Wave equations in conformally separable, accreting, rotating black holes

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The Teukolsky wave equations governing fields of spin 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2 are generalized to the case of conformally separable solutions for accreting, rotating black holes.

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I. INTRODUCTION

While the exterior geometry of an astrophysically realistic black hole is well modeled by the Kerr metric, the region below the event horizon remains more elusive. The inner horizon of a Kerr black hole is subject to the Poisson-Israel [1,2] "mass inflation" instability, in which incident outgoing and ingoing streams of accreting matter or radiation drive an exponentially growing curvature. The outcome of the inflationary instability, from both a classical and quantum perspective, is a subject of active research [3–25].

The Kerr solution [26,27] and its electrovac cousins [28,29] are strictly stationary, strictly separable, and axisymmetric. References [30–32] have generalized these spacetimes to allow for self-similar accretion, leading to conformally stationary, conformally separable, axisymmetric solutions for the interior structure of accreting, rotating black holes. Whereas strict separability posits that all geodesics are Hamilton-Jacobi separable, conformal separability posits only that null geodesics are separable. The hypothesis of conformal separability imposes all the usual constraints on the possible form of the separable line element [33,34] except that the overall conformal factor is permitted to be arbitrary. The further hypothesis of conformal stationarity posits that the spacetime has a conformal timelike Killing vector, or in other words, that the spacetime grows self-similarly, by accretion, and that the accretion rate is small.

The energy-momentum tensor that sources the conformally stationary, conformally separable solutions fits that of a collisionless fluid containing a combination of outgoing and ingoing components. It is remarkable that the entire system of Einstein equations (and Maxwell equations, if the black hole is charged), coupled to the equations of a collisionless fluid, are jointly separable and solvable.

The conformally separable solutions are approximate, holding in the asymptotic limit of small accretion rate. It is helpful to emphasize the precise sense in which the conformally separable solutions are approximate. The standard Λ-Kerr-Newman line element can be derived from three assumptions: time-translation symmetry, azimuthal symmetry, and the separability of the Hamilton-Jacobi equation for geodesics of particles of rest mass m. More specifically, the Λ-Kerr-Newman line element follows from assuming that the Hamilton-Jacobi equation separates "in the simplest possible way" [33], which requires that the particle action S be a separated sum [this is Eq. (28) of Ref. [31]]

$$S = \frac{1}{2} m^2 \lambda - Et + L\phi + S_x(x) + S_y(y), \qquad (1)$$

in which λ is an affine parameter, E and L are a conserved energy and angular momentum associated with time t translation and azimuthal ϕ symmetry, and $S_x(x)$ and $S_y(y)$ are functions of two other coordinates, x and y, respectively. The conformally separable line element [Eq. (2)] follows from relaxing the assumption of time-translation symmetry to conformal time-translation symmetry, meaning that the metric depends on a conformal time coordinate t only through an overall conformal factor e^{vt} with accretion rate v, and from relaxing the assumption of strict Hamilton-Jacobi separability to conformal separability, meaning that Eq. (1) is required to hold only for massless particles, $m = 0$. With the line element in Eq. (2) in hand, it is then a laborsome matter to separate the Einstein (and Maxwell) equations systematically. That separation works only if certain terms are treated as negligible, which Ref. [31] shows is valid in the asymptotic limit of small accretion rate v. The proof that the negligible terms can in fact be neglected is again laborsome.

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Since the accretion rate is small, the geometry of the conformal solutions is well approximated by the Kerr (or more general electrovac) geometry down to just above the inner horizon. However, even in the limit of a tiny accretion rate, hyper-relativistic counterstreaming between outgoing and ingoing streams just above the inner horizon ignites and then drives the Poisson-Israel mass inflation instability. During inflation, the proper counterstreaming density and pressure, along with the Weyl curvature, exponentiate to huge (super-Planckian, if quantum gravity did not intervene) values. The inflationary instability is the nonlinear realization of the infinite blueshift at the inner horizon first pointed out by Ref. [35].

The end result of mass inflation is the formation of a singularity near where the inner horizon should have been. Perturbation analyses in the Kerr geometry find this singularity to be either weak and null [4] or strong and spacelike [11,36]. The conformally stationary, conformally separable solutions predict a singularity of the latter type. Whereas a weak, null singularity is derived from the assumption that the black hole remains isolated into the indefinite future after an initial collapse or accretion event, real astronomical black holes are not isolated, with ongoing accretion from the cosmic microwave background, dark matter, gas, and ambient astronomical bodies [37]. As a result of the continued accretion, mass inflation stalls at exponentially large curvature, at which point the spacetime collapses (in the sense that the conformal factor shrinks to an exponentially tiny scale) toward a spacelike singularity. This is the general behavior predicted by the nonlinear, dynamical solutions of Refs. [30–32].

The conformally separable solutions are unrealistic in the sense that they require a special incident accretion flow, which is constant in time and constant in angle over the inner horizon. This would seem to diminish the physical relevance of the solutions. It appears, however, that the solutions have more general application to arbitrary accretion flows, as long as the accretion rate is small, as is true most of the time in most astronomical black holes. The reason is that during the initial, inflationary phase, the Kerr (or more general electrovac) geometry remains essentially unchanged while outgoing and ingoing streams focus and blueshift exponentially along the principal null directions. During this inflationary phase, tidal forces (second gradients of the metric) grow (exponentially) large, but the metric itself barely budges. Eventually, the tidal force starts to backreact on the spacetime, leading to collapse in the transverse directions, and stretching along the principal directions. Inflation and collapse occur over such a tiny proper time that what happens at one point on the inner horizon is causally disconnected from what happens at another point. Different causal patches on the inner horizon evolve essentially independently of each other. The conformally separable solutions [30–32] fail deep into the collapse regime, where rotational motions reassert themselves. Reference [38] has explored numerically what happens at that point, finding that inflation and collapse is followed by Belinskii-Khalatnikov-Lifshitz (BKL) [39–43] oscillatory collapse to a spacelike singular surface. The conformally separable inflation and collapse regimes prove to be just the first two Kasner epochs in a succession of Kasner epochs separated by BKL bounces.

The purpose of the present paper is to derive the wave equations of massless, neutral fields of various spin ($0, \frac{1}{2}, 1, \frac{3}{2}$, and 2) in conformally separable black hole spacetimes, extending the well-known Teukolsky solutions for stationary rotating black holes [27,44–47]. The problem is by itself of some interest: do the wave equations in rotating black holes remain separable if the condition of strict separability is relaxed to conformal separability? The answer is yes. The generalization of the Teukolsky master equation [27,44,45] to the conformally separable solutions is given by Eq. (60) with the potentials in Eq. (61).

The conformal wave equations are also relevant for analyzing quantum effects near the inner horizons of astronomically realistic black holes. Semiclassically, the quantum backreaction to the metric is governed by the renormalized vacuum expectation value of a field's energy-momentum tensor, which is built out of solutions to the wave equation. For Kerr black holes, this quantity has recently been calculated by Ref. [20] at the inner horizon, revealing singular behavior at that null boundary. The full story of what happens at the inner horizon of an astronomically realistic rotating black hole is a topic of active research [4,6,9,20,21,24,25]. One of the big outstanding questions is whether quantum effects transform a weak, null singularity into a strong, spacelike singularity, even in the absence of accretion from outside.

This paper considers only uncharged black holes, even though conformally separable solutions exist also for charged black holes [32]. The problem is that electromagnetic and gravitational perturbations become inextricably linked in the presence of a finite electric field, preventing separation of the wave equations. In his epic monograph, Chandrasekhar [46] was able to separate the coupled electromagnetic and gravitational perturbations for the case of a Reissner-Nordström (charged, nonrotating) black hole, but not for a charged, rotating black hole.

The plan of this paper is as follows. Section II reviews the conformally separable black hole solutions, and it defines the Newman-Penrose and spinor frames with respect to which the wave equations are separable. Section III summarizes the wave equations for massless fields of arbitrary spin. The summary in Sec. III is based on the wave equations derived in Secs. IV–VIII for each of the spins $s = 0, \frac{1}{2}, 1, \frac{3}{2}$, and 2.

Appendix A takes a deeper dive into the derivation of wave equations in general spacetimes. Appendix B gives a relation between differential operators needed to write down the expressions in Eq. (116) for the boost-weight-0 component $\tilde{F}_0$ of an electromagnetic wave. Appendix C translates between the notation of Chandrasekhar [46] and the present paper.

In this paper, mid-latin indices $k, l, \ldots$ run over tetrad vector indices, while early latin indices $a, b, \ldots$ run over chiral spinor indices. Greek indices $\kappa, \lambda, \ldots$ are coordinate indices. The units are such that the speed of light and Newton's gravitational constant are unity, $c = G = 1$.

II. CONFORMALLY SEPARABLE ROTATING BLACK HOLES

A. Line element

This section summarizes results from Refs. [30,31].

The conformally separable line element is, in conformal coordinates $\{x, t, y, \phi\}$,

$$ds^2 = \rho^2 \left[\frac{dx^2}{\Delta_x} - \frac{\Delta_x}{(1-\omega_x\omega_y)^2}(dt - \omega_y d\phi)^2 + \frac{dy^2}{\Delta_y} + \frac{\Delta_y}{(1-\omega_x\omega_y)^2}(d\phi - \omega_x dt)^2 \right]. \tag{2}$$

The line element (2) defines not only a metric, but also an orthonormal tetrad (see Sec. II B), whose null directions are chosen to align with the principal null directions. The line element (2) is Kerr (or more general electrovac), with two differences elaborated below: the conformal factor ρ [Eq. (6)], and the radial horizon function Δ_x [Eqs. (15) and (22c)]. Separation of the Einstein equations is most natural with respect to the radial and angular coordinates x and y, which are related to the usual Boyer-Lindquist radius r and polar angle θ by

$$r \equiv a \cot(ax), \qquad \cos\theta \equiv -y, \tag{3}$$

where a is the angular momentum per unit mass of the black hole, with a positive for right-handed rotation about the axis of rotation. Note that

$$\frac{\partial}{\partial x} = -R^2 \frac{\partial}{\partial r}, \qquad R \equiv \sqrt{r^2 + a^2}. \tag{4}$$

The quantities ω_x and ω_y in the conformally separable line element [Eq. (2)] take their usual Kerr forms, and they are functions only of the radial and angular coordinates, respectively:

$$\omega_x = \frac{a}{R^2}, \qquad \omega_y = a \sin^2\theta. \tag{5}$$

Physically, ω_x is the angular velocity of the principal null frame through the coordinates, while ω_y is the specific angular momentum of light rays on the principal null congruence.

The first difference in the line element (2) between stationary (Λ-Kerr-Newman) and conformally separable black hole spacetimes is in the conformal factor ρ. In stationary black hole spacetimes, the conformal factor ρ is separable, its square being a sum $\rho^2 = \rho_s^2 = r^2 + a^2\cos^2\theta$ of radial and angular parts. In the conformally separable spacetimes, the conformal factor ρ is a product of the Λ-Kerr-Newman separable factor ρ_s, a time-dependent factor e^{vt}, and an inflationary factor $e^{-\xi(x)}$:

$$\rho = \rho_s\rho_i, \qquad \rho_s = \sqrt{r^2 + a^2\cos^2\theta}, \qquad \rho_i = e^{vt-\xi(x)}. \tag{6}$$

It is useful to define the complex combination

$$\bar{\rho} \equiv r - ia\cos\theta, \tag{7}$$

whose complex conjugate is $\bar{\rho}^* = r + ia\cos\theta$, and whose absolute value is the separable conformal factor, $|\bar{\rho}| = \sqrt{\bar{\rho}\bar{\rho}^*} = \rho_s$. The conformally separable solutions apply in the limit of a small but nonvanishing accretion rate,

$$v \to 0. \tag{8}$$

The conformally separable spacetimes possess a Killing vector $\partial/\partial\phi$ associated with azimuthal symmetry, a conformal Killing vector $\partial/\partial t$ associated with conformal time-translation invariance, and a traceless conformal Killing tensor K^{mn} [31],

$$K^{mn} \equiv \frac{1}{2}\rho^2 \mathrm{diag}(1, -1, 1, 1), \tag{9}$$

which satisfies the conformal Killing equation

$$D_{(k}K_{mn)} - \frac{1}{3}\eta_{(km}D^l K_{n)l} = 0. \tag{10}$$

The connection between the existence of a conformal Killing tensor and the separability of wave equations is discussed by Ref. [48].

There is a gauge freedom in the choice of the zero point and the scaling of conformal time t. The proper time far from the black hole yet well inside any cosmological horizon defines the Kerr time t_{Kerr}:

$$t_{\mathrm{Kerr}} \equiv \int e^{vt} dt \propto \frac{e^{vt}}{v}. \tag{11}$$

The proper mass M of the black hole increases proportionally to the conformal factor, and thus linearly with Kerr time. It is natural to scale conformal time, and hence v, so that

$$M = v t_{\mathrm{Kerr}}. \tag{12}$$

With this choice, the accretion rate v is just equal to the dimensionless rate at which the proper mass of the black hole increases, as measured by a distant observer,

$$v = \dot{M}. \tag{13}$$

The second difference in the line element (2) between stationary (Λ-Kerr-Newman) and conformally separable black hole spacetimes is in the radial horizon function Δ_x. The conformally separable spacetimes share with their separable cousins the property that the radial horizon function Δ_x and the angular polar function Δ_y are functions only of the radial coordinate x and the angular coordinate y, respectively. Indeed, the polar function Δ_y for the conformally separable spacetimes is unchanged from the Λ-Kerr-Newman spacetimes, and is, including a possible cosmological constant Λ,

$$\Delta_y = \sin^2\theta\left(1 + \frac{\Lambda a^2 \cos^2\theta}{3}\right). \tag{14}$$

The analysis in this paper holds both outside and inside the outer horizon. The conformally separable geometry is the standard Λ-Kerr geometry down to just above the inner horizon, where the usual Teukolsky wave equations are recovered. In the Λ-Kerr region outside the outer horizon and down to just above the inner horizon, the horizon function Δ_x is

$$\Delta_x \underset{r \gg r_-}{\rightarrow} \frac{1}{R^2}\left(1 - \frac{2M_\bullet r}{R^2} - \frac{\Lambda r^2}{3}\right), \tag{15}$$

where $M_\bullet$ is the conformal mass of the black hole, a constant, and Λ is the cosmological constant.

However, the conformally separable solutions undergo violent inflation from just above the inner horizon inward. Just above the inner horizon, the horizon function is negative and tending to zero, $\Delta_x \to -0$. The radial coordinate x is then timelike, and increasing inward, the direction of increasing proper time, while the time coordinate t is spacelike, and increasing in the outgoing direction (at the inner horizon, outgoing particles accreted in the past encounter ingoing particles accreted in the future, so it is natural to choose outgoing particles inside the black hole to be moving forward in t, while ingoing particles move backward in t). The conformally separable solutions are characterized by two small parameters u, v which specify the rates $u \pm v$ of accretion of outgoing (+) and ingoing (−) collisionless streams incident on the inner horizon (do not confuse these accretion rates with the ingoing and outgoing tetrad indices u and v introduced in Sec. II B). Positivity of both accretion rates requires

$$u > v > 0. \tag{16}$$

The conformally separable solutions are valid in the limit of small but finite u and v.

Solution of the Einstein equations for the conformally separable line element [Eq. (2)] driven by outgoing and ingoing null streams leads to evolutionary equations for the inflationary exponent ξ and the horizon function Δ_x near the inner horizon [30,31]. First, define radial and angular tortoise coordinates x^* and y^* by

$$x^* \equiv \int \frac{dx}{-\Delta_x}, \qquad y^* \equiv \int \frac{dy}{\Delta_y}. \tag{17}$$

Outside the outer horizon r_+, the radial tortoise coordinate x^* increases outward, from $-\infty$ at the horizon r_+, to

$$x^* \to \begin{cases} +\infty & \text{at } r \to \infty \quad \text{if } \Lambda = 0, \\ +\infty & \text{at } r \to r_\Lambda \quad \text{if } \Lambda > 0, \\ \text{finite} & \text{at } r \to \infty \quad \text{if } \Lambda < 0. \end{cases} \tag{18}$$

Inside the outer horizon, the tortoise coordinate x^* increases inward, from $-\infty$ at the horizon r_+. Define the dimensionless quantity $U(x)$ by

$$U \equiv \frac{d\xi}{dx^*}. \tag{19}$$

The Einstein equations near the inner horizon lead to [31]

$$\frac{dU}{dx^*} = 2(U^2 - v^2), \tag{20a}$$

$$\frac{d\ln\Delta_x}{dx^*} = 3U - \Delta'_x, \tag{20b}$$

where $\Delta'_x \equiv d\Delta_x/dx|_{x_-}$, a constant, is the (positive) derivative of the electrovac horizon function at the inner horizon $x = x_-$. The initial condition for U well above the inner horizon is

$$U = u. \tag{21}$$

The differential equations of Eq. (20) with the initial condition in Eq. (21) solve to give the inflationary exponent ξ, the horizon function Δ_x, and the radial coordinate x near the inner horizon:

$$e^\xi = \left(\frac{U^2 - v^2}{u^2 - v^2}\right)^{1/4}, \tag{22a}$$

$$e^{vx^*} = \left[\frac{(U - v)(u + v)}{(U + v)(u - v)}\right]^{1/4}, \tag{22b}$$

$$\Delta_x = -e^{3\xi - \Delta'_x x^*}, \tag{22c}$$

$$x - x_- = -\int \frac{\Delta_x dU}{2(U^2 - v^2)}. \tag{22d}$$

Equation (22d) says that the radial coordinate x is essentially frozen at its inner horizon value x_- throughout inflation and collapse.

B. Tetrad

The line element [Eq. (2)] defines not only a metric $g_{\kappa\lambda}$, which is an inner product of coordinate tangent vectors $\boldsymbol{e}_\kappa$, but also, through

$$\boldsymbol{e}_\kappa \cdot \boldsymbol{e}_\lambda = g_{\kappa\lambda} = \eta_{kl} e^k{}_\kappa e^l{}_\lambda = e^k{}_\kappa \boldsymbol{\gamma}_k \cdot e^l{}_\lambda \boldsymbol{\gamma}_l, \tag{23}$$

a vierbein matrix $e^k{}_\kappa$, and a corresponding orthonormal tetrad $\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3\}$, whose inner products form the Minkowski metric, $\boldsymbol{\gamma}_k \cdot \boldsymbol{\gamma}_l = \eta_{kl}$. The orthonormal tetrad behaves smoothly across horizons provided that the time axis $\boldsymbol{\gamma}_0$ is chosen to be timelike and future-pointing both outside ($\Delta_x > 0$) and inside ($\Delta_x < 0$) the horizon, while the radial axis $\boldsymbol{\gamma}_1$ is outgoing both outside and inside the horizon:

$$\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3\} = \begin{cases} \{\boldsymbol{\gamma}_t, \boldsymbol{\gamma}_x, \boldsymbol{\gamma}_\theta, \boldsymbol{\gamma}_\phi\} & \Delta_x > 0 \\ \{\boldsymbol{\gamma}_x, \boldsymbol{\gamma}_t, \boldsymbol{\gamma}_\theta, \boldsymbol{\gamma}_\phi\} & \Delta_x < 0 \end{cases}. \tag{24}$$

The sign of the vierbein coefficient $e^1{}_x$ is negative outside the horizon [because the spacelike radial coordinate x decreases outwards, Eq. (4)], while $e^0{}_x$ is positive inside the horizon [because the timelike radial coordinate x increases inwards].

