Colliding plane fronted waves and a gravitoelectromagnetic searchlight

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We present a formulation of Einstein-Maxwell vacuum fields due to plane fronted electromagnetic waves sharing their wave fronts with gravitational waves. This is based on a recent geometrical reconstruction of plane fronted wave fields by the authors that clearly identifies the cases in which the wave fronts collide or do not collide. In the former case our construction suggests an explicit example of a searchlight beam, accompanied by gravitational radiation, which sweeps across the sky. This gravitoelectromagnetic searchlight and its properties are described in detail.

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I. INTRODUCTION

We describe a formulation of Einstein-Maxwell vacuum fields of plane fronted electromagnetic waves sharing their wave fronts with gravitational waves. This follows our recent reconstruction [1,2] of plane fronted waves that enables a clear distinction between plane waves with colliding wave fronts (Kundt [3] waves) and waves with noncolliding wave fronts (pp waves [4,5]). Our new formulation suggests a simple example in which the gravitational and electromagnetic radiation fields vanish if the waves are not colliding. This is an example of a searchlight beam, accompanied by gravitational radiation, sweeping across the sky. We refer to it as a gravitoelectromagnetic searchlight. Such waves are the asymptotic limit of spherical waves emitted by an isolated source [6]. The wave fronts are colliding or not colliding depending upon the motion of the source. The paper is organized as follows: in Sec. II we give a detailed description of our formulation of plane fronted waves in the context of Einstein-Maxwell theory. This is followed in Sec. III by the gravitoelectromagnetic searchlight, which is an explicit solution of the Einstein-Maxwell equations which exploits variables which arise naturally in our geometrical construction of plane fronted waves. The solution is also described in coordinates closely associated with rectangular Cartesians and time as this makes the solution more surveyable. This leads to further properties of the solution described in Sec. IV by making use of tensor fields on a background Minkowskian space time. The paper ends with a discussion of our results in Sec. V.

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II. EINSTEIN-MAXWELL VACUUM FIELDS

We use units for which the gravitational constant $G = 1$ and the speed of light in a vacuum $c = 1$ and we choose the sign conventions of Synge [7]. Thus in a local coordinate system $\{x^i\}$ we obtain from the Riemann curvature tensor components R_{ijkm} the Ricci tensor components $R_{jk} = g^{im}R_{ijkm}$. The Ricci scalar $R = g^{jk}R_{jk}$, the Einstein tensor components $G_{jk} = R_{jk} - \frac{1}{2}g_{jk}R$ and the Einstein-Maxwell vacuum field equations

$$G_{jk} = -\kappa E_{jk} \quad \text{with} \quad E_{jk} = F_{jp}F_k{}^p - \frac{1}{4}g_{jk}F_{mp}F^{mp}, \tag{1}$$

with the Maxwell tensor $F_{ij} = -F_{ji}$ satisfying Maxwell's vacuum field equations

$$F^{ij}{}_{;j} = 0 \quad \text{with} \quad F_{ij;k} + F_{ki;j} + F_{jk;i} = 0. \tag{2}$$

In (1) $\kappa = 8\pi$, $E_{jk} = E_{kj}$ with $E^j{}_j = 0$ is the electromagnetic energy-momentum tensor, indices are raised with g^{ij} and lowered with g_{ij} and g^{ij} is defined by $g^{ij}g_{jk} = \delta^i_k$ and $g_{ij} = g_{ji}$ are the components of the metric tensor. The semicolon denotes covariant differentiation with respect to the Riemannian connection calculated with the metric tensor g_{ij}. We shall denote by A^i the components of a four potential from which the components of the Maxwell tensor can be obtained via

$$F_{ij} = A_{j;i} - A_{i;j} = A_{j,i} - A_{i,j}, \tag{3}$$

with the comma denoting partial differentiation with respect to the coordinates x^i. With F_{ij} given by (3) the second equation in (2) is satisfied. The first equation in (2) can be written in terms of the four potential as ([7], p. 357)

$$g^{jk}A_{i;j;k} + R_{ij}A^j = 0 \quad \text{provided} \quad A^i{}_{;i} = 0. \tag{4}$$

This relies on the Ricci identities in Synge's form

$$A_{i;j;k} - A_{i;k;j} = A_p R^p{}_{ijk}, \tag{5}$$

which reveals his convention for the definition of the Riemann curvature tensor. Making use of this, following the substitution of (3) into the first of Maxwell's equations (2), results in (4). As a final preliminary we note the components of the Weyl conformal curvature tensor C_{ijkm} are given by

$$C_{ijkm} = R_{ijkm} + \frac{1}{2}(g_{ik}R_{jm} + g_{jm}R_{ik} - g_{im}R_{jk} - g_{jk}R_{im}) + \frac{1}{6}R(g_{im}g_{jk} - g_{ik}g_{jm}). \tag{6}$$

