Noether's second theorem in teleparallel gravity

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Gauge symmetries in teleparallel gravity, together with the identities among the dynamical equations they provide, are analyzed in relation to the way they condition the coupling between matter and gravity. Particularly, the coupling of fermionic matter seems to be excluded in a wide range of teleparallel theories.

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I. INTRODUCTION

One of the distinctive features of Maxwell and Einstein equations is the *automatic conservation*,

$$\partial_\mu \partial_\nu (\sqrt{|g|} F^{\mu\nu}) \equiv 0, \qquad G^{\mu\nu}{}_{;\mu} \equiv 0, \tag{1}$$

which are off-shell identities evidencing that the dynamical equations are not independent; they imply a restriction on the number of genuine degrees of freedom (dof) the dynamical equations govern. Noether's second theorem traces the automatic conservation to a gauge symmetry of the action. The electromagnetic Lagrangian $L_{\mathrm{em}} \propto |g|^{1/2} F^{\mu\nu} F_{\mu\nu}$ is made of the field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which is invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \xi. \tag{2}$$

Thus the action is insensitive to the *local* variations $\delta A_\mu = \partial_\mu \xi$, not only on the solution to the dynamical equations but on arbitrary field evolutions. These variations are unable to generate dynamics; as a consequence, the resulting dynamical equations cannot be independent. Function $\xi(x)$ in Eq. (2) is called the *generator* of the gauge transformation. The gauge freedom of the potential A_μ can be completely fixed by means of two gauge conditions (for instance, $\vec{\nabla} \cdot \vec{A} = 0$ and $A_0 = 0$). Then the gauge symmetry (2) involves two spurious dof, so leaving the potential with two genuine dof (transversal modes). The reason why two spurious dof are associated with just one gauge generator comes from the fact that $\partial_0 A_0$ is not present in $F_{\mu\nu}$. Thus A_0 lacks a kinetic term in the Lagrangian, which means that A_0 is not a genuine dof. Besides, since A_0 is differentiated only along spatial directions, the variation of the action with respect to A_0 does not result in a dynamical equation for other components of A_μ but in the *constraint* $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_o$, which implies one additional spurious dof. So, let us review how the automatic conservation $(\sqrt{|g|} F^{\mu\nu})_{,\nu\mu} \equiv 0$ emerges from the gauge symmetry. The general variation of an action $S[A_\mu]$ is

$$\delta S = \int d^4x \left[\frac{\delta L}{\delta A_\mu} \delta A_\mu + \partial_\nu \left(\frac{\partial L}{\partial(\partial_\nu A_\mu)} \delta A_\mu \right) \right] \tag{3}$$

where $\delta L/\delta A_\mu$ stands for the Euler-Lagrange derivative $\partial L/\partial A_\mu - \partial_\nu (\partial L/\partial(\partial_\nu A_\mu))$. For the gauge symmetry (2), it is $\delta A_\mu = \partial_\mu \xi$ and $\delta S_{\mathrm{em}} \equiv 0$; thus, after an integration by parts one gets[1]

$$\int d^4x\, \xi \partial_\mu \left(\frac{\delta L_{\mathrm{em}}}{\delta A_\mu} \right) \equiv \int d^4x\, \partial_\mu \left[\xi \partial_\nu \left(\frac{\partial L_{\mathrm{em}}}{\partial(\partial_\nu A_\mu)} \right) + \xi \frac{\delta L_{\mathrm{em}}}{\delta A_\mu} \right]. \tag{4}$$

If the symmetry (2) were valid only for a specific function $\xi(x)$, then Eq. (4) would express a conservation law on-shell (i.e., when the Euler-Lagrange equations $\delta L_{\mathrm{em}}/\delta A_\mu = 0$ are fulfilled). The bracket in the right-hand side (rhs) of Eq. (4) would be then the (divergenceless) current associated with the conserved charge, as stated in Noether's first theorem. However the gauge symmetry (2) is valid no matter what the function $\xi(x)$ is. This essential feature puts the Eq. (4) in the context of Noether's second theorem. While the left-hand side (lhs) of Eq. (4) depends on the behavior of an arbitrary function $\xi(x)$ in the entire region of integration, the rhs just depends on its behavior on the

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[1]To write the rhs of Eq. (4), we have used that $\partial L_{\mathrm{em}}/\partial(\partial_\nu A_\mu)$ is antisymmetric in μ, ν.

boundary. Therefore, Eq. (4) makes sense only if the integrands on each side are identically zero *off-shell*. The conservation law we could deduce in this context is, to use Emmy Noether's words, *improper* [1]: the zero divergence of the current in the rhs of Eq. (4) is not a consequence of the dynamics but a mere off-shell identity.[2] The off-shell vanishing of $\partial_\mu(\delta L_{\mathrm{em}}/\delta A_\mu)$ in Eq. (4) yields the automatic conservation of Eq. (1). On the other hand, if the field is not free but sources are present, then the automatic conservation will force them to be conserved.

In the next sections we will discuss gauge symmetries in theories of gravity that admit a teleparallel formulation, the identities at the level of the dynamical equations they provide, and the way in which these identities restrict the coupling of the matter to gravity. Sections II and III are devoted to the invariance associated with diffeomorphisms. Section IV considers the invariance of general relativity under local Lorentz transformations of the tetrad. Section V studies the compatibility of the Dirac field with the identities coming from the gauge symmetries. Section VI displays the conclusions.

