Consistency between causality and complementarity guaranteed by the Robertson inequality in quantum field theory

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It has long been debated whether gravity should be quantized or not. Recently, the authors in [Sci. Rep. 6, 22777 (2016); Proc. Natl. Acad. Sci. U.S.A. **106**, 3035 (2009)] discussed the inconsistency between causality and complementarity in a Gedankenexperiment involving the quantum superposition of massive/ charged bodies, and Belenchia *et al.* [Phys. Rev. D **98**, 126009 (2018); Int. J. Mod. Phys. D **28**, 1943001 (2019)] resolved the inconsistency by requiring the quantum radiation and vacuum fluctuations of gravitational/electromagnetic field. Stimulated by their works, we reanalyze the consistency between the two physical properties, causality and complementarity, according to the quantum field theory. In this analysis, we consider a Gedankenexperiment inspired by [Sci. Rep. **6**, 22777 (2016); Proc. Natl. Acad. Sci. U.S.A. **106**, 3035 (2009); Phys. Rev. D **98**, 126009 (2018); Int. J. Mod. Phys. D **28**, 1943001 (2019)], in which two charged particles coupled with a photon field are in a superposition of two trajectories. First, we observe that causality is satisfied by the retarded propagation of the photon field. Next, by introducing an inequality between visibility and which-path information, we show that the quantum radiation and vacuum fluctuations of the photon field ensure complementarity. We further find that the Robertson inequality associated with the photon field leads to the consistency between causality and complementarity in our Gedankenexperiment. Finally, we mention that a similar feature appears in the quantum field of gravity.

I. INTRODUCTION

The unification of quantum mechanics and general relativity is a fundamental unsolved problem in theoretical physics. Despite all the efforts that have been made, the exact theory of quantum gravity has not yet been completed. Moreover, we do not even know whether gravity really follows the principle of quantum mechanics or not [1–3]. Recently, testing the quantum nature of gravity has attracted significant interest in theoretical physics, stimulated by the proposal by Bose *et al.* [4], and Marlleto and Vedral [5]. The Bose-Marlleto-Vedral (BMV) proposal suggests that quantum entanglement due to the Newtonian potential between two masses can be an evidence of quantum gravity, which can be tested by a tabletop experiment (see also [6]). Inspired by their works [4,5], Newtonian entanglement was evaluated in experimental proposals for matter-wave interferometry [7,8], mechanical oscillators [9,10], optomechanical systems [11–14], hybrid systems [15–18], etc. However, there is room for arguments to understand what the detection of the Newtonian entanglement means, e.g.,

how the Newtonian entanglement is related to the quantum field theory of gravity and gravitons [19–24].

We revisit the entanglement generation in the BMV proposal in the framework of the quantum field theory by focusing on a paradox in a Gedankenexperiment, which was previously analyzed in Refs. [24–28]. In the Gedankenexperiment (see Fig. 1), Alice prepares a particle in a superposition of spatially localized states separated by a distance L and starts to recombine her particle at a time $t = t_0$ to observe its interference. The recombination process is performed during a time T_A. Bob, who is at a distance D $(\gg L)$ from Alice, can choose whether to release a particle at the time $t = t_0$. When Bob released his particle, after a time T_B, he measures his particle to determine the strength of the Newtonian/Coulomb force induced by Alice's particle and gains information about which path her particle took. The actions of Alice and Bob after the time $t = t_0$ occur in spacelike separated regions $(D > T_A$ and $D > T_B)$. If Bob acquires any which-path information from his measurement, the state of his particle must be entangled with Alice's particle. This leads to the correlation between Alice and Bob. Then, because of the correlation due to the entanglement, Alice's particle cannot be in a perfect coherent superposition when Bob measures

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his released particle. This is the result of complementarity. However, when Bob does not release his particle, Alice's particle can maintain perfect coherence. Bob's choice affects the coherence of Alice's particle. Since Alice and Bob perform their actions in a spacelike separated region, it is impossible for Bob's measurement to have any effect on Alice's result owing to causality. This leads to the apparent violation of causality or complementarity. This paradox was first discussed in Refs. [25,26], and the authors in Refs. [24,27,28] claimed that the paradox can be resolved by Alice's limitation in maintaining coherence due to the emission of entangling gravitons/photons during the process of recombination of her particle and Bob's limitation in acquiring which-path information due to the vacuum fluctuations of gravitational/electromagnetic field (for a brief review, see Sec. II). The most important implication made by the above mentioned authors is that the existence of a quantum gravitational field and gravitons may be necessary to solve the paradox.

In this study, we reanalyze the paradox rigorously by estimating the feasibility of the measurements by Alice and Bob. We use the theoretical model developed in [29], in which we investigated entanglement generation between a pair of charged particles in a superposition of spatially localized states based on quantum electrodynamics. We demonstrate that the causality in our model is automatically satisfied by the retarded propagation of the photon field. Furthermore, by estimating the visibility measured by Alice and the distinguishability in Bob's measurement, we show that the complementarity in our model is protected by the radiation and vacuum fluctuations of the photon field. Additionally, we prove that the complementarity is guaranteed by the Robertson inequality for the photon field, which reflects the noncommutativity of a quantized field. From the analogy between electromagnetic dynamics and general relativity, we mention that a similar feature may appear in quantum gravitational fields.

The remainder of this paper is organized as follows. In Sec. II, we briefly review the paradox in the Gedankenexperiment by following Refs. [27,28]. In Sec. III, we demonstrate that causality is not violated. In Sec. IV, we show that complementarity is satisfied for two charged particles coupled with a photon field. Section V is devoted to the summary and conclusion. In Appendix A, we derive Eqs. (11) and (20). In Appendix B, we prove the inequality in visibility and distinguishability. In Appendix C, we present the proof of the statement in (35). Throughout this study, we used the natural units with $c = \hbar = 1$.

II. A BRIEF REVIEW OF THE GEDANKENEXPERIMENT

In this section, we review the paradox of the Gedankenexperiment addressed in Refs. [24–28]. As is shown in Fig. 1, Alice and Bob are separated by a distance

D. Their particles interact via the Newtonian/Coulomb potential. Alice's particle with a spin is in a superposition of spatially localized states separated by a distance L, which was prepared through a Stern-Gerlach apparatus, and an interference experiment is performed during a time T_A. In contrast, Bob chooses whether his particle is released or trapped at a time $t = t_0$. If Bob releases his particle, it moves under the gravitational/electromagnetic potential created by Alice's particle. After a time T_B, he measures the position of his particle.

Assuming the regimes $D > T_A$ and $D > T_B$, in which Alice and Bob perform their actions in spacelike separated regions, we can consider the following two incompatible arguments.

(i) If causality holds, Alice can observe the interference pattern of her particle regardless of whether Bob measures his particle.

(ii) If complementarity holds, Bob's measurement of his particle should lead to the decoherence of Alice's particle.

Arguments (i) and (ii) seem to contradict each other, and thus the paradox appears.

The authors in Refs. [24,27,28] claimed that this paradox is resolved by the quantum radiation of gravitons/photons emitted by massive/charged particles and the vacuum fluctuations of gravitational/electromagnetic fields. The quantum radiation from Alice's particle causes the decoherence of her particle, and then the interference experiment fails. In other words, this entangling radiation limits the maintenance of coherence in Alice's experiment. The presence of the vacuum fluctuations limits the ability to obtain the which-path information of Alice's particle for Bob's measurement. The two effects, the decoherence due to quantum radiation and the limitation of which-path information due to vacuum fluctuations are key to resolving this paradox [24,27,28].