We now consider a novel form for the line element of the space-time model for plane fronted gravitational waves sharing their wave fronts with electromagnetic waves, which has been constructed in [1,2]:

$$ds^2 = g_{ij}dx^i dx^j = 2|d\zeta - \beta(u)v\,du|^2 + 2qdu(dv + Hdu), \tag{7}$$

with

$$q(\zeta, \bar{\zeta}, u) = \beta(u)\bar{\zeta} + \bar{\beta}(u)\zeta + \frac{1}{2}\gamma(u). \tag{8}$$

The coordinate ζ is complex with complex conjugate $\bar{\zeta}$ while u, v are two real coordinates. The function $\beta(u)$ is an arbitrary complex-valued function of the coordinate u with complex conjugate denoted by $\bar{\beta}(u)$, $\gamma(u)$ is an arbitrary real-valued function of u, and $H(\zeta, \bar{\zeta}, u)$ is a real-valued function to be determined by the field equations. The hypersurfaces $u =$ constant are null and generated by the shear-free, expansion-free, geodesic integral curves of the null vector field $\partial/\partial v$. It is demonstrated explicitly in [1,2] that these null hypersurfaces intersect, and thus the wave fronts collide, if and only if $\beta(u) \neq 0$. Labeling the coordinates $x^i = (\zeta, \bar{\zeta}, v, u)$ for $i = 1, 2, 3, 4$ we find that the only nonvanishing components of the Riemann curvature tensor are

$$R_{1424} = -qH_{\zeta\bar{\zeta}}, \quad R_{1414} = -qH_{\zeta\zeta}, \quad R_{2424} = -qH_{\bar{\zeta}\bar{\zeta}}, \tag{9}$$

with the subscripts here, and in the sequel, denoting partial derivatives. The Ricci tensor has only one nonvanishing component

$$R_{44} = 2qH_{\zeta\bar{\zeta}}. \tag{10}$$

Consequently the Ricci scalar vanishes and the nonvanishing components of the Weyl conformal curvature tensor are

$$C_{1414} = -qH_{\zeta\zeta} \quad \text{and} \quad C_{2424} = -qH_{\bar{\zeta}\bar{\zeta}}. \tag{11}$$

Next we consider a four potential on the space-time with line element (7) given by the one form

$$A = f(\zeta, \bar{\zeta}, u)du, \tag{12}$$

with f a real-valued function of its arguments. This leads to a candidate for Maxwell field F_{ij} having nonvanishing components

$$F_{14} = f_\zeta \quad \text{and} \quad F_{24} = f_{\bar{\zeta}}, \tag{13}$$

with the subscripts denoting partial derivatives as before. Now Maxwell's vacuum field equations (2) are satisfied provided f satisfies

$$f_{\zeta\bar{\zeta}} = 0, \tag{14}$$

and thus f_ζ is an analytic function of ζ with an arbitrary dependence on u. The electromagnetic energy-momentum tensor in (1) has one nonvanishing component

$$E_{44} = 2f_\zeta f_{\bar{\zeta}}. \tag{15}$$

With (10) and (15) the Einstein-Maxwell field equations (1) provide the following single equation

$$H_{\zeta\bar{\zeta}} = -\kappa q^{-1} f_\zeta f_{\bar{\zeta}}. \tag{16}$$

In order to survey what we have here it is useful to introduce the null tetrad k^i, l^i, m^i, $\bar{m}^i$ given in the coordinates $x^i = (\zeta, \bar{\zeta}, v, u)$ for $i = 1, 2, 3, 4$ by

$$k^i = \delta^i_3, \qquad m^i = \delta^i_2, \qquad \bar{m}^i = \delta^i_1,$$
$$l^i = q^{-1}\beta v\delta^i_1 + q^{-1}\bar{\beta}v\delta^i_2 - q^{-1}H\delta^i_3 + q^{-1}\delta^i_4, \tag{17}$$

and the three complex bivectors

$$L^{ij} = \bar{m}^i l^j - \bar{m}^j l^i, \tag{18}$$

$$M^{ij} = m^i \bar{m}^j - \bar{m}^i m^j + k^i l^j - k^j l^i, \tag{19}$$

$$N^{ij} = m^i k^j - m^j k^i. \tag{20}$$

Each of these bivectors is self-dual in the sense that ${}^*L^{ij} = iL^{ij}$, ${}^*M^{ij} = iM^{ij}$ and ${}^*N^{ij} = iN^{ij}$ with $i = \sqrt{-1}$ and ${}^*L^{ij} = \frac{1}{2}\eta^{ijkm}L_{km}$ etc. with $\eta_{ijkm} = \sqrt{-g}\epsilon_{ijkm}$,