II. TELEPARALLEL GRAVITY

General relativity (GR) also exhibits a gauge symmetry; the transformation of the metric tensor [2]

$$\mathbf{g} \to \mathbf{g} + \pounds_\xi \mathbf{g}, \quad \text{or} \quad g_{\mu\nu} \to g_{\mu\nu} + 2\xi_{(\mu;\nu)}, \tag{5}$$

changes the Einstein-Hilbert action by a boundary term; so these variations of the metric do not generate dynamics. Thus, Einstein equations are not independent but are linked by the automatic conservation (1). Since the gauge transformations (5) come with four arbitrary generator functions $\xi_\mu(x)$, they suppress eight spurious dof; thus only two genuine dof remain among the ten components of the metric.[3]

General relativity and other theories of gravity can be formulated in terms of the tetrad field $\{\mathbf{E}^a(x)\}$—the basis of the cotangent space—and its dual basis $\{\mathbf{e}_a(x)\}$,

$$e^\mu_a E^a_\nu = \delta^\mu_\nu, \qquad e^\mu_a E^b_\mu = \delta^b_a. \tag{6}$$

The tetrad relates to the metric through the orthonormality condition

$$g_{\mu\nu} e^\mu_a e^\nu_b = \eta_{ab}, \qquad g_{\mu\nu} = \eta_{ab} E^a_\mu E^b_\nu, \tag{7}$$

where $\eta_{ab} = \mathrm{diag}\{1,-1,-1,-1\}$ is the Minkowski symbol.

Like $\mathbf{F} = d\mathbf{A}$ in electromagnetism, $\mathbf{T}^a = d\mathbf{E}^a$ is the fundamental magnitude in *teleparallel gravity*. Although $\mathbf{T}^a$ is invariant under the transformations $\mathbf{E}^a \to \mathbf{E}^a + d\xi^a$, this is not a symmetry in teleparallel theories because teleparallel Lagrangians are also made of vectors $\{\mathbf{e}_a\}$ and the volume $E = \det[E^a_\mu] = |g|^{1/2}$. The set of four 2-forms $\mathbf{T}^a$ can be read as a *torsion*, provided the Weitzenböck connection $\Gamma^\rho_{\lambda\mu} = e^\rho_b E^b_{\mu,\lambda}$ is adopted:

$$T^\rho{}_{\lambda\mu} = e^\rho_a T^a{}_{\lambda\mu} = e^\rho_a(\partial_\lambda E^a_\mu - \partial_\mu E^a_\lambda) = \Gamma^\rho_{\lambda\mu} - \Gamma^\rho_{\mu\lambda}. \tag{8}$$

Weitzenböck connection proves to be flat (so, parallelism is *absolute*; it does not depend on the path), and cancels out the covariant derivative of the tetrad (then, it is a *metric* connection).

The *teleparallel equivalent of general relativity* (TEGR) is defined by Lagrangian density

$$L_{\mathrm{TEGR}} = (2\kappa)^{-1} E S_\rho{}^{\lambda\mu} T^\rho{}_{\lambda\mu}, \tag{9}$$

where $S_\rho{}^{\lambda\mu}$ is

$$2S_\rho{}^{\lambda\mu} = \underbrace{\frac{1}{2}(T_\rho{}^{\lambda\mu} - 2T^{[\lambda\mu]}{}_\rho)}_{\text{contorsion } K^{\lambda\mu}{}_\rho} + 2T^{[\lambda}\delta^{\mu]}_\rho, \tag{10}$$

and $T^\mu = T_\lambda{}^{\lambda\mu} = K^{\lambda\mu}{}_\lambda$ is the *torsion vector*. On the other hand, the Einstein-Hilbert Lagrangian is $L_{\mathrm{EH}} = -(2\kappa)^{-1}|g|^{1/2}R$, where the Levi-Civita curvature R depends on second derivatives of the metric. TEGR and GR are dynamically equivalent because, once the relation (7) is used to write R in terms of the tetrad, their Lagrangians differ in a divergence

$$L_{\mathrm{TEGR}} = L_{\mathrm{EH}} + \partial_\nu(\kappa^{-1} E\, T^\nu). \tag{11}$$

Any teleparallel theory like TEGR, whose Lagrangian density L is homogeneous of degree 2 in first derivatives of the tetrad, accepts the following form for L:

$$L = \frac{1}{4\kappa} E\, E^a{}_{\nu,\mu} E^b{}_{\lambda,\rho} e^\mu_c e^\nu_e e^\rho_d e^\lambda_f M_{ab}{}^{cedf}. \tag{12}$$

In TEGR the symbol $M_{ab}{}^{cedf}$ is [3]

$$M_{ab}{}^{cedf} = 2\eta_{ab}\eta^{c[d}\eta^{f]e} - 4\delta^{[d}_a\eta^{f][c}\delta^{e]}_b + 8\delta^{[c}_a\eta^{e][d}\delta^{f]}_b, \tag{13}$$

but other combinations of its three terms are proposed in alternative theories of gravity known as new general relativity (NGR) [4–8]. Note that the antisymmetrized

[2]See Noether's analysis of Hilbert's assertion about the nonexistence of a proper concept of gravitational energy in general relativity. Noether points out that the space-time translations in general relativity are just a subgroup of an infinity dimensional symmetry group (diffeomorphisms) for which her second theorem applies.

