Black lens in a bubble of nothing

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Applying the inverse scattering method to static and biaxisymmetric Einstein equations, we construct a nonrotating black lens inside a bubble of nothing whose horizon is topologically lens space, $L(n,1)=S^3/\mathbb{Z}_n$. Using this solution, we discuss whether a static black lens can be in equilibrium by the force balance between the expansion and gravitational attraction.

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I. INTRODUCTION

The studies on higher-dimensional black hole solutions to the Einstein equations have played important roles in the microscopic derivation of black hole entropy [1] and in fundamental research on the scenario of large extra dimensions [2] through black hole production in an accelerator. Although recent developments in solution-generation techniques have enabled us to find various exact solutions of higher-dimensional black holes, our knowledge about them is still incomplete. For example, according to the topology theorem for five-dimensional black holes [3–6], the topology of the spatial cross section of the event horizon must be either a sphere (S^3), ring ($S^1\times S^2$), or lens space ($L(p,q)$) if a spacetime with two commuting rotational Killing vector fields and a timelike Killing vector field is asymptotically flat. For the first two topologies, the exact solutions to the vacuum Einstein equations [7–10] have already been found. For lens-space topology, however, it has been difficult to find a regular vacuum solution since the resultant solutions have naked singularities.

The inverse scattering method (ISM) is perhaps one of the most powerful tools to obtain exact solutions of the Einstein equations with $(D-2)$ Killing isometries, where D is a spacetime dimension. In particular, combined with the rod structure [11,12], this method has allowed for the derivation of five-dimensional vacuum black hole solutions. The first example of the construction of black hole solutions using the ISM was the rederivation of the five-dimensional Myers-Perry black hole solution [13]. Next, a black ring with S^2 rotation (first derived in Refs. [14,15]) was rederived using the ISM [16], but it turned out that the generation of a black ring with S^1 rotation has a serious problem on how to choose the seed, namely an easy choice for the seed always results in the generation of a singular solution. A suitable seed to derive the black ring with S^1 rotation was first considered in Refs. [17,18]. Subsequently, the more general black ring solution with both S^1 and S^2 rotations was constructed by Pomeransky and Sen'kov [10]. Moreover, the ISM was used to find vacuum solutions with multiple horizons, such as black Saturns [19], black di-rings [20,21],[1] and bicycling black rings (orthogonal black di-rings) [22,23].

Using the ISM, some authors attempted to construct asymptotically flat black lens solutions to the five-dimensional vacuum Einstein equations. First, Evslin [24] attempted to construct a static black lens with the lens space topology $L(n^2+1,1)$ but found that curvature singularities cannot be eliminated, whereas both conical and orbifold singularities can be removed. Moreover, by using the ISM, Chen and Teo [25] constructed black hole solutions with the horizon topology $L(n,1)=S^3/\mathbb{Z}_n$, but these solutions must have either conical singularities or naked curvature singularities. Thus, the major obstacle in constructing a black lens solution is always the existence of naked singularities. However, a sudden breakthrough has come from supersymmetric solutions. Kunduri and Lucietti [26] succeeded in deriving the first regular supersymmetric solution of an asymptotically flat black lens with the horizon topology $L(2,1)=S^3/\mathbb{Z}_2$. This solution was further generalized to the more general supersymmetric black lens with the horizon topology $L(n,1)=S^2/\mathbb{Z}_n (n\geq 3)$ [27,28]. Building upon the work of Kunduri and Lucietti, the authors

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[1]After this solution was first constructed using the Bäcklund transformation in Ref. [20], it was reconstructed using the ISM in Ref. [21].

of Ref. [29] attempted to construct the vacuum solution of a black lens with $L(2,1)$ without singularities, but the solution has unavoidable closed timelike curves (CTCs). Thereafter, Ref. [30] discussed the nonexistence of vacuum black lenses.