$g = \det(g_{ij})$, and ϵ_{ijkm} the totally skewsymmetric Levi-Civita permutation symbol in four dimensions with $\epsilon_{1234} = 1$. Any self-dual complex bivector can be expressed as a linear combination of these three self-dual bivectors. In particular we have from (13)

$$F_{ij} - i^* F_{ij} = 2q^{-1} f_\zeta N_{ij}. \tag{21}$$

The covariant vector field k_i and the complex bivector N_{ij} satisfy the important equations in the current context

$$k_{i;j} = -q^{-1}\bar{\beta} m_i k_j - q^{-1}\beta \bar{m}_i k_j, \tag{22}$$

given in [1,2], and

$$N_{ij;k} = -q^{-1}\beta M_{ij} k_k, \tag{23}$$

so that if $\beta = 0$ then k_i and N_{ij} are covariantly constant. Using (23) we confirm that (21) satisfies Maxwell's equations in the form

$$(F^{ij} - i^* F^{ij})_{;j} = -2q^{-1} f_{\zeta\bar{\zeta}} k^i = 0. \tag{24}$$

It also follows from (21) that

$$(F_{ij} - i^* F_{ij}) k^j = 0, \tag{25}$$

indicating that we have here a purely radiative Maxwell field with propagation direction k^i in space-time. Next we can write (11) in the form

$$C^{ijkm} - i^* C^{ijkm} = -2q^{-1} H_{\zeta\zeta} N^{ij} N^{km}. \tag{26}$$

Using (23) we obtain from this

$$(C^{ijkm} - i^* C^{ijkm})_{;m} = 2q^{-1} H_{\zeta\zeta\bar{\zeta}} N^{ij} k^k. \tag{27}$$

The Bianchi identities and the field equations (1) result in

$$C^{ijkm}{}_{;m} = \frac{1}{2}\kappa(E^{ki;j} - E^{kj;i}), \tag{28}$$

with

$$E^{ki} = 2q^{-2} f_\zeta f_{\bar{\zeta}} k^k k^i. \tag{29}$$

With the use of (22) we find that

$$C^{ijkm}{}_{;m} = -\kappa q^{-1}(q^{-1} f_\zeta f_{\bar{\zeta}})_\zeta N^{ij} k^k - \kappa q^{-1}(q^{-1} f_\zeta f_{\bar{\zeta}})_{\bar{\zeta}} \bar{N}^{ij} k^k. \tag{30}$$

Since ${}^*N^{ij} = iN^{ij}$ we have

$${}^*C^{ijkm}{}_{;m} = -\kappa i q^{-1}(q^{-1} f_\zeta f_{\bar{\zeta}})_\zeta N^{ij} k^k + \kappa i q^{-1}(q^{-1} f_\zeta f_{\bar{\zeta}})_{\bar{\zeta}} \bar{N}^{ij} k^k. \tag{31}$$

Hence the left hand side of (27) can be written

$$(C^{ijkm} - i^* C^{ijkm})_{;m} = -2\kappa q^{-1}(q^{-1} f_\zeta f_{\bar{\zeta}})_\zeta N^{ij} k^k. \tag{32}$$

Therefore (27) is automatically satisfied since it reduces to the derivative with respect to ζ of the field equation (16). Finally we see from (26) that

$$(C^{ijkm} - i^* C^{ijkm}) k_m = 0, \tag{33}$$

indicating pure gravitational radiation with propagation direction k^i in space-time.

III. A GRAVITOELECTROMAGNETIC SEARCHLIGHT

In the geometrical construction of (7), described in [1,2], the function $q(\zeta, \bar{\zeta}, u)$ given in (8) emerges naturally. We can use it to provide a simple explicit example of the Einstein-Maxwell fields described above by taking $f = g_0 q$ in the potential 1-form (12) with g_0 as a real constant. Then the Maxwell field $F_{ij} = 0$ except for $F_{14} = g_0\bar{\beta}$ and $F_{24} = g_o\beta$ while (14) is satisfied. The Einstein-Maxwell field equation (16) now reads

$$H_{\zeta\bar{\zeta}} = -\kappa g_0^2 q^{-1}|\beta|^2, \tag{34}$$

and we solve this with $H = -\kappa g_0^2 q \log q$. Now the Weyl conformal curvature tensor $C_{ijkm} = 0$ except for $C_{1414} = 2g_0^2\bar{\beta}^2$ and $C_{2424} = 2g_0^2\beta^2$. In this case (21) and (26) read

