(\psi_{\omega lm}^{\mathrm{int}})^2$ and $|\psi_{\omega lm,r_*}^{\mathrm{int}}|^2 + \tilde{\omega}^2|\psi_{\omega lm}^{\mathrm{int}}|^2 - 2\tilde{\omega}\omega_+$ vanish, and $|\psi_{\omega lm,r_*}^{\mathrm{int}}|^2 + \tilde{\omega}^2|\psi_{\omega lm}^{\mathrm{int}}|^2 + 2\tilde{\omega}\omega_+$ simplifies to $4\omega_+^2$. Equation (B36) then reduces to

$$\langle T_{vv}^+ \rangle_{\mathrm{bare}}^U = \hat{\sum}_+ \omega_+ \left(|\rho_{\omega lm}^{\mathrm{up}}|^2 \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) - 1 \right] + 1 \right) \tag{B44}$$

and Eq. (B35) to

$$\langle T_{uu}^+ \rangle_{\mathrm{bare}}^U = \hat{\sum}_+ \omega_+ \coth\left(\frac{\pi\omega_+}{\kappa_+}\right). \tag{B45}$$

5. The inner horizon

We now turn to the IH. At $r \to r_-$, the radial function $\psi_{\omega lm}^{\mathrm{int}}$ behaves asymptotically as given in Eq. (3.23), namely, $\psi_{\omega lm}^{\mathrm{int}} \simeq A_{\omega lm} e^{i\omega_- r_*} + B_{\omega lm} e^{-i\omega_- r_*}$.

---

[36]In the corresponding coordinate systems, g_{vv} and g_{uu} vanish on approaching the horizons as δr^2 (where $\delta r \equiv r - r_\pm$ denotes the distance to the corresponding horizon). Thus, for the $g_{yy}\langle \bar{T}_\mu^\mu \rangle_{\mathrm{bare}}^U$ term to vanish there, we assume that the trace diverges at a sufficiently slow rate as $\delta r \to 0$. (This is, indeed, the case in the RN counterpart—see Eq. (15) in Ref. [36], where the trace divergence rate is weaker than $1/\delta r$.)

We substitute $\tilde{\omega} = \omega_-$ in Eqs. (B36) and (B35). Then, using the Wronskian relation in Eq. (3.34), we find in the $r \to r_-$ limit

$$\omega_-^2 |\psi_{\omega lm}^{\mathrm{int}}|^2 + 2\omega_+\omega_- + |\psi_{\omega lm,r_*}^{\mathrm{int}}|^2 = 4\omega_-^2 |B_{\omega lm}|^2, \quad \text{(B46)}$$

$$\omega_-^2 |\psi_{\omega lm}^{\mathrm{int}}|^2 - 2\omega_+\omega_- + |\psi_{\omega lm,r_*}^{\mathrm{int}}|^2 = 4\omega_-^2 |A_{\omega lm}|^2, \quad \text{(B47)}$$

and

$$\omega_-^2 (\psi_{\omega lm}^{\mathrm{int}})^2 + (\psi_{\omega lm,r_*}^{\mathrm{int}})^2 = 4\omega_-^2 A_{\omega lm} B_{\omega lm}. \quad \text{(B48)}$$

For $\langle T_{vv}^- \rangle_{\mathrm{bare}}^U$, this yields

$$\langle T_{vv}^- \rangle_{\mathrm{bare}}^U = \hat{\sum}_- \frac{\omega_-^2}{\omega_+} \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right)(|A_{\omega lm}|^2 + |\rho_{\omega lm}^{\mathrm{up}}|^2 |B_{\omega lm}|^2) \right.$$
$$+ 2\,\mathrm{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho_{\omega lm}^{\mathrm{up}} A_{\omega lm} B_{\omega lm})$$
$$\left. + (1 - |\rho_{\omega lm}^{\mathrm{up}}|^2)|B_{\omega lm}|^2 \right]. \tag{B49}$$