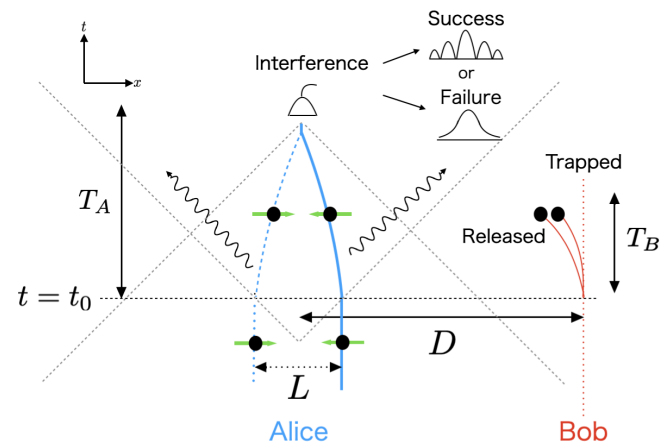In the following two sections, we reanalyze the consistency between causality and complementarity by

FIG. 1. Setup for the Gedankenexperiment introduced by [27,28].

assuming a situation similar to that in Fig. 1. This is an extension of a previous study [29], which investigated the effect of vacuum fluctuations of a photon field on the electromagnetic version of the BMV proposal. This work is based on the quantum electromagnetic dynamics; however, our result can be reinterpreted for the quantized gravitational field, as discussed in Sec. V.

III. CONSISTENCY OF CAUSALITY

In this section, we show that Bob's particle does not affect Alice's particle because of the causality satisfied for $D > T_A$ and $D > T_B$. We first introduce the model of two charged particles (Alice's particle and Bob's particle) coupled with a photon field developed in Ref. [29]. The total Hamiltonian of our system is composed of the local Hamiltonians of each charged particle $\hat{H}_A$ and $\hat{H}_B$, the free Hamiltonian of the photon field $\hat{H}_{ph}$, and the interaction term $\hat{V}$ as

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_{ph} + \hat{V},$$
$$\hat{V} = \int d^3x (\hat{J}_A^\mu(\boldsymbol{x}) + \hat{J}_B^\mu(\boldsymbol{x}))\hat{A}^\mu(\boldsymbol{x}), \qquad (1)$$

where $\hat{J}_A^\mu$ and $\hat{J}_B^\mu$ are the current operators of each particle coupled with the photon field operator $\hat{A}^\mu$. We consider the following initial condition

$$|\Psi(0)\rangle = \frac{1}{2}|C\rangle_A (|\uparrow\rangle_A + |\downarrow\rangle_A)|C\rangle_B (|\uparrow\rangle_B + |\downarrow\rangle_B)|\alpha\rangle_{ph}, \quad (2)$$

where $|\uparrow\rangle_j (|\downarrow\rangle_j)$ are the spin degrees of freedom of the charged particle j with $j = A, B$, and $|C\rangle_A$ and $|C\rangle_B$ denote the localized particle wave function of A and B, respectively. The photon field is in a coherent state $|\alpha\rangle_{ph}$ with $|\alpha\rangle_{ph} = \hat{D}(\alpha)|0\rangle_{ph}$. $|0\rangle_{ph}$ is the vacuum state satisfying $\hat{a}_\mu(\boldsymbol{k})|0\rangle_{ph} = 0$ for annihilation operator of the photon field $\hat{a}_\mu(\boldsymbol{k})$, and $\hat{D}(\alpha)$ is the unitary operator called a displacement operator defined as

$$\hat{D}(\alpha) = \exp\left[\int d^3k(\alpha^\mu(\boldsymbol{k})\hat{a}_\mu^\dagger(\boldsymbol{k}) - \text{H.c.})\right], \qquad (3)$$

where the complex function $\alpha^\mu(\boldsymbol{k})$ characterizes the amplitude and phase of initial photon field. The form of the complex function $\alpha^\mu(\boldsymbol{k})$ is restricted by the auxiliary condition in the BRST formalism [29]. The coherent state $|\alpha\rangle_{ph}$ is interpreted as a state in which there is a mode of the electromagnetic field following Gauss's law due to the presence of charged particles (See Appendix A of Ref. [29]). For $t < 0$, the charged particles A and B are localized around each trajectory, whose states are described by $|C\rangle_A$ and $|C\rangle_B$, respectively. Then the photon field for $t < 0$ is not in a quantum superposition and behaves
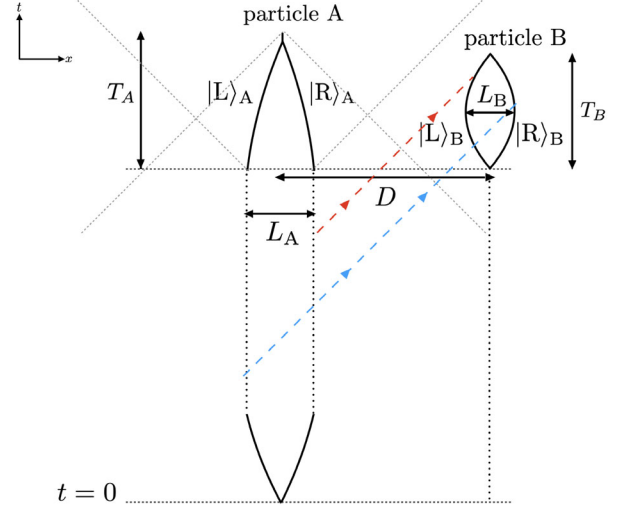

FIG. 2. Configuration of our model. L_A and L_B are each separation of a spatial superposition of particles A and B, and D is a distance between Alice's system and Bob's system. T_A is a timescale recombining particle A, and particle B in Bob's system is superposed during a time T_B. Here, we assume the regimes $D > T_A$ and $D > T_B$. Particle A takes the right or left path $|R\rangle_A$ ($|L\rangle_A$) and induces the retarded photon field along each path (as shown in the dashed red or blue line). The retarded field affects particle B moving the left or right path $|L\rangle_B$ and $|R\rangle_B$.

classically. In this case the states of A and B are uncorrelated with the photon field. Now, we assume that each particle is manipulated through an inhomogeneous magnetic field ($|C\rangle_j|\uparrow\rangle_j \to |\psi_L\rangle_j|\uparrow\rangle_j, |C\rangle_j|\downarrow\rangle_j \to |\psi_R\rangle_j|\downarrow\rangle_j$) to create spatially superposed states with $|\psi_L\rangle_j|\uparrow\rangle_j$, and $|\psi_R\rangle_j|\downarrow\rangle_j$, which is understood as the Stern–Gerlach effect discussed in [4,27]. In our Gedankenexperiment shown in Fig. 2, each particle is spatially superposed at different times. In the following, $|C\rangle_j|\uparrow\rangle_j$ and $|C\rangle_j|\downarrow\rangle_j$ are represented by $|L\rangle_j$ and $|R\rangle_j$ with $j = A, B$ for simplicity. The initial state is rewritten as

$$|\Psi(0)\rangle = \frac{1}{2}(|L\rangle_A + |R\rangle_A)(|L\rangle_B + |R\rangle_B)|\alpha\rangle_{ph}, \quad (4)$$