Whenever dealing with massless fields, it is advantageous to use the Newman-Penrose double-null tetrad formalism. The double-null Newman-Penrose basis tetrad $\{\boldsymbol{\gamma}_v, \boldsymbol{\gamma}_u, \boldsymbol{\gamma}_+, \boldsymbol{\gamma}_-\}$ corresponding to the orthonormal tetrad $\{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3\}$ is

$$\boldsymbol{\gamma}_{\substack{v\\u}} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{\gamma}_0 \pm \boldsymbol{\gamma}_1), \qquad \boldsymbol{\gamma}_\pm \equiv \frac{1}{\sqrt{2}}(\boldsymbol{\gamma}_2 \pm i\boldsymbol{\gamma}_3). \tag{25}$$

The choice in Eq. (24) of orthonormal axes ensures that the Newman-Penrose basis vectors [Eq. (25)] behave smoothly across horizons, with $\boldsymbol{\gamma}_v$ outgoing and $\boldsymbol{\gamma}_u$ ingoing both outside and inside the outer horizon. The tetrad metric of the Newman-Penrose basis vectors is ($k, l = v, u, +, -$)

$$\gamma_{kl} \equiv \boldsymbol{\gamma}_k \cdot \boldsymbol{\gamma}_l = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{26}$$

The transverse Newman-Penrose basis vectors $\boldsymbol{\gamma}_\pm$ defined by Eq. (25) are complex conjugates of each other,

$$\boldsymbol{\gamma}_- = \boldsymbol{\gamma}_+^*. \tag{27}$$

Consequently, the directed derivatives $\partial_\pm$ in the $\boldsymbol{\gamma}_\pm$ directions are complex conjugates,

$$\partial_- = \partial_+^*. \tag{28}$$

More generally, covariant tetrad-frame derivatives $D_\pm$ in the $\boldsymbol{\gamma}_\pm$ directions are complex conjugates,

$$D_- = D_+^*. \tag{29}$$

Boosting the tetrad frame by rapidity η in the v-u plane boosts the outgoing $\boldsymbol{\gamma}_v$ and ingoing $\boldsymbol{\gamma}_u$ basis vectors by

$$\boldsymbol{\gamma}_v \to e^{\eta}\boldsymbol{\gamma}_v, \qquad \boldsymbol{\gamma}_u \to e^{-\eta}\boldsymbol{\gamma}_u, \tag{30}$$

while spatially rotating the tetrad frame by angle ζ in the $+$-$-$ plane rotates the $\boldsymbol{\gamma}_+$ and $\boldsymbol{\gamma}_-$ basis vectors by

$$\boldsymbol{\gamma}_+ \to e^{-i\zeta}\boldsymbol{\gamma}_+, \qquad \boldsymbol{\gamma}_- \to e^{i\zeta}\boldsymbol{\gamma}_-. \tag{31}$$

Tetrad-frame tensors inherit their transformation properties from the tetrad-frame basis vectors $\boldsymbol{\gamma}_k$. A tensor of boost weight σ is multiplied by $e^{\sigma\eta}$ under a Lorentz boost by rapidity η in the v-u plane, while a tensor of spin weight ς is multiplied by $e^{-i\varsigma\zeta}$ under a right-handed spatial rotation by angle ζ in the $+$-$-$ plane [49]. The boost and spin weights of a tetrad-frame tensor can be determined by inspection, according to the rules

boost weight

$$= \text{number of } v \text{ minus number of } u \text{ covariant indices,} \tag{32a}$$

spin weight

$$= \text{number of } + \text{ minus number of } - \text{ covariant indices.} \tag{32b}$$

Contravariant indices count oppositely to covariant indices.

C. Spinors

To deal with fields of half-integral spin, it is necessary to introduce a matrix representation of the tetrad $\boldsymbol{\gamma}_k$. These matrices are commonly called Dirac γ matrices (which accounts for the notation $\boldsymbol{\gamma}_k$ for the tetrad basis vectors). In the chiral representation where the chiral operator is diagonal [Eq. (35) below], the Dirac γ matrices are the 4×4 real unitary ($\boldsymbol{\gamma}^k = \boldsymbol{\gamma}_k^\dagger$) matrices

$$\boldsymbol{\gamma}_v = \begin{pmatrix} 0 & \sigma_v \\ -\sigma_u & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma}_u = \begin{pmatrix} 0 & \sigma_u \\ -\sigma_v & 0 \end{pmatrix}, \tag{33a}$$

$$\boldsymbol{\gamma}_+ = \begin{pmatrix} 0 & \sigma_+ \\ \sigma_+ & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma}_- = \begin{pmatrix} 0 & \sigma_- \\ \sigma_- & 0 \end{pmatrix}, \tag{33b}$$

where σ_k are the Newman-Penrose Pauli matrices

$$\sigma_v \equiv \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_u \equiv \sqrt{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{34a}$$

$$\sigma_+ \equiv \sqrt{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- \equiv \sqrt{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{34b}$$

The Lorentz-invariant chiral operator γ_5 is $-i$ times the pseudoscalar I,

$$\gamma_5 \equiv -iI \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3 = \boldsymbol{\gamma}_v \wedge \boldsymbol{\gamma}_u \wedge \boldsymbol{\gamma}_+ \wedge \boldsymbol{\gamma}_- = -\frac{i}{4!}\varepsilon^{klmn}\boldsymbol{\gamma}_k\boldsymbol{\gamma}_l\boldsymbol{\gamma}_m\boldsymbol{\gamma}_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{35}$$

the sign convention for the totally antisymmetric tensor being $\varepsilon^{klmn} = -\varepsilon_{klmn} = [klmn]$ in a locally inertial frame.

The Dirac γ matrices act by matrix multiplication on Dirac spinors, which are spin-$\frac{1}{2}$ spinors in $3+1$ spacetime dimensions. A Dirac spinor ψ is a complex linear combination of chiral basis spinors $\boldsymbol{\epsilon}_a$,

$$\psi = \psi^a \boldsymbol{\epsilon}_a, \tag{36}$$

in which $\boldsymbol{\epsilon}_a$ is the foursome of chiral basis spinors

$$\boldsymbol{\epsilon}_a = \{\boldsymbol{\epsilon}_{\Uparrow\uparrow}, \boldsymbol{\epsilon}_{\Downarrow\downarrow}, \boldsymbol{\epsilon}_{\Uparrow\downarrow}, \boldsymbol{\epsilon}_{\Downarrow\uparrow}\}. \tag{37}$$

In the chiral representation, the chiral basis spinors $\boldsymbol{\epsilon}_a$ are column vectors with 1 in the ath place, zero elsewhere; for example $\boldsymbol{\epsilon}_{\Uparrow\uparrow} = \mathrm{col}(1,0,0,0)$. The basis spinors with boost and spin aligned, $\boldsymbol{\epsilon}_{\Uparrow\uparrow}$ and $\boldsymbol{\epsilon}_{\Downarrow\downarrow}$, are right-handed, with positive chirality, while spinors with boost and spin antialigned, $\boldsymbol{\epsilon}_{\Uparrow\downarrow}$ and $\boldsymbol{\epsilon}_{\Downarrow\uparrow}$, are left-handed, with negative chirality. The basis spinors satisfy a Lorentz-invariant, antisymmetric Dirac scalar product

$$\boldsymbol{\epsilon}_a \cdot \boldsymbol{\epsilon}_b = \varepsilon_{ab} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{38}$$

The spinor indices $\{\Uparrow, \Downarrow, \uparrow, \downarrow\}$ are spinor analogs of the Newman-Penrose vector indices $\{v, u, +, -\}$. Boosting the tetrad frame by rapidity η in the v-u plane boosts basis spinors with boost index $\Uparrow$ and $\Downarrow$ by [compare Eq. (30)]

$$\boldsymbol{\epsilon}_\Uparrow \to e^{\eta/2}\boldsymbol{\epsilon}_\Uparrow, \qquad \boldsymbol{\epsilon}_\Downarrow \to e^{-\eta/2}\boldsymbol{\epsilon}_\Downarrow, \tag{39}$$

while spatially rotating the tetrad frame by angle ζ in the $+$-$-$ plane rotates basis spinors with spin indices $\uparrow$ and $\downarrow$ by [compare Eq. (31)]

$$\boldsymbol{\epsilon}_\uparrow \to e^{-i\zeta/2}\boldsymbol{\epsilon}_\uparrow, \qquad \boldsymbol{\epsilon}_\downarrow \to e^{i\zeta/2}\boldsymbol{\epsilon}_\downarrow. \tag{40}$$

The boost and spin weights of a spinor tensor can be determined by inspection, according to the rules

boost weight

$$= \frac{1}{2}\text{ number of } \Uparrow \text{ minus number of } \Downarrow \text{ covariant indices,} \tag{41a}$$

spin weight

$$= \frac{1}{2}\text{number of } \uparrow \text{ minus number of } \downarrow \text{ covariant indices.} \tag{41b}$$

Contravariant indices count oppositely to covariant indices.

The spin-$\frac{3}{2}$ fields considered in Sec. VII carry both vector and spinor indices. The boost and spin weights of a vector-spinor tensor are just the sums of the boost and spin weights of the vector and spinor indices [Eqs. (32) and (41)].

III. WAVE EQUATIONS

The bulk of this paper, Secs. IV–VIII, is devoted to deriving wave equations for massless fields for each of the spins $s = 0, \frac{1}{2}, 1, \frac{3}{2}$, and 2 (in this paper, s is always positive). This section summarizes the results, giving expressions valid for any of the spins $0, \frac{1}{2}, 1, \frac{3}{2}$, or 2.

For brevity and clarity, many of the equations in this paper carry upper and lower indices, such as ^v_u and $\pm$ in Eq. (44); unless otherwise stated, such equations should be interpreted in the "natural" way as pairs of equations in which all upper indices apply to the upper equation, and all lower indices apply to the lower equation.

Appendix A takes a deeper dive into the derivation of the wave equations in a general spacetime, clarifying the origin of the wave equations in the conformally separable spacetimes considered in this paper.

A. Petrov type D

The conformally separable black hole spacetimes are Petrov type D, meaning that the only nonvanishing component of the Weyl tensor is its boost- and spin-weight-zero component. The right-handed boost/spin-weight-zero Weyl component $\tilde{C}_0 \equiv \tilde{C}_{vuvu}$ defined by Eq. (151) is

$$\rho^2\tilde{C}_0 = \frac{1}{12}\left(\frac{d^2(R^4\Delta_x)}{dr^2} + \frac{d^2\Delta_y}{dy^2}\right) + \frac{1}{2}\bar{\rho}\left[\frac{\partial}{\partial r}\left(R^4\Delta_x\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial y}\left(\Delta_y\frac{\partial}{\partial y}\right)\right]\bar{\rho}^{-1}. \tag{42}$$

The left-handed component is the same with $\bar{\rho} \to \bar{\rho}^*$ on the second line, so it is the complex conjugate of the right-handed component.

Reference [50] has emphasized that all vacuum spacetimes of Petrov type D satisfy decoupled wave equations that can be solved by separation of variables. The conformally separable black hole spacetimes are type D, but they are not vacuum; rather, the conformally separable spacetimes are sourced by collisionless outgoing and ingoing streams that produce an exponentially growing proper energy-momentum in the center-of-mass frame.

B. Frequency and azimuthal mode number

Conformal time-translation symmetry and axisymmetry of the background imply that wave amplitudes ψ can be expanded in modes of definite conformal frequency w and azimuthal mode number m:

$$\psi = e^{-i(wt+m\phi)}\Psi(x,y). \tag{43}$$

The signs of w and m accord with the convention that positive frequency w corresponds to positive energy, and that a wave of azimuthal angular momentum m varies as $e^{-im\phi}$ under a right-handed spatial rotation.

Periodicity requires m to be integral for integral spin s, or half-integral for half-integral spin s. Solutions with real frequency w define normal modes. The frequency w can also be complex. Of particular interest in gravitational wave astronomy are quasinormal modes, a discrete spectrum of complex frequencies for each angular mode ℓm, corresponding to long-lived modes of decay of a perturbed black hole [27,51–53].

Acting on modes ψ, Eq. (43), of definite (possibly complex) frequency w and azimuthal number m, the Newman-Penrose directed derivatives ∂_k are related to coordinate derivatives by

$$\sqrt{2}\rho\partial_{\substack{v\\u}}\psi = \frac{\substack{+\\ \pm}\,1}{\sqrt{|\Delta_x|}}\left(\mp\Delta_x\frac{\partial}{\partial x} + i\alpha_x\right)\psi, \tag{44a}$$

$$\sqrt{2}\rho\partial_{\pm}\psi = \frac{1}{\sqrt{\Delta_y}}\left(\Delta_y\frac{\partial}{\partial y} \pm \alpha_y\right)\psi, \tag{44b}$$

where the initial $\substack{+\\ \pm}$ sign in Eq. (44a) is $+$ for ∂_v, with $+$ outside and $-$ inside the horizon for ∂_u, and where α_x and α_y are defined by

$$\alpha_x \equiv w + m\omega_x, \qquad \alpha_y \equiv w\omega_y + m. \tag{45}$$

It is evident that the radial differential operators $\rho\partial_{\substack{v\\u}}$ are purely radial, in the sense that they involve a derivative with respect to the radial coordinate x and not the angular coordinate y, while the angular differential operators $\rho\partial_{\pm}$ are purely angular, in the sense that they involve a derivative with respect to the angular coordinate y and not the radial coordinate x.

C. Chirality

The chiral operator γ_5, Eq. (35), is a Lorentz invariant pseudoscalar. Waves of massless fields of nonzero spin can be decomposed into a sum of independently evolving right- and left-handed chiral components, which are eigenstates of γ_5 with eigenvalues $+1$ and -1, respectively. Physically, the right- and left-handed chiral components correspond to waves in which the spin axis is, respectively, aligned and antialigned with the boost axis.

A wave of given chirality and spin s has $2s+1$ components, with boost weights $\sigma = -s, -s+1, \ldots, s$, and spin weights ς either equal ($\varsigma = \sigma$, right-handed chirality) or opposite ($\varsigma = -\sigma$, left-handed chirality) to the boost weight. The $2s+1$ different components are coupled by equations of motion, so they are not independent of each other, but rather oscillate in harmony. In flat space far from the black hole, a component of spin s and boost weight σ falls off with radius as, according to Eq. (69),

$$\psi_\sigma \sim r^{-1-s\mp\sigma}, \tag{46}$$

where the $\mp$ sign is $-$ for outgoing waves, $+$ for ingoing waves—that is, the sign of σ is opposite to the direction of motion (the boost direction). Only the largest component survives far from the black hole, satisfying $\psi_\sigma \sim r^{-1}$. The large component is called the propagating component of the wave. Each of the outgoing and ingoing components has either of two chiralities, with spin weight ς equal (right-handed) or opposite (left-handed) to the boost weight σ.

D. Boost and spin raising and lowering operators

The boost and spin weight of a field can be read off from its covariant chiral indices [Eqs. (32) and (41)]. Operating on a field with one of the Newman-Penrose directed derivatives, ∂_k yields an object whose boost weight (if $k = v$ or u) or spin weight (if $k = +$ or $-$) differs by ± 1 from that of the field. In effect, the Newman-Penrose directed derivatives "raise" and "lower" the boost and spin weights of a field. However, the directed radial derivatives ∂_v and ∂_u do not commute with the directed angular derivatives ∂_+ and ∂_-.

It is advantageous to define modified versions of the Newman-Penrose directed derivatives with the property that the radial (boost) derivatives commute with the angular (spin) derivatives, besides having the property that, like the directed derivatives $\rho\partial_k$ [Eq. (44)], when acting on modes of definite frequency w and azimuthal number m, the radial derivatives are purely radial, and the angular derivatives are purely angular. Define therefore boost and spin raising and lowering operators ${}_\sigma\eth_k$ by [49,54,55] (see Appendix A for an exposition of how these operators are defined in a general spacetime)

$$ {}_\sigma\eth_{\substack{v\\u}} \equiv \substack{+\\ \pm}\sqrt{2}(R^2\sqrt{|\Delta_x|})^{\pm\sigma}\rho\partial_{\substack{v\\u}}(R^2\sqrt{|\Delta_x|})^{\mp\sigma}, \tag{47a} $$

$$ {}_\varsigma\eth_\pm \equiv \pm\sqrt{2}(\sqrt{\Delta_y})^{\pm\varsigma}\rho\partial_\pm(\sqrt{\Delta_y})^{\mp\varsigma}, \tag{47b} $$

where the initial $\substack{+\\ \pm}$ sign in Eq. (47a) is $+$ for the raising operator ${}_\sigma\eth_v$, with $+$ outside and $-$ inside the horizon for the lowering operator ${}_\sigma\eth_u$. The operators in Eq. (47) are constructed so that the boost raising and lowering operators ${}_\sigma\eth_v$ and ${}_\sigma\eth_u$ commute with the spin raising and lowering operators ${}_\varsigma\eth_+$ and ${}_\varsigma\eth_-$, for arbitrary boosts σ and spins ς. The initial signs in Eq. (47) ensure that the raising and lowering operators are Hermitian conjugates of each other [Eq. (48)]. The $\sqrt{2}$ factor brings the operators to conventional normalization. The definition [Eq. (47a)] of the boost raising and lowering operators holds both outside the horizon, where the horizon function is positive, $\Delta_x > 0$, and inside the horizon, where the horizon function is negative, $\Delta_x < 0$.

The boost/spin index σ or ς on the boost/spin raising and lowering operators ${}_\sigma\eth_{\substack{v\\u}}$ or ${}_\varsigma\eth_\pm$ is often suppressed in this paper, since it equals the boost/spin weight of the object being operated on. The boost and spin raising and lowering operators, respectively, raise and lower by 1 the boost and spin weight of the object they are operating on.

Acting on modes [Eq. (43)] of given frequency w and azimuthal mode m, the boost raising and lowering operators in Eq. (47a) connecting adjacent boost weights are Hermitian conjugates with respect to the integration measure dr over a suitable integration interval; and likewise, the spin raising and lowering operators in Eq. (47b) connecting adjacent spin weights are Hermitian conjugates with respect to the integration measure dy over the interval $[-1, 1]$:

$$ {}_\sigma\eth_v^\dagger = {}_{\sigma+1}\eth_u, \tag{48a} $$

$$ {}_\varsigma\eth_+^\dagger = {}_{\varsigma+1}\eth_-. \tag{48b} $$

Hermitian conjugacy of the spin raising and lowering operators follows from the fact that for any differentiable functions χ and ψ of the same definite (possibly complex) frequency w and azimuthal mode m, an integration by parts shows that

$$ \int \chi({}_\varsigma\eth_+\psi)dy = [\sqrt{\Delta_y}\chi\psi] + \int({}_{\varsigma+1}\eth_-\chi)\psi dy. \tag{49} $$

The surface term vanishes if the integration interval is $[-1, 1]$, since the polar function Δ_y vanishes at these limits. Similarly, Hermitian conjugacy of the boost raising and lowering operators follows from an analogous integration by parts,

$$ \int \chi({}_\sigma\eth_v\psi)dr = \mp[R^2\sqrt{|\Delta_x|}\chi\psi] + \int({}_{\sigma+1}\eth_u\chi)\psi dr, \tag{50} $$

where the sign $\mp$ on the surface term is $-$ outside the horizon ($\Delta_x > 0$) and $+$ inside the horizon ($\Delta_x < 0$). The surface term vanishes at horizons, where $\Delta_x = 0$, and also at infinity, provided that the functions χ and ψ decrease sufficiently rapidly at infinity.

E. Wave operators

As remarked after Eq. (46), the propagating component of a spin-s wave is the component with the most negative boost weight σ along the direction of motion, $\sigma = -s$ in the outgoing direction, and $\sigma = +s$ in the ingoing direction. For propagating waves of right-handed chirality, the spin weight equals the boost weight, $\varsigma = \sigma$, and the spin-s wave equations derived in Secs. IV–VIII are of the form

$$ ({}_{\pm s}\Box_{\substack{vu\\uv}} - {}_{\pm s}\Box_{\substack{+-\\-+}})\hat{\psi}_{\pm s} = 0, \tag{51} $$

where the wave operators ${}_\sigma\Box_{kl}$ are defined by Eq. (57). For left-handed chirality, the spin weight is the negative of the boost weight, $\varsigma = -\sigma$, and ${}_{\pm s}\Box_{\substack{+-\\-+}} \to {}_{\mp s}\Box_{\substack{-+\\+-}}$ in the wave equation (51). Components with general boost weights σ satisfy wave equations similar to Eq. (51), but with the addition of a term proportional to the boost/spin-weight-zero component $\tilde{C}_0$ of the Weyl tensor [Eq. (42)],

$$ ({}_\sigma\Box_{\substack{vu\\uv}} - {}_\sigma\Box_{\substack{+-\\-+}} + c_{s|\sigma|}\rho^2\tilde{C}_0)\hat{\psi}_\sigma = 0, \tag{52} $$

in which $c_{s|\sigma|}$ is a constant that depends on the spin s and on the absolute value $|\sigma|$ of the boost weight of the component [see Eqs. (109c), (109d), (139b)–(139e), and (156c), (156d)]. The constant $c_{s|\sigma|}$ vanishes if $|\sigma| = s$, as is true for propagating waves and their complements of opposite boost, in which case the wave equation reduces to Eq. (51). If $\rho^2\tilde{C}_0$ were a separated sum of radial and angular coordinates x and y, then the wave equations in Eq. (52) for general boost weights would indeed become separable, but the term on the second line of the expression (42) for $\rho^2\tilde{C}_0$ is a mixed function of radial and angular coordinates, preventing separability.