$$F_{ij} - i^* F_{ij} = 2g_0 q^{-1}\bar{\beta} N_{ij} \tag{35}$$

and

$$C_{ijkm} - i^* C_{ijkm} = 2\kappa g_0^2 q^{-2}\bar{\beta}^2 N_{ij} N_{km}, \tag{36}$$

and so we have the Weyl double copy (cf. [8])

$$C_{ijkm} - i^* C_{ijkm} = \frac{1}{2}\kappa(F_{ij} - i^* F_{ij})(F_{km} - i^* F_{km}). \tag{37}$$

The potential 1-form (12) now reads

$$A = g_0 q du, \tag{38}$$

and the line element (7) reads

$$ds^2 = 2|d\zeta - \beta(u) v du|^2 + 2q du(dv - \kappa g_0^2 q \log q du). \tag{39}$$

This gravitoelectromagnetic searchlight can be written in a more surveyable form by introducing coordinates $X^i = (X, Y, Z, T)$ with $i = 1, 2, 3, 4$ via the transformations [1,2]

$$X + iY = \sqrt{2}\zeta - \sqrt{2}l(u)v, \tag{40}$$

$$Z + T = 2\{\bar{l}(u)\zeta + l(u)\bar{\zeta}\} - 2l(u)\bar{l}(u)v + n(u), \tag{41}$$

$$Z - T = v, \tag{42}$$

with $dl/du = \beta$ and $dn/du = \gamma$. Applying these transformations to the line element (39) results in

$$\begin{aligned} ds^2 &= (dX)^2 + (dY)^2 + (dZ)^2 - (dT)^2 - 2\kappa g_0^2 q^2 \log q du^2 \\ &= \eta_{ij} dX^i dX^j - 2\kappa g_0^2 q^2 \log q du^2, \end{aligned} \tag{43}$$

where $\eta_{ij} = \mathrm{diag}(1, 1, 1, -1)$ is the Minkowskian metric tensor in rectangular Cartesian coordinates and time. Solving (40)–(42) for $(\zeta, \bar{\zeta}, v, u)$ in terms of X^i is straightforward except for the equation giving the coordinate u as a function of X^i. This equation is obtained by substituting (40) and (42) into (41) to arrive at

$$\begin{aligned} &\sqrt{2}(\bar{l} + l)X + i\sqrt{2}(\bar{l} - l)Y + (2l\bar{l} - 1)Z \\ &\quad - (2l\bar{l} + 1)T + n = 0, \end{aligned} \tag{44}$$

giving $u(X^i)$ implicitly. Defining

$$a^i(u) = (\sqrt{2}(\bar{l} + l), i\sqrt{2}(\bar{l} - l), 2l\bar{l} - 1, 2l\bar{l} + 1), \tag{45}$$

we have

$$\eta_{ij} a^i a^j = a_j a^j = 0, \tag{46}$$

and (44) takes the form

$$a_i(u) X^i + n(u) = 0. \tag{47}$$

We thus see that the hypersurfaces $u = \text{constant}$ are null hyperplanes in Minkowskian space-time. These hyperplanes intersect provided $\dot{a}^i = da^i/du \neq 0 (\Leftrightarrow \beta \neq 0)$. From (47) we have

$$du = u_{,i} dX^i = -W a_i dX^i \quad \text{with} \quad W = (\dot{a}_j X^j + \gamma)^{-1}. \tag{48}$$

With $q(\zeta, \bar{\zeta}, u)$ given by (8) we find, using (40) and (42),

$$\begin{aligned} 2q &= \sqrt{2}(\bar{\beta} + \beta)X + i\sqrt{2}(\bar{\beta} - \beta)Y \\ &\quad + 2(\bar{\beta} l + \beta \bar{l})(Z - T) + \gamma \\ &= \dot{a}^1 X + \dot{a}^2 Y + \dot{a}^3 Z - \dot{a}^4 T + \gamma, \end{aligned} \tag{49}$$

so that

$$2q = \dot{a}_j X^j + \gamma = W^{-1}. \tag{50}$$

Hence we have

$$q du = -\frac{1}{2} a_i dX^i, \tag{51}$$

and so the line element (43) written entirely in the coordinates X^i reads

$$ds^2 = \left(\eta_{ij} + \frac{1}{2}\kappa g_0^2 \log W a_i a_j\right) dX^i dX^j. \tag{52}$$