We note that $|R\rangle_A$ ($|R\rangle_B$) and $|L\rangle_A$ ($|L\rangle_B$) are the states of wave packets localized around classical trajectories. After each particle has passed through an inhomogeneous magnetic field, the states $|L\rangle_j$ and $|R\rangle_j$ are regarded as the localized states of the particle $j = A, B$ around the left trajectory and the right trajectory shown in Fig. 2, respectively. We assume that the current operators $\hat{J}_{iI}^\mu(x) = e^{i\hat{H}_0 t}\hat{J}_i^\mu(0, \boldsymbol{x})e^{-i\hat{H}_0 t}$ in the interaction picture with respect to $\hat{H}_0 = \hat{H}_A + \hat{H}_B + \hat{H}_{ph}$ are approximated using the classical currents as

$$\hat{J}_{AI}^\mu(x)|P\rangle_A \approx J_{AP}^\mu(x)|P\rangle_A, \quad \hat{J}_{BI}^\mu(x)|Q\rangle_B \approx J_{BQ}^\mu(x)|Q\rangle_B, \quad (5)$$

$$J_{\mathrm{AP}}^{\mu}(x) = e_{\mathrm{A}} \int d\tau \frac{dX_{\mathrm{AP}}^{\mu}}{d\tau} \delta^{(4)}(x - X_{\mathrm{AP}}(\tau)),$$

$$J_{\mathrm{BQ}}^{\mu}(x) = e_{\mathrm{B}} \int d\tau \frac{dX_{\mathrm{BQ}}^{\mu}}{d\tau} \delta^{(4)}(x - X_{\mathrm{BQ}}(\tau)), \tag{6}$$

where $X_{\mathrm{AP}}^{\mu}(\tau)$ and $X_{\mathrm{BQ}}^{\mu}(\tau)$ with $\mathrm{P}, \mathrm{Q} = \mathrm{R}, \mathrm{L}$ represent the trajectories of each particle with coupling constants e_{A} and e_{B}. Note that these approximations are valid for the following two assumptions [29,30]: the first assumption is that the de Brogile wavelength is smaller than the wave-packet width of particle. The second assumption is that the Compton wavelength λ_{C} of the charged particle is much shorter than the wavelength of photon field λ_{ph} (for example, the wavelength of photon field emitted from charged particle) ($\lambda_{\mathrm{C}} \ll \lambda_{\mathrm{ph}}$). The initial state evolves as follows:

$$\begin{aligned}
|\Psi(T)\rangle &= \exp[-i\hat{H}T]|\Psi(0)\rangle \\
&= e^{-i\hat{H}_0 T} \mathrm{T} \exp\left[-i \int_0^T dt \hat{V}_{\mathrm{I}}(t)\right] |\Psi(0)\rangle \\
&\approx e^{-i\hat{H}_0 T} \frac{1}{2} \sum_{\mathrm{P,Q=R,L}} |\mathrm{P}\rangle_{\mathrm{A}} |\mathrm{Q}\rangle_{\mathrm{B}} \hat{U}_{\mathrm{PQ}} |\alpha\rangle_{\mathrm{ph}} \\
&= \frac{1}{2} \sum_{\mathrm{P,Q=R,L}} |\mathrm{P_f}\rangle_{\mathrm{A}} |\mathrm{Q_f}\rangle_{\mathrm{B}} e^{-i\hat{H}_{\mathrm{ph}} T} \hat{U}_{\mathrm{PQ}} |\alpha\rangle_{\mathrm{ph}}, \tag{7}
\end{aligned}$$

where $T (> T_{\mathrm{A}})$ is the total timescale while particle A is spatially superposed. We used the approximations given by (5) in the third line. $|\mathrm{P_f}\rangle_{\mathrm{A}} = e^{-i\hat{H}_{\mathrm{A}} T} |\mathrm{P}\rangle_{\mathrm{A}}$ and $|\mathrm{Q_f}\rangle_{\mathrm{B}} = e^{-i\hat{H}_{\mathrm{B}} T} |\mathrm{Q}\rangle_{\mathrm{B}}$ with $\mathrm{P}, \mathrm{Q} = \mathrm{R}, \mathrm{L}$ are the states of charged particles A and B, which moved along the trajectories P and Q, respectively. The unitary operator $\hat{U}_{\mathrm{PQ}}$ is given by

$$\hat{U}_{\mathrm{PQ}} = \mathrm{T} \exp\left[-i \int_0^T dt \int d^3x (J_{\mathrm{AP}}^{\mu} + J_{\mathrm{BQ}}^{\mu}) \hat{A}_{\mu}^{\mathrm{I}}(x)\right], \tag{8}$$

where T denotes the time ordering, and $\hat{A}_{\mu}^{\mathrm{I}}$ is the photon field operator in the interaction picture. For convenience, we rewrite the state given in (7) as

$$\begin{aligned}
|\Psi(T)\rangle &= \frac{1}{2} \sum_{\mathrm{P,Q=R,L}} |\mathrm{P_f}\rangle_{\mathrm{A}} |\mathrm{Q_f}\rangle_{\mathrm{B}} e^{-i\hat{H}_{\mathrm{ph}} T} \hat{U}_{\mathrm{PQ}} |\alpha\rangle_{\mathrm{ph}} \\
&= \frac{1}{\sqrt{2}} |\mathrm{R_f}\rangle_{\mathrm{A}} |\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}} + \frac{1}{\sqrt{2}} |\mathrm{L_f}\rangle_{\mathrm{A}} |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}, \tag{9}
\end{aligned}$$

where we defined

$$|\Omega_{\mathrm{P}}\rangle_{\mathrm{B,ph}} = \frac{1}{\sqrt{2}} \sum_{\mathrm{Q=R,L}} |\mathrm{Q_f}\rangle_{\mathrm{B}} e^{-i\hat{H}_{\mathrm{ph}} T} \hat{U}_{\mathrm{PQ}} |\alpha\rangle_{\mathrm{ph}}. \tag{10}$$

The vector $|\Omega_{\mathrm{P}}\rangle_{\mathrm{B,ph}}$ describes the composite state of particle B and the photon field when particle A moves along the trajectory P. The quantum state of particle A is obtained by tracing out the degrees of freedom of particle B and the photon field:

$$\begin{aligned}
\rho_{\mathrm{A}} &= \mathrm{Tr}_{\mathrm{B,ph}}[|\Psi(T)\rangle\langle\Psi(T)|] \\
&= \frac{1}{2} \begin{pmatrix} 1 & \frac{1}{2} e^{-\Gamma_{\mathrm{A}} + i\Phi_{\mathrm{A}}} \left(e^{-i \int d^4 x (J_{\mathrm{AR}}^{\mu} - J_{\mathrm{AL}}^{\mu}) A_{\mathrm{BR}\mu}} + e^{-i \int d^4 x (J_{\mathrm{AR}}^{\mu} - J_{\mathrm{AL}}^{\mu}) A_{\mathrm{BL}\mu}} \right) \\ * & 1 \end{pmatrix}, \tag{11}
\end{aligned}$$

where we used the basis $\{|\mathrm{R_f}\rangle_{\mathrm{A}}, |\mathrm{L_f}\rangle_{\mathrm{A}}\}$ to represent the density operator, and $*$ is the complex conjugate of the (R, L) component. $A_{\mathrm{BQ}}^{\mu}(\mathrm{Q} = \mathrm{R}, \mathrm{L})$ is the retarded photon field caused by charged particle B,

$$A_{\mathrm{BQ}}^{\mu}(x) = \int d^4 y G^{\mathrm{r},\mu}{}_{\nu}(x, y) J_{\mathrm{BQ}}^{\nu}(y), \tag{12}$$

with the retarded Green's function,

$$G_{\mu\nu}^{\mathrm{r}}(x, y) = -i[\hat{A}_{\mu}^{\mathrm{I}}(x), \hat{A}_{\nu}^{\mathrm{I}}(y)] \theta(x^0 - y^0). \tag{13}$$