The scaled wave amplitude $\hat{\psi}_\sigma$ (with a hat) in Eq. (51) is related to the native wave amplitude ψ_σ by

$$ \psi_\sigma = \begin{cases} f_s\hat{\psi}_\sigma & \text{right chirality} \\ f_s^*\hat{\psi}_\sigma & \text{left chirality} \end{cases}, \tag{53} $$

where (note that s is positive)

$$ f_s \equiv \frac{1}{\bar{\rho}^s\rho_{\mathrm{i}}^{s+1}}, \tag{54} $$

with ρ_i, the inflationary conformal factor, defined by Eq. (6), and $\bar{\rho}$, the complex conformal factor, defined by Eq. (7). The scaling factors f_s and f_s^* for right- and left-handed chiralities are complex conjugates of each other.

The wave equations (51) admit separated solutions ($\varsigma = \pm\sigma$ right-/left-handed)

$$\hat{\psi}_\sigma = e^{-i(wt+m\phi)} X_\sigma(x) Y_\varsigma(y), \tag{55}$$

with the equations (51) separating as

$$({}_\sigma\Box_{\substack{vu\\uv}} - \lambda_\sigma)\hat{\psi}_\sigma = 0, \qquad ({}_\sigma\Box_{\substack{+-\\-+}} - \lambda_\sigma)\hat{\psi}_\sigma = 0, \tag{56}$$

for some eigenvalues λ_σ. For normal modes, the frequency w is real, in which case the angular wave operators $\Box_{+-}$ and $\Box_{-+}$ [Eq. (57)], are Hermitian, so the eigenvalues λ_σ are real. For quasinormal modes, the frequency w is complex, and the eigenvalues λ_σ are complex. As demonstrated in Secs. IV–VIII for spins $s = 0, \frac{1}{2}, 1, \frac{3}{2}$, and 2, in terms of the boost and spin raising and lowering operators $\eth_k$ defined by Eq. (47), the radial and angular spin-s wave operators $\Box_{kl}$ acting on waves $\hat{\psi}_\sigma$ of boost σ are (spin weight $\varsigma = +\sigma$ right-handed, $\varsigma = -\sigma$ left-handed)

$$\begin{aligned} {}_\sigma\Box_{\substack{vu\\uv}}\hat{\psi}_\sigma \equiv \Big[&-\mathrm{sgn}(\Delta_x)\,{}_{\sigma\mp 1}\eth_{\substack{v\\u}}\,{}_\sigma\eth_{\substack{u\\v}} + 4i\left(\sigma \mp \frac{1}{2}\right)wr \\ &+ \frac{1}{3}\left(\sigma \mp \frac{1}{2}\right)(\sigma \mp 1)\frac{d^2(R^4\Delta_x)}{dr^2}\Big]\hat{\psi}_\sigma, \end{aligned} \tag{57a}$$

$$\begin{aligned} {}_\varsigma\Box_{\substack{+-\\-+}}\hat{\psi}_\varsigma \equiv \Big[&{}_{\varsigma\mp 1}\eth_\pm\,{}_\varsigma\eth_\mp - 4\left(\varsigma \mp \frac{1}{2}\right)way \\ &- \frac{1}{3}\left(\varsigma \mp \frac{1}{2}\right)(\varsigma \mp 1)\frac{d^2\Delta_y}{dy^2}\Big]\hat{\psi}_\varsigma. \end{aligned} \tag{57b}$$

Note that the indices σ or ς on the radial and angular wave operators ${}_\sigma\Box$ and ${}_\varsigma\Box$ are always equal to the boost/spin weights σ and ς of the field $\hat{\psi}$ that they operate on, so they could be omitted for brevity. For normal modes, for which by definition the frequency w is real, each of the angular operators $\Box_{+-}$ and $\Box_{-+}$ is Hermitian thanks to the Hermitian conjugacy of the angular operators $\eth_+$ and $\eth_-$ [Eq. (48)]. The radial operators $\Box_{vu}$ and $\Box_{uv}$, on the other hand, are Hermitian only if the imaginary term $4i(\sigma \mp \frac{1}{2})$ in Eq. (57a) vanishes; thus, $\Box_{vu}$ is Hermitian only for $\sigma = +\frac{1}{2}$, while $\Box_{uv}$ is Hermitian only for $\sigma = -\frac{1}{2}$.

Equations (A7) and (A8) in Appendix A give expressions for the difference $\Box_{\substack{vu\\uv}} - \Box_{\substack{+-\\-+}}$ of radial and angular wave operators [Eq. (57)] which enter the wave equations (51) and (52).

Operating on separated solutions [Eq. (55)], the boost and spin raising and lowering operators [Eq. (47)] yield

$${}_\sigma\eth_{\substack{u\\v}}\hat{\psi}_\sigma = e^{-i(wt+m\phi)} Y_\varsigma(y)\left({}_\sigma\eth_{\substack{u\\v}} - \frac{i\alpha_x}{\sqrt{|\Delta_x|}}\right)X_\sigma(x), \tag{58a}$$

$${}_\varsigma\eth_\mp\hat{\psi}_\varsigma = e^{-i(wt+m\phi)} X_\sigma(x)\left({}_\varsigma\eth_\mp + \frac{\alpha_y}{\sqrt{\Delta_y}}\right)Y_\varsigma(y), \tag{58b}$$

where α_x and α_y are defined by Eq. (45). In terms of coordinate derivatives, the boost and spin raising and lowering operators acting on the functions $X_\sigma(x)$ of boost weight σ and $Y_\varsigma(y)$ of spin weight ς are, with x^* and y^* being the tortoise coordinates defined by Eq. (17),

$$\eth_{\substack{v\\u}} X_\sigma(x) = \frac{1}{\sqrt{|\Delta_x|}}\left(\pm\frac{d}{dx^*} - \frac{\sigma}{2}\frac{d\ln(R^4|\Delta_x|)}{dx^*}\right)X_\sigma(x), \tag{59a}$$

$$\eth_\pm Y_\varsigma(y) = \frac{1}{\sqrt{\Delta_y}}\left(\pm\frac{d}{dy^*} - \frac{\varsigma}{2}\frac{d\ln\Delta_y}{dy^*}\right)Y_\varsigma(y). \tag{59b}$$

In terms of coordinate derivatives, the wave equations (56) acting on separated solutions [Eq. (55)] for propagating outgoing ($\sigma = -s$) and ingoing ($\sigma = s$) right-handed ($\varsigma = \sigma$) and left-handed ($\varsigma = -\sigma$) waves reduce to

$$\frac{1}{R\Delta_x}\left[\frac{d^2}{dx^{*2}} + \left(\alpha_x - \frac{i\sigma}{2}\frac{d\ln|\Delta_x|}{dx^*}\right)^2 + V_\sigma\right]RX_\sigma = 0, \tag{60a}$$

$$-\frac{1}{\Delta_y}\left[\frac{d^2}{dy^{*2}} - \left(\alpha_y - \frac{\varsigma}{2}\frac{d\ln\Delta_y}{dy^*}\right)^2 + W_\varsigma\right]Y_\varsigma = 0, \tag{60b}$$

where the radial and angular potentials V_σ and W_ς are

$$\begin{aligned} V_\sigma = \Big[&\frac{1}{6}(1+2\sigma^2)\frac{d^2\Delta_x}{dx^2} - \frac{1}{3}(1-4\sigma^2)a^2\Delta_x \\ &- 2i\sigma\frac{d\alpha_x}{dx} - \lambda_{\sigma\varsigma}\Big]\Delta_x, \end{aligned} \tag{61a}$$

$$W_\varsigma = \left[\frac{1}{6}(1+2\varsigma^2)\frac{d^2\Delta_y}{dy^2} - 2\varsigma\frac{d\alpha_y}{dy} + \lambda_{\sigma\varsigma}\right]\Delta_y, \tag{61b}$$

the derivatives of α_x and α_y defined by Eq. (45) being

$$\frac{d\alpha_x}{dx} = \frac{2mar}{R^2}, \qquad \frac{d\alpha_y}{dy} = -2way. \tag{62}$$

Equation (60) with the potentials in Eq. (61) constitute the generalization of the Teukolsky master equation [27,44,45] to the conformally separable solutions for accreting, rotating black holes. The angular eigenfunctions Y_ς are unchanged from those of Λ-Kerr(-Newman) (stationary) black holes.

For Kerr(-Newman) (zero cosmological constant), the angular eigenfunctions can be expressed as spin-weighted spheroidal harmonics [56] (with parameter $c = -wa$), or as confluent Heun functions [57].

For spherical black holes, the angular eigenfunctions Y_ς reduce to spin-weighted spherical harmonics, whose eigenvalues are

$$\lambda_\varsigma = \ell(\ell+1) + \frac{1}{3}(1-\varsigma^2) \quad \text{(spherical)}, \tag{63}$$

with harmonic number $\ell = \ell_0, \ell_0 + 1, \ldots$ starting from $\ell_0 = \max(|\varsigma|, |m|)$.

The wave equations (60) with Eq. (61) satisfy some discrete symmetries. If the frequency w is real (or if w is complex, but left unconjugated, so that α_x and α_y are unconjugated), then the wave equation (60a) for $X_{-\sigma}$ is the complex conjugate of that for X_σ. The wave equations (60) are unchanged if the boost and spin weights σ and ς are flipped, and at the same time the signs of the frequency w and azimuthal number m, hence α_x and α_y, are flipped:

$$\sigma \to -\sigma, \qquad \varsigma \to -\varsigma, \qquad w \to -w, \qquad m \to -m. \tag{64}$$

F. Asymptotics

The radial wave equation (60a) simplifies whenever the radial potential V_σ is negligible,

$$V_\sigma \sim 0. \tag{65}$$

This happens if the horizon function Δ_x, Eq. (15), is small, and the angular momenta [the azimuthal number m, and the eigenvalue λ_ς of the angular wave equation (60b)] are not too large. The horizon function Δ_x goes to zero at horizons (inner, outer, or cosmological), and at spatial infinity in asymptotically flat space (which happens if the cosmological constant is zero). In the case of the inner horizon, the term $\Delta_x d^2\Delta_x/dx^2$ in the potential V_σ grows large during inflation and collapse, and it must be retained; this case is deferred to the end of Sec. III F.

When the potential V_σ is negligible, the Wentzel—Kramers—Brillouin (WKB) solution of the radial wave equation (60a) gives

$$RX_\sigma \sim e^{\pm i(wx^* + m\omega_x^*)}|\Delta_x|^{\pm\sigma/2}, \tag{66}$$

where ω_x^* is an ω_x-weighted tortoise coordinate,

$$\omega_x^* \equiv -\int \frac{\omega_x dx}{\Delta_x}. \tag{67}$$

Consider first spatial infinity in asymptotically flat (Minkowski) space. In this case, the horizon function [Eq. (15)] falls off as $\Delta_x \sim 1/r^2$ as $r \to \infty$, so that the WKB solution [Eq. (66)] for the radial wave X_σ is

$$X_\sigma \sim e^{\pm i(wx^* + m\omega_x^*)} r^{-1\mp\sigma} \quad r \to \infty. \tag{68}$$

The Weyl tensor also goes to zero at infinity, so the wave equations (52) for arbitrary boost weights σ become separable, and the WKB solution in Eq. (68) holds for arbitrary boost weight. The upper sign in Eq. (68) is for an outgoing wave, where $\psi \sim e^{-i\omega(t-x^*)}$, while the lower sign is for an ingoing wave, where $\psi \sim e^{-i\omega(t+x^*)}$. The scaling factor f_s, Eq. (53), contributes an additional factor of r^{-s}. The net wave function ψ_σ, Eq. (55), of a component of boost weight σ far from the black hole is

$$\psi_\sigma \sim r^{-1-s} \begin{cases} r^{-\sigma} e^{-i[w(t-x^*) + m(\phi - \omega_x^*)]} & \text{outgoing} \\ r^{+\sigma} e^{-i[w(t+x^*) + m(\phi + \omega_x^*)]} & \text{ingoing} \end{cases}. \tag{69}$$

The propagating component is the one that falls off most slowly at infinity, so the propagating wave has $\sigma = -s$ for an outgoing wave and $\sigma = +s$ for an ingoing wave, as already claimed following Eq. (46). For the propagating wave, the radial factor is $\psi_\sigma \sim r^{-1}$. Equation (69) holds for both right-handed ($\varsigma = +\sigma$) and left-handed ($\varsigma = -\sigma$) chiralities.

Now, consider waves in the vicinity of the outer horizon, or the cosmological horizon if the cosmological constant Λ is positive. Near a horizon, the propagating wave functions ψ_σ are ($\sigma = -s$ outgoing, $\sigma = +s$ ingoing)

$$\psi_\sigma \sim |\Delta_x|^{-s/2} \begin{cases} e^{-i[w(t-x^*) + m(\phi - \omega_x^*)]} & \text{outgoing} \\ e^{-i[w(t+x^*) + m(\phi + \omega_x^*)]} & \text{ingoing} \end{cases}, \tag{70}$$

which diverge as $|\Delta_x|^{-s/2}$ for both outgoing and ingoing waves. A tensor of boost weight σ is multiplied by $e^{\sigma\eta}$ under a boost by rapidity η in the v-u plane. The rapidity η is positive for an outward boost, and negative for an inward boost. The divergence of the propagating components at the outer horizon can be removed by an outward boost of the outgoing wave by boost factor $e^\eta = |\Delta_x|^{-1/2}$, and by an inward boost of the ingoing wave by boost factor $e^\eta = |\Delta_x|^{1/2}$:

$$\psi_\sigma \sim \begin{cases} e^{-i[w(t-x^*) + m(\phi - \omega_x^*)]} & \text{out wave, out frame} \\ e^{-i[w(t-x^*) + m(\phi - \omega_x^*)]}|\Delta_x|^{-s} & \text{out wave, in frame} \\ e^{-i[w(t+x^*) + m(\phi + \omega_x^*)]} & \text{in wave, in frame} \\ e^{-i[w(t+x^*) + m(\phi + \omega_x^*)]}|\Delta_x|^{-s} & \text{in wave, out frame} \end{cases}. \tag{71}$$

Equation (71) says that a (suitably boosted) outgoing observer sees an outgoing wave as having constant amplitude near the horizon, and similarly an ingoing observer sees an ingoing wave as having constant amplitude; but an outgoing observer sees an ingoing wave, and an ingoing observer sees an outgoing wave, boosted by a diverging factor $|\Delta_x|^{-s}$.

The apparent divergence of outgoing waves seen by an ingoer, and of ingoing waves seen by an outgoer, should be interpreted with care. Near the outer horizon of a black hole, outgoing waves, whether outside or inside the horizon, always move away from the horizon, so an ingoer always sees the amplitude of an outgoing wave near the outer horizon as smaller than when the outgoing wave was emitted. Similarly, near a cosmological horizon, ingoing waves, whether outside or inside the horizon, always move away from the horizon, so again an outgoer always sees the amplitude of an ingoing wave near the cosmological horizon as smaller than when the outgoing wave was emitted. Thus, observers near the outer horizon or cosmological horizon do not see any actual divergence.

An asymptotic analysis applies also in the inflationary/collapse regime near the inner horizon, where the horizon function Δ_x is small (negative) and the angular momenta (m and λ_ς) are not too large; but the $\Delta_x d^2\Delta_x/dx^2$ term in the potential V_σ grows large during inflation and collapse, and must be retained. The Einstein equations (20) allow the radial wave equation (60a) to be recast in terms of the inflationary variable U defined by Eq. (19),

$$\left[\left(2(U^2-v^2)\frac{d}{dU}\right)^2+\left(\alpha_x-\frac{i\sigma}{2}(3U-\Delta'_x)\right)^2 + (1+2\sigma^2)(U^2-v^2)\right]X_\sigma=0, \tag{72}$$

the term on the second line being the $\Delta_x d^2\Delta_x/dx^2$ term in the potential V_σ [Eq. (61a)]. The solutions to Eq. (72) can be expressed in terms of hypergeometric functions ${}_2F_1$, but the expressions are unenlightening. A more insightful approach is to take the $v\to 0$ limit of Eq. (72), in which case the solutions are Whittaker functions $M_{\kappa,\lambda}(z)$ and $W_{\kappa,\lambda}(z)$, which are scaled versions of Kummer confluent hypergeometric functions ${}_1F_1$ [58]. The solutions to Eq. (72) with $v\to 0$ are [where $(M|W)$ in the following denotes either of the two Whittaker functions M or W]

$$X_\sigma=(M|W)_{\frac{3\sigma}{4},\frac{\sigma}{4}}\left(\frac{-2i\alpha_x+\sigma\Delta'_x}{2U}\right), \tag{73}$$

the tortoise coordinate x^* in this case reducing to

$$x^*=\text{constant}-\frac{1}{2U} \tag{74}$$

from Eq. (20a).

The radial wave solutions [Eq. (73)] may seem abstruse, but it is worth noting that Whittaker functions are solutions to the radial wave equation for fields of spin s in flat (Minkowski) spacetime. Specifically, waves of spin s and boost weight σ in flat spacetime are ($\varsigma=+\sigma$ right-handed, $\varsigma=-\sigma$ left-handed)

$$\psi_\sigma=r^{-1-s}e^{-i(wt+m\phi)}Y_\varsigma(y)\begin{cases}M_{\sigma,\ell+\frac{1}{2}}(2iwr) & \text{outgoing}\\ W_{\sigma,\ell+\frac{1}{2}}(2iwr) & \text{ingoing}\end{cases}$$
$$\underset{r\to\infty}{\to} r^{-1-s}Y_\varsigma(y)\begin{cases}r^{-\sigma}e^{-i[w(t-r)+m\phi]} & \text{outgoing}\\ r^{+\sigma}e^{-i[w(t+r)+m\phi)]} & \text{ingoing}\end{cases}. \tag{75}$$

Note that in flat spacetime, α_x as defined by Eq. (45) reduces to the temporal frequency $\alpha_x=w$, and $Y_\varsigma(y)e^{-im\phi}$ reduce to standard spin-weighted spherical harmonics.

G. Cosmological constant

The outgoing and ingoing hyper-relativistic streams that drive inflation near the inner horizon in the conformally separable solutions carry an exponentially growing proper energy-momentum. But because the streams are null, the trace of their energy-momentum is zero. The only possible nonvanishing contribution to the Ricci scalar R in the conformally separable solutions is from a cosmological constant Λ, which contributes a Ricci scalar

$$R=4\Lambda. \tag{76}$$

In the real Universe, the cosmological constant Λ is tiny compared to the energy density of any astronomical black hole, so it is negligible in practice. It is nevertheless useful for the sake of completeness to consider the possibility of a cosmological constant in the wave equations.