Supersymmetric black lenses carry mass, electric charge (saturating the Bogomol'nyi-Prasad-Sommerfield bound), two angular momenta, and magnetic fluxes [18,26]. As was discussed in Ref. [18], there exists no limit such that all of the magnetic fluxes vanish. Therefore, as for the supersymmetric solutions, the existence of the magnetic fluxes seems to play an essential role in supporting the horizon of a black lens. In general, however, it is not clear whether such magnetic fluxes are necessarily needed to construct a black lens. Recently, a different type of solution within a class of generalized Weyl solutions—static black hole binaries and black rings in expanding bubbles of nothing—was studied in Ref. [31], although equilibrium configuration of black holes in bubble had been studied only in the context of Kaluza-Klein theory [32–34]. As is well known, an asymptotically flat, static black ring cannot be in equilibrium since the horizon collapses due to the self-gravitational force. However, the black ring in Ref. [31] is allowed to be in static equilibrium by the balance between the expanding force of a bubble and the gravitational force, so it has no conical singularities. This solution leads a simple, interesting question: is a nonrotating black lens in a bubble of nothing allowed to be in equilibrium? Studying such a solution may show us what (except for magnetic fluxes) is needed to obtain a regular black lens. Thus, the goal of this paper is to investigate whether an expanding bubble of nothing admits the existence of a black lens in equilibrium. In this paper, to derive such a solution, we apply the ISM to the five-dimensional vacuum Einstein equations with staticity and biaxisymmetry, and construct a one-soliton solution by considering a static black ring inside of a bubble (as in Ref. [31]) as a seed solution. Note that our procedure in the ISM is entirely the same as the work of Chen and Teo where the seed solution was chosen as a static black ring, namely, the only different point is the seed solution.

The remainder of the paper is organized as follows. In Sec. II, under the assumptions of staticity and biaxisymmetry, we present a vacuum solution of a nonrotating black lens with the horizon topology $L(n,1)$ in bubble of nothing as a one-soliton solution in five dimensions by using the ISM. In Sec. III we impose the boundary conditions such that the spacetime has no curvature, conical singularities, or orbifold singularities on the axis and horizon. In Sec. IV we further impose that there are no CTCs in the domain of communication. In Sec. V we discuss whether a nonrotating black lens in a bubble of nothing indeed exists. In Sec. VI we confirm the limit of our solution to the Chen-Teo static solution. Finally, Sec. VII is devoted to a summary and discussion of our results.

II. STATIC BLACK RING IN A BUBBLE OF NOTHING AS A SEED SOLUTION

In general, the metric for a stationary and biaxisymmetric spacetime can be written in canonical coordinates as

$$ds^2 = g_{ab}dx^a dx^b + f(d\rho^2 + dz^2), \quad (a,b = t,\phi,\psi), \quad (1)$$

where g_{ab} and f depend on only (ρ, z). The following constraint condition must be satisfied:

$$\det(g_{ab}) = -\rho^2. \quad (2)$$

According to the procedure of Chen and Teo [25], we construct the static black lens in a bubble of nothing using the ISM, where a static black ring as the seed solution is replaced with a black ring in a bubble of nothing [31] (see Fig. 1 for the rod structure).

Therefore, let us start with the exact solution to the five-dimensional vacuum Einstein equations of a black ring in a bubble of nothing, whose metric is given by

$$G_0 = \mathrm{diag}\left(-\rho^2\frac{\mu_1}{\mu_0\mu_2}, \frac{\mu_0\mu_2}{\mu_1\mu_3}, \mu_3\right), \quad (3)$$

$$f_0 = C_f\frac{\mu_3 W_{01}^2 W_{03} W_{12}^2 W_{23}}{W_{02}^2 W_{13} W_{00} W_{11} W_{22} W_{33}}, \quad (4)$$

where for $i,\ j = 0,\ 1,\ 2,\ 3$,

$$\mu_i := \sqrt{\rho^2 + (z-z_i)^2} - (z-z_i), \quad (5)$$

$$\bar{\mu}_i := -\frac{\rho^2}{\mu_i}, \quad (6)$$

$$W_{ij} := \rho^2 + \mu_i\mu_j. \quad (7)$$

First, let us remove a trivial antisoliton with the Belinsky-Sakharov (BZ) vector (0,0,1) at $z = z_3$:

$$g_0 = \mathrm{diag}\left(1, 1, -\frac{\bar{\mu}_3^2}{\rho^2}\right)G_0 = \mathrm{diag}\left(1, 1, -\frac{\rho^2}{\mu_3^2}\right)G_0. \quad (8)$$