The quantities Γ_{A} and Φ_{A} are

$$\begin{aligned}
\Gamma_{\mathrm{A}} = \frac{1}{4} \int d^4 x d^4 y (J_{\mathrm{AR}}^{\mu}(x) - J_{\mathrm{AL}}^{\mu}(x))(J_{\mathrm{AR}}^{\mu}(y) \\
- J_{\mathrm{AL}}^{\mu}(y)) \langle \{\hat{A}_{\mu}^{\mathrm{I}}(x), \hat{A}_{\mu}^{\mathrm{I}}(y)\} \rangle, \tag{14}
\end{aligned}$$

$$\begin{aligned}
\Phi_{\mathrm{A}} = \int d^4 x (J_{\mathrm{AR}}^{\mu}(x) - J_{\mathrm{AL}}^{\mu}(x)) A_{\mu}(x) - \frac{1}{2} \int d^4 x d^4 y (J_{\mathrm{AR}}^{\mu}(x) \\
- J_{\mathrm{AL}}^{\mu}(x))(J_{\mathrm{AR}}^{\nu}(y) + J_{\mathrm{AL}}^{\nu}(y)) G_{\mu\nu}^{\mathrm{r}}(x, y), \tag{15}
\end{aligned}$$

where $\langle \cdot \rangle$ denotes the vacuum expectation value and $A_{\mu}(x)$ is defined in Appendix A. The derivation of the density operator ρ_{A} is presented in Appendix A. The quantity Γ_{A} characterizes the decoherence effect due to the radiation of the on-shell photon emitted by particle A [24,29]. The result (11) with the retarded photon field A_{BQ}^{μ} of particle B implies that the effect of particle B can propagate to Alice's system. However, in the spacelike case $D > T_{\mathrm{A}}$ and $D > T_{\mathrm{B}}$ (see Fig. 2), the photon field induced by particle B does not reach particle A, i.e., $A_{\mathrm{BQ}}^{\mu}(x) = 0$. Thus, the density operator (11) becomes

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & e^{-\Gamma_A + i\Phi_A} \\ e^{-\Gamma_A - i\Phi_A} & 1 \end{pmatrix}. \tag{16}$$

This result indicates that the process of charged particle B during the time T_B does not affect the interference experiment on charged particle A by causality. Note that, given the law of charge conservation, we also have to consider the contribution from charged particle B before the time T_B. Even by considering this, we can see that the density operator ρ_A does not depend on influences from spacelike separated regions. In the derivation of the above equations, for simplicity, we only discussed the contribution from particle B during the time T_B. In the next section, we confirm that the paradox does not appear from the viewpoints of *visibility* and *distinguishability*.

IV. CONSISTENCY OF COMPLEMENTARITY

In this section, we introduce the visibility $\mathcal{V}_A$ of charged particle A and the distinguishability $\mathcal{D}_B$ which quantifies the which-path information of particle A acquired through charged particle B. These two quantities are useful for expressing complementarity. Additionally, we discuss the relationship with the Robertson inequality in the last subsection. According to Refs. [31,32], the visibility $\mathcal{V}_A$ and the distinguishability $\mathcal{D}_B$ satisfy the inequality,

$$\mathcal{V}_A^2 + \mathcal{D}_B^2 \le 1. \tag{17}$$

This inequality expresses the complementarity: if the distinguishability is unity, $D_B = 1$, the visibility $\mathcal{V}_A$ vanishes, and if the visibility is unity, $\mathcal{V}_A = 1$, the distinguishability $\mathcal{D}_B$ vanishes. In Appendix B, we present a simple proof of the above inequality by using the definitions of visibility and distinguishability described in the next subsection A.

A. Visibility and distinguishability

We introduce the visibility $\mathcal{V}_A$ of charged particle A defined as

$$\mathcal{V}_A = 2|_A\langle L_f|\rho_A|R_f\rangle_A|, \tag{18}$$

where ρ_A is the reduced density operator of particle A given in Eq. (11). The visibility $\mathcal{V}_A$ describes the extent to which the coherence of charged particle A remains when Alice performs an interference experiment. Using Eq. (11), we have

$$\mathcal{V}_A = e^{-\Gamma_A}\left|\cos\left(\frac{\Phi_{AB}}{2}\right)\right|, \tag{19}$$

where $\Phi_{AB} = \int d^4x(J_{AR}^\mu - J_{AL}^\mu)\Delta A_{B\mu}$ with $\Delta A_B^\mu = A_{BR}^\mu - A_{BL}^\mu$. For the case $D > T_A$ and $D > T_B$, the retarded photon field induced by charged particle B during time T_B is

zero ($A_{BQ}^\mu = 0$, with Q = R, L). Then, the visibility is simply written as $\mathcal{V}_A = e^{-\Gamma_A}$ with Γ_A, which quantifies the decoherence effect due to the radiation of photon field emitted from particle A.

Next, we introduce the distinguishability computed from the state of charged particle B. Tracing over particle A and the photon field from the state given in (7), we obtain the state of particle B:

$$\begin{aligned} \mathrm{Tr}_{A,ph}[|\Psi(T)\rangle\langle\Psi(T)|] &= \frac{1}{2}\mathrm{Tr}_{ph}[|\Omega_R\rangle_{B,ph}\langle\Omega_R|] \\ &\quad + \frac{1}{2}\mathrm{Tr}_{ph}[|\Omega_L\rangle_{B,ph}\langle\Omega_L|] \\ &= \frac{1}{2}\rho_{BR} + \frac{1}{2}\rho_{BL}, \end{aligned} \tag{20}$$

where we defined $\rho_{BP} = \mathrm{Tr}_{ph}[|\Omega_P\rangle_{B,ph}\langle\Omega_P|]$ with P = R, L in the second line. The density operator ρ_{BP} describes the state of particle B when particle A moves along the trajectory P. The distinguishability $\mathcal{D}_B$ which characterizes how Bob can distinguish the trajectory of particle A from the state of particle B is defined as

$$\mathcal{D}_B = \frac{1}{2}\mathrm{Tr}_B|\rho_{BR} - \rho_{BL}|, \tag{21}$$

where $\mathrm{Tr}|\hat{O}| = \sum_i |\lambda_i|$ is given by the eigenvalues λ_i of a Hermitian operator $\hat{O}$. The distinguishability is nothing but the trace distance between the density operators ρ_{BR} and ρ_{BL} [33]. If the distinguishability vanishes, $\mathcal{D}_B = 0$, and the two density operators ρ_{BR} and ρ_{BL} are identical. This means that Bob cannot know which trajectory particle A has taken from the state of particle B. However, if $\mathcal{D}_B = 1$, the density operators ρ_{BR} and ρ_{BL} are orthogonal to each other ($\rho_{BR}\rho_{BL} = 0$). Then, by measuring the state of particle B, Bob can guess which trajectory particle A has passed through. In this sense, the distinguishability $\mathcal{D}_B$ quantifies the amount of which path information of particle A. The general property of the trace distance is presented in [33], and the meaning of the distinguishability mentioned above was discussed in [32].

Using the expression for the density operator ρ_{BP} presented in Appendix A, we obtain the eigenvalues of the density operator $\rho_{BR} - \rho_{BL}$ as

$$\begin{aligned} \lambda_\pm &= \pm\frac{1}{2}|e^{-\Gamma_B + i\Phi_B - i\int d^4x(J_{BR}^\mu - J_{BL}^\mu)A_{R\mu}} \\ &\quad - e^{-\Gamma_B + i\Phi_B - i\int d^4x(J_{BR}^\mu - J_{BL}^\mu)A_{L\mu}}| \\ &= \pm e^{-\Gamma_B}\left|\sin\left(\frac{1}{2}\int d^4x(J_{BR}^\mu - J_{BL}^\mu)\Delta A_{A\mu}\right)\right|, \end{aligned} \tag{22}$$

where $\Delta A_A^\mu = A_{AR}^\mu - A_{AL}^\mu$ with

$$A^\mu_{AP}(x) = \int d^4 y G^{r,\mu}{}_\nu(x,y) J^\nu_{AP}(y), \qquad (23)$$

and Γ_B and Φ_B are

$$\Gamma_B = \frac{1}{4} \int d^4 x d^4 y (J^\mu_{BR}(x) - J^\mu_{BL}(x))(J^\mu_{BR}(y)$$
$$- J^\mu_{BL}(y))\langle\{\hat{A}^I_\mu(x), \hat{A}^I_\mu(y)\}\rangle, \qquad (24)$$

$$\Phi_B = \int d^4 x (J^\mu_{BR}(x) - J^\mu_{BL}(x)) A_\mu(x) - \frac{1}{2} \int d^4 x d^4 y (J^\mu_{BR}(x)$$
$$- J^\mu_{BL}(x))(J^\nu_{BR}(y) + J^\nu_{BL}(y)) G^r_{\mu\nu}(x,y). \qquad (25)$$