We note in passing that $W(X^i)$ given by (48) is, with respect to the Minkowskian metric, a wave function and its gradient is orthogonal to a^i. With a comma denoting partial differentiation with respect to X^i the first and second derivatives of W are

$$W_{,i} = (\ddot{a}_j X^j + \dot{\gamma}) W^3 a_i - W^2 \dot{a}_i, \tag{53}$$

and

$$\begin{aligned} W_{,ij} &= W^3(a_i \ddot{a}_j + \ddot{a}_i a_j + 2\dot{a}_i \dot{a}_j) \\ &\quad - 3(\ddot{a}_k X^k + \dot{\gamma}) W^4 (a_i \dot{a}_j + \dot{a}_i a_j) \\ &\quad + \{3(\ddot{a}_k X^k + \dot{\gamma})^2 W^5 - (\dddot{a}_k X^k + \ddot{\gamma}) W^4\} a_i a_j. \end{aligned} \tag{54}$$

Hence

$$W_{,i} a^i = 0, \qquad W_{,i} \dot{a}^i = -W^2 \dot{a}_i \dot{a}^i = W^2 a_i \ddot{a}^i, \tag{55}$$

so that, in particular, the gradient of W is orthogonal to a^i with respect to the Minkowskian metric, and

$$\Box W = \eta^{ij} W_{,ij} = 2W^3(a_i \ddot{a}^i + \dot{a}_i \dot{a}^i) = 0, \tag{56}$$

so that W is a Minkowskian wave function.

In coordinates X^i the gravitoelectromagnetic searchlight has a metric tensor of the Kerr-Schild form

$$\begin{aligned} g_{ij} &= \eta_{ij} + \frac{1}{2}\kappa g_0^2 \log W a_i(u) a_j(u) \\ \Rightarrow g^{ij} &= \eta^{ij} - \frac{1}{2}\kappa g_0^2 \log W a^i a^j, \end{aligned} \tag{57}$$

with $a^i = \eta^{ij} a_j = g^{ij} a_j$. W is given by (48) and $u(X^i)$ implicitly by (47). The electromagnetic four potential is

$$A^i = -\frac{1}{2} g_0 a^i(u). \tag{58}$$

Calculating directly with these we see that the Maxwell field is

$$F^{ij} = \frac{1}{2} g_0 W(\dot{a}^i a^j - \dot{a}^j a^i) \Rightarrow F^{ij} a_j = 0. \tag{59}$$

Denoting by a semicolon covariant differentiation with respect to the Riemannian connection calculated with the metric tensor (57) and noting from the metric form that $g = \det(g_{ij}) = \det(\eta_{ij}) = -1$ we satisfy the gauge condition

$$A^i{}_{;i} = \frac{1}{\sqrt{-g}} (\sqrt{-g} A^i)_{,i} = A^i{}_{,i} = \frac{1}{2} g_0 W \dot{a}^i a_i = 0. \tag{60}$$

Also we confirm that Maxwell's equations are satisfied since

$$F^{ij}{}_{;j} = \frac{1}{\sqrt{-g}} (\sqrt{-g} F^{ij})_{,j} = F^{ij}{}_{,j}$$
$$= \frac{1}{2} g_0 W_{,j} a^j \dot{a}^i + \frac{1}{2} g_0 (W^2 a_j \ddot{a}^j - W_{,j} \dot{a}^j) a^i = 0, \tag{61}$$

with the final equality following from (55). The electromagnetic energy-momentum tensor is given by

$$E_{ij} = \frac{1}{4} g_0^2 W^2 \dot{a}_p \dot{a}^p a_i a_j. \tag{62}$$

A calculation of the Riemann curvature tensor components starting with the metric tensor (57) results in

$$R_{ijkm} = \frac{1}{4} \kappa g_0^2 W^2 (\dot{a}_i a_j - a_i \dot{a}_j)(\dot{a}_k a_m - a_k \dot{a}_m), \tag{63}$$

from which the Ricci tensor components are found to be

$$R_{jk} = g^{im} R_{ijkm} = -\frac{1}{4} \kappa g_0^2 W^2 \dot{a}_m \dot{a}^m a_j a_k. \tag{64}$$