[3]In Eq. (5) $\pounds$ denotes the Lie-derivative; the semicolon stands for the Levi-Civita covariant derivative. In a chart where the Levi-Civita connection locally vanishes, it is $g_{\mu\nu} \to g_{\mu\nu} + 2g_{\lambda(\mu}\xi^\lambda{}_{,\nu)}$, which coincides with the infinitesimal transformation of the metric components under the change of coordinates $x^\lambda \to x^\lambda + \xi^\lambda$.

indices of $M_{ab}{}^{cedf}$ make the derivatives of the tetrad enter the Lagrangian only through the components of the Weitzenböck torsion.

Lagrangian density (12) takes its most elegant form when written in terms of the anholonomy or commutation coefficients f^a_{bc},

$$\pounds_{\mathbf{e}_b}\mathbf{e}_c \equiv [\mathbf{e}_b, \mathbf{e}_c] = f^a_{bc}\mathbf{e}_a \Rightarrow f^a_{bc} = -(\pounds_{\mathbf{e}_b}\mathbf{E}^a)(\mathbf{e}_c) = -2e^\mu_c e^\nu_b E^a_{[\mu,\nu]} = e^\mu_c e^\nu_b T^a{}_{\mu\nu} \tag{14}$$

(Eq. (6) is used to solve f^a_{bc}). Thus the Lagrangian is written in terms of scalar objects valued in the tangent space,

$$L = \frac{1}{16\kappa} E f^a_{ce} f^b_{df} M_{ab}{}^{cedf} \coloneqq (2\kappa)^{-1} E\, T, \tag{15}$$

since no coordinate indices are left in its constituent parts (cf. Ref. [9,10]).

Teleparallel dynamics is invariant under the infinitesimal local transformation [the analogous of (5)]

$$\mathbf{E}^a \to \mathbf{E}^a + \pounds_\xi \mathbf{E}^a, \qquad \mathbf{e}_a \to \mathbf{e}_a + \pounds_\xi \mathbf{e}_a = \mathbf{e}_a + [\xi, \mathbf{e}_a], \tag{16}$$

which keeps the duality (6). In fact, on the one hand Eq. (14) implies that[4]

$$\delta f^a_{bc} = \delta(e^\mu_c e^\nu_b (d\mathbf{E}^a)_{\mu\nu}) = \pounds_\xi f^a_{bc} = \xi^\mu f^a_{bc,\mu}; \tag{17}$$

then it is $\delta T = \xi^\mu T_{,\mu}$. On the other hand, the change of the volume is

$$\delta E = E e^\lambda_a \delta E^a_\lambda = E e^\lambda_a (\xi^\nu E^a_{\lambda,\nu} + E^a_\nu \xi^\nu_{,\lambda}) = (E\xi^\nu)_{,\nu}. \tag{18}$$

The behavior of the Lagrangian (15) under the transformation (16) is then

$$\delta L = \delta(E\,T) = (E\xi^\nu T)_{,\nu} = (L\,\xi^\nu)_{,\nu}. \tag{19}$$

Equation (19) implies the action gets a boundary term under the transformation (16), so the dynamics is not affected. This conclusion can be extended to other teleparallel formulations, like $f(T)$ gravity; in fact,

$$\delta[Ef(T)] = f(T)\delta E + Ef'(T)\delta T = f(T)(E\xi^\nu)_{,\nu} + Ef'(T)\delta T = (Ef(T)\xi^\nu)_{,\nu}. \tag{20}$$

The result $\delta L = (L\,\xi^\nu)_{,\nu}$ is common to any theory of the spacetime geometry whose Lagrangian scalar is made of a chart-independent combination of the commutation coefficients. Thus teleparallel actions are not strictly gauge invariant under the transformation (16), but δS turns out to be[5]

$$\delta S|_{\delta E^a_\mu = (\pounds_\xi \mathbf{E}^a)_\mu} = \int d^4x \left[\frac{\delta L}{\delta E^a_\mu}\delta E^a_\mu + \partial_\nu\left(\frac{\partial L}{\partial(\partial_\nu E^a_\mu)}\delta E^a_\mu\right)\right]_{\delta E^a_\mu = (\pounds_\xi \mathbf{E}^a)_\mu} = \int d^4x\, \partial_\nu (L\,\xi^\nu), \tag{21}$$

i.e.,

$$\int d^4x \frac{\delta L}{\delta E^a_\mu}(\pounds_\xi \mathbf{E}^a)_\mu = \int d^4x\, \text{divergence}. \tag{22}$$

The components of $\pounds_\xi \mathbf{E}^a$ can be written in different ways,

$$(\pounds_\xi \mathbf{E}^a)_\mu = \xi^\lambda E^a_{\mu,\lambda} + E^a_\lambda \xi^\lambda{}_{,\mu} = (E^a_\lambda \xi^\lambda)_{,\mu} + \xi^\lambda (E^a_{\mu,\lambda} - E^a_{\lambda,\mu}) = (E^a_\lambda \xi^\lambda)_{,\mu} + \xi^\lambda T^a{}_{\lambda\mu}; \tag{23}$$

we will replace the last one in Eq. (22) which, after integration by parts, turns out to be

$$\int d^4x \xi^\lambda (-E^a_\lambda \partial_\mu + T^a{}_{\lambda\mu}) \frac{\delta L}{\delta E^a_\mu} = \int d^4x\, \text{divergence}. \tag{24}$$

In this equation the lhs depends on the behavior of the *arbitrary* infinitesimal vector field ξ in the entire region of integration, while the rhs only depends on the behavior of ξ and its first derivatives at the boundary. Therefore, Eq. (24) makes sense only if both integrands are identically zero off-shell, which leads to the automatic conservation

$$(E^a_\lambda \partial_\mu - T^a{}_{\lambda\mu}) \frac{\delta L}{\delta E^a_\mu} \equiv 0. \tag{25}$$