The quantity Γ_B characterizes the dephasing effect induced by the vacuum fluctuations of the photon field around particle B (see subsection B or Refs. [29,34,35]). The distinguishability is computed as

$$\mathcal{D}_B = \frac{1}{2}(|\lambda_+| + |\lambda_-|) = e^{-\Gamma_B}\left|\sin\left(\frac{\Phi_{BA}}{2}\right)\right|, \qquad (26)$$

where $\Phi_{BA} = \int d^4 x (J^\mu_{BR} - J^\mu_{BL})\Delta A_{A\mu}$, and therefore, the inequality (17) is expressed as

$$\mathcal{V}^2_A + \mathcal{D}^2_B = e^{-2\Gamma_A}\cos^2\left(\frac{\Phi_{AB}}{2}\right) + e^{-2\Gamma_B}\sin^2\left(\frac{\Phi_{BA}}{2}\right) \le 1. \qquad (27)$$

For the case $D > T_A$ and $D > T_B$, the retarded photon field of particle B vanishes ($A^\mu_{BP} = 0$), which leads to $\Phi_{AB} = 0$, and we have

$$\mathcal{V}^2_A + \mathcal{D}^2_B = e^{-2\Gamma_A} + e^{-2\Gamma_B}\sin^2\left(\frac{\Phi_{BA}}{2}\right) \le 1. \qquad (28)$$

This inequality is consistent with the existence of the quantum radiation emitted from particle A ($\Gamma_A > 0$) and the

vacuum fluctuations of the photon field around particle B ($\Gamma_B > 0$) when the causality holds. If we can remove the two effects ($\Gamma_A = \Gamma_B = 0$), this inequality would be violated as long as the retarded photon field of particle A does not vanish ($A^\mu_{AP} \ne 0$ and then $\Phi_{BA} \ne 0$). Hence, if the two effects vanish, then complementarity is violated, and the paradox would appear. In the following subsection, we will discuss that the inequality (28) is never violated by the Robertson inequality associated with the photon field.

B. Relationship with uncertainty relation

In Refs. [29,34,35], the quantity Γ_i ($i = A, B$) was evaluated as the dephasing effect due to the vacuum fluctuations of the photon field,

$$\langle 0|e^{i\hat{\phi}_i}|0\rangle = e^{-\langle 0|\hat{\phi}^2_i|0\rangle/2} = e^{-\Gamma_i}, \qquad (29)$$

with the operators $\hat{\phi}_A$ and $\hat{\phi}_B$ defined by

$$\hat{\phi}_A = \int d^4 x (J^\mu_{AR}(x) - J^\mu_{AL}(x))\hat{A}^I_\mu(x),$$
$$\hat{\phi}_B = \int d^4 x (J^\mu_{BR}(x) - J^\mu_{BL}(x))\hat{A}^I_\mu(x), \qquad (30)$$

where $\hat{A}^I_\mu$ is the photon field operator in the interaction picture, and J^μ_{AP} and J^μ_{BQ} are the charged currents of each particle. The operators $\hat{\phi}_A$ and $\hat{\phi}_B$ describe the phase shifts due to the quantum fluctuations of the photon field. The variances of $\hat{\phi}_A$ and $\hat{\phi}_B$ are related to the quantities Γ_A and Γ_B as follows:

$$(\Delta\phi_A)^2 = \langle 0|\hat{\phi}^2_A|0\rangle - (\langle 0|\hat{\phi}_A|0\rangle)^2 = 2\Gamma_A,$$
$$(\Delta\phi_B)^2 = \langle 0|\hat{\phi}^2_B|0\rangle - (\langle 0|\hat{\phi}_B|0\rangle)^2 = 2\Gamma_B. \qquad (31)$$

In the following equations, we show that the product of Γ_A and Γ_B has a lower bound given by the quantity Φ_{BA}. To observe this, we focus on the commutation relation of the operators $\hat{\phi}_A$ and $\hat{\phi}_B$,

$$[\hat{\phi}_A, \hat{\phi}_B] = \int d^4 x d^4 y (J^\mu_{AR}(x) - J^\mu_{AL}(x))(J^\nu_{BR} - J^\nu_{BL}(y))[\hat{A}^I_\mu(x), \hat{A}^I_\nu(y)]$$

$$= \int d^4 x d^4 y (J^\mu_{AR}(x) - J^\mu_{AL}(x))(J^\nu_{BR} - J^\nu_{BL}(y))[\hat{A}^I_\mu(x), \hat{A}^I_\nu(y)]\theta(x^0 - y^0)$$

$$+ \int d^4 x d^4 y (J^\mu_{AR}(x) - J^\mu_{AL}(x))(J^\nu_{BR} - J^\nu_{BL}(y))[\hat{A}^I_\mu(x), \hat{A}^I_\nu(y)]\theta(y^0 - x^0)$$

$$= i \int d^4 x (J^\mu_{AR} - J^\mu_{AL})\Delta A_{B\mu} - i \int d^4 x (J^\mu_{BR} - J^\mu_{BL})\Delta A_{A\mu}$$

$$= -i\Phi_{BA}, \qquad (32)$$

where we inserted the step functions $\theta(x^0 - y^0) + \theta(y^0 - x^0)$ in the second line, and we changed variables as $x^\mu \leftrightarrow y^\mu$ and indices as $\mu \leftrightarrow \nu$ of the second term in the third line. Note that the first term $i\Phi_{AB} = i \int d^4x (J^\mu_{AR} - J^\mu_{AL})\Delta A_{B\mu}$ in the third line vanished by assuming the case $D > T_A$ and $D > T_B$, where there is no retarded propagation of photon field from Bob's system to Alice's system. This commutation relation shows that the operators $\hat{\phi}_A$ and $\hat{\phi}_B$ do not commute with each other because the influence of particle A causally propagates to particle B from the far past (the red or blue line in Fig. 2) and then $\Phi_{BA} \neq 0$. Using this commutation relation, we obtain the following Robertson inequality as

$$(\Delta\phi_A)^2 (\Delta\phi_B)^2 \geq \frac{1}{4}|\langle 0|[\hat{\phi}_A, \hat{\phi}_B]|0\rangle|^2 = \frac{1}{4}\Phi^2_{BA}. \quad (33)$$

From (31), we get the inequality among Γ_A, Γ_B, and Φ_{BA},

$$\Gamma_A \Gamma_B \geq \frac{1}{16}\Phi^2_{BA}. \quad (34)$$

This means that the quantities Γ_A and Γ_B do not vanish simultaneously if $\Phi_{BA} \neq 0$. Additionally, we can show that the Robertson inequality (34) is a sufficient condition for the inequality (28):

$$\Gamma_A \Gamma_B \geq \frac{1}{16}\Phi^2_{BA} \Rightarrow e^{-2\Gamma_A} + e^{-2\Gamma_B}\sin^2\left(\frac{\Phi_{BA}}{2}\right) \leq 1. \quad (35)$$

The proof of this statement is presented in Appendix C. This result implies that the Robertson inequality among Γ_A, Γ_B, and Φ_{BA}, which reflects the noncommutative property of the photon field, guarantees the complementarity described by the inequality between the visibility $\mathcal{V}_A$ and the distinguishability $\mathcal{D}_B$.