For spin half, one, or two, $s=\frac{1}{2}$, 1, or 2, the wave equations for propagating waves remain separable in the presence of a cosmological constant, the only effect of the cosmological constant being on the functional form of the horizon function Δ_x, Eq. (15). For spins $s=0$ or $\frac{3}{2}$, the effect of a cosmological constant (besides modifying the horizon function Δ_x) is to replace the difference of radial and angular wave operators in the wave equations by

$$\Box_{\substack{vu\\uv}}-\Box_{\substack{+-\\-+}}\to\Box_{\substack{vu\\uv}}-\Box_{\substack{+-\\-+}}+\chi_s\rho^2\Lambda \tag{77}$$

for some spin-dependent constant χ_s. In the Λ-Kerr-(Newman) regime away from the inner horizon, the conformal factor is separable, $\rho^2\to\rho_s^2=r^2+a^2y^2$ [Eq. (6)], and the wave equations remain separable if the radial and angular wave operators $\Box_{\substack{vu\\uv}}$ and $\Box_{\substack{+-\\-+}}$ are adjusted by $+\chi_s r^2\Lambda$ and $-\chi_s a^2y^2\Lambda$, respectively. This adjustment fails in the inflationary regime near the inner horizon, because the conformal factor $\rho=\rho_s\rho_i$ ceases to be separable. However, the conformally separable solutions hold in the asymptotic limit [Eq. (8)] of small accretion rate. In this limit, the conformal factor can be effectively separated as $\rho^2=(\rho^2-a^2y^2)+a^2y^2$, and the radial and angular wave operators $\Box_{\substack{vu\\uv}}$ and $\Box_{\substack{+-\\-+}}$ defined by Eq. (57) adjusted as

$$\Box_{\substack{vu\\uv}} \to \Box_{\substack{vu\\uv}} + \chi_s(\rho^2 - a^2y^2)\Lambda,$$
$$\Box_{\substack{+-\\-+}} \to \Box_{\substack{+-\\-+}} - \chi_s a^2 y^2 \Lambda, \tag{78}$$

once again yielding separable wave equations. The adjustment in Eq. (78) works in the Λ-Kerr(-Newman) regime where the conformal factor ρ is separable; in the early inflationary regime, where the derivatives $\partial\xi/\partial x$ and $\partial^2\xi/\partial x^2$ of the inflationary factor grow exponentially huge but the inflationary factor $\xi(x)$ itself remains sensibly equal to zero, so the inflationary conformal factor is still equal to one, $\rho_{\rm i} = 1$; and in the late inflationary and collapse phases, where $\xi(x)$ grows exponentially huge but the angular coordinate y is frozen at a constant value, its value on the inner horizon.

A possible cosmological constant Λ is retained for completeness throughout this paper. Throughout this paper, the radial and angular wave operators $\Box_{\substack{vu\\uv}}$ and $\Box_{\substack{+-\\-+}}$ are as defined by Eq. (57), without the adjustment in Eq. (78) from a cosmological constant.

IV. SPIN-0 WAVES

The wave equation for a minimally coupled massless scalar field φ is

$$D^k D_k \varphi = 0. \tag{79}$$

Denote the scalar field $\varphi \equiv \psi_0$ and define a scaled scalar $\hat{\varphi}$ by Eq. (53):

$$\varphi \equiv f_0 \hat{\varphi}, \tag{80}$$

with $f_0 \equiv 1/\rho_{\rm i} = e^{-vt+\xi(x)}$ [Eq. (54)]. The d'Alembertian $D^k D_k$ for the scalar field φ can be written in terms of the spin $s = 0$ wave operators $\Box_{kl}$ defined by Eq. (57),

$$\rho^2 D^k D_k \varphi = f_0\left(\Box_{vu} - \Box_{+-} + \frac{1}{6}\rho^2 R\right)\hat{\varphi}, \tag{81}$$

where R is the Ricci scalar. A more detailed exposition of the derivation of wave equations is given in Appendix A; the scalar d'Alembertian is given by Eq. (A10). As discussed in Sec. III G, the only possible contribution to the Ricci scalar in the conformally separable black hole spacetimes is from a cosmological constant Λ, in which case $R = 4\Lambda$. If the cosmological constant is nonzero, there are two possibilities that yield a separable spin-0 wave equation: The first is to adjust the radial and angular wave operators $\Box$ per Eq. (78). The second is to take the Ricci scalar over to the left-hand side of Eq. (81), in which case the equation becomes the conformally coupled scalar wave equation [59]

$$\rho^2\left(D^k D_k - \frac{1}{6}R\right)\varphi = f_0(\Box_{vu} - \Box_{+-})\hat{\varphi} = 0. \tag{82}$$

Equation (82) establishes the correctness of the claimed wave equation (51) for the case of zero spin, $s = 0$.

V. SPIN-$\frac{1}{2}$ WAVES

The wave equation for massless spin-$\frac{1}{2}$ waves is the massless Dirac equation

$$\boldsymbol{D}\psi = 0, \tag{83}$$

where ψ is a Dirac spinor [Eq. (36)], $\boldsymbol{D}$ is the covariant derivative

$$\boldsymbol{D} \equiv \boldsymbol{\gamma}^k D_k, \tag{84}$$

and $\boldsymbol{\gamma}_k$ are Dirac γ matrices given by Eq. (33). To make the boost and spin weights transparent, according to the rules in Eq. (32), it is convenient to work with the covariant components ψ_a of the spinor,

$$\psi = \psi_a \boldsymbol{\epsilon}^a. \tag{85}$$

This is consistent with the convention that, for example, $\boldsymbol{\gamma}^v \eth_v$ with covariant $\eth_v$, and not $\boldsymbol{\gamma}_v \eth^v$ with contravariant $\eth^v$, is a raising operator. The covariant components ψ_a are related to the contravariant components ψ^a of the expansion [Eq. (36)] by

$$\psi_a = \begin{pmatrix} \psi_{\Uparrow\uparrow} \\ \psi_{\Downarrow\downarrow} \\ \psi_{\Uparrow\downarrow} \\ \psi_{\Downarrow\uparrow} \end{pmatrix} = \varepsilon_{ab}\psi^b = \begin{pmatrix} \psi^{\Downarrow\downarrow} \\ -\psi^{\Uparrow\uparrow} \\ -\psi^{\Downarrow\uparrow} \\ \psi^{\Uparrow\downarrow} \end{pmatrix}. \tag{86}$$

The top two components $\psi_{\Uparrow\uparrow}$ and $\psi_{\Downarrow\downarrow}$ are right-handed, with boost and spin weights $\sigma = \varsigma = \pm\frac{1}{2}$, while the bottom two components $\psi_{\Uparrow\downarrow}$ and $\psi_{\Downarrow\uparrow}$ are left-handed, with boost and spin weights $\sigma = -\varsigma = \pm\frac{1}{2}$.

The massless spin-$\frac{1}{2}$ wave equation (83) can be written

$$\boldsymbol{\gamma}^k\left(\partial_k + \frac{1}{2}\boldsymbol{\Gamma}_k\right)\psi = 0, \tag{87}$$

where the tetrad-frame connection $\boldsymbol{\Gamma}_k$ is the set of four bivectors

$$\boldsymbol{\Gamma}_k = \Gamma_{klm}\boldsymbol{\gamma}^l \wedge \boldsymbol{\gamma}^m \tag{88}$$

implicitly summed over distinct antisymmetric pairs of indices lm. The tetrad-frame connections Γ_{klm} are also called Lorentz connections, or Ricci rotation coefficients, or spin coefficients in the context of the Newman-Penrose formalism.

Massless spin-$\frac{1}{2}$ waves of opposite chirality do not mix. The right- and left-handed chiral components $\tilde{\psi}$ of the

Dirac spinor are (the tilde on $\tilde{\psi}$ signifies a right- or left-handed chiral component)

$$\tilde{\psi} \equiv \frac{1}{2}(1 \pm \gamma_5)\psi, \tag{89}$$

where the $\pm$ sign indicates $+$ for right-handed, and $-$ for left-handed. The right- and left-handed spinor fields each have two distinct nonvanishing complex components, with boost weights $\sigma = +1/2$ and $\sigma = -1/2$, respectively, and spin weights $\varsigma = \sigma$ for right-handed, $\varsigma = -\sigma$ for left-handed:

$$\tilde{\psi}_\sigma = \begin{pmatrix} \tilde{\psi}_{+1/2} \\ \tilde{\psi}_{-1/2} \end{pmatrix} \equiv \begin{cases} \begin{pmatrix} \tilde{\psi}_{\Uparrow\uparrow} \\ \tilde{\psi}_{\Downarrow\downarrow} \end{pmatrix} & \text{right} \\ \begin{pmatrix} \tilde{\psi}_{\Uparrow\downarrow} \\ \tilde{\psi}_{\Downarrow\uparrow} \end{pmatrix} & \text{left} \end{cases}. \tag{90}$$

Define the scaled right- and left-handed spinors $\hat{\psi}_\sigma$ (with hats) by

$$\tilde{\psi}_\sigma = \begin{cases} f_{\frac{1}{2}}\hat{\psi}_\sigma & \text{right} \\ f^*_{\frac{1}{2}}\hat{\psi}_\sigma & \text{left} \end{cases}, \tag{91}$$

in accordance with Eq. (53). In terms of the scaled spinors $\hat{\psi}_\sigma$, the spin-$\frac{1}{2}$ wave equation (87) is [Eq. (A11) in Appendix A gives these equations in a general spacetime]

$$\boldsymbol{D}\psi = \frac{1}{\rho}\begin{pmatrix} -f^*_{\frac{1}{2}}(\eth_v\hat{\psi}_{\Downarrow\uparrow} + \eth_+\hat{\psi}_{\Uparrow\downarrow}) \\ f^*_{\frac{1}{2}}(\eth_-\hat{\psi}_{\Downarrow\uparrow} \mp \eth_u\hat{\psi}_{\Uparrow\downarrow}) \\ f_{\frac{1}{2}}(\eth_v\hat{\psi}_{\Downarrow\downarrow} + \eth_-\hat{\psi}_{\Uparrow\uparrow}) \\ -f_{\frac{1}{2}}(\eth_+\hat{\psi}_{\Downarrow\downarrow} \mp \eth_u\hat{\psi}_{\Uparrow\uparrow}) \end{pmatrix} = 0, \tag{92}$$

where the $\mp$ sign in the second and last rows indicates $-$ outside the horizon ($\Delta_x > 0$), and $+$ inside the horizon ($\Delta_x < 0$). The γ matrices [Eq. (33)], and hence the covariant derivative $\boldsymbol{D} \equiv \boldsymbol{\gamma}^k D_k$, connect spinors of opposite chirality, so the top two components of the wave equation (92) are equations for the left-handed spinor components, while the bottom two are for the right-handed spinor components. Equation (92) shows that the two left-handed spinor components are coupled to each other, and the two right-handed spinor components are coupled to each other, but the left-handed spinor is decoupled from the right-handed spinor. Since the boost operators $\eth_v$ and $\eth_u$ commute with the spin operators $\eth_+$ and $\eth_-$, the four components in the column vector in the wave equation (92) can be combined in pairs to yield wave equations for each spinor component separately:

$$(\Box_{vu} - \Box_{+-})\hat{\psi}_{\Uparrow\uparrow} = 0, \tag{93a}$$

$$(\Box_{uv} - \Box_{-+})\hat{\psi}_{\Downarrow\downarrow} = 0, \tag{93b}$$

$$(\Box_{vu} - \Box_{-+})\hat{\psi}_{\Uparrow\downarrow} = 0, \tag{93c}$$

$$(\Box_{uv} - \Box_{+-})\hat{\psi}_{\Downarrow\uparrow} = 0, \tag{93d}$$

where the spin-$\frac{1}{2}$ wave operators $\Box_{kl}$ are defined by Eq. (57). Equation (93) establishes the correctness of the claimed wave equation (51) for the case of spin half, $s = \frac{1}{2}$. The wave equations in Eq. (93) admit separated solutions of the form

$$\hat{\psi}_\sigma = e^{-i(wt+m\phi)}X_\sigma(x)Y_\varsigma(y), \tag{94}$$

where σ and ς each run over boost and spin weights $\pm\frac{1}{2}$. The corresponding eigenvalues are $\lambda_{\sigma\varsigma}$.

The Teukolsky-Starobinski [60,61] identities follow immediately from the simple form $\Box_{kl} = \eth_k\eth_l$ modulo a sign [Eq. (57)] of the spin-$\frac{1}{2}$ wave operators:

$$(\Box_{uv}\eth_u - \eth_u\Box_{vu})\hat{\psi}_{\Uparrow\uparrow} = 0, \tag{95a}$$

$$(\Box_{vu}\eth_v - \eth_v\Box_{uv})\hat{\psi}_{\Downarrow\downarrow} = 0, \tag{95b}$$

$$(\Box_{-+}\eth_- - \eth_-\Box_{+-})\hat{\psi}_{\Uparrow\uparrow} = 0, \tag{95c}$$

$$(\Box_{+-}\eth_+ - \eth_+\Box_{-+})\hat{\psi}_{\Downarrow\downarrow} = 0. \tag{95d}$$

The identities in Eq. (95) are for the right-handed components; a similar set of identities holds for the left-handed components. The first Teukolsky-Starobinski identity [Eq. (95a)] shows that $\eth_u\hat{\psi}_{\Uparrow\uparrow}$ is an eigenfunction of $\Box_{uv}$ with eigenvalue $\lambda_{\Uparrow\uparrow}$, while the last Teukolsky-Starobinski identity [Eq. (95d)] shows that $\eth_+\hat{\psi}_{\Downarrow\downarrow}$ is an eigenfunction of $\Box_{+-}$ with eigenvalue $\lambda_{\Downarrow\downarrow}$. But the fourth row of Eq. (92) shows that $\eth_u\hat{\psi}_{\Uparrow\uparrow}$ equals $\eth_+\hat{\psi}_{\Downarrow\downarrow}$ modulo a sign, so it follows that the eigenvalues must be equal, $\lambda_{\Uparrow\uparrow} = \lambda_{\Downarrow\downarrow} = \lambda$.

The eigenvalue λ is real and positive. This follows from the fact that the spin-$\frac{1}{2}$ wave operators are positive definite, being the product of an operator and its Hermitian conjugate [Eq. (48)]:

$$\Box_{vu} = \eth_v\eth_u = \eth_u^\dagger\eth_u, \qquad \Box_{+-} = \eth_+\eth_- = \eth_-^\dagger\eth_-. \tag{96}$$

Thus, λ can be written as the square of some real number μ,

$$\mu^2 = \lambda. \tag{97}$$

The third row of Eq. (92) shows that $X_{-1/2}$ raised once is proportional to $X_{+1/2}$, and that $Y_{+1/2}$ lowered once is proportional to $Y_{-1/2}$. Similarly, the last row of Eq. (92) shows that $X_{+1/2}$ lowered once is proportional to $X_{-1/2}$, and that $Y_{-1/2}$ raised once is proportional to $Y_{+1/2}$. With a convenient choice of relative normalization, the eigenfunctions are related by

$$ð_v X_{-1/2} = -\mu X_{+1/2}, \qquad ð_u X_{+1/2} = \mathrm{sgn}(\Delta_x)\mu X_{-1/2}, \tag{98a}$$

$$ð_+ Y_{-1/2} = \mu Y_{+1/2}, \qquad ð_- Y_{+1/2} = \mu Y_{-1/2}. \tag{98b}$$

Equation (98) specifies the relation between the boost/spin-weight $\pm\frac{1}{2}$ components of an eigenmode of given chirality, right or left. Modes of opposite chirality evolve independently of each other.

VI. SPIN-1 WAVES

Wave equations for massless, neutral spin-1 waves—electromagnetic waves—are provided by Maxwell's equations,

$$\boldsymbol{D}\boldsymbol{F} = \boldsymbol{j}, \tag{99}$$

where $\boldsymbol{D}$ is the covariant derivative defined by Eq. (84), $\boldsymbol{F}$ is the electromagnetic field bivector

$$\boldsymbol{F} \equiv F_{kl}\boldsymbol{\gamma}^k \wedge \boldsymbol{\gamma}^l \tag{100}$$

implicitly summed over distinct antisymmetric pairs of indices kl, and $\boldsymbol{j}$ is the electric current. The electromagnetic units here are Heaviside; in Gaussian units, the right-hand side of Maxwell's equation (99) would be $4\pi\boldsymbol{j}$. In the present case, the source current is taken to vanish,

$$\boldsymbol{j} = 0. \tag{101}$$

The assumption of vanishing source current requires that the black hole be uncharged. The single equation (99) embodies both source-free (magnetic) and source (electric) parts of Maxwell's equations.

Like massless Dirac spinors, massless spin-1 waves decompose into two distinct chiralities that do not mix. The right- and left-handed chiral components $\tilde{\boldsymbol{F}}$ of the electromagnetic field bivector are (the tilde on $\tilde{\boldsymbol{F}}$ signifies a right- or left-handed chiral component)

$$\tilde{\boldsymbol{F}} \equiv \frac{1}{2}(1 \pm \gamma_5)\boldsymbol{F}, \tag{102}$$

where the $\pm$ sign is $+$ for right-handed, and $-$ for left-handed. Equation (102) implies that the components $\tilde{F}_{kl}$ of the right- and left-handed electromagnetic field $\tilde{\boldsymbol{F}}$ are

$$\tilde{F}_{kl} = \frac{1}{2}(F_{kl} \mp i\varepsilon_{kl}{}^{mn}F_{mn}) \tag{103}$$

(implicitly summed over distinct antisymmetric indices mn) with $-$ for right-handed, and $+$ for left-handed. Maxwell's equations (99) with zero source current are then

$$D^k\tilde{F}_{kl} = 0. \tag{104}$$

The right- and left-handed electromagnetic field $\tilde{F}_{kl}$ bivectors each have three distinct nonvanishing complex components, with boost weights $\sigma = +1$, 0, and -1, respectively, and spin weights $\varsigma = \sigma$ for right-handed, and $\varsigma = -\sigma$ for left-handed:

$$\tilde{F}_\sigma \equiv \begin{pmatrix} \tilde{F}_{+1} \\ \tilde{F}_0 \\ \tilde{F}_{-1} \end{pmatrix} \equiv \begin{cases} \begin{pmatrix} \tilde{F}_{v+} \\ \tilde{F}_{vu} \\ \tilde{F}_{u-} \end{pmatrix} = \begin{pmatrix} F_{v+} \\ \frac{1}{2}(F_{vu} - F_{+-}) \\ F_{u-} \end{pmatrix} & \text{right} \\ \begin{pmatrix} \tilde{F}_{v-} \\ \tilde{F}_{vu} \\ \tilde{F}_{u+} \end{pmatrix} = \begin{pmatrix} F_{v-} \\ \frac{1}{2}(F_{vu} + F_{+-}) \\ F_{u+} \end{pmatrix} & \text{left} \end{cases}. \tag{105}$$

The three components $\tilde{F}_\sigma$ with boost weights $\sigma = +1, 0, -1$ are commonly [46] denoted ϕ_0, ϕ_1, ϕ_2 [Eq. (C4)], but the notation in Eq. (105) makes manifest the boost weights of the components. The propagating component is, Eq. (46), $\tilde{F}_{-1} = \phi_2$ outgoing, and $\tilde{F}_{+1} = \phi_0$ ingoing.