By rearranging this result, or using the first of the forms of $(\pounds_\xi \mathbf{E}^a)_\mu$ in Eq. (23), we can also write

$$\partial_\mu \left(E^a_\lambda \frac{\delta L}{\delta E^a_\mu}\right) - E^a_{\mu,\lambda} \frac{\delta L}{\delta E^a_\mu} \equiv 0. \tag{26}$$

These identities about the tensor density $E^a_\lambda \delta L/\delta E^a_\mu$, which possesses the structure of the dynamical equations, express the content of Noether's second theorem in teleparallel gravity regarding the gauge symmetry (16). They can be used to write the Levi-Civita divergence of the tensor $E^{-1}E^a_\lambda \delta L/\delta E^a_\mu$, to give the identity a more familiar look. We will need the relation between Weitzenböck and Levi-Civita connections, Γ and $\{\}$, which is given by the contorsion tensor introduced in Eq. (10):

$$K^\rho{}_{\mu\lambda} = \Gamma^\rho{}_{\lambda\mu} - \left\{ {\rho \atop \lambda\mu} \right\} \Rightarrow \left\{ {\rho \atop \lambda\mu} \right\} = e^\rho_b E^b_{\mu,\lambda} - K^\rho{}_{\mu\lambda}, \qquad \left\{ {\mu \atop \lambda\mu} \right\} = e^\mu_b E^b_{\mu,\lambda} = E^{-1}\partial_\lambda E, \tag{27}$$

[4]Lie and exterior derivatives commute.

[5]δE^a_μ in the Euler-Lagrange derivative involves the differentiation of e^ν_b. Duality (6) implies that $e^\nu_a \delta E^a_\mu = -E^a_\mu \delta e^\nu_a$.

since $K^{\mu}{}_{\mu\lambda} = 0$. Thus, the Levi-Civita divergence of a tensor is

$$C^{\mu}{}_{\lambda;\mu} = \partial_\mu C^{\mu}{}_{\lambda} + \left\{ \begin{matrix} \mu \\ \mu\rho \end{matrix} \right\} C^{\rho}{}_{\lambda} - \left\{ \begin{matrix} \rho \\ \mu\lambda \end{matrix} \right\} C^{\mu}{}_{\rho}$$
$$= E^{-1}\partial_\mu (E\, C^{\mu}{}_{\lambda}) - (e^{\rho}_{b} E^{b}_{\mu,\lambda} - K^{\rho}{}_{\mu\lambda}) C^{\mu}{}_{\rho}, \quad (28)$$

where we have used the symmetry of the Levi-Civita connection. By replacing the tensor density $EC^{\mu}{}_{\lambda}$ with $E^{a}_{\lambda}\delta L/\delta E^{a}_{\mu}$, and using Eq. (26) to cancel out terms, one obtains

$$\left(E^{-1} E^{a}_{\lambda} \frac{\delta L}{\delta E^{a}_{\mu}}\right)_{;\mu} - K^{\rho}{}_{\mu\lambda} E^{-1} E^{a}_{\rho} \frac{\delta L}{\delta E^{a}_{\mu}} \equiv 0, \quad (29)$$

which is an equivalent way of writing the automatic conservation (26).

III. THE ACTION OF MATTER

The automatic conservation is not only important as a tracer of the spurious dof in a theory. It also forces the sources to obey a conservation law with its same form. In electrodynamics, the automatic conservation $(\sqrt{|g|}F^{\mu\nu})_{,\nu\mu} \equiv 0$ constrains the charges to be conserved: $(\sqrt{|g|}j^{\mu})_{,\mu} = 0$. In turn, charge conservation is the key to have the gauge invariance of the coupling term $\sqrt{|g|}A_\mu j^\mu$ in the full field-charge action, since it changes as

$$\sqrt{|g|}A_\mu j^\mu \to \sqrt{|g|}(A_\mu + \xi_{,\mu}) j^\mu$$
$$= \sqrt{|g|}A_\mu j^\mu - \xi(\sqrt{|g|}j^\mu)_{,\mu} + \text{divergence}. \quad (30)$$

So, the conservation of the sources and the gauge symmetry of the coupling term are two features that go together.

In GR, the automatic conservation $G^{\mu\nu}{}_{;\mu} \equiv 0$ forces the energy-momentum tensor of the sources to satisfy $\mathfrak{T}^{\mu\nu}{}_{;\mu} = 0$. This means that the automatic conservation in theories of gravity essentially determines the dynamics of the sources; it fixes the evolution of the matter energy-momentum in each geometry.[6] Like in electromagnetism, the coupling matter-gravity must exhibit the symmetry that gives rise to the automatic conservation; only then will the dynamics of the sources be compatible with the dictates of the automatic conservation. This is a severe restriction to the form of the action of matter, which is entirely a matter-gravity coupling action because it is necessarily formulated in a geometric background. This symmetry requirement is overcome by writing the action of matter in terms of geometric objects, as is the case with the gravity action. In other words the action must be invariant under diffeomorphisms (see Note 3). Actually the action of matter must come with *all* the gauge symmetries of the gravity action to provide a consistent set of dynamical equations.