V. CONCLUSION

In this study, we revisited the resolution of the paradox proposed by Belenchia *et al.* [27,28] in the system of a photon field interacting with two charged particles in the superposition states of two trajectories. The analysis based on the quantum field theory explicitly demonstrated the intuitively legitimate result that causality holds and that operations on Bob's system at a spacelike distance do not affect Alice's interference experiment at all by deriving Alice's reduced density operator. On the other hand, to find the validity of complementarity, we first derived visibility and distinguishability, which represent the degree of success of Alice's interference experiment and the degree of distinction of Bob's quantum state, respectively. Then, we argued that there is an inequality between these quantities, which is guaranteed by the Robertson inequality

associated with the noncommutative property of the photon field (the quantized electromagnetic field). This inequality describes the limit of complementarity in resolving this paradox.

Thus, to resolve this paradox, the fact that the photon field has a noncommutative property is the most important factor in our analysis. This conclusion is applicable to gravitational interactions. A similar analysis of the gravitational version of the present paper should be performed explicitly in future work, but the results will be inferred with reference to our analysis, as follows. Let us consider the massive particles A and B. According to the analogy in Sec. IV, the phase shifts induced by the quantum fluctuations of gravitational field can be described as follows:

$$\hat{\phi}^g_A = \int d^4x (T^{\mu\nu}_{AR}(x) - T^{\mu\nu}_{AL}(x))\hat{h}^I_{\mu\nu}(x),$$

$$\hat{\phi}^g_B = \int d^4x (T^{\mu\nu}_{BR}(x) - T^{\mu\nu}_{BL}(x))\hat{h}^I_{\mu\nu}(x), \quad (36)$$

where $\hat{h}^I_{\mu\nu}$ is the linearized quantum gravitational field in the interaction picture which is the perturbation from the Minkowski spacetime, and $T^{\mu\nu}_{iP}$ ($i = A, B$ and $P = R, L$) is the energy-momentum tensor of each massive particle. Hence the decoherence (dephasing) effects due to the vacuum fluctuations can be characterized by

$$\Gamma^g_A = \frac{1}{2}\langle 0|(\hat{\phi}^g_A)^2|0\rangle, \qquad \Gamma^g_B = \frac{1}{2}\langle 0|(\hat{\phi}^g_B)^2|0\rangle, \quad (37)$$

and are limited by the phase shift induced by the retarded gravitational field owing to the Robertson inequality:

$$\Gamma^g_A \Gamma^g_B \geq \frac{1}{16}(\Phi^g_{BA})^2, \quad (38)$$

where Φ^g_{BA} is defined by

$$\Phi^g_{BA} \equiv \int d^4x (T^{\mu\nu}_{BR} - T^{\mu\nu}_{BL})\Delta h^A_{\mu\nu}, \quad (39)$$

with the retarded gravitational field,

$$\Delta h^A_{\mu\nu}(x) = \int d^4y (T^{\rho\sigma}_{AR}(y) - T^{\rho\sigma}_{AL}(y))G^r_{\mu\nu\rho\sigma}(x, y). \quad (40)$$

Note that the function $G^r_{\mu\nu\rho\sigma}(x, y)$ is the retarded Green's function, and the detailed formula is given in [36,37]. In the gravitational version of our analysis, the consistency between causality and complementarity is guaranteed by the Robertson inequality. Repeating the discussion of

Belenchia *et al.* [24,27,28], we suggest that the quantities Γ_A^g and Γ_B^g do not vanish at the same time so that either Γ_A^g or Γ_B^g must be caused by the on-shell gravitational radiation from Alice's particle A and the vacuum fluctuation of the gravitational field around Bob's particle B. This shows the necessity of the noncommutative property of the gravitational field related to the Robertson inequality.

Note added.—Recently the authors of Ref. [38] revisited the same paradox by assuming a simple theoretical model so that Alice with a spin and Bob with a continuous variable are coupled to each other through a quantized scalar field. They focused on the quantity $\langle \Psi_\downarrow | \Psi_\uparrow \rangle_{\phi,B} = e^{-\gamma_A} \delta_\epsilon(M)$, which denotes the interference term of Alice's state after tracing out the states of the scalar field ϕ and Bob's states. The quantity $e^{-\gamma_A}$ represents the decoherence due to the vacuum fluctuations of the scalar field ϕ, while $\delta_\epsilon(M)$ is an overlap of the wave function of Bob's system with M described with the retarded Green's function propagating from Bob to Alice. Therefore, γ_A and M in their study [38] correspond to Γ_A and Φ_{AB}, respectively. Therefore, $e^{-\gamma_A} \delta_\epsilon(M)$ corresponds to the visibility function (19) in the present study. The primary purpose of our study in the present paper is to demonstrate that the consistency between causality and complementarity is guaranteed by the Robertson inequality of the quantized field in the positions of Alice and Bob. This reflects the existence of a gravitational field with quantum noncommutativity.

APPENDIX A: DERIVATION OF THE DENSITY OPERATORS ρ_A AND ρ_{BP}

In this appendix, we derive the expression of the density operators ρ_A and ρ_{BP}. To do this, we compute $\mathrm{Tr}_{ph}[|\Omega_P\rangle_{B,ph}\langle\Omega_{P'}|]$ as follows:

$$\mathrm{Tr}_{ph}[|\Omega_P\rangle_{B,ph}\langle\Omega_{P'}|] = \frac{1}{2}\sum_{Q,Q'=R,L}|Q_f\rangle_B\langle Q'_f|_{ph}\langle\alpha|\hat{U}_{P'Q'}^\dagger\hat{U}_{PQ}|\alpha\rangle_{ph}$$
$$= \frac{1}{2}\sum_{Q,Q'=R,L}e^{-\Gamma_{P'Q'PQ}+i\Phi_{P'Q'PQ}}|Q_f\rangle_B\langle Q'_f|,$$
$$(A1)$$

where $_{ph}\langle\alpha|\hat{U}_{P'Q'}^\dagger\hat{U}_{PQ}|\alpha\rangle_{ph} = e^{-\Gamma_{P'Q'PQ}+i\Phi_{P'Q'PQ}}$ with the quantities,

$$\Gamma_{P'Q'PQ} = \frac{1}{4}\int d^4x\int d^4y(J_{P'Q'}^\mu(x) - J_{PQ}^\mu(x))(J_{P'Q'}^\nu(y) - J_{PQ}^\nu(y))\langle\{\hat{A}_\mu^I(x),\hat{A}_\nu^I(y)\}\rangle, \quad (A2)$$

$$\Phi_{P'Q'PQ} = \int d^4x(J_{P'Q'}^\mu(x) - J_{PQ}^\mu(x))A_\mu(x)$$
$$- \frac{1}{2}\int d^4x\int d^4y(J_{P'Q'}^\mu(x) - J_{PQ}^\mu(x))(J_{P'Q'}^\nu(y) + J_{PQ}^\nu(y))G_{\mu\nu}^r(x,y), \quad (A3)$$

was obtained in Appendix in [29]. $J_{PQ}^\mu = J_{AP}^\mu + J_{BQ}^\mu$ is given by the currents J_{AP}^μ and J_{BQ}^μ of charged particles A and B, respectively. The field $A_\mu(x)$ in (A3) is the coherent photon field defined as

$$A_\mu(x) = \int\frac{d^3k}{(2\pi)^{3/2}\sqrt{2k^0}}(\alpha_\mu(\boldsymbol{k})e^{ik_\nu x^\nu} + \text{c.c.}), \quad (A4)$$

and the complex function $\alpha_\mu(\boldsymbol{k})$ satisfies

$$k^\mu\alpha_\mu(\boldsymbol{k}) = -\frac{\tilde{J}^0(\boldsymbol{k})}{\sqrt{2k^0}} \quad (A5)$$

to guarantee the Becchi-Rouet-Stora-Tyutin (BRST) condition (Appendix in [29]). Note that $\tilde{J}^0(\boldsymbol{k}) = \tilde{J}_A^0(\boldsymbol{k}) + \tilde{J}_B^0(\boldsymbol{k})$ is the eigenvalue of the Fourier transform of the charged current $\hat{\tilde{J}}^0(\boldsymbol{k}) = \hat{\tilde{J}}_A^0(\boldsymbol{k}) + \hat{\tilde{J}}_B^0(\boldsymbol{k})$ at the initial time $t = 0$. The function $\langle\{\hat{A}_\mu^I(x),\hat{A}_\mu^I(y)\}\rangle$ is the two-point function of the vacuum. We can compute the reduced density operator ρ_A of the particle A in the basis $\{|R_f\rangle_A, |L_f\rangle_A\}$ as