The equations (62) and (64) confirm that the Einstein-Maxwell field equations $R_{ij} = -\kappa E_{ij}$ are satisfied. With A^i given by (58), satisfying the gauge condition (60), and the Ricci tensor (64), it is straightforward to check that the Einstein-Maxwell field equations in the form of (4) are satisfied. Finally the Weyl conformal curvature tensor components in coordinates X^i are

$$C_{ijkm} = \frac{1}{4} \kappa g_0^2 W^2 \{ S_{ik} a_j a_m + S_{jm} a_i a_k - S_{im} a_j a_k - S_{jk} a_i a_m \}, \tag{65}$$

with

$$S_{ij} = \dot{a}_i \dot{a}_j - \frac{1}{2} \dot{a}_p \dot{a}^p \eta_{ij} = S_{ji} \Rightarrow S_{ij} a^j = -\frac{1}{2} \dot{a}_p \dot{a}^p a_i. \tag{66}$$

Consequently we have the radiative property [the partner for (59)]

$$C_{ijkm} a^m = 0. \tag{67}$$

IV. FIELDS ON MINKOWSKIAN SPACE-TIME

It follows from (61) that F^{ij} given by (59) is not only a Maxwell field on the space-time with line element (57) but is also a Maxwell field on Minkowskian space-time with metric tensor components η_{ij} in coordinates $X^i = (X, Y, Z, T)$. This Maxwell field on Minkowskian space-time determines an electric three-vector $\mathbf{E} = (E^1, E^2, E^3)$ and a magnetic three-vector $\mathbf{B} = (B^1, B^2, B^3)$ given, respectively, by

$$E^\alpha = F^{\alpha 4}, \qquad B^\alpha = {}^*F^{\alpha 4}, \tag{68}$$

with greek indices taking values 1, 2, 3. With $a^i(u)$ in (45) we have from (59):

$$\mathbf{B} + i\mathbf{E} = 2ig_0 W(2l\bar{l} + 1)\bar{\beta}\mathbf{m}, \tag{69}$$

with the three-vector $\mathbf{m}$ given by

$$\mathbf{m} = \left(\frac{1 - 2l^2}{\sqrt{2}(2l\bar{l} + 1)}, \frac{i(1 + 2l^2)}{\sqrt{2}(2l\bar{l} + 1)}, \frac{2l}{2l\bar{l} + 1} \right). \tag{70}$$

The formula (69) with (70) is a special case of a formula satisfied by plane fronted electromagnetic waves in general given in [1,2]. The histories in Minkowskian space-time of the plane fronted waves are the null hyperplanes $u(X^i) =$ constant given by (47). The wave velocity (see [9]) has components

$$v^\alpha = -\frac{u_{,4} u_{,\alpha}}{u_{,\beta} u_{,\beta}}, \tag{71}$$

which, as a consequence of (48), read as (see Fig. 1)

$$\mathbf{v} = (a^4)^{-1} \mathbf{a}, \tag{72}$$

with $a^i = (\mathbf{a}, a^4)$ by (45). The plane fronted waves, of course, travel with the speed of light so that $\mathbf{v} \cdot \mathbf{v} = 1$. With $u_{,4} = -W a_4 = W a^4$ we find that

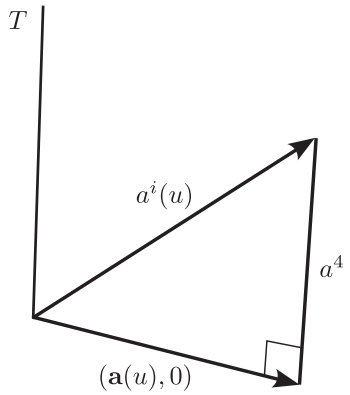

FIG. 1. $a^i(u)$ is resolved into $(\mathbf{a}(u), 0)$ in the observer's space and a vector of magnitude $a^4(u)$ parallel to the T axis. The three velocity of the waves is $\mathbf{v}(u) = (a^4)^{-1}\mathbf{a}$ in the observer's space.

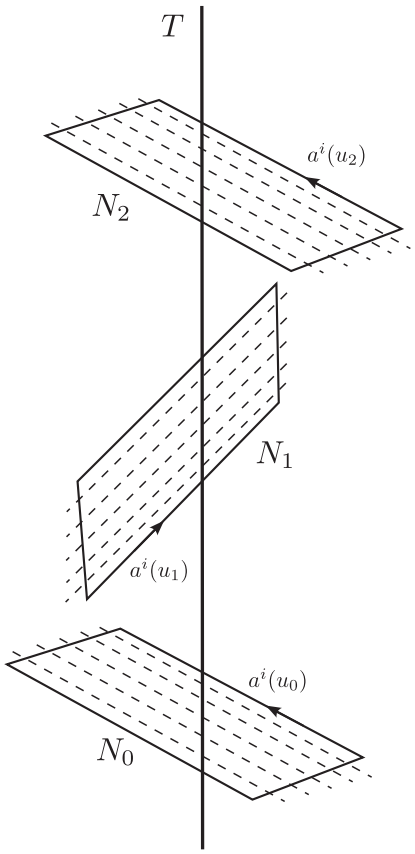

FIG. 2. The intersecting null hyperplane histories $N_0(u_0)$, $N_1(u_1)$, $N_2(u_2)$ of some plane wave fronts as the gravitoelectromagnetic searchlight sweeps across the sky with angular velocity $|\omega| = W(\dot{a}_m \dot{a}^m)^{1/2}$ in the observer's space. The generators of the null hyperplanes [the integral curves of $a^i(u)$ in each case] are indicated.