In the case of the teleparallel gauge symmetry (16), the action of matter must satisfy Eq. (29) on-shell. It is expected that the matter will keep to this consistency requirement by conserving its energy-momentum. This is so for any type matter that couples to the metric but not to its derivatives, like scalar or spin 1 matter. In fact, the absence of derivatives of the tetrad implies that

$$E^{a}_{\lambda} \frac{\delta L_{\text{mat}}}{\delta E^{a}_{\mu}} = E^{a}_{\lambda} \frac{\partial L_{\text{mat}}}{\partial E^{a}_{\mu}} = E^{a}_{\lambda} \frac{\partial L_{\text{mat}}}{\partial g_{\rho\nu}} \frac{\partial g_{\rho\nu}}{\partial E^{a}_{\mu}}$$
$$= -\sqrt{|g|}\mathfrak{T}^{\rho\nu}\eta_{ab}\delta^{\mu}_{\rho} E^{a}_{\lambda} E^{b}_{\nu} = -E\, \mathfrak{T}^{\mu}_{\lambda}. \quad (31)$$

where $\mathfrak{T}^{\mu\lambda} = -2|g|^{-1/2}\partial L_{\text{mat}}/\partial g_{\mu\lambda}$ is the *metric* energy-momentum tensor (the source of Einstein equations). By replacing this result in Eq. (29) one gets

$$\mathfrak{T}^{\mu}{}_{\lambda;\mu} - K^{\rho}{}_{\mu\lambda}\mathfrak{T}^{\mu}{}_{\rho} = 0. \quad (32)$$

At first sight this result seems to contradict the equivalence principle, since one expects to recover the special-relativity law $\mathfrak{T}^{\mu}{}_{\lambda,\mu} = 0$ in a chart where the Levi-Civita connection locally vanishes and the freely falling particles locally move on straight curves (their parametric equations are linear in the affine parameter). However the issue is solved by noting that the contorsion tensor is antisymmetric in its two first indices ($K^{\mu\nu}{}_{\lambda} = -K^{\nu\mu}{}_{\lambda}$) and the metric energy-momentum tensor is symmetric. Thus the second term on the lhs of Eq. (32) is zero, and the equivalence principle is safe (cf. Ref. [5]).

IV. LORENTZ GAUGE INVARIANCE IN TEGR

The antisymmetric part of TEGR dynamical equations is identically zero, as stressed in several publications [5,12–15]. This is an identity exclusive to TEGR, that comes from the TEGR gauge invariance under local Lorentz transformation of the tetrad; so it also falls within the framework of Noether's second theorem. As an object of the vector representation $(\frac{1}{2},\frac{1}{2})$ of the Lorentz group, the tetrad in Eq. (7) transforms as

$$\mathbf{E}^{a'} = \Lambda^{a'}{}_{a}\mathbf{E}^{a}, \quad \mathbf{e}_a = \mathbf{e}_{a'}\Lambda^{a'}{}_{a}, \quad \Lambda^{a'}{}_{a} = \exp\left[\frac{1}{2}\sigma_{gh}(M^{gh})^{a'}{}_{a}\right], \quad (33)$$

where the six generators M^{gh} are

$$(M^{gh})^{a}{}_{b} = 2\eta^{a[g}\delta^{h]}_{b}. \quad (34)$$

If the parameters $\sigma_{gh} = \sigma_{[gh]}$ are infinitesimal, the Lorentz transformation becomes

[6]See Sec. 20.6 in Ref. [11] for nice examples and discussions about this issue.

$$\delta_L E^a_\mu = \sigma_{gh}\eta^{a[g}\delta^{h]}_b E^b_\mu = \sigma^a{}_b E^b_\mu,$$
$$\delta_L e^\nu_b = -e^\nu_a \sigma_{gh}\eta^{a[g}\delta^{h]}_b = -e^\nu_a \sigma^a{}_b. \qquad (35)$$

Einstein-Hilbert action is invariant under *local* Lorentz transformations of the tetrad (parameters σ_{gh} become arbitrary functions) because it depends purely on the components of the metric (7), which are locally Lorentz invariant. Instead TEGR action is *pseudoinvariant*, since S_{TEGR} gets a boundary term after the transformation due to the divergence term in Eq. (11). The pseudoinvariance implies that the general structure of $\delta_L S_{\mathrm{TEGR}}$ [see Eqs. (3) and (24)] is not identically zero in this case, but it is

$$\delta_L S_{\mathrm{TEGR}} = \int d^4x \left[\frac{\delta L_{\mathrm{TEGR}}}{\delta E^a_\mu}\delta_L E^a_\mu + \partial_\nu\left(\frac{\partial L_{\mathrm{TEGR}}}{\partial(\partial_\nu E^a_\mu)}\delta_L E^a_\mu\right)\right]$$
$$\equiv \delta_L \int d^4x\, \partial_\nu(\kappa^{-1} E T^\nu). \qquad (36)$$

Therefore it results

$$\int d^4x \frac{\delta L_{\mathrm{TEGR}}}{\delta E^a_\mu}\delta_L E^a_\mu$$
$$\equiv \int d^4x\, \partial_\nu\left(\kappa^{-1} E\, \delta_L T^\nu - \frac{\partial L_{\mathrm{TEGR}}}{\partial(\partial_\nu E^a_\mu)}\delta_L E^a_\mu\right) \qquad (37)$$

(Lorentz matrices are unimodular, so $E = \det E^a_\mu$ is Lorentz invariant). The lhs in Eq. (37) depends on the behavior of the *free* parameters $\sigma_{gh}(x)$ in the entire region of integration; instead, the rhs depends on their values at the boundary. This result makes sense only if each side identically vanishes whatever the parameters σ_{gh} are. Thus we obtain six off-shell identities among the TEGR dynamical equations

$$\frac{\delta L_{\mathrm{TEGR}}}{\delta E^a_\mu}\eta^{a[g}E^{h]}_\mu \equiv 0. \qquad (38)$$

Equation (38) shows that the antisymmetric part of TEGR dynamical equations is identically zero. The way the matter couples to gravity in TEGR (and GR) must be consistent with this identity. The full action $S_{\mathrm{TEGR}} + S_{\mathrm{matter}}$ must be (pseudo) invariant under local Lorentz transformations of the tetrad field, otherwise the dynamical equations would be sourced by an (incompatible) nonsymmetric energy-momentum tensor. As shown above, this point is not a problem for matter fields that couple to the metric.