Focus on the right-handed electromagnetic field; the left-handed field is quite similar. Define the scaled right-handed electromagnetic field tensor $\hat{F}_\sigma$ (with a hat instead of a tilde over the F) by

$$\tilde{F}_\sigma = f_1\hat{F}_\sigma, \tag{106}$$

in accordance with Eq. (53). In terms of the scaled electromagnetic field $\hat{F}_{kl}$ and the boost and spin raising and lowering operators [Eq. (47)], Maxwell's equations (104) are [Eq. (A13) in Appendix A give these equations in a general spacetime]

$$D^k\begin{pmatrix} \tilde{F}_{kv} \\ \tilde{F}_{ku} \\ \tilde{F}_{k+} \\ \tilde{F}_{k-} \end{pmatrix} = \frac{f_1}{\sqrt{2}\rho}\begin{pmatrix} \bar{\rho}^{-1}ð_v\bar{\rho}\hat{F}_0 + \bar{\rho}ð_-\bar{\rho}^{-1}\hat{F}_{+1} \\ \mp\bar{\rho}^{-1}ð_u\bar{\rho}\hat{F}_0 - \bar{\rho}ð_+\bar{\rho}^{-1}\hat{F}_{-1} \\ \bar{\rho}^{-1}ð_+\bar{\rho}\hat{F}_0 \mp \bar{\rho}ð_u\bar{\rho}^{-1}\hat{F}_{+1} \\ \bar{\rho}^{-1}ð_-\bar{\rho}\hat{F}_0 - \bar{\rho}ð_v\bar{\rho}^{-1}\hat{F}_{-1} \end{pmatrix} = 0, \tag{107}$$

where the $\mp$ sign in the middle two rows is $-$ outside the horizon ($\Delta_x > 0$), and $+$ inside the horizon ($\Delta_x < 0$). Equation (107) shows that the three components of the right-handed electromagnetic field evolve not independently, but rather in harmony with each other. Combining Eq. (107) in pairs yields pairs of equations for each of the three components $\hat{F}_\sigma$:

$$\bar{\rho}^{-1}(\eth_v\bar{\rho}^2\eth_u \pm \eth_+\bar{\rho}^{-2}\eth_-)\bar{\rho}^{-1}\hat{F}_{+1} = 0, \tag{108a}$$

$$\bar{\rho}^{-1}(\eth_u\bar{\rho}^2\eth_v \pm \eth_-\bar{\rho}^2\eth_+)\bar{\rho}^{-1}\hat{F}_{-1} = 0, \tag{108b}$$

$$\eth_v^2\hat{F}_{-1} + \eth_-^2\hat{F}_{+1} = 0, \tag{108c}$$

$$\eth_u^2\hat{F}_{+1} + \eth_+^2\hat{F}_{-1} = 0, \tag{108d}$$

$$\bar{\rho}(\eth_v\bar{\rho}^{-2}\eth_u \pm \eth_+\bar{\rho}^{-2}\eth_-)\bar{\rho}\hat{F}_0 = 0, \tag{108e}$$

$$\bar{\rho}(\eth_u\bar{\rho}^{-2}\eth_v \pm \eth_-\bar{\rho}^{-2}\eth_+)\bar{\rho}\hat{F}_0 = 0. \tag{108f}$$

The top and bottom pairs of lines in Eq. (108) can also be written in terms of the radial and angular wave operators $\Box$ defined by Eq. (57):

$$(\Box_{vu} - \Box_{+-})\hat{F}_{+1} = 0, \tag{109a}$$

$$(\Box_{uv} - \Box_{-+})\hat{F}_{-1} = 0, \tag{109b}$$

$$(\Box_{vu} - \Box_{+-} - 2\rho^2\tilde{C}_0)\hat{F}_0 = 0, \tag{109c}$$

$$(\Box_{uv} - \Box_{-+} - 2\rho^2\tilde{C}_0)\hat{F}_0 = 0, \tag{109d}$$

where $\tilde{C}_0$ is the spin-0 component of the right-handed Weyl tensor [Eq. (42)]. The top pair of lines in Eq. (109) establish the correctness of the claimed wave equation (51) for the case of electromagnetic waves, $s = 1$. The top pair admit separated solutions for the boost-weight ±1 components of the right-handed ($\varsigma = \sigma$) electromagnetic field,

$$\hat{F}_\sigma = e^{-i(wt+m\phi)}X_\sigma(x)Y_\varsigma(y). \tag{110}$$

The corresponding eigenvalues are $\lambda_{\sigma\varsigma}$.

The spin-1 wave operators $\Box_{kl}$ can be checked to satisfy the Teukolsky-Starobinsky [60,61] identities

$$(\Box_{uv}\eth_u^2 - \eth_u^2\Box_{vu})\hat{F}_{+1} = 0, \tag{111a}$$

$$(\Box_{vu}\eth_v^2 - \eth_v^2\Box_{uv})\hat{F}_{-1} = 0, \tag{111b}$$

$$(\Box_{-+}\eth_-^2 - \eth_-^2\Box_{+-})\hat{F}_{+1} = 0, \tag{111c}$$

$$(\Box_{+-}\eth_+^2 - \eth_+^2\Box_{-+})\hat{F}_{-1} = 0. \tag{111d}$$

The identities in Eq. (111) are for the right-handed components; a similar set of identities holds for the left-handed components. The first Teukolsky-Starobinski identity [Eq. (111a)] shows that $\eth_u^2\hat{F}_{+1}$ is an eigenfunction of $\Box_{vu}$ with eigenvalue λ_{v+}, while the fourth Teukolsky-Starobinski identity [Eq. (111d)] shows that $\eth_+^2\hat{F}_{-1}$ is an eigenfunction of $\Box_{+-}$ with eigenvalue λ_{u-}. But Eq. (108d) shows that $\eth_u^2\hat{F}_{+1}$ equals $\eth_+^2\hat{F}_{-1}$ modulo a minus sign, so it follows that the eigenvalues must be equal, $\lambda_{v+} = \lambda_{u-} = \lambda$.

The middle pair of lines in Eq. (108) imply that Y_{+1} lowered twice is proportional to Y_{-1}, and that Y_{-1} raised twice is proportional to Y_{+1}. A similar statement holds for X_{+1} and X_{-1}. Lowering Y_{+1} twice, then raising the result twice yields some constant μ^2 times Y_{+1}. By adjusting the relative normalization of Y_{+1} and Y_{-1}, the result of Y_{+1} lowered twice can be taken to be μY_{-1}, and that of Y_{-1} raised twice to be μY_{+1}. The middle pair of lines in Eq. (108) imply that the radial constant of proportionality is the negative of the angular constant. Thus,

$$\eth_u^2 X_{+1} = -\mu X_{-1}, \qquad \eth_v^2 X_{-1} = -\mu X_{+1}, \tag{112a}$$

$$\eth_-^2 Y_{+1} = \mu Y_{-1}, \qquad \eth_+^2 Y_{-1} = \mu Y_{+1}. \tag{112b}$$

Solving for μ^2 in

$$(\eth_+^2\eth_-^2 - \mu^2)Y_{+1} = 0, \tag{113}$$

given that $\tilde{F}_{+1}$ satisfies $(\Box_{+-} - \lambda)\tilde{F}_{+1} = 0$, yields the standard result [according to Eqs. (35) and (52) of Ref. [46]]

$$\mu^2 = \lambda^2 - 4aw(aw + m). \tag{114}$$

Since the raising and lowering operators are Hermitian conjugates [Eq. (48b)], the operator on the left-hand side of Eq. (113) can be written

$$\eth_+^2\eth_-^2 = (\eth_-^2)^\dagger\eth_-^2, \tag{115}$$

which is positive definite, being the product of an operator and its Hermitian conjugate. It follows that the eigenvalue μ^2 is real and positive, and consequently μ is real.

Expressions for the scaled boost-weight-0 component $\hat{F}_0$ of the electromagnetic field follow from any of the four lines of Eq. (107). Consider the first line of Eq. (107), which expresses a certain derivative of $\hat{F}_0$ in terms of a certain derivative of the boost-weight-1 component $\hat{F}_{+1}$. This first line of Eq. (107) can be solved for $\hat{F}_0$ by expressing the radial part of the separated solution $\hat{F}_{+1}$ as $X_{+1} = -\mu^{-1}\eth_v^2 X_{-1}$ [Eq. (112a)], and by using the relation (B1). The first and last of the four lines of Eq. (107) redundantly yield one expression for $\hat{F}_0$, while the second and third lines redundantly yield a second expression. The two expressions are, with $\hat{F}_0 \equiv e^{-i(wt+m\phi)}\mathring{F}_0(x,y)$,

$$\mathring{F}_0(x,y) = \mp\frac{1}{\mu}\left[\eth_u\eth_+ - \frac{1}{\bar{\rho}}\left(\mp R^2\sqrt{|\Delta_x|}\eth_+ + ia\sqrt{\Delta_y}\eth_u\right)\right]X_{+1}Y_{-1} \tag{116a}$$

$$= \frac{1}{\mu}\left[\eth_v\eth_- + \frac{1}{\bar\rho}\left(\mp R^2\sqrt{|\Delta_x|}\eth_- + ia\sqrt{\Delta_y}\eth_v\right)\right]X_{-1}Y_{+1} \tag{116b}$$

where the $\mp$ signs are $-$ outside the horizon, $\Delta_x > 0$, and $+$ inside the horizon, $\Delta_x < 0$. Unlike the boost-weight ±1 components [Eq. (110)], the boost-weight-0 function $\mathring{F}_0(x, y)$ is not a separated product of radial and angular coordinates x and y.

VII. SPIN-$\frac{3}{2}$ WAVES

Spin-$\frac{3}{2}$ fields arise in supersymmetric local gauge theories, where the generators of the gauge group are taken to be spinors [62–64]. The signature feature of supersymmetry is that it transforms bosonic (integral spin) fields into fermionic (half-integral spin) fields and vice versa. Spin-$\frac{3}{2}$ fields in Λ-Kerr-Newman black holes have been considered by Refs. [65–67]. The supersymmetric gauge connection defines the gravitino potential $\boldsymbol{\psi}$, a vector of spinors,

$$\boldsymbol{\psi} \equiv \psi_k\boldsymbol{\gamma}^k \equiv \psi_{ka}\boldsymbol{\gamma}^k \otimes \boldsymbol{\epsilon}^a, \tag{117}$$

where $\boldsymbol{\gamma}^k$ and $\boldsymbol{\epsilon}^a$ are vector and spinor basis elements, respectively (Sec. II C). Each of the four vector components of the gravitino potential $\boldsymbol{\psi}$ is a spinor ψ_k,

$$\psi_k \equiv \psi_{ka}\boldsymbol{\epsilon}^a. \tag{118}$$

The gauge-covariant supersymmetric derivative is

$$\boldsymbol{D} + \boldsymbol{\psi} = (D_k + \psi_k)\boldsymbol{\gamma}^k, \tag{119}$$

where $\boldsymbol{D}$ [Eq. (84)] is the usual general-relativistic covariant derivative. A vector-spinor $\boldsymbol{\psi}$ has $4 \times 4 = 16$ complex components, whose irreducible parts under Lorentz transformations comprise a four-component spin-$\frac{1}{2}$ part and a 12-component spin-$\frac{3}{2}$ part. The spin-$\frac{1}{2}$ parts of the vector-spinor $\boldsymbol{\psi}$ are removed, leaving only the spin-$\frac{3}{2}$ parts of the gravitino, by imposing the four conditions (here the Dirac γ matrices act by matrix multiplication on the spinors ψ_k):

$$\boldsymbol{\gamma}^k\psi_k = 0. \tag{120}$$

A supersymmetric gauge transformation by a spinor λ transforms the gravitino potential ψ_k as

$$\psi_k \to \psi_k + D_k\lambda. \tag{121}$$

Recall that the covariant derivative acting on a spinor is $D_k\lambda = (\partial_k + \frac{1}{2}\boldsymbol{\Gamma}_k)\lambda$ [Eq. (87)]. The supersymmetric gauge freedom can be removed by imposing some gauge condition. When dealing with waves, a convenient choice of gauge is the analog of the Lorenz gauge of electromagnetism,

$$D^k\psi_k = 0. \tag{122}$$

The gauge condition (122) removes four of the 12 complex degrees of freedom of the spin-$\frac{3}{2}$ gravitino potential, leaving it with eight physical degrees of freedom. The massless gravitino potential decomposes further into right- and left-handed chiral parts, each with four physical degrees of freedom (the tilde on $\tilde\psi$ signifies a right-handed or left-handed chiral component):

$$\tilde\psi_k \equiv \frac{1}{2}(1 \pm \gamma_5)\psi_k \equiv \frac{1}{2}(1 \pm \gamma_5)\psi_{ka}\boldsymbol{\epsilon}^a, \tag{123}$$

where the $\pm$ sign is $+$ for right-handed, and $-$ for left-handed.

The commutator of the supersymmetric gauge-covariant derivative defines the curvature,

$$[\boldsymbol{D} + \boldsymbol{\psi}, \boldsymbol{D} + \boldsymbol{\psi}] \equiv [D_k + \psi_k, D_l + \psi_l]\boldsymbol{\gamma}^k \wedge \boldsymbol{\gamma}^l, \tag{124}$$

implicitly summed over distinct antisymmetric kl. The curvature in Eq. (124) has both a bosonic part, the Riemann curvature tensor $\boldsymbol{R}$, and a fermionic part, the gravitino field $\boldsymbol{\Psi}$:

$$[\boldsymbol{D} + \boldsymbol{\psi}, \boldsymbol{D} + \boldsymbol{\psi}] = \boldsymbol{R} + \boldsymbol{\Psi}. \tag{125}$$

The Riemann tensor $\boldsymbol{R}$ is a bivector of bivectors, while the gravitino field $\boldsymbol{\Psi}$ is a bivector of spinors:

$$\boldsymbol{R} \equiv \boldsymbol{R}_{kl}\boldsymbol{\gamma}^k \wedge \boldsymbol{\gamma}^l \equiv R_{klmn}(\boldsymbol{\gamma}^k \wedge \boldsymbol{\gamma}^l) \otimes (\boldsymbol{\gamma}^m \wedge \boldsymbol{\gamma}^n), \tag{126a}$$

$$\boldsymbol{\Psi} \equiv \Psi_{kl}\boldsymbol{\gamma}^k \wedge \boldsymbol{\gamma}^l \equiv \Psi_{kla}(\boldsymbol{\gamma}^k \wedge \boldsymbol{\gamma}^l) \otimes \boldsymbol{\epsilon}^a, \tag{126b}$$

with implicit summation over distinct antisymmetric bivector indices kl (and mn). Compared to its usual general relativistic expression, the Riemann tensor $\boldsymbol{R}$ contains an additional part $[\boldsymbol{\psi}, \boldsymbol{\psi}] = (\{\psi_k\psi_l\}\cdot)\boldsymbol{\gamma}^k \wedge \boldsymbol{\gamma}^l$ proportional to a square of the gravitino potential $\boldsymbol{\psi}$. The factor $\{\psi_k\psi_l\}\cdot$, a symmetric outer product of like-handed spinors (in four spacetime dimensions) is dictated by the requirement that the spinor product transform like a bivector ($\psi_k\cdot \equiv \psi_k^\top\varepsilon$, with ε the spinor metric [Eq. (38)], denotes the row spinor associated with the column spinor ψ_k). The contribution $[\boldsymbol{\psi}, \boldsymbol{\psi}]$ plays no role in the present paper, since the gravitino field vanishes in the background spacetime, so $[\boldsymbol{\psi}, \boldsymbol{\psi}]$ is of quadratic order in the gravitino field and can be neglected to linear order of wave amplitudes. Each of the six bivector components Ψ_{kl} of the gravitino field $\boldsymbol{\Psi}$ is a spinor,

$$\Psi_{kl} \equiv D_k\psi_l - D_l\psi_k \equiv (D_k\psi_{la} - D_l\psi_{ka})\boldsymbol{\epsilon}^a. \tag{127}$$

As usual, the covariant derivative D_k acts on both vector and spinor components.

Like massless fields of other nonzero spin, massless spin-$\frac{3}{2}$ waves of opposite chirality do not mix. The right- and left-handed chiral components $\tilde{\Psi}_{kl}$ of the gravitino field (the tilde on $\tilde{\Psi}$ signifies a right- or left-handed chiral component) are

$$\tilde{\Psi}_{kl} \equiv \frac{1}{2}(1 \pm \gamma_5)(\Psi_{kl} \mp i\varepsilon_{kl}{}^{mn}\Psi_{mn}) \tag{128}$$

implicitly summed over distinct antisymmetric indices mn, where the upper and lower signs are, respectively, right- and left-handed. It can be shown that

$$\boldsymbol{\gamma}^k\tilde{\Psi}_{kl} = \boldsymbol{\gamma}_l D^k\tilde{\psi}_k, \tag{129}$$

so imposing the Lorenz gauge condition [Eq. (122)] ensures that

$$\boldsymbol{\gamma}^k\tilde{\Psi}_{kl} = 0. \tag{130}$$

A bivector spinor Ψ has $6 \times 4 = 24$ components, but the projection in Eq. (128) projects both spinor and bivector parts into their chiral components, leaving each chiral component with six complex components. The Lorenz gauge condition [Eq. (122)] removes two components from each chirality, leaving each with four physical components, which is as it should be. The right- and left-handed components of the gravitino field $\tilde{\Psi}_{kl}$ subject to the gauge condition (122) are conveniently labeled $\tilde{\Psi}_\sigma$ by their boost weights $\sigma = +\frac{3}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ (spin weights $\varsigma = \sigma$ right-handed, $\varsigma = -\sigma$ left-handed):

$$\tilde{\Psi}_\sigma \equiv \begin{pmatrix} \tilde{\Psi}_{+3/2} \\ \tilde{\Psi}_{+1/2} \\ \tilde{\Psi}_{-1/2} \\ \tilde{\Psi}_{-3/2} \end{pmatrix} \equiv \begin{cases} \begin{pmatrix} \tilde{\Psi}_{v+\Uparrow\uparrow} \\ \tilde{\Psi}_{vu\Uparrow\uparrow} \\ \tilde{\Psi}_{uv\Downarrow\downarrow} \\ \tilde{\Psi}_{u-\Downarrow\downarrow} \end{pmatrix} & \text{right} \\ \begin{pmatrix} \tilde{\Psi}_{v-\Uparrow\downarrow} \\ \tilde{\Psi}_{vu\Uparrow\downarrow} \\ \tilde{\Psi}_{uv\Downarrow\uparrow} \\ \tilde{\Psi}_{u+\Downarrow\uparrow} \end{pmatrix} & \text{left} \end{cases}. \tag{131}$$

The boost $\pm\frac{3}{2}$ components are gauge invariant. The condition in Eq. (130) that follows from the Lorenz gauge condition (122) imposes on the boost $\pm\frac{1}{2}$ components the conditions

$$\tilde{\Psi}_{1/2} = \begin{cases} \tilde{\Psi}_{vu\Uparrow\uparrow} = \tilde{\Psi}_{v+\Downarrow\downarrow} & \text{right} \\ \tilde{\Psi}_{vu\Uparrow\downarrow} = -\tilde{\Psi}_{v-\Downarrow\uparrow} & \text{left} \end{cases},$$
$$\tilde{\Psi}_{-1/2} = \begin{cases} \tilde{\Psi}_{uv\Downarrow\downarrow} = -\tilde{\Psi}_{u-\Uparrow\uparrow} & \text{right} \\ \tilde{\Psi}_{uv\Downarrow\uparrow} = \tilde{\Psi}_{u+\Uparrow\downarrow} & \text{left} \end{cases}. \tag{132}$$

For brevity, denote the supersymmetric covariant derivative by $\mathcal{D}_k \equiv D_k + \Psi_k$. Wave equations for the chiral gravitino fields $\tilde{\boldsymbol{\Psi}}$ follow from the Jacobi identity (also known as Bianchi identities):

$$\mathcal{D}_{[k}[\mathcal{D}_l, \mathcal{D}_{m]}] = [\mathcal{D}_{[k}, \mathcal{D}_l]\mathcal{D}_{m]}. \tag{133}$$

The brackets around indices mean to antisymmetrize over bracketed indices. The fermionic part of the Jacobi identity [Eq. (133)] is

$$D_{[k}\Psi_{lm]} = \frac{1}{2}\boldsymbol{R}_{[kl}\psi_{m]} \tag{134}$$

[the right-hand side of Eq. (134) comes from the Riemann operator acting on the spinor indices a of the gravitino potential ψ_{ma}; the possible contribution $R_{[klm]n}\psi^n$ from the Riemann operator acting on the vector indices m of ψ_{ma} vanishes to linear order because the vanishing of torsion in the background spacetime implies $R_{[klm]n} = 0$ in the background]. Applied to the right- and left-handed gravitino fields, the curl on the left-hand side of Eq. (134) can be replaced by a divergence,

$$D_{[+}\tilde{\Psi}_{-v]} = \mp D^k\tilde{\Psi}_{kv}, \qquad D_{[v}\tilde{\Psi}_{u+]} = \pm D^k\tilde{\Psi}_{k+}, \tag{135a}$$

$$D_{[+}\tilde{\Psi}_{-u]} = \pm D^k\tilde{\Psi}_{ku}, \qquad D_{[v}\tilde{\Psi}_{u-]} = \mp D^k\tilde{\Psi}_{k-}, \tag{135b}$$

where the upper and lower signs are right- and left-handed, respectively.