$$\frac{\partial \mathbf{v}}{\partial T} = \frac{\partial \mathbf{v}}{\partial u} u_{,4} = 2W(\bar{\beta}\mathbf{m} + \beta\bar{\mathbf{m}}), \tag{73}$$

with $\bar{\mathbf{m}}$ as the complex conjugate of $\mathbf{m}$. Among the useful properties of the complex three-vector $\mathbf{m}$ ($\mathbf{m}\cdot\bar{\mathbf{m}} = 1$, $\mathbf{m}\cdot\mathbf{m} = 0 = \bar{\mathbf{m}}\cdot\bar{\mathbf{m}}$ and $\mathbf{m}\times\bar{\mathbf{m}} = i\mathbf{v}$) are the vector products

$$\mathbf{m} = i\mathbf{m}\times\mathbf{v} \quad \text{and} \quad \bar{\mathbf{m}} = -i\bar{\mathbf{m}}\times\mathbf{v}. \tag{74}$$

Consequently it follows from (69) that $|\mathbf{E}|^2 = |\mathbf{B}|^2$, $\mathbf{E}\cdot\mathbf{B} = 0$, $\mathbf{E}\times\mathbf{B} = |\mathbf{E}|^2\mathbf{v}$, and $\mathbf{E}$, $\mathbf{B}$, $\mathbf{v}$ form a right-handed orthogonal triad. In addition (73) can be written as

$$\frac{\partial \mathbf{v}}{\partial T} = \omega\times\mathbf{v} \quad \text{with} \quad \omega = 2iW(\bar{\beta}\mathbf{m} - \beta\bar{\mathbf{m}}), \tag{75}$$

so that $|\omega|^2 = \omega\cdot\omega = 8W^2|\beta|^2$. Hence provided that $\beta \neq 0$ [so that the null hyperplanes (47) with $u =$ constant intersect and the wave fronts collide], then ω is the angular velocity with which the searchlight sweeps across the sky as described by Pirani [10] (see Fig. 2). If $\beta = 0$, then there is no searchlight. The searchlight we are considering here can also project gravitational radiation and is then a gravitoelectromagnetic searchlight. We achieve this in the present context by considering (57) as a perturbation of Minkowskian space-time written as

$$g_{ij} = \eta_{ij} + \gamma_{ij}, \tag{76}$$

where

$$\gamma_{ij} = \frac{1}{2}\kappa g_0^2 \log W a_i(u) a_j(u) \tag{77}$$

is a tensor field on Minkowskian space-time. Using (46), (48), and (53) we find that γ_{ij} satisfies the gauge condition

$$\eta^{jk}\gamma_{ij,k} = 0. \tag{78}$$

The linearized Riemann curvature tensor components are

$$L_{ijkm} = \frac{1}{2}(\gamma_{im,jk} + \gamma_{jk,im} - \gamma_{ik,jm} - \gamma_{jm,ik}), \tag{79}$$

and so, making use of (54), the components of the linearized Ricci tensor are

$$\begin{aligned} L_{jk} &= \eta^{im}L_{ijkm} = \frac{1}{2}\eta^{im}\gamma_{jk,im} \\ &= -\frac{1}{4}\kappa g_0^2 W^2 \dot{a}_m \dot{a}^m a_j a_k = -\kappa E_{jk}, \end{aligned} \tag{80}$$

with E_{jk} given by (62). On account of the Weyl double copy (37) the linearized Weyl tensor is given in coordinates X^i by

$$C^{ijkm} - i^*C^{ijkm} = \frac{1}{2}\kappa(F^{ij} - i^*F^{ij})(F^{km} - i^*F^{km}). \tag{81}$$