Unlike gauge transformations (2) and (16), the transformation (35) does not contain derivatives of the parameters σ^{ag}. Therefore, the identities (38) do not have the appearance of automatic conservation; they are six relations among the TEGR dynamical equations (not their derivatives). Neither the counting of the spurious dof develops like in the electromagnetic case, since the TEGR Lorentz gauge symmetry does not come from the absence of a kinetic term. The six off-shell identities (38) actually suppress six spurious dof, which must be added to the eight spurious dof coming from the automatic conservation (like in electromagnetism, each component of ξ in Eq. (16) involves two spurious dof). In sum, the sixteen elements of the tetrad E^a_μ contain only $16 - 2\times 4 - 6 = 2$ genuine dof.

V. DIRAC FIELD

We will now consider the Dirac field, which couples not to the metric but to the tetrad field. So, we will study how the Dirac field accommodates to the identities coming from the gravity sector. Dirac Lagrangian

$$L_D = E(i\bar\psi\gamma^c e^\nu_c \partial_\nu\psi - m\bar\psi\psi) \qquad (39)$$

is a scalar density under changes of chart, since ψ behaves like a scalar field in such case, and is invariant under *global* Lorentz transformations. As an object of the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group, the Dirac spinor has four complex components ψ^α which transform as [16]

$$\psi^{\alpha'} = U^{\alpha'}{}_\alpha \psi^\alpha, \quad \bar\psi_{\beta'} = (\psi^\dagger\gamma^0)_{\beta'} = \bar\psi_\beta U^\beta{}_{\beta'},$$
$$\text{where}\ U^\beta{}_{\alpha'} U^{\alpha'}{}_\alpha = \delta^\beta_\alpha, \quad U^{\alpha'}{}_\alpha = \exp\left[\frac{1}{2}\sigma_{gh}(S^{gh})^{\alpha'}{}_\alpha\right]. \qquad (40)$$

$\sigma_{gh} = \sigma_{[gh]}$ are the six parameters characterizing the set of Lorentz transformations.[7] The generators S^{gh} are

$$S^{gh} = \frac{1}{4}[\gamma^g, \gamma^h], \qquad (41)$$

and Dirac matrices γ^a satisfy the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}. \qquad (42)$$

As generators of different representations of the same group, S^{gh} in Eq. (41) and M^{gh} in Eq. (34) both obey the Lorentz algebra. The same parameters σ_{gh} must be used to have the same Lorentz transformation in each representation. Thus, those parameters used to rotate a vector 360° make the spinor to change the sign. The invariance of the Dirac equation

$$i\gamma^c e^\nu_c \partial_\nu\psi - m\psi = 0, \qquad i e^\nu_c \partial_\nu\bar\psi\gamma^c + m\bar\psi = 0, \qquad (43)$$

under Lorentz transformations implies that the Dirac matrices transform as [17]

$$\gamma^{c'} = U\gamma^c \Lambda^{c'}{}_c U^{-1}, \quad \text{or} \quad \Lambda^d{}_{c'}\gamma^{c'} = U\gamma^d U^{-1}, \qquad (44)$$

which moves the Dirac matrices to another representation of the Clifford algebra (42). This means that the change of

[7] α, β are spinor labels. They are also labels for the components of the Dirac matrices γ^c, which have been omitted.

basis Λ in the tangent space is compensated by the respective linear transformation U in the space of spinors, so leaving the Dirac equation invariant.

Let us compute $E^a_\lambda \delta L_D/\delta E^a_\mu$ to test the consistency of Dirac action with the automatic conservation (26). Since L_D does not contain derivatives of the tetrad, it is[8]

$$E^a_\lambda \frac{\delta L_D}{\delta E^a_\mu} = E^a_\lambda \frac{\partial L_D}{\partial E^a_\mu} = L_D \delta^\mu_\lambda - Ei\bar{\psi}\gamma^c e^\mu_c \partial_\lambda \psi = -E\mathfrak{T}^\mu{}_\lambda, \qquad (45)$$

where $\mathfrak{T}^\mu{}_\lambda$ is the canonical energy-momentum tensor of the Dirac field. The automatic conservation (26) compels the Dirac dynamics to make zero the quantity $(E\mathfrak{T}^\mu{}_\lambda)_{,\mu} - E^a_{\mu,\lambda} e^\rho_a E\mathfrak{T}^\mu{}_\rho$. By using Eq. (43) it becomes

$$(E\mathfrak{T}^\mu{}_\lambda)_{,\mu} - E^a_{\mu,\lambda} e^\rho_a E\mathfrak{T}^\mu{}_\rho = i\bar{\psi}\gamma^c (E\, e^\mu_c)_{,\mu} \partial_\lambda \psi. \qquad (46)$$

Therefore, the compatibility with the automatic conservation could be reached in a gauge fixed scheme where the Lorenz gauge $(Ee^\mu_c)_{,\mu} = 0$ (partially) fixes the gauge freedom (16).[9]