$$
\begin{aligned}
\rho_A &= \mathrm{Tr}_{B,ph}[|\Psi(T)\rangle\langle\Psi(T)|]\\
&= \frac{1}{2}\sum_{P,P'=R,L}{}_{B,ph}\langle\Omega_{P'}|\Omega_P\rangle_{B,ph}|P_f\rangle_A\langle P'_f|\\
&= \frac{1}{4}\sum_{P,P'=R,L}\sum_{Q=R,L}e^{-\Gamma_{P'QPQ}+i\Phi_{P'QPQ}}|P_f\rangle_A\langle P'_f|\\
&= \frac{1}{2}\begin{pmatrix}1 & \frac{1}{2}\left(e^{-\Gamma_{RRLR}+i\Phi_{RRLR}}+e^{-\Gamma_{RLLL}+i\Phi_{RLLL}}\right)\\ * & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & \frac{1}{2}e^{-\Gamma_A+i\Phi_A}\left(e^{-i\int d^4x(J^\mu_{AR}-J^\mu_{AL})A_{BR\mu}}+e^{-i\int d^4x(J^\mu_{AR}-J^\mu_{AL})A_{BL\mu}}\right)\\ * & 1\end{pmatrix},
\end{aligned}\tag{A6}
$$

where ${}_{B,ph}\langle\Omega_{P'}|\Omega_P\rangle_{B,ph}=\mathrm{Tr}_B[\mathrm{Tr}_{ph}[|\Omega_P\rangle_{B,ph}\langle\Omega_{P'}|]]$, and $*$ is the complex conjugate of the (R,L) component. Note that

$$\Gamma_{RRLR}=\Gamma_{RLLL}=\Gamma_A,\tag{A7}$$

$$\Phi_{RRLR}=\Phi_A-\int d^4x(J^\mu_{AR}(x)-J^\mu_{AL}(x))A_{BR\mu}(x),\tag{A8}$$

$$\Phi_{RLLL}=\Phi_A-\int d^4x(J^\mu_{AR}(x)-J^\mu_{AL}(x))A_{BL\mu}(x).\tag{A9}$$

Here, Γ_A and Φ_A are defined by Eqs. (14) and (15), and the retarded field A^μ_{BQ} is given in (12). The reduced density operators ρ_{BR} and ρ_{BL} in the basis $\{|R_f\rangle_B,|L_f\rangle_B\}$ are given as

$$
\begin{aligned}
\rho_{BR} &= \mathrm{Tr}_{ph}[|\Omega_R\rangle_{B,ph}\langle\Omega_R|]\\
&= \frac{1}{2}\sum_{Q,Q'=R,L}e^{-\Gamma_{RQ'RQ}+i\Phi_{RQ'RQ}}|Q_f\rangle_B\langle Q'_f|,\\
&= \frac{1}{2}\begin{pmatrix}1 & e^{-\Gamma_{RRRL}+i\Phi_{RRRL}}\\ * & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & e^{-\Gamma_B+i\Phi_B-i\int d^4x(J^\mu_{BR}-J^\mu_{BL})A_{R\mu}}\\ * & 1\end{pmatrix},
\end{aligned}\tag{A10}
$$

and

$$
\begin{aligned}
\rho_{BL} &= \mathrm{Tr}_{ph}[|\Omega_L\rangle_{B,ph}\langle\Omega_L|]\\
&= \frac{1}{2}\sum_{Q,Q'=R,L}e^{-\Gamma_{LQ'LQ}+i\Phi_{LQ'LQ}}|Q_f\rangle_B\langle Q'_f|,\\
&= \frac{1}{2}\begin{pmatrix}1 & e^{-\Gamma_{LRLL}+i\Phi_{LRLL}}\\ * & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & e^{-\Gamma_B+i\Phi_B-i\int d^4x(J^\mu_{BR}-J^\mu_{BL})A_{L\mu}}\\ * & 1\end{pmatrix},
\end{aligned}\tag{A11}
$$

where we used

$$\Gamma_{RRRL}=\Gamma_{LRLL}=\Gamma_B,\tag{A12}$$

$$\Phi_{RRRL}=\Phi_B-\int d^4x(J^\mu_{BR}(x)-J^\mu_{BL}(x))A_{AR\mu}(x),\tag{A13}$$

$$\Phi_{LRLL}=\Phi_B-\int d^4x(J^\mu_{BR}(x)-J^\mu_{BL}(x))A_{AL\mu}(x),\tag{A14}$$

where Γ_B and Φ_B are defined in Eqs. (24) and (25), respectively. The retarded field A^μ_{PQ} is given in (23).

APPENDIX B: PROOF OF THE INEQUALITY BETWEEN VISIBILITY AND DISTINGUISHABILITY

We prove the inequality (17) between visibility and distinguishability. First, we derive the visibility for the state given in (9). The visibility of charged particle A is calculated as

$$
\begin{aligned}
\mathcal{V}_A &= 2|_A\langle L_f|\rho_A|R_f\rangle_A|\\
&= 2|\mathrm{Tr}_{B,ph}[_A\langle L_f|\Psi(T)\rangle\langle\Psi(T)|R_f\rangle_A]|\\
&= |_{B,ph}\langle\Omega_R|\Omega_L\rangle_{B,ph}|\equiv|\alpha|.
\end{aligned}\tag{B1}
$$

We next evaluate the distinguishability of charged particle B. For a trace distance $\mathcal{D}(\rho,\sigma)$ with arbitrary density operators ρ and σ, we use the fact that the trace-preserving quantum operations are contractive [33]:

$$\mathcal{D}(\mathcal{E}(\rho),\mathcal{E}(\sigma))\leq\mathcal{D}(\rho,\sigma),\tag{B2}$$

where $\mathcal{E}$ is a trace-preserving quantum operation. This inequality means that the operation $\mathcal{E}$ makes it difficult to distinguish between the two quantum states ρ and σ, i.e., the trace distance does not increase. Then, the distinguishability is bounded as

$$\mathcal{D}_{\mathrm{B}} = \frac{1}{2}\mathrm{Tr}_{\mathrm{B}}|\rho_{\mathrm{BR}} - \rho_{\mathrm{BL}}|$$

$$= \frac{1}{2}\mathrm{Tr}_{\mathrm{B}}|\mathrm{Tr}_{\mathrm{ph}}[|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}|] - \mathrm{Tr}_{\mathrm{ph}}[|\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}|]|$$

$$\leq \frac{1}{2}\mathrm{Tr}_{\mathrm{B}}||\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}||, \quad \text{(B3)}$$

where the inequality (B2) was used in the third line because the partial trace is a trace-preserving quantum operation. To obtain the eigenvalues of the operator