Focus on the right-handed gravitino field; the left-handed field is quite similar. Define the scaled right-handed gravitino field $\hat{\Psi}_{kl}$ (with a hat instead of a tilde over the Ψ) by

$$\tilde{\Psi}_{kl} = f_{3/2}\hat{\Psi}_{kl} \tag{136}$$

in accordance with Eq. (53). The Jacobi identity [Eq. (134)] provides six independent equations governing the four components of the right-handed gravitino field. The left-hand side of the Jacobi identity in Eq. (134) is [Eq. (A16) in Appendix A gives these equations in a general spacetime]

$$D^k \begin{pmatrix} \hat{\Psi}_{kv\Uparrow\uparrow} \\ \hat{\Psi}_{k+\Uparrow\uparrow} \\ \hat{\Psi}_{k-\Uparrow\uparrow} \\ \hat{\Psi}_{k+\Downarrow\downarrow} \\ \hat{\Psi}_{k-\Downarrow\downarrow} \\ \hat{\Psi}_{ku\Downarrow\downarrow} \end{pmatrix} = \frac{f_{3/2}}{\sqrt{2}\rho} \begin{pmatrix} \bar{\rho}^{-2}\eth_v\bar{\rho}^2\hat{\Psi}_{+1/2} + \bar{\rho}^2\eth_-\bar{\rho}^{-2}\hat{\Psi}_{+3/2} \\ \bar{\rho}^{-2}\eth_+\bar{\rho}^2\hat{\Psi}_{+1/2} \mp \bar{\rho}^2\eth_u\bar{\rho}^{-2}\hat{\Psi}_{+3/2} \\ \eth_v\hat{\Psi}_{-1/2} + \eth_-\hat{\Psi}_{+1/2} \\ \mp\eth_u\hat{\Psi}_{+1/2} + \eth_+\hat{\Psi}_{-1/2} \\ \bar{\rho}^{-2}\eth_-\bar{\rho}^2\hat{\Psi}_{-1/2} - \bar{\rho}^2\eth_v\bar{\rho}^{-2}\hat{\Psi}_{-3/2} \\ \mp\bar{\rho}^{-2}\eth_u\bar{\rho}^2\hat{\Psi}_{-1/2} - \bar{\rho}^2\eth_+\bar{\rho}^{-2}\hat{\Psi}_{-3/2} \end{pmatrix}, \tag{137}$$

where the upper sign of $\pm$ or $\mp$ is outside the horizon, $\Delta_x > 0$, while the lower sign is inside the horizon, $\Delta_x < 0$. Since the radial operators $\eth_{\substack{v\\u}}$ commute with the angular operators $\eth_\pm$, equations for each of the four components $\hat{\Psi}_\sigma$ can be obtained by combining Eq. (137) in pairs. Derivatives of the right-hand side of the Jacobi identity [Eq. (134)] turn the gravitino potential ψ into the gravitino field Ψ, yielding contributions proportional to chiral components of the Riemann tensor that are nonvanishing in the background spacetime—namely, the spin-0 component $\tilde{C}_0$ of the Weyl tensor [Eq. (42)], and the Ricci scalar R, whose only nonvanishing contribution in the conformally separable spacetimes is from the cosmological constant, where $R = 4\Lambda$. The result is six equations for the four components $\hat{\Psi}_\sigma$ of the right-handed gravitino field:

$$\left(\mp\bar{\rho}^{-2}\eth_v\bar{\rho}^4\eth_u\bar{\rho}^{-2} - \bar{\rho}^{-2}\eth_+\bar{\rho}^4\eth_-\bar{\rho}^{-2} + \rho^2\left(2\tilde{C}_0 - \frac{1}{12}R\right)\right)\hat{\Psi}_{+3/2} = 0, \tag{138a}$$

$$\left(\mp\bar{\rho}^2\eth_u\bar{\rho}^{-4}\eth_v\bar{\rho}^2 - \bar{\rho}^2\eth_-\bar{\rho}^{-4}\eth_+\bar{\rho}^2 + \rho^2\left(2\tilde{C}_0 - \frac{1}{12}R\right)\right)\hat{\Psi}_{+1/2} = 0, \tag{138b}$$

$$\left(\mp\eth_v\eth_u - \eth_+\eth_- + \rho^2\left(-4\tilde{C}_0 - \frac{1}{12}R\right)\right)\hat{\Psi}_{+1/2} = 0, \tag{138c}$$

$$\left(\mp\eth_u\eth_v - \eth_-\eth_+ + \rho^2\left(-4\tilde{C}_0 - \frac{1}{12}R\right)\right)\hat{\Psi}_{-1/2} = 0, \tag{138d}$$

$$\left(\mp\bar{\rho}^2\eth_v\bar{\rho}^{-4}\eth_u\bar{\rho}^2 - \bar{\rho}^2\eth_+\bar{\rho}^{-4}\eth_-\bar{\rho}^2 + \rho^2\left(2\tilde{C}_0 - \frac{1}{12}R\right)\right)\hat{\Psi}_{-1/2} = 0, \tag{138e}$$

$$\left(\mp\bar{\rho}^{-2}\eth_u\bar{\rho}^4\eth_v\bar{\rho}^{-2} - \bar{\rho}^{-2}\eth_-\bar{\rho}^4\eth_+\bar{\rho}^{-2} + \rho^2\left(2\tilde{C}_0 - \frac{1}{12}R\right)\right)\hat{\Psi}_{-3/2} = 0. \tag{138f}$$

Equations (138a)–(138f) can be recast in terms of the wave operators defined by Eq. (57) as

$$\left(\Box_{vu} - \Box_{+-} - \frac{1}{3}\rho^2\Lambda\right)\hat{\Psi}_{+3/2} = 0, \tag{139a}$$

$$\left(\Box_{uv} - \Box_{-+} - 4\rho^2\tilde{C}_0 - \frac{1}{3}\rho^2\Lambda\right)\hat{\Psi}_{+1/2} = 0, \tag{139b}$$

$$\left(\Box_{vu} - \Box_{+-} - 4\rho^2\tilde{C}_0 - \frac{1}{3}\rho^2\Lambda\right)\hat{\Psi}_{+1/2} = 0, \tag{139c}$$

$$\left(\Box_{uv} - \Box_{-+} - 4\rho^2\tilde{C}_0 - \frac{1}{3}\rho^2\Lambda\right)\hat{\Psi}_{-1/2} = 0, \tag{139d}$$

$$\left(\Box_{vu} - \Box_{+-} - 4\rho^2\tilde{C}_0 - \frac{1}{3}\rho^2\Lambda\right)\hat{\Psi}_{-1/2} = 0, \tag{139e}$$

$$\left(\Box_{uv} - \Box_{-+} - \frac{1}{3}\rho^2\Lambda\right)\hat{\Psi}_{-3/2} = 0. \tag{139f}$$

The wave equations (139a) and (139f) for $\hat{\Psi}_{+3/2}$ and $\hat{\Psi}_{-3/2}$ are separable, after the adjustment in Eq. (78) to the wave operators if the cosmological constant Λ is nonvanishing. The separated solutions can be written [Eq. (55)], and the corresponding eigenvalues are $\lambda_{\sigma\varsigma}$.

The Teukolsky-Starobinski identities for the gravitino field do not work out quite so nicely in the conformally separable spacetimes as they do in vacuum spacetimes. The Teukolsky-Starobinski identities for the gravitino field are

$$(\Box_{uv}\eth_u^3 - \eth_u^3\Box_{vu})\hat{\Psi}_{+3/2} = \frac{1}{6}|\Delta_x|^{3/2}K_x\hat{\Psi}_{+3/2}, \tag{140a}$$

$$(\Box_{vu}\eth_v^3 - \eth_v^3\Box_{uv})\hat{\Psi}_{-3/2} = -\frac{1}{6}|\Delta_x|^{3/2}K_x\hat{\Psi}_{-3/2}, \tag{140b}$$

$$(\Box_{-+}\eth_-^3 - \eth_-^3\Box_{+-})\hat{\Psi}_{+3/2} = \frac{1}{6}\Delta_y^{3/2}K_y\hat{\Psi}_{+3/2}, \tag{140c}$$

$$(\Box_{+-}\eth_+^3 - \eth_+^3\Box_{-+})\hat{\Psi}_{-3/2} = -\frac{1}{6}\Delta_y^{3/2}K_y\hat{\Psi}_{-3/2}, \tag{140d}$$

where K_x and K_y are radial and angular functions defined by

$$K_x \equiv R^4\frac{d^5R^4\Delta_x}{dr^5}, \qquad K_y \equiv \frac{d^5\Delta_y}{dy^5}. \tag{141}$$

The radial and angular functions K_x and K_y vanish in Λ-Kerr (-Newman) spacetimes, where $R^4\Delta_x$ and Δ_y are quartic in r and y, respectively, but they do not vanish in the conformally separable spacetimes. The wave operators $\Box$ in the Teukolsky-Starobinski identities [Eq. (140)] are for the wave operators defined by Eq. (57), *not* adjusted for a cosmological constant per the modification in Eq. (78). We have not found a comparably simple set of identities if the wave operators are adjusted for a cosmological constant per Eq. (78). For this reason, the cosmological constant is taken to be zero, $\Lambda = 0$, in the remainder of this Sec. VII.

For Kerr spacetime (without a charge and without a cosmological constant), the Teukolsky-Starobinski identities [Eq. (140a)] show that $X_{+3/2}$ lowered three times (with $ð_u^3$) is proportional to $X_{-3/2}$, and $X_{-3/2}$ raised three times (with $ð_v^3$) is proportional to $X_{+3/2}$; and similarly $Y_{+3/2}$ lowered three times (with $ð_-^3$) is proportional to $Y_{-3/2}$, and $Y_{-3/2}$ raised three times (with $ð_+^3$) is proportional to $Y_{+3/2}$. With a convenient choice of relative normalization, the eigenfunctions are related by

$$ð_u^3 X_{+3/2} = -\mu_x X_{-3/2}, \qquad ð_v^3 X_{-3/2} = \mathrm{sgn}(\Delta_x)\mu_x X_{+3/2}, \tag{142a}$$

$$ð_-^3 Y_{+3/2} = \mu_y Y_{-3/2}, \qquad ð_+^3 Y_{-3/2} = \mu_y Y_{+3/2}. \tag{142b}$$

Solving for μ_x^2 and μ_y^2 in

$$(ð_v^3 ð_u^3 + \mathrm{sgn}(\Delta_x)\mu_x^2) X_{+3/2} = 0, \tag{143a}$$

$$(ð_+^3 ð_-^3 - \mu_y^2) Y_{+3/2} = 0, \tag{143b}$$

given that $\Psi_{+3/2}$ satisfies $(\Box_{vu} - \lambda)\Psi_{+3/2} = (\Box_{+-} - \lambda)\Psi_{+3/2} = 0$, shows that μ_x and μ_y are related to the eigenvalue λ by, for Kerr,

$$\mu_x = \mu_y = \lambda^3 - \lambda\left[\frac{1}{12} + 4aw(aw+m)\right] + \frac{1}{108} + \frac{2}{3}aw(4aw+m). \tag{144}$$

The eigenvalues agree with those in Eqs. (12) and (15) of Ref. [67] with the substitutions (there $\leftrightarrow$ here) $\lambda\!\!\!^{-} = 2\lambda - \frac{1}{3}$ and $\sigma = w$.

VIII. SPIN-2 WAVES

Wave equations for massless spin-2 waves, gravitational waves, follow from the Bianchi identities, which imply the Weyl evolution equations

$$D^k C_{klmn} = J_{lmn}, \tag{145}$$

where C_{klmn} is the Weyl tensor, the traceless part of the Riemann tensor, and J_{lmn} is the Weyl current, defined in terms of the Einstein tensor G_{mn} and its trace G by

$$J_{lmn} \equiv \frac{1}{2}(D_m G_{ln} - D_n G_{lm}) - \frac{1}{6}(\gamma_{ln} D_m G - \gamma_{lm} D_n G). \tag{146}$$

Like the Weyl tensor, the Weyl current J_{lmn} is traceless and satisfies the cyclic symmetry $J_{[lmn]} = 0$. It satisfies the conservation law

$$D^l J_{lmn} = 0, \tag{147}$$

which can be thought of as the gravitational analog of Maxwell's equations.

Like massless fields of other nonzero spin, massless spin-2 waves of opposite chirality do not mix. The right- and left-handed chiral components $\tilde{C}_{klmn}$ of the gravitational field constitute the complex self-dual Weyl tensor (the tilde on $\tilde{C}$ signifies a right- or left-handed chiral component):

$$\tilde{C}_{klmn} \equiv \frac{1}{4}(\delta_k^p \delta_l^q \mp i\varepsilon_{kl}{}^{pq})(\delta_m^r \delta_n^s \mp i\varepsilon_{mn}{}^{rs}) C_{pqrs}, \tag{148}$$

implicitly summed over distinct antisymmetric indices pq and rs, the upper and lower signs being right- and left-handed, respectively. The wave equations for the right- and left-handed fields follow from the Weyl evolution equations (145):

$$D^k \tilde{C}_{klmn} = \tilde{J}_{lmn}, \tag{149}$$

where $\tilde{J}_{lmn}$ is the complex Weyl current

$$\tilde{J}_{lmn} \equiv \frac{1}{2}(\delta_m^r \delta_n^s \mp i\varepsilon_{mn}{}^{rs}) J_{lrs}, \tag{150}$$

the upper and lower signs again being right- and left-handed, respectively.

The complex Weyl tensor $\tilde{C}_{klmn}$ is a traceless, symmetric, complex 3×3 matrix of bivectors, with five complex degrees of freedom. It is convenient to label the components $\tilde{C}_\sigma$ by their boost weights $\sigma = +2, +1, 0, -1, -2$ (spin weights $\varsigma = \sigma$ right-handed, $\varsigma = -\sigma$ left-handed):

$$\tilde{C}_\sigma \equiv \begin{pmatrix} \tilde{C}_{+2} \\ \tilde{C}_{+1} \\ \tilde{C}_0 \\ \tilde{C}_{-1} \\ \tilde{C}_{-2} \end{pmatrix} \equiv \begin{cases} \begin{pmatrix} \tilde{C}_{v+v+} \\ \tilde{C}_{vuv+} \\ \tilde{C}_{vuvu} \\ \tilde{C}_{uvu-} \\ \tilde{C}_{u-u-} \end{pmatrix} & \text{right,} \\ \begin{pmatrix} \tilde{C}_{v-v-} \\ \tilde{C}_{vuv-} \\ \tilde{C}_{vuvu} \\ \tilde{C}_{uvu+} \\ \tilde{C}_{u+u+} \end{pmatrix} & \text{left.} \end{cases} \tag{151}$$

The five components of Eq. (151) are commonly [46] denoted (minus) Ψ_0, Ψ_1, Ψ_2, Ψ_3, and Ψ_4 [Eq. (C5)], but the notation in Eq. (151) makes manifest their transformation properties. The propagating component is [Eq. (46)] $\tilde{C}_{-2} = -\Psi_4$ outgoing, and $\tilde{C}_{+2} = -\Psi_0$ ingoing.

Focus on the right-handed gravitational field; the left-handed field is quite similar. Define the scaled Weyl tensor $\hat{C}_\sigma$ and scaled Weyl current $\hat{J}_{lmn}$ (with hats instead of tildes over the C and J) by

$$\tilde{C}_\sigma = f_2 \hat{C}_\sigma, \qquad \tilde{J}_{lmn} = f_2 \hat{J}_{lmn}, \tag{152}$$

in accordance with Eq. (53). The left-hand side of the complex Weyl evolution equation [Eq. (149)] has eight nonvanishing components [Eq. (A18) in Appendix A give these equations in a general spacetime]:

$$D^k \begin{pmatrix} \hat{C}_{kvv+} \\ \hat{C}_{k+v+} \\ \hat{C}_{k-v+} \\ \hat{C}_{kuv+} \\ \hat{C}_{k+u-} \\ \hat{C}_{kvu-} \\ \hat{C}_{kuu-} \\ \hat{C}_{k-u-} \end{pmatrix} = \begin{pmatrix} -3\Gamma_{+vv}\hat{C}_0 \\ -3\Gamma_{+v+}\hat{C}_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3\Gamma_{-uu}\hat{C}_0 \\ -3\Gamma_{-u-}\hat{C}_0 \end{pmatrix} + \frac{f_2}{\sqrt{2}\rho} \begin{pmatrix} \bar\rho^{-3}\eth_v\bar\rho^3\hat{C}_{+1} + \bar\rho^3\eth_-\bar\rho^{-3}\hat{C}_{+2} \\ \bar\rho^{-3}\eth_+\bar\rho^3\hat{C}_{+1} \mp \bar\rho^3\eth_u\bar\rho^{-3}\hat{C}_{+2} \\ \bar\rho^{-1}\eth_v\bar\rho\hat{C}_0 + \bar\rho\eth_-\bar\rho^{-1}\hat{C}_{+1} \\ \bar\rho^{-1}\eth_+\bar\rho\hat{C}_0 \mp \bar\rho\eth_u\bar\rho^{-1}\hat{C}_{+1} \\ \pm\bar\rho^{-1}\eth_u\bar\rho\hat{C}_0 - \bar\rho\eth_+\bar\rho^{-1}\hat{C}_{-1} \\ -\bar\rho^{-1}\eth_-\bar\rho\hat{C}_0 - \bar\rho\eth_v\bar\rho^{-1}\hat{C}_{-1} \\ \pm\bar\rho^{-3}\eth_u\bar\rho^3\hat{C}_{-1} - \bar\rho^3\eth_+\bar\rho^{-3}\hat{C}_{-2} \\ -\bar\rho^{-3}\eth_-\bar\rho^3\hat{C}_{-1} - \bar\rho^3\eth_v\bar\rho^{-3}\hat{C}_{-2} \end{pmatrix}. \tag{153}$$

The nonderivative terms to the immediate right of the equals sign in Eq. (153) involve products of boost- and spin-weight ± 2 components of Lorentz connections, also known as shears, with the boost- and spin-weight-0 Weyl tensor $\tilde{C}_0$ [Eq. (42)]. As Ref. [46] has emphasized, although the shears vanish in the conformally separable background, their derivatives [Eq. (A20)] yield boost/spin-weight ± 2 Weyl components $\tilde{C}_{\pm 2}$ that contribute to the wave equations for those components. For example, the difference of $\bar\rho^{-3}\eth_+\bar\rho^3$ times the first row of Eq. (153) and $\bar\rho^{-3}\eth_v\bar\rho^3$ times the second row yields

$$(\bar\rho^{-3}\eth_v\bar\rho^6\eth_u\bar\rho^{-3} \pm \bar\rho^{-3}\eth_+\bar\rho^6\eth_-\bar\rho^{-3} + 6\rho^2\tilde{C}_0)\hat{C}_{+2} \\ = \bar\rho^{-3}(\eth_v\bar\rho^3\hat{J}_{+v+} - \eth_+\bar\rho^3\hat{J}_{vv+}), \tag{154}$$

with the $6\rho^2\tilde{C}_0$ term coming from the difference of derivatives of the shears Γ_{+vv} and Γ_{+v+} [top row of Eq. (A20a) with $\psi_{s\sigma\varsigma} = \tilde{C}_0$]. Remarkably, the combination of derivatives of Weyl currents $\hat{J}_{lmn}$ on the right-hand side of Eq. (154) vanishes for the conformally separable solutions, despite the fact that neither of the Weyl currents vanishes individually. This proves to be true for all of the wave equations derived from combining equations in pairs from Eq. (153): in all cases, the combinations of derivatives of Weyl currents from the right-hand sides of the Weyl evolution equations (149) vanish for the conformally separable spacetimes, despite the fact that none of the Weyl currents vanishes individually. The shear derivatives contribute precisely what is needed to make the wave equations for the boost/weight ± 2 components separable [Eqs. (156a) and (156f)], but the same shear derivatives complicate the wave equations for the boost/weight-spin ± 1 components that follow from combining the top and bottom pairs from Eq. (153), respectively, so these equations are omitted from Eq. (155), which follows.