Tetrad $\mathbf{e}_c = e^\nu_c \partial_\nu$ is the geometric object (independent of the chart) that provides the Dirac equation (43) with the information about the gravitational-inertial field, thus taking the role that the metric plays for bosonic fields. If gravity is governed by TEGR (or GR) dynamics, then the tetrad will be determined *modulo* local Lorentz transformations. However Dirac Lagrangian (39) is not invariant under local Lorentz transformations because $\partial_\nu \psi$ does not transform as a spinor. Therefore local Lorentz transformations are not allowed in Dirac theory unless a covariant derivative $D_\nu \psi$ be introduced. The covariant derivative $D_\nu \psi$ must transform as a spinor,

$$(D_\nu \psi)^{\alpha'} = U^{\alpha'}{}_\alpha (D_\nu \psi)^\alpha, \qquad (47)$$

which will require a (to be determined) connection term [18],[10]

$$D_\nu \psi = \left(\partial_\nu + \frac{1}{2}\Omega_{gh\nu} S^{gh}\right)\psi. \qquad (48)$$

Thus,

$$\begin{aligned}(D_\nu \psi)' &= \left(\partial_\nu + \frac{1}{2}\Omega_{g'h'\nu} S^{g'h'}\right)\psi' \\ &= U\left(\partial_\nu + U^{-1}(\partial_\nu U) + \frac{1}{2}\Omega_{g'h'\nu} U^{-1} S^{g'h'} U\right)\psi. \qquad (49)\end{aligned}$$

According to Eq. (44) it is

$$S^{g'h'} = \Lambda^{g'}{}_g \Lambda^{h'}{}_h U S^{gh} U^{-1}, \qquad (50)$$

therefore

$$(D_\nu \psi)' = U\left(\partial_\nu + U^{-1}(\partial_\nu U) + \frac{1}{2}\Omega_{g'h'\nu} \Lambda^{g'}{}_g \Lambda^{h'}{}_h S^{gh}\right)\psi. \qquad (51)$$

To satisfy Eq. (47) the parenthesis in the rhs should be equal to D_ν. For infinitesimal transformations we have

$$U^{-1}(\partial_\nu U) = \frac{1}{2}\sigma_{gh,\nu} S^{gh}. \qquad (52)$$

This shows that a proper connection in Eq. (51) is one that transforms as

$$\Omega_{g'h'\nu} \Lambda^{g'}{}_g \Lambda^{h'}{}_h = \Omega_{gh\nu} - \sigma_{gh,\nu}, \quad \text{i.e.,} \quad \delta_L \Omega_{gh\nu} = -\sigma_{gh,\nu}. \qquad (53)$$

We will try with

$$\Omega_{gh\nu} = -\eta_{d[g}\, e^\rho_{h]} E^d_{\rho,\nu} + \cdots \qquad (54)$$

since, according to Eq. (33), it fulfills the Eq. (53):

$$\begin{aligned}\delta_L(-\eta_{d[g}\, e^\rho_{h]} E^d_{\rho,\nu}) &= -\eta_{d[g}\, e^\rho_{h]} (\delta_L E^d_\rho)_{,\nu} \\ &= -\eta_{d[g}\, e^\rho_{h]}\, \sigma^d_{b,\nu} E^b_\rho = -\sigma_{gh,\nu}. \qquad (55)\end{aligned}$$

Equation (54) only shows the compensation term we need to build a covariant derivative D_ν. However we must care the good behavior of $\Omega_{gh\nu}$ in the manifold index ν. Then, $E^d_{\rho,\nu}$ in Eq. (54) has to be replaced with a covariant derivative; its respective affine connection must be local Lorentz invariant not to disturb the behavior (53). Thus we are left with the Levi-Civita connection [21,22], since it depends just on the (locally Lorentz invariant) metric tensor. By using Eq. (27), the Levi-Civita covariant derivative of the tetrad turns out to be

$$\begin{aligned}E^d_{\rho;\nu} &= E^d_{\rho,\nu} - \left\{ \begin{matrix} \lambda \\ \nu\rho \end{matrix} \right\} E^d_\lambda = E^d_{\rho,\nu} - (\Gamma^\lambda_{\nu\rho} - K^\lambda{}_{\rho\nu}) E^d_\lambda \\ &= E^d_{\rho,\nu} - (e^\lambda_f E^f_{\rho,\nu} - K^\lambda{}_{\rho\nu}) E^d_\lambda = K^\lambda{}_{\rho\nu} E^d_\lambda \qquad (56)\end{aligned}$$

Thus, it is[11]

[8] We have used $\partial E/\partial E^a_\mu = E e^\mu_a$, and $\partial e^\nu_b/\partial E^a_\mu = -e^\mu_b e^\nu_a$ (see Note 5).

[9] $(Ee^\mu_c)_{,\mu} = Ee^\mu_{c;\mu}$ relates to the torsion vector since $T_\lambda = -E^c_\lambda e^\mu_{c;\mu}$.

[10] For a historical account see [19]. For fermion coupling in the broader context of general (linear) affine geometries see [20].