$|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}|$, we define the orthonormal basis $\{|u_{\mathrm{A}}\rangle, |u_{\mathrm{B}}\rangle\}$ using the Gram-Schmidt orthonormalization as:

$$|u_{\mathrm{A}}\rangle = |\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}, \quad |u_{\mathrm{B}}\rangle = \frac{|\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}} - \alpha|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}}{\sqrt{1 - |\alpha|^2}}, \quad \text{(B4)}$$

where the overlap α is defined in (B1). In this basis, the operator $|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}|$ can be rewritten as

$$|\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}| = |u_{\mathrm{A}}\rangle\langle u_{\mathrm{A}}| - (\alpha|u_{\mathrm{A}}\rangle + \sqrt{1 - |\alpha|^2}|u_{\mathrm{B}}\rangle)(\alpha^*\langle u_{\mathrm{A}}| + \sqrt{1 - |\alpha|^2}\langle u_{\mathrm{B}}|)$$

$$= \begin{pmatrix} 1 - |\alpha|^2 & \alpha\sqrt{1 - |\alpha|^2} \\ \alpha^*\sqrt{1 - |\alpha|^2} & -(1 - |\alpha|^2) \end{pmatrix}, \quad \text{(B5)}$$

in the orthonormal basis $\{|u_{\mathrm{A}}\rangle, |u_{\mathrm{B}}\rangle\}$. Thus, the eigenvalues of this matrix $\lambda_{\mathrm{A,B}}$ are

$$\lambda_{\mathrm{A}} = \sqrt{1 - |\alpha|^2}, \quad \lambda_{\mathrm{B}} = -\sqrt{1 - |\alpha|^2}, \quad \text{(B6)}$$

and the distinguishability $\mathcal{D}_{\mathrm{B}}$ is suppressed by the sum of these eigenvalues as follows:

$$\mathcal{D}_{\mathrm{B}} \leq \frac{1}{2}\mathrm{Tr}_{\mathrm{B}}||\Omega_{\mathrm{R}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{R}}| - |\Omega_{\mathrm{L}}\rangle_{\mathrm{B,ph}}\langle\Omega_{\mathrm{L}}||$$

$$= \frac{1}{2}(|\lambda_{\mathrm{A}}| + |\lambda_{\mathrm{B}}|) = \sqrt{1 - |\alpha|^2}. \quad \text{(B7)}$$

Substituting (B1) into (B7), we find the relationship

$$\mathcal{V}_{\mathrm{A}}^2 + \mathcal{D}_{\mathrm{B}}^2 \leq 1. \quad \text{(B8)}$$

Therefore, the visibility of charged particle A and the distinguishability of charged particle B follow the inequality (17).

APPENDIX C: PROOF OF THE STATEMENT IN (35)

We first numerically demonstrate the statement in (35). Using the Robertson inequality (34), $\Gamma_{\mathrm{A}}\Gamma_{\mathrm{B}} \geq \Phi_{\mathrm{AB}}^2/16$, we have

$$1 - e^{-2\Gamma_{\mathrm{A}}} - e^{-2\Gamma_{\mathrm{B}}}\sin^2\left(\frac{\Phi_{\mathrm{BA}}}{2}\right)$$

$$\geq 1 - e^{-2\Gamma_{\mathrm{A}}} - e^{-\Phi_{\mathrm{BA}}^2/8\Gamma_{\mathrm{A}}}\sin^2\left(\frac{\Phi_{\mathrm{BA}}}{2}\right) = f(X, Y), \quad \text{(C1)}$$

where we defined the function $f(X, Y)$ with $X = e^{-2\Gamma_{\mathrm{A}}}$ and $Y = e^{-\Phi_{\mathrm{BA}}^2/8\Gamma_{\mathrm{A}}}$ as follows:

$$f(X, Y) = 1 - X - Y\sin^2(\sqrt{\log X \log Y}). \quad \text{(C2)}$$

As it is sufficient to consider that $\Gamma_{\mathrm{A}} > 0$ and $\Phi_{\mathrm{BA}} > 0$, we can assume that $0 < X < 1$ and $0 < Y < 1$.

Figure 3 shows the behavior of the function $f(X, Y)$, which is positive in the regions $0 < X < 1$ and $0 < Y < 1$. Since the function $f(X, Y)$ is positive, the inequality $e^{-2\Gamma_{\mathrm{A}}} + e^{-2\Gamma_{\mathrm{B}}}\sin^2(\Phi_{\mathrm{BA}}/2) \leq 1$ in (28) is satisfied. Hence, the Robertson inequality (34) is the sufficient condition for the inequality (28), and the statement in (35) holds. In the following, we show that the function $f(X, Y)$ is always positive in an analytic manner.

Proof. Now let derive the partial derivatives to find the gradient for $f(X, Y)$, and the results are

$$\frac{\partial f(X, Y)}{\partial X} = -1 - \frac{Y\log Y\sin(\sqrt{\log X \log Y})\cos(\sqrt{\log X \log Y})}{X\sqrt{\log X \log Y}}, \quad \text{(C3)}$$
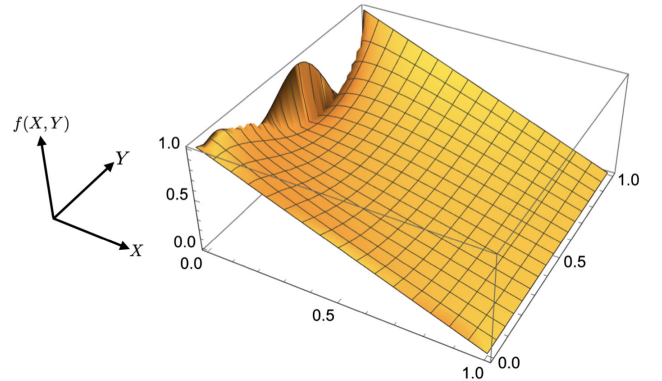

FIG. 3. Behavior of the function $f(X, Y)$ where the region $0 < X < 1$ and $0 < Y < 1$.

$$\frac{\partial f(X,Y)}{\partial Y} = -\left(\frac{\log X \cos(\sqrt{\log X \log Y})}{\sqrt{\log X \log Y}} + \sin(\sqrt{\log X \log Y})\right)\sin(\sqrt{\log X \log Y}). \tag{C4}$$

We are looking for the gradient is zero:

$$0 = \log X \cos(\sqrt{\log X \log Y}) + \sqrt{\log X \log Y}\sin(\sqrt{\log X \log Y}), \tag{C5}$$

and

$$0 = -X\sqrt{\log X \log Y} - Y \log Y \sin(\sqrt{\log X \log Y})\cos(\sqrt{\log X \log Y})$$
$$= -X(\log X \log Y) - Y \log Y((\sqrt{\log X \log Y})\sin(\sqrt{\log X \log Y}))\cos(\sqrt{\log X \log Y}), \tag{C6}$$

where we multiplied by the factor $\sqrt{\log X \log Y}$ in the second line. Substituting (C5) into (C6), we obtain the following condition

$$0 = (\log X \log Y)(-X - Y\sin^2(\sqrt{\log X \log Y}) + Y). \tag{C7}$$

Case 1: $\log X \log Y = 0$, i.e., $X = 1$ or $Y = 1$. When $X = 1$, by definition of the function $f(X,Y)$, we have

$$f(1,Y) = 0, \tag{C8}$$

where we used $\log 1 = 0$ and $\sin 0 = 0$ for arbitrary value Y. Note that when $Y \to 0$, then $\sqrt{\log X \log Y}$ is nontrivial. However, due to $Y \to 0$, $f(1,Y)$ becomes 0. When $Y = 1$,

$$f(X,1) = 1 - Y > 0, \tag{C9}$$

where we used $\log 1 = 0$ and $\sin 0 = 0$ for arbitrary values X. Note that when $X \to 0$, then $\sqrt{\log X \log Y}$ is also nontrivial. However, in this case, $f(X,Y)$ is

$$\lim_{X\to 0} f(X,Y)|_{Y=1} = 1 - \sin^2(\sqrt{\log X \log Y}) > 0. \tag{C10}$$

Thus, in case 1, $f(X,Y)$ is always positive.

Case 2: $-X - Y\sin^2(\sqrt{\log X \log Y}) + Y = 0$. Then $f(X,Y)$ becomes

$$f(X,Y) = 1 - X - Y\sin^2(\sqrt{\log X \log Y})$$
$$= 1 - X > 0. \tag{C11}$$

Thus, in case 2, $f(X,Y)$ is also always positive. In either case, $f(X,Y) \geq 0$, so the result is proven.

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