Combining Eq. (153) in pairs so as to eliminate one of the two spin components on each row yields six second-order differential equations:

$$(\mp\bar\rho^{-3}\eth_v\bar\rho^6\eth_u\bar\rho^{-3} - \bar\rho^{-3}\eth_+\bar\rho^6\eth_-\bar\rho^{-3} + 6\rho^2\tilde{C}_0)\hat{C}_{+2} = 0, \tag{155a}$$

$$(\mp\bar\rho^{-1}\eth_v\bar\rho^2\eth_u\bar\rho^{-1} - \bar\rho^{-1}\eth_+\bar\rho^2\eth_-\bar\rho^{-1})\hat{C}_{+1} = 0, \tag{155b}$$

$$(\mp\bar\rho\eth_v\bar\rho^{-2}\eth_u\bar\rho - \bar\rho\eth_+\bar\rho^{-2}\eth_-\bar\rho)\hat{C}_0 = 0, \tag{155c}$$

$$(\mp\bar\rho\eth_u\bar\rho^{-2}\eth_v\bar\rho - \bar\rho\eth_-\bar\rho^{-2}\eth_+\bar\rho)\hat{C}_0 = 0, \tag{155d}$$

$$(\mp\bar\rho^{-1}\eth_u\bar\rho^2\eth_v\bar\rho^{-1} - \bar\rho^{-1}\eth_-\bar\rho^2\eth_+\bar\rho^{-1})\hat{C}_{-1} = 0, \tag{155e}$$

$$(\mp\bar\rho^{-3}\eth_u\bar\rho^6\eth_v\bar\rho^{-3} - \bar\rho^{-3}\eth_-\bar\rho^6\eth_+\bar\rho^{-3} + 6\rho^2\tilde{C}_0)\hat{C}_{-2} = 0. \tag{155f}$$

Equations (155a)–(155f) can be recast in terms of the wave operators defined by Eq. (57) as

$$(\Box_{vu} - \Box_{+-})\hat{C}_{+2} = 0, \tag{156a}$$

$$(\Box_{vu} - \Box_{+-})\hat{C}_{+1} = 0, \tag{156b}$$

$$(\Box_{uv} - \Box_{-+} - 2\rho^2\tilde{C}_0)\hat{C}_0 = 0, \tag{156c}$$

$$(\Box_{vu} - \Box_{+-} - 2\rho^2\tilde{C}_0)\hat{C}_0 = 0, \tag{156d}$$

$$(\Box_{uv} - \Box_{-+})\hat{C}_{-1} = 0, \tag{156e}$$

$$(\Box_{uv} - \Box_{-+})\hat{C}_{-2} = 0. \tag{156f}$$

Note the similarity of the gravitational wave equations (156e)–(156b) with boost weights ± 1 and 0 to the electromagnetic wave equations (109). The gravitational wave equations for not only the boost-weight ± 2 components but also the boost-weight ± 1 components are separable. Reference [46] discusses this on p. 435, where Chandrasekhar says that the separability of the boost-weight ± 1 wave equations is a gauge choice associated with the freedom of Lorentz transformations of the tetrad frame; but in the present case, the tetrad frame is chosen to be aligned with the principal null directions, and that choice leads to separable wave equations for $\tilde{C}_{\pm 1}$.

The Teukolsky-Starobinski identities for the boost-weight ± 1 components $\tilde{C}_{\pm 1}$ of the gravitational field are the same as those in Eq. (111) for the electromagnetic field, and the relations between separated factors $X_{\pm 1}$ and $Y_{\pm 1}$ for the gravitational field are the same as those in Eq. (113) for the electromagnetic field.

The Teukolsky-Starobinski identities for the boost-weight ± 2 components $\tilde{C}_{\pm 2}$ of the gravitational field are

$$(\Box_{uv}\eth_u^4 - \eth_u^4\Box_{vu})\hat{C}_{+2} = -K_x^{1/2}R^3|\Delta_x|^{3/2}\eth_u(K_x^{1/2}R^{-3}\hat{C}_{+2}), \tag{157a}$$

$$(\Box_{vu}\eth_v^4 - \eth_v^4\Box_{uv})\hat{C}_{-2} = K_x^{1/2}R^3|\Delta_x|^{3/2}\eth_v(K_x^{1/2}R^{-3}\hat{C}_{-2}), \tag{157b}$$

$$(\Box_{-+}\eth_-^4 - \eth_-^4\Box_{+-})\hat{C}_{+2} = -K_y^{1/2}\Delta_y^{3/2}\eth_-(K_y^{1/2}\hat{C}_{+2}), \tag{157c}$$

$$(\Box_{+-}\eth_+^4 - \eth_+^4\Box_{-+})\hat{C}_{-2} = K_y^{1/2}\Delta_y^{3/2}\eth_+(K_y^{1/2}\hat{C}_{-2}), \tag{157d}$$

where R is the radial coordinate [Eq. (4)] (not the Ricci scalar), and K_x and K_y are the radial and angular functions defined by Eq. (141). As remarked following Eq. (141), the functions K_x and K_y vanish in Λ-Kerr(-Newman) spacetimes, but they do not vanish in the conformally separable spacetimes. For the remainder of this Sec. VIII, the spacetime is taken to be Λ-Kerr.

For Λ-Kerr spacetimes, with a convenient choice of relative normalization, the boost- and spin-weight ± 2 eigenfunctions are related by

$$\eth_u^4 X_{+2} = \mu_x X_{-2}, \qquad \eth_v^4 X_{-2} = \mu_x X_{+2}, \tag{158a}$$

$$\eth_-^4 Y_{+2} = \mu_y Y_{-2}, \qquad \eth_+^4 Y_{-2} = \mu_y Y_{+2}. \tag{158b}$$

Solving for μ_x^2 and μ_y^2 in

$$(\eth_v^4\eth_u^4 - \mu_x^2)X_{+2} = 0, \tag{159a}$$

$$(\eth_+^4\eth_-^4 - \mu_y^2)Y_{+2} = 0, \tag{159b}$$

given that $\tilde{C}_2$ satisfies $(\Box_{vu} - \lambda)\tilde{C}_{+2} = (\Box_{+-} - \lambda)\tilde{C}_{+2} = 0$, shows that μ_x and μ_y are related to the eigenvalue λ by, for Λ-Kerr,

$$\mu_x^2 = \mu_y^2 + 144w^2M_\bullet^2, \tag{160a}$$

$$\begin{aligned}\mu_y^2 = {}& \left[\lambda^2 - 1 - 4aw(aw+m) - \frac{1}{3}a^2\Lambda\left(14 + \frac{1}{3}a^2\Lambda\right)\right] \\ & \times\left[\lambda^2 - 1 - 36aw(aw+m) + \frac{1}{3}a^2\Lambda\left(42 - \frac{1}{3}a^2\Lambda\right)\right] \\ & + 32\left(\lambda - 1 + \frac{1}{3}a^2\Lambda\right)\left[aw(4aw+m)\right. \\ & \left. - \frac{1}{3}a^2\Lambda(aw+m)(4aw+3m)\right] \\ & - \frac{16}{3}a^2\Lambda\left[8aw(aw+m) - \frac{49}{3}a^2\Lambda\right].\end{aligned} \tag{160b}$$

Equations (160a) and (160b) agree with Eq. (61) from p. 440 of Ref. [46], which gives the case $\Lambda = 0$, with the translations (there $\leftrightarrow$ here) being $\lambda\!\!\!^- = \lambda - \frac{1}{2}$ and $\mathcal{C} = \mu_x$, $D = \mu_y$, $\sigma = w$.

IX. CONCLUSIONS

The wave equations in the conformally separable solutions of Refs. [30,31] for accreting, rotating, uncharged black holes are solved for massless fields of spin 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2, resulting in the generalized Teukolsky wave equations (60) with the potentials (61), generalizing the well-known Teukolsky wave equations for stationary black holes [27,44–47]. As is well known, massless waves resolve into independently evolving right- and left-handed chiralities. A wave of given chirality and spin s has $2s+1$ components, with boost weights $\sigma = -s, -s+1, \ldots, s$, and spin weights ς either equal ($\varsigma = \sigma$, right-handed chirality) or opposite ($\varsigma = -\sigma$, left-handed chirality) to the boost weight. The $2s+1$ different components are coupled by equations of motion, so they are not independent of each other, but rather oscillate in harmony. The propagating components of a wave have boost weight $\sigma = -s$ for outgoing waves, and $\sigma = +s$ for ingoing waves. The wave equations for components with $\sigma = \pm s$, which include the propagating components, are separable in the conformally separable solutions, as they are in stationary solutions for black holes. In addition, the wave equations for boost-weight ± 1 components of gravitational waves ($s = 2$) are separable.

The Teukolsky-Starobinsky identities [60,61] carry through essentially unchanged (with a modified horizon function) for fields of spin $\frac{1}{2}$ and 1, but they are more complicated for fields of spin $\frac{3}{2}$ and 2.

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APPENDIX A: WAVE EQUATIONS IN A GENERAL SPACETIME

This appendix takes a deeper dive into the derivation of wave equations in a general spacetime. Subsections A 1–A 5 give the equations of motion that lead to wave equations for spins $s = 0, \frac{1}{2}, 1, \frac{3}{2}$, and 2. Subsection A 6 gives a general expression for derivatives of shear needed in the wave equations for gravitational waves.

Massless waves are described by independently evolving chiral fields with all-right-handed or all-left-handed bivector and spinor indices. For example, spin-$\frac{3}{2}$ waves have a bivector index and a spinor index, and those indices are either both right-handed or both left-handed [Eq. (128)]. Similarly, spin-2 waves have two bivector indices, and those indices are either both right-handed or both left-handed [Eq. (148)]. Equations of motion yield linear differential equations that relate components of adjacent boost (and spin) weight to each other. The right- and left-handed differential equations involve purely right- or purely left-handed Lorentz connections. Right- and left-handed Lorentz connections are defined by (the tilde on $\tilde{\Gamma}$ signifies a right- or left-handed chiral component)

$$\tilde{\Gamma}_{klp} \equiv \frac{1}{2}(\Gamma_{klp} \mp i\varepsilon_{kl}{}^{mn}\Gamma_{mnp}). \tag{A1}$$

In a completely general spacetime (not necessarily a conformally separable spacetime), the most general right- or left-handed linear differential operators with the properties that (i) they involve purely right- or left-handed Lorentz connections, and (ii) they raise and lower the boost/spin weight by 1, are the radial ${}_{s\sigma}\mathcal{D}_{\substack{u\\v}}$ and angular ${}_{s\varsigma}\mathcal{D}_{\pm}$ operators defined by

$${}_{s\sigma}\mathcal{D}_{\substack{v\\u}} \equiv \partial_{\substack{v\\u}} \pm \sigma\Gamma_{\substack{uvv\\vuu}} + \begin{cases} \mp\sigma\Gamma_{\substack{+-v\\-+u}} + (s\pm\sigma+1)\Gamma_{\substack{+v-\\-u+}} & \text{right} \\ \mp\sigma\Gamma_{\substack{-+v\\+-u}} + (s\pm\sigma+1)\Gamma_{\substack{-v+\\+u-}} & \text{left} \end{cases}, \tag{A2a}$$

$${}_{s\varsigma}\mathcal{D}_{\pm} \equiv \partial_{\pm} \mp \varsigma\Gamma_{\substack{-++\\+--}} + \begin{cases} \pm\varsigma\Gamma_{\substack{uv+\\vu-}} + (s\pm\varsigma+1)\Gamma_{\substack{+vu\\-uv}} & \text{right} \\ \pm\varsigma\Gamma_{\substack{uv+\\vu-}} + (s\pm\varsigma+1)\Gamma_{\substack{+uv\\-vu}} & \text{left} \end{cases}. \tag{A2b}$$

Note that the boost and spin weights of the various terms of each operator agree, as they must, per Eq. (32).

Explicit calculation shows that the index s of the operators in Eq. (A2) denotes the spin of the field, with $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ for scalar, spinor, electromagnetic, gravitino, and gravitational fields, respectively. The index σ denotes the boost weight of the field, which ranges over the $2s+1$ components $-s, -s+1, \ldots, s$, while $\varsigma = \pm\sigma$ denotes the spin weight of the field, with $+$ for right-handed, and $-$ for left-handed. For the most part, the spin s and boost/spin weight σ and ς indices can be suppressed, because they equal the spin and boost/spin weight of the field they are acting on. Equations for the various spins are (A10), (A11) and (A12), (A13) and (A14), (A16) and (A17), and (A18) and (A19).

The operators $\mathcal{D}$ defined by Eq. (A2) do not commute with each other, but (primed) operators defined by

$${}_{s\sigma}\mathcal{D}'_{\substack{v\\u}} \equiv {}_{s\sigma}\mathcal{D}_{\substack{v\\u}} + \begin{cases} -\Gamma_{\substack{-+v\\+-u}} + \Gamma_{\substack{-v+\\+u-}} & \text{right} \\ -\Gamma_{\substack{+-v\\-+u}} + \Gamma_{\substack{+v-\\-u+}} & \text{left} \end{cases}, \tag{A3a}$$

$${}_{s\varsigma}\mathcal{D}'_{\pm} \equiv {}_{s\varsigma}\mathcal{D}_{\pm} + \begin{cases} \Gamma_{\substack{uv+\\vu-}} - \Gamma_{\substack{+uv\\-vu}} & \text{right} \\ \Gamma_{\substack{vu+\\uv-}} - \Gamma_{\substack{+vu\\-uv}} & \text{left} \end{cases} \tag{A3b}$$

have the property that, in the conformally separable spacetimes, the radial and angular $\mathcal{D}$ and $\mathcal{D}'$ operators do commute:

$${}_{s\varsigma}\mathcal{D}'_{\pm}{}_{s\sigma}\mathcal{D}_{\substack{v\\u}} - {}_{s\sigma}\mathcal{D}'_{\substack{v\\u}}{}_{s\varsigma}\mathcal{D}_{\pm} = 0, \tag{A4}$$

in both right- and left-handed versions. The existence of radial/angular operators $\mathcal{D}'$ that commute with angular/radial operators $\mathcal{D}$ is crucial to forming wave equations for individual components of particular boost and spin weight. In the conformally separable spacetimes, the $\mathcal{D}$ and $\mathcal{D}'$ operators are related by

$${}_{s\sigma}\mathcal{D}'_k = \rho^{-1}{}_{s\sigma}\mathcal{D}_k\rho \tag{A5}$$

in both right- and left-handed versions, with ρ being the conformal factor [Eq. (6)].

In conformally separable spacetimes, the raising and lowering operators ${}_{s\sigma}\eth_k$ defined by Eq. (47) are related to the ${}_{s\sigma}\mathcal{D}_k$ operators by, in the right-handed case,

$$\sqrt{2}\rho_{s\sigma}\mathcal{D}_k = (\pm)f_s\bar{\rho}^{\mp 2\sigma-1}{}_{\sigma}\eth_k\bar{\rho}^{\pm 2\sigma+1}f_s^{-1}, \tag{A6}$$

where f_s is given by Eq. (54), and the initial $(\pm)$ sign is $+$ for $k = v$ or $+$, or $k = u$ outside the horizon; $-$ for $k = -$, or $k = u$ inside the horizon; while the $\pm$ sign multiplying 2σ is $+$ for indices $k = v$ or $+$, and $-$ for $k = u$ or $-$. The left-handed relation is the complex conjugate of the right-handed relation in Eq. (A6), obtained by replacing $f_s \to f_s^*$ and $\bar{\rho} \to \bar{\rho}^*$.

Wave equations are obtained by taking the differential equations linear in the operators $\mathcal{D}$ [Eq. (A2)] that follow from the equations of motion, applying operators $\mathcal{D}'$ [Eq. (A3)], and differencing the resulting second-order differential equations in such a way that the commutation

[Eq. (A4)] leads to cancellation of terms. The resulting equations can be expressed in terms of the difference of the radial and angular wave operators $\Box$ defined by Eq. (57). For arbitrary spins s and boost weights σ, the difference $\Box_{\substack{vu\\uv}} - \Box_{\substack{+-\\-+}}$ of radial and angular wave operators, expressed in terms of the $\mathcal{D}$ and $\mathcal{D}'$ operators, and then in terms of the ð raising and lowering operators [Eq. (47)] are, for right-handed ($\varsigma = +\sigma$) modes,

$$\begin{aligned}({}_\sigma\Box_{\substack{vu\\uv}} - {}_\sigma\Box_{\substack{+-\\-+}})\hat\psi_\sigma &= 2\rho^2 f_s^{-1}\left[(-{}_{s,\sigma\mp1}\mathcal{D}'_{\substack{v\\u}}{}_{s,\sigma}\mathcal{D}_{\substack{u\\v}} + {}_{s,\sigma\mp1}\mathcal{D}'_{\pm}{}_{s,\sigma}\mathcal{D}_{\mp}) + 2\left(\sigma\mp\frac{1}{2}\right)(\sigma\mp1)\tilde C_0\right] f_s\hat\psi_\sigma \\ &= \left[\bar\rho^{-(\pm2\sigma-1)}(-\mathrm{sgn}(\Delta_x){}_{\sigma\mp1}\eth_{\substack{v\\u}}\bar\rho^{2(\pm2\sigma-1)}{}_\sigma\eth_{\substack{u\\v}} - {}_{\sigma\mp1}\eth_\pm\bar\rho^{2(\pm2\sigma-1)}{}_\sigma\eth_\mp)\bar\rho^{-(\pm2\sigma-1)}\right. \\ &\quad\left. + 4\rho^2\left(\sigma\mp\frac{1}{2}\right)(\sigma\mp1)\tilde C_0\right]\hat\psi_\sigma. \qquad \text{(A7)}\end{aligned}$$

The equivalent result for left-handed modes ($\varsigma = -\sigma$) is obtained by flipping angular indices $+ \leftrightarrow -$, and complex-conjugating $f_s \to f_s^*$ and $\bar\rho \to \bar\rho^*$:

$$\begin{aligned}({}_\sigma\Box_{\substack{vu\\uv}} - {}_\sigma\Box_{\substack{-+\\+-}})\hat\psi_\sigma &= 2\rho^2 (f_s^*)^{-1}\left[(-{}_{s,\sigma\mp1}\mathcal{D}'_{\substack{v\\u}}{}_{s,\sigma}\mathcal{D}_{\substack{u\\v}} + {}_{s,\sigma\pm1}\mathcal{D}'_{\mp}{}_{s,\sigma}\mathcal{D}_{\pm}) + 2\left(\sigma\mp\frac{1}{2}\right)(\sigma\mp1)\tilde C_0\right] f_s^*\hat\psi_\sigma \\ &= \left[(\bar\rho^*)^{-(\pm2\sigma-1)}(-\mathrm{sgn}(\Delta_x){}_{\sigma\mp1}\eth_{\substack{v\\u}}(\bar\rho^*)^{2(\pm2\sigma-1)}{}_\sigma\eth_{\substack{u\\v}} - {}_{\sigma\pm1}\eth_\mp(\bar\rho^*)^{2(\pm2\sigma-1)}{}_\sigma\eth_\pm)(\bar\rho^*)^{-(\pm2\sigma-1)}\right. \\ &\quad\left. + 4\rho^2\left(\sigma\mp\frac{1}{2}\right)(\sigma\mp1)\tilde C_0\right]\hat\psi_\sigma. \qquad \text{(A8)}\end{aligned}$$

1. Spin-0 waves

In a general spacetime, the d'Alembertian operator that goes in the scalar wave equation (79) is

$$D^k D_k\varphi = (-{}_{1,-1}\mathcal{D}'_v\partial_u - {}_{1,+1}\mathcal{D}'_u\partial_v + {}_{1,-1}\mathcal{D}'_+\partial_- + {}_{1,+1}\mathcal{D}'_-\partial_+)\varphi, \qquad \text{(A9)}$$

which holds true for both right- and left-handed versions of the $\mathcal{D}'$ operators [Eq. (A3)]. In conformally separable spacetimes, Eq. (A9) can be recast as

$$D^k D_k\varphi = 2\left(-{}_{0,\mp1}\mathcal{D}'_{\substack{v\\u}}{}_{0,0}\mathcal{D}_{\substack{u\\v}} + {}_{0,\mp1}\mathcal{D}'_{\pm}{}_{0,0}\mathcal{D}_\mp + \tilde C_0 + \frac{1}{12}R\right)\varphi, \qquad \text{(A10)}$$

which again holds true for both right- and left-handed versions of the $\mathcal{D}$ and $\mathcal{D}'$ operators. Note that whereas the index s on ${}_{s\sigma}\mathcal{D}'_k$ is $s = 1$ in the general equation (A9), the index is $s = 0$ on ${}_{s\sigma}\mathcal{D}_k$ and ${}_{s\sigma}\mathcal{D}'_k$ in the conformally separable version [Eq. (A10)].