[11] $D_\nu(\bar{\psi}\psi) = \partial_\nu(\bar{\psi}\psi)$ because $\bar{\psi}\psi$ is a local Lorentz invariant; then it is $D_\nu\bar{\psi} = \partial_\nu\bar{\psi} - \frac{1}{2}\Omega_{gh\nu}\bar{\psi}S^{gh}$. Besides, $\overline{D_\nu\psi} = \overline{\partial_\nu\psi} + \frac{1}{2}\Omega_{gh\nu}\overline{S^{gh}\psi} = \partial_\nu\bar{\psi} + \frac{1}{2}\Omega_{gh\nu}\psi^\dagger (S^{gh})^\dagger \gamma^0 = D_\nu\bar{\psi}$, since it is $\gamma^0\gamma^0 = \mathbf{1}$, $\gamma^0\gamma^{a\dagger}\gamma^0 = \gamma^a$, and $(S^{gh})^\dagger = -\gamma^0 S^{gh}\gamma^0$.

$$\Omega_{gh\nu} = -\eta_{d[g}\, e^{\rho}_{h]} K^{\lambda}{}_{\rho\nu} E^{d}_{\lambda} = -K_{ghl} E^{l}_{\nu},$$
$$D_\nu \psi = \left(\partial_\nu - \frac{1}{2} K_{ghl} E^l_\nu S^{gh}\right)\psi \tag{57}$$

The set of six independent 1-forms $\mathbf{\Omega}_{gh} = -K_{ghl}\mathbf{E}^l$ is the Levi-Civita spin connection. $\mathbf{\Omega}_{gh}$ cannot be made zero by choosing a locally inertial chart since, as a geometric object, it does not depend on the coordinates. Instead, $\mathbf{\Omega}_{gh}$ is affected by local Lorentz transformations, because contorsion K_{ghl} is tensorial only under *global* Lorentz transformations. However the local transformations are unable to make K_{ghl} zero, since they contain just 6 parameters to make 24 components zero. While locally inertial charts are useful for viewing geodesics as straight lines and bosonic field equations as in special relativity, the effects of the spin connection on the Dirac field cannot be suppressed by changing coordinates or the tetrad [23]. The spin connection term in the Dirac Lagrangian implies a contribution of gravity to the fermion mass [24]; constitutes a form of nonminimal coupling.

Leaving aside the issue of compatibility between the covariantized Dirac Lagrangian and the automatic conservation (26), which would require a locally Lorentz invariant gauge fixing of the tetrad, let us pass to examine the consequences of the identity (38) that reflects the (pseudo) invariance of TEGR under local Lorentz transformations of the tetrad. For the original Lagrangian (39) it is

$$\frac{\delta L_D}{\delta E^a_\mu}\eta^{a[g}E^{h]}_\mu = -E\mathfrak{T}^{\mu}{}_{\lambda} e^{\lambda}_a \eta^{a[g}E^{h]}_\mu = -E\mathfrak{T}^{[gh]}. \tag{58}$$

Since $\mathfrak{T}^{gh}$ is nonsymmetric (even on-shell, when L_D vanishes); then the result (58) implies that Dirac Lagrangian (39) is not compatible with TEGR gravity [25]. Dirac canonical energy-momentum tensor cannot be a source of TEGR (or GR). The covariant derivative (48) is unable to modify this feature; it just adds to $\mathfrak{T}^{gh}$ terms that do not contain derivatives of the Dirac field which will not alter the nonsymmetric character of $\mathfrak{T}^{gh}$.

VI. CONCLUSIONS

We have discussed the identities emerging from the gauge symmetry (5) in teleparallel theories of gravity; they constitute the automatic conservation of Eq. (25) (equivalently, (26) or (29)). Scalar and spin 1 matter are compatible sources in teleparallel gravity since their energy-momentum tensors are (on-shell) conserved. Even the Dirac field, which couples to the tetrad rather than the metric, becomes compatible with the automatic conservation if a (locally) gauge fixed tetrad is adopted. Contrarily, the Dirac field is not compatible with the identities coming from the symmetry under local Lorentz transformations of the tetrad that characterizes TEGR (or GR), and makes the antisymmetric part of its dynamical equations vanish [see Eq. (38)]. Although the Levi-Civita spin connection (57) was introduced to endow the Dirac equation with invariance under local Lorentz transformations, this does not means that the Dirac Lagrangian thus fits the TEGR gauge symmetry (33). To emulate this symmetry, the matter action must be invariant under local Lorentz transformations of the tetrad *alone*, as happens with scalar and spin 1 matter. The gauge symmetry of the Dirac action requires, instead, the transformation of *both* the tetrad and the spinor. This difference between bosonic and fermionic matter is rooted in the fact that bosonic matter couples to the metric, which is itself local Lorentz invariant (no derivatives of the metric are needed in bosonic Lagrangians; the introduction of the Levi-Civita affine connection is innocuous indeed). Fermionic matter, in turn, couples to the tetrad. The common replacement $\gamma^c e^\nu_c = g^{\mu\nu}\gamma_\mu$ is misleading; it pretends a coupling to the metric but hides the coupling to the tetrad in γ_μ. In sum, the Levi-Civita spin connection $\Omega_{gh\nu}$ was introduced to make sense of the Dirac theory in a gravitational context where the tetrad is determined modulo local Lorentz transformations; however, the problem of considering fermionic matter as the source of this type of gravity theories is not solved in this way. Instead of forcing the coupling between TEGR (or GR) and Dirac theory, we might consider moving toward teleparallel theories of gravity preserving the six dof associated with Lorentz transformations of the tetrad [26], while keeping the dynamics of the metric close to that of GR. Only the global Lorentz symmetry would survive in such case; thus no covariant derivative D_ν would be necessary. Current teleparallel theories of modified gravity would not be appropriate for this purpose because they contain remnant local Lorentz symmetries [27,28].

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