In terms of the electric part of the linearized Weyl tensor $E^{\alpha\beta} = C^{\alpha 4\beta 4}$ and the magnetic part $B^{\alpha\beta} = {}^*C^{\alpha 4\beta 4}$ we obtain from this the gravitational partner for (69) namely

$$B^{\alpha\beta} + iE^{\alpha\beta} = 2i\kappa g_0^2 W^2 \bar{\beta}^2 (2l\bar{l} + 1)^2 m^\alpha m^\beta, \tag{82}$$

with $(m^\alpha) = \mathbf{m}$. The plane fronted gravitational waves described by this tensor share their wave fronts with the electromagnetic waves above and their wave fronts undergo the rotation implied by (73) making the searchlight a gravitoelectromagnetic searchlight. Using (69) and (82) we deduce that

$$E^{\alpha\beta} = \frac{1}{2}\kappa(E^{\alpha}E^{\beta} - B^{\alpha}B^{\beta}) = E^{\beta\alpha} \Rightarrow E^{\alpha\alpha} = 0, \tag{83}$$

and

$$B^{\alpha\beta} = \frac{1}{2}\kappa(E^{\alpha}B^{\beta} + E^{\beta}B^{\alpha}) = B^{\beta\alpha} \Rightarrow B^{\alpha\alpha} = 0. \tag{84}$$

Using the algebraic relations following (74) above involving **E**, **B**, and **v** we obtain the corresponding relations involving $E^{\alpha\beta}$, $B^{\alpha\beta}$, and **v**:

$$E^{\alpha\beta}E^{\beta\sigma} = B^{\alpha\beta}B^{\beta\sigma}, \tag{85}$$

$$E^{\alpha\beta}B^{\beta\sigma} + B^{\alpha\beta}E^{\beta\sigma} = 0, \tag{86}$$

and

$$\epsilon_{\sigma\rho\lambda}E^{\alpha\rho}B^{\beta\lambda} = E^{\alpha\rho}E^{\rho\beta}v^{\sigma} = B^{\alpha\rho}B^{\rho\beta}v^{\sigma}, \tag{87}$$

where $\epsilon_{\sigma\rho\lambda}$ is the three-dimensional Levi-Civita permutation symbol.

V. DISCUSSION

The electromagnetic field and the gravitational field of our explicit model of a gravitoelectromagnetic searchlight are given by (35) and (36), respectively. Their existence depends upon the nonvanishing of the complex valued function $\beta(u)$. This in turn means that the plane fronted waves, both electromagnetic and gravitational, must have colliding wave fronts (see [1,2]). A useful property of these fields is given by the so-called Weyl double copy in the form of (37). In the context of tensor fields on Minkowskian space-time in Sec. IV this leads to simple relationships (85) and (86) between the electric and magnetic three-vectors of the electromagnetic field and the so-called electric and magnetic parts of the gravitational field. In our treatment of the gravitoelectromagnetic searchlight in coordinates X^i (in Sec. III), an interesting function W, which appears first in (48), plays a key role. Its most important attribute is that it is a wave function on Minkowskian space-time, which we have utilized in confirming that (4) is satisfied by the searchlight field and also in deriving (80). A more general scenario than that considered here emerges from the asymptotic limit (in the manner of [6]) of the Liénard-Wiechert electromagnetic field. The limiting case is algebraically general but contains a radiation part (just as the Liénard-Wiechert field does). A challenging open question is to find a corresponding gravitoelectromagnetic model in Einstein-Maxwell theory.

[1] P. A. Hogan and D. Puetzfeld, Gravitational waves with colliding or noncolliding wave fronts, Phys. Rev. D **103**, 124064 (2021).

[2] P. A. Hogan and D. Puetzfeld, *Exact Space–Time Models of Gravitational Waves* (Springer, Cham, 2022), ISBN 978-3-031-16825-3.

[3] W. Kundt, The plane fronted gravitational waves, Z. Phys. **163**, 77 (1961).

[4] J. Ehlers and W. Kundt, Exact solutions of the gravitational field equations, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley and Sons, Inc., New York, 1962), p. 49.

[5] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd. ed. (Cambridge University Press, Cambridge, 2003), ISBN 978-0-511-53518-5.

[6] P. A. Hogan and D. Puetzfeld, Plane fronted limit of spherical electro-magnetic and gravitational waves, Phys. Rev. D **104**, 124081 (2021).

[7] J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960).

[8] H. Godazgar, M. Godazgar, R. Monteiro, D. P. Veiga, and C. N. Pope, Weyl Double Copy for Gravitational Waves, Phys. Rev. Lett. **126**, 101103 (2021).

[9] J. L. Synge, *Relativity: The Special Theory* (North–Holland Publishing Company, Amsterdam, 1956).

[10] F. A. E. Pirani, Introduction to gravitational radiation theory, in *Lectures on General Relativity*, Brandeis Lectures (Prentice–Hall, New Jersey, 1965), Vol. 1, p. 249.