2. Spin-$\frac{1}{2}$ waves

In a general spacetime, the Dirac equation (83) for the components ψ_σ of the spinor field [Eq. (105)], expressed with respect to a Newman-Penrose tetrad in terms of the differential operators defined by Eq. (A2), are, for the right-handed components,

$$\frac{1}{\sqrt2}(\boldsymbol{D}\psi)_{\substack{\Downarrow\uparrow\\\Uparrow\downarrow}} = {}_{\mp1/2}\mathcal{D}_{\substack{v\\u}}\psi_{\substack{\Uparrow\uparrow\\\Downarrow\downarrow}} - {}_{\pm1/2}\mathcal{D}_\mp\psi_{\substack{\Downarrow\downarrow\\\Uparrow\uparrow}} = 0, \qquad \text{(A11)}$$

and for the left-handed components,

$$\frac{1}{\sqrt2}(\boldsymbol{D}\psi)_{\substack{\Downarrow\downarrow\\\Uparrow\uparrow}} = -{}_{\mp1/2}\mathcal{D}_{\substack{v\\u}}\psi_{\substack{\Uparrow\downarrow\\\Downarrow\uparrow}} - {}_{\mp1/2}\mathcal{D}_\pm\psi_{\substack{\Downarrow\uparrow\\\Uparrow\downarrow}} = 0. \qquad \text{(A12)}$$

3. Spin-1 waves

In a general spacetime, Maxwell's equations (104) for the components $\tilde{F}_\sigma$ of the electromagnetic field [Eq. (105)], expressed with respect to a Newman-Penrose tetrad in terms of the differential operators defined by Eq. (A2), are, for the right-handed components ($\varsigma = +\sigma$),

$$D^k \tilde{F}_{\substack{kv\\ku}} = \Gamma_{\substack{-vv\\+uu}} \tilde{F}_{\mp 1} \pm {}_0\mathcal{D}_{\substack{v\\u}} \tilde{F}_0 - {}_{\pm 1}\mathcal{D}_{\mp} \tilde{F}_{\pm 1} = \tilde{j}_{\substack{u\\v}}, \tag{A13a}$$

$$D^k \tilde{F}_{\substack{k-\\k+}} = \Gamma_{\substack{+v+\\-u-}} \tilde{F}_{\mp 1} \pm {}_0\mathcal{D}_{\pm} \tilde{F}_0 - {}_{\pm 1}\mathcal{D}_{\substack{u\\v}} \tilde{F}_{\pm 1} = \tilde{j}_{\pm}, \tag{A13b}$$

and for the left-handed components ($\varsigma = -\sigma$),

$$D^k \tilde{F}_{\substack{kv\\ku}} = \Gamma_{\substack{-vv\\+uu}} \tilde{F}_{\mp 1} \pm {}_0\mathcal{D}_{\substack{v\\u}} \tilde{F}_0 - {}_{\mp 1}\mathcal{D}_{\pm} \tilde{F}_{\pm 1} = \tilde{j}_{\substack{u\\v}}, \tag{A14a}$$

$$D^k \tilde{F}_{\substack{k-\\k+}} = \Gamma_{\substack{-v-\\+u+}} \tilde{F}_{\mp 1} \pm {}_0\mathcal{D}_{\mp} \tilde{F}_0 - {}_{\pm 1}\mathcal{D}_{\substack{u\\v}} \tilde{F}_{\pm 1} = \tilde{j}_{\pm}. \tag{A14b}$$

The units of the currents j_k are Heaviside; in Gaussian units, the currents would be multiplied by 4π. Conservation of electric current is expressed by

$$D^k j_k = -{}_{+1}\mathcal{D}'_u j_v - {}_{-1}\mathcal{D}'_v j_u + {}_{-1}\mathcal{D}'_+ j_- + {}_{+1}\mathcal{D}'_- j_+ = 0. \tag{A15}$$

4. Spin-$\frac{3}{2}$ waves

Wave equations for the spin-$\frac{3}{2}$ gravitino field $\boldsymbol{\Psi}$ follow from the Jacobi identity [Eq. (134)]. In a general spacetime, the left-hand sides of the Jacobi identity (134) in terms of the differential operators defined by Eq. (A2) are, for the right-handed components,

$$\frac{1}{\sqrt{2}}(\boldsymbol{D}\tilde{\Psi}_{\substack{vu\\uv}})_{\substack{\Downarrow\uparrow\\\Uparrow\downarrow}} = D^k \tilde{\Psi}_{\substack{kv\Uparrow\uparrow\\ku\Downarrow\downarrow}} = D^k \tilde{\Psi}_{\substack{k-\Uparrow\uparrow\\k+\Downarrow\downarrow}} = \Gamma_{\substack{+vv\\-uu}} \tilde{\Psi}_{\mp 3/2} \pm {}_{\mp 1/2}\mathcal{D}_{\substack{v\\u}} \tilde{\Psi}_{\mp 1/2} \mp {}_{\pm 1/2}\mathcal{D}_{\mp} \tilde{\Psi}_{\pm 1/2} + \Gamma_{\substack{-u-\\+v+}} \tilde{\Psi}_{\pm 3/2}, \tag{A16a}$$

$$\frac{1}{\sqrt{2}}(\boldsymbol{D}\tilde{\Psi}_{\substack{v+\\u-}})_{\substack{\Downarrow\uparrow\\\Uparrow\downarrow}} = D^k \tilde{\Psi}_{\substack{kv\Downarrow\downarrow\\ku\Uparrow\uparrow}} = \mp 2\Gamma_{\substack{+vv\\-uu}} \tilde{\Psi}_{\mp 1/2} \pm {}_{\pm 1/2}\mathcal{D}_{\substack{v\\u}} \tilde{\Psi}_{\pm 1/2} - {}_{\pm 3/2}\mathcal{D}_{\mp} \tilde{\Psi}_{\pm 3/2}, \tag{A16b}$$

$$-\frac{1}{\sqrt{2}}(\boldsymbol{D}\tilde{\Psi}_{\substack{v+\\u-}})_{\substack{\Uparrow\downarrow\\\Downarrow\uparrow}} = D^k \tilde{\Psi}_{\substack{k+\Downarrow\downarrow\\k-\Uparrow\uparrow}} = \mp 2\Gamma_{\substack{+v+\\-u-}} \tilde{\Psi}_{\mp 1/2} \pm {}_{\pm 1/2}\mathcal{D}_{\pm} \tilde{\Psi}_{\pm 1/2} - {}_{\pm 3/2}\mathcal{D}_{\substack{u\\v}} \tilde{\Psi}_{\pm 3/2}, \tag{A16c}$$

and for the left-handed components,

$$-\frac{1}{\sqrt{2}}(\boldsymbol{D}\tilde{\Psi}_{\substack{vu\\uv}})_{\substack{\Downarrow\downarrow\\\Uparrow\uparrow}} = D^k \tilde{\Psi}_{\substack{kv\Uparrow\downarrow\\ku\Downarrow\uparrow}} = \pm D^k \tilde{\Psi}_{\substack{k+\Downarrow\uparrow\\k-\Uparrow\downarrow}} = \Gamma_{\substack{-vv\\+uu}} \tilde{\Psi}_{\mp 3/2} \pm {}_{\mp 1/2}\mathcal{D}_{\substack{v\\u}} \tilde{\Psi}_{\mp 1/2} \pm {}_{\mp 1/2}\mathcal{D}_{\pm} \tilde{\Psi}_{\pm 1/2} - \Gamma_{\substack{+u+\\-v-}} \tilde{\Psi}_{\pm 3/2}, \tag{A17a}$$

$$\frac{1}{\sqrt{2}}(\boldsymbol{D}\tilde{\Psi}_{\substack{v-\\u+}})_{\substack{\Downarrow\downarrow\\\Uparrow\uparrow}} = D^k \tilde{\Psi}_{\substack{kv\Downarrow\uparrow\\ku\Uparrow\downarrow}} = \pm 2\Gamma_{\substack{-vv\\+uu}} \tilde{\Psi}_{\mp 1/2} \pm {}_{\pm 1/2}\mathcal{D}_{\substack{v\\u}} \tilde{\Psi}_{\pm 1/2} - {}_{\mp 3/2}\mathcal{D}_{\pm} \tilde{\Psi}_{\pm 3/2}, \tag{A17b}$$

$$\frac{1}{\sqrt{2}}(\boldsymbol{D}\tilde{\Psi}_{\substack{v-\\u+}})_{\substack{\Uparrow\uparrow\\\Downarrow\downarrow}} = D^k \tilde{\Psi}_{\substack{k-\Downarrow\uparrow\\k+\Uparrow\downarrow}} = \pm 2\Gamma_{\substack{-v-\\+u+}} \tilde{\Psi}_{\mp 1/2} \pm {}_{\mp 1/2}\mathcal{D}_{\mp} \tilde{\Psi}_{\pm 1/2} - {}_{\pm 3/2}\mathcal{D}_{\substack{u\\v}} \tilde{\Psi}_{\pm 3/2}. \tag{A17c}$$

5. Spin-2 waves

In a general spacetime, the Weyl evolution equations (149) in terms of the differential operators defined by Eq. (A2) are, for right-handed components,

$$D^k \tilde{C}_{\substack{k-v+\\k+u-}} = -2\Gamma_{\substack{+vv\\-uu}} \tilde{C}_{\mp 1} + {}_0\mathcal{D}_{\substack{v\\u}} \tilde{C}_0 - {}_{\pm 1}\mathcal{D}_{\mp} \tilde{C}_{\pm 1} + \Gamma_{\substack{-u-\\+v+}} \tilde{C}_{\pm 2} = \tilde{J}_{\substack{-v+\\+u-}}, \tag{A18a}$$

$$D^k \tilde{C}_{\substack{kuv+\\kvu-}} = -2\Gamma_{\substack{+v+\\-u-}} \tilde{C}_{\mp1} + {}_0\mathcal{D}_{\pm} \tilde{C}_0 - {}_{\pm1}\mathcal{D}_{\substack{u\\v}} \tilde{C}_{\pm1} + \Gamma_{\substack{-uu\\+vv}} \tilde{C}_{\pm2} = \tilde{J}_{\substack{uv+\\vu-}}, \tag{A18b}$$

$$D^k \tilde{C}_{\substack{kvv+\\kuu-}} = -3\Gamma_{\substack{+vv\\-uu}} \tilde{C}_0 + {}_{\pm1}\mathcal{D}_{\substack{u\\v}} \tilde{C}_{\pm1} - {}_{\pm2}\mathcal{D}_{\mp} \tilde{C}_{\pm2} = \tilde{J}_{\substack{vv+\\uu-}}, \tag{A18c}$$

$$D^k \tilde{C}_{\substack{k+v+\\k-u-}} = -3\Gamma_{\substack{+v+\\-u-}} \tilde{C}_0 + {}_{\pm1}\mathcal{D}_{\pm} \tilde{C}_{\pm1} - {}_{\pm2}\mathcal{D}_{\substack{u\\v}} \tilde{C}_{\pm2} = \tilde{J}_{\substack{+v+\\-u-}}, \tag{A18d}$$

and for left-handed components,

$$D^k \tilde{C}_{\substack{k+v-\\k-u+}} = 2\Gamma_{\substack{-vv\\+uu}} \tilde{C}_{\mp1} + {}_0\mathcal{D}_{\substack{v\\u}} \tilde{C}_0 + {}_{\mp1}\mathcal{D}_{\pm} \tilde{C}_{\pm1} + \Gamma_{\substack{+u+\\-v-}} \tilde{C}_{\pm2} = \tilde{J}_{\substack{+v-\\-u+}}, \tag{A19a}$$

$$D^k \tilde{C}_{\substack{kuv-\\kvu+}} = 2\Gamma_{\substack{-v-\\+u+}} \tilde{C}_{\mp1} + {}_0\mathcal{D}_{\mp} \tilde{C}_0 + {}_{\pm1}\mathcal{D}_{\substack{u\\v}} \tilde{C}_{\pm1} + \Gamma_{\substack{+uu\\-vv}} \tilde{C}_{\pm2} = \tilde{J}_{\substack{uv-\\vu+}}, \tag{A19b}$$

$$D^k \tilde{C}_{\substack{kvv-\\kuu+}} = -3\Gamma_{\substack{-vv\\+uu}} \tilde{C}_0 - {}_{\pm1}\mathcal{D}_{\substack{v\\u}} \tilde{C}_{\pm1} - {}_{\mp2}\mathcal{D}_{\pm} \tilde{C}_{\pm2} = \tilde{J}_{\substack{vv-\\uu+}}, \tag{A19c}$$

$$D^k \tilde{C}_{\substack{k-v-\\k+u+}} = -3\Gamma_{\substack{-v-\\+u+}} \tilde{C}_0 - {}_{\mp1}\mathcal{D}_{\mp} \tilde{C}_{\pm1} - {}_{\pm2}\mathcal{D}_{\substack{u\\v}} \tilde{C}_{\pm2} = \tilde{J}_{\substack{-v-\\+u+}}. \tag{A19d}$$

6. Derivatives of shear

The Newman-Penrose formalism makes a $2+2$ split of the tangent space of spacetime into a radial subspace (null indices v and u) and an angular subspace (angular indices $+$ and $-$). Conformally separable black-hole spacetimes are shear-free, meaning that the eight Lorentz connections of the form Γ_{aza}, where a and z are from opposite spaces, are all zero. Although the shears all vanish in the unperturbed background, their derivatives yield spin ±2 components of the Riemann tensor [compare the last of Eqs. (3) and (4) on p. 431 of Ref. [46]]

$${}_{s,\varsigma\pm1}\mathcal{D}'_{\pm}(\Gamma_{\substack{+vv\\-uu}}\psi_{s\sigma\varsigma}) - {}_{s,\sigma\pm1}\mathcal{D}'_{\substack{v\\u}}(\Gamma_{\substack{+v+\\-u-}}\psi_{s\sigma\varsigma}) - \Gamma_{\substack{+vv\\-uu}}s\varsigma\mathcal{D}_{\pm}\psi_{s\sigma\varsigma} + \Gamma_{\substack{+v+\\-u-}}s\sigma\mathcal{D}_{\substack{v\\u}}\psi_{s\sigma\varsigma} = R_{\substack{v+v+\\u-u-}}\psi_{s\sigma\varsigma} \quad \text{right}, \tag{A20a}$$

$${}_{s,\varsigma\mp1}\mathcal{D}'_{\mp}(\Gamma_{\substack{-vv\\+uu}}\psi_{s\sigma\varsigma}) - {}_{s,\sigma\pm1}\mathcal{D}'_{\substack{v\\u}}(\Gamma_{\substack{-v-\\+u+}}\psi_{s\sigma\varsigma}) - \Gamma_{\substack{-vv\\+uu}}s\varsigma\mathcal{D}_{\mp}\psi_{s\sigma\varsigma} + \Gamma_{\substack{-v-\\+u+}}s\sigma\mathcal{D}_{\substack{v\\u}}\psi_{s\sigma\varsigma} = R_{\substack{v-v-\\u+u+}}\psi_{s\sigma\varsigma} \quad \text{left}. \tag{A20b}$$

Equation (A20) is valid in an arbitrary spacetime for arbitrary spin s and boost/spin weight σ and ς. The case relevant to gravitational waves, Eq. (154), has $\psi_{s\sigma\varsigma} = \tilde{C}_0$, for which $s = \sigma = \varsigma = 0$.

APPENDIX B: A RELATION AMONG DIFFERENTIAL OPERATORS

In the conformally separable spacetimes considered in this paper, the differential operators $\eth_k$ defined by Eq. (47), acting on any arbitrary (not necessarily separable) function of the coordinates $\{x, t, y, \phi\}$, satisfy the following cubic relations:

$$\bar{\rho}\eth_l\bar{\rho}^{-1}\eth_k^2 = \bar{\rho}^{-1}\eth_k\bar{\rho}\left[\eth_k\eth_l - \frac{1}{\bar{\rho}}(\pm_k \mathrm{sgn}(\Delta_x)R^2\sqrt{|\Delta_x|}\eth_l \pm_l ia\sqrt{\Delta_y}\eth_k)\right], \tag{B1a}$$

$$\bar{\rho}\eth_k\bar{\rho}^{-1}\eth_l^2 = \bar{\rho}^{-1}\eth_l\bar{\rho}\left[\eth_k\eth_l - \frac{1}{\bar{\rho}}(\pm_k \mathrm{sgn}(\Delta_x)R^2\sqrt{|\Delta_x|}\eth_l \pm_l ia\sqrt{\Delta_y}\eth_k)\right], \tag{B1b}$$

where the index k is either of v or u, and the index l is either of $+$ or $-$. The $\pm_k$ sign is $+$ or $-$ as $k = v$ or u, while the $\pm_l$ sign is $+$ or $-$ as $l = +$ or $-$. The relations in Eq. (B1) lead to the expressions of Eq. (116) for the boost-weight-0 component $\tilde{F}_0$ of the electromagnetic field.

APPENDIX C: COMPARISON TO CHANDRASEKHAR [46] NOTATION

Chandrasekhar's null directions $\boldsymbol{l}$, $\boldsymbol{n}$, $\boldsymbol{m}$, $\bar{\boldsymbol{m}}$ [Eq. (283) on p. 41] correspond to

$$\boldsymbol{l} = \boldsymbol{\gamma}_v, \qquad \boldsymbol{n} = \boldsymbol{\gamma}_u, \qquad \boldsymbol{m} = \boldsymbol{\gamma}_+, \qquad \bar{\boldsymbol{m}} = \boldsymbol{\gamma}_-. \tag{C1}$$

Chandrasekhar's null indices 1,2,3,4 are

$$1 = v, \qquad 2 = u, \qquad 3 = +, \qquad 4 = -. \tag{C2}$$

Chandrasekhar's spin coefficients γ_{klm} [Eq. (253) on p. 37] coincide with the Lorentz connections Γ_{klm}:

$$\gamma_{klm} = \Gamma_{klm}. \tag{C3}$$

Chandrasekhar's electromagnetic fields ϕ_i [Eq. (324) on p. 51] are

$$\phi_0 = F_{+1} \equiv \tilde{F}_{v+}, \tag{C4a}$$

$$\phi_1 = \tilde{F}_0 \equiv \frac{1}{2}(F_{vu} - F_{+-}), \tag{C4b}$$

$$\phi_2 = \tilde{F}_{-1} \equiv F_{u-}, \tag{C4c}$$

and his gravitational fields Ψ_i [Eq. (294) on p. 43] are

$$-\Psi_0 = \tilde{C}_{+2} \equiv C_{v+v+}, \tag{C5a}$$

$$-\Psi_1 = \tilde{C}_{+1} \equiv C_{vuv+} = C_{-+v+}, \tag{C5b}$$

$$\begin{aligned} -\Psi_2 = \tilde{C}_0 \equiv C_{v+-u} &= \frac{1}{2}(C_{vuvu} - C_{vu+-}) \\ &= \frac{1}{2}(C_{+-+-} - C_{vu+-}), \end{aligned} \tag{C5c}$$

$$-\Psi_3 = \tilde{C}_{-1} \equiv C_{uvu-} = C_{+-u-}, \tag{C5d}$$

$$-\Psi_4 = \tilde{C}_{-2} \equiv C_{u-u-}. \tag{C5e}$$

Chandrasekhar's operators $\mathcal{D}_n$, $\mathcal{D}_n'$, $\mathcal{L}_n$, $\mathcal{L}_n'$ [Eq. (3) on p. 383], are related to the raising and lowering operators [Eq. (47)] of the present paper by

$$\begin{aligned} \mathcal{D}_n &= \frac{1}{R^2\sqrt{\Delta_x}}\,{}_{-2n}\eth_v, \qquad & \mathcal{D}_n' &= -\frac{1}{R^2\sqrt{\Delta_x}}\,{}_{2n}\eth_u, \\ \mathcal{L}_n &= -{}_n\eth_-, & \mathcal{L}_n' &= {}_{-n}\eth_+. \end{aligned} \tag{C6}$$

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