Turning now to $\langle T_{uu}^- \rangle_{\mathrm{bare}}^U$, we note again that Eq. (B30) differs from Eq. (B31) by merely taking $\tilde{\omega} \mapsto -\tilde{\omega}$. This amounts here to taking $\omega_- \mapsto -\omega_-$, which in turn interchanges Eq. (B46) and Eq. (B47). Consequently, $\langle T_{uu}^- \rangle_{\mathrm{bare}}^U$ is obtained by interchanging $A_{\omega lm}$ and $B_{\omega lm}$ in Eq. (B49),

$$\langle T_{uu}^- \rangle_{\mathrm{bare}}^U = \hat{\sum}_- \frac{\omega_-^2}{\omega_+} \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right)(|B_{\omega lm}|^2 \right.$$
$$+ |\rho_{\omega lm}^{\mathrm{up}}|^2 |A_{\omega lm}|^2)$$
$$+ 2\,\mathrm{cosech}\left(\frac{\pi\omega_+}{\kappa_+}\right)\Re(\rho_{\omega lm}^{\mathrm{up}} A_{\omega lm} B_{\omega lm})$$
$$\left. + (1 - |\rho_{\omega lm}^{\mathrm{up}}|^2)|A_{\omega lm}|^2 \right]. \tag{B50}$$

Note how, through relations (B46)–(B48), all oscillatory factors (of the form $e^{\pm i\omega_- r_*}$, as appear in the asymptotic behavior of $\psi_{\omega lm}^{\mathrm{int}}$ at the IH) are canceled out—and, as a consequence, the individual mode contribution to the flux components have a well-defined limiting value at the IH, which depends only on the scattering parameters $\rho_{\omega lm}^{\mathrm{up}}$, $A_{\omega lm}$, and $B_{\omega lm}$. In this respect, the flux components are simpler than other $T_{\alpha\beta}$ components at the IH limit.

Equation (B41) for the renormalized difference also applies at the IH [in coordinates $(u, v, \theta, \varphi_-)$], yielding

$$\langle T_{uu}^- \rangle_{\mathrm{ren}}^U - \langle T_{vv}^- \rangle_{\mathrm{ren}}^U$$
$$= \hat{\sum}_- \omega_- \left[\coth\left(\frac{\pi\omega_+}{\kappa_+}\right) - 1 \right](1 - |\rho_{\omega lm}^{\mathrm{up}}|^2). \tag{B51}$$

It may be interesting to explore how this quantity behaves in the limit of extremal Kerr, $a \to M$. In this limit

$\kappa_+ \to 0^+$ and hence $\coth(\pi\omega_+/\kappa_+)$ reduces to $\mathrm{sign}(\omega_+)$, leaving only a contribution from negative ω_+ in the integrand. Therefore, recalling that (for positive ω) a negative ω_+ requires $m > 0$ and $\omega < m\Omega_+$, we obtain

$$
\langle T^-_{uu} \rangle_{\mathrm{ren}} - \langle T^-_{vv} \rangle_{\mathrm{ren}}
$$

$$
= \frac{\hbar}{8\pi^2 M^2} \sum_{l=1}^{\infty} \sum_{m=1}^{l} \int_0^{m/2M} [S^\omega_{lm}(\theta)]^2 \omega_- (|\rho^{\mathrm{up}}_{\omega lm}|^2 - 1) \mathrm{d}\omega,
$$

$$
(a \to M), \tag{B52}
$$

where we have used the fact that, in the extremal limit, $\Omega_\pm = 1/2M$.

Clearly, the rhs of Eq. (B52) vanishes on the pole, where only $m = 0$ modes could contribute, so hereafter we concentrate on $\theta \neq 0, \pi$.

Recalling that in the superradiance band ($0 < \omega < m\Omega_+$ for $m > 0$) we have $\omega_+ < 0$ and $|\rho^{\mathrm{up}}_{\omega lm}| > 1$ (see Sec. III D), and that in the extremal limit $\omega_- = \omega_+$, we readily see that in the last equation the integrand is negative throughout the domain of integration. Thus, $\langle T^-_{uu} \rangle^U_{\mathrm{ren}} - \langle T^-_{vv} \rangle^U_{\mathrm{ren}}$ is strictly negative (off the pole) in the extremal limit (see also footnote 23).

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