In turn, let us add back a nontrivial antisoliton with $(0, -a, 1)$; then, we can obtain a new one-soliton solution, a solution of *a static black lens in a bubble of nothing*,

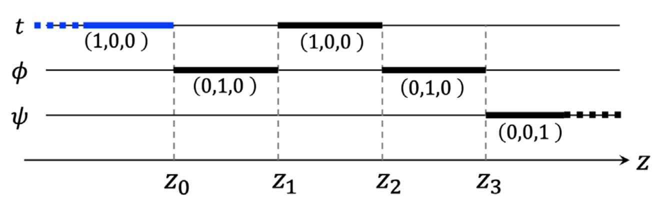

FIG. 1. Rod structure of the black ring inside a bubble of nothing.

$$g_1 = -\rho^2 \frac{\mu_1}{\mu_0\mu_2} dt^2 + \frac{\mu_0\mu_2(\mu_1 W_{03}^2 W_{23}^2 + 4a^2 z_3^2 \mu_0\mu_2\mu_3^2 W_{13}^2)}{\mu_1\mu_3(\mu_1 W_{03}^2 W_{23}^2 - 4a^2 z_3^2 \rho^2 \mu_0\mu_2 W_{13}^2)} d\phi^2 - 2a z_3 \frac{2\mu_0\mu_2 W_{03} W_{13} W_{23} W_{33}}{\mu_3(\mu_1 W_{03}^2 W_{23}^2 - 4a^2 z_3^2 \rho^2 \mu_0\mu_2 W_{13}^2)} d\phi d\psi$$
$$+ \frac{\mu_1\mu_3^2 W_{03}^2 W_{23}^2 + 4a^2 z_3^2 \rho^4 \mu_0\mu_2 W_{13}^2}{\mu_3(\mu_1 W_{03}^2 W_{23}^2 - 4a^2 z_3^2 \rho^2 \mu_0\mu_2 W_{13}^2)} d\psi^2, \quad (9)$$

and

$$f_1 = f_0 \frac{\mu_1 W_{03}^2 W_{23}^2 - 4a^2 z_3^2 \rho^2 \mu_0\mu_2 W_{13}^2}{\mu_1 W_{03}^2 W_{23}^2}. \quad (10)$$

It is easy to confirm that in the limit of $a \to 0$, this solution coincides with the static black ring in a bubble of nothing [31]. One should note that t, ψ are dimensionless and ϕ has the dimension of length. In the following section, after an appropriate coordinate transformation, we will impose the periodicity of ϕ, ψ so that conical singularities do not exist on axes of symmetry.

III. BOUNDARY CONDITIONS ON THE RODS

In order to impose the appropriate boundary conditions so that the solution has the rod structure in Fig. 2 and no conical or orbifold singularities, let us introduce the new parameters $b \coloneqq z_3 a$ and new coordinates (ϕ', ψ'), defined by

$$\frac{\partial}{\partial\phi'} = \sqrt{\frac{z_{30}}{z_{30} - 2b^2}} \left(\frac{\partial}{\partial\phi} + \frac{b}{z_{30}} \frac{\partial}{\partial\psi} \right),$$
$$\frac{\partial}{\partial\psi'} = \sqrt{\frac{z_{30}}{z_{30} - 2b^2}} \left(\frac{\partial}{\partial\psi} + 2b \frac{\partial}{\partial\phi} \right). \quad (11)$$

Then, the angular components are written as

$$g_{\phi'\phi'} = \frac{z_{30}}{z_{30} - 2b^2} \left(g_{\phi\phi} + \frac{b^2}{z_{30}^2} g_{\psi\psi} + \frac{2b}{z_{30}} g_{\phi\psi} \right), \quad (12)$$

$$g_{\psi'\psi'} = \frac{z_{30}}{z_{30} - 2b^2} (g_{\psi\psi} + 4b^2 g_{\phi\phi} + 4b g_{\phi\psi}), \quad (13)$$

$$g_{\phi'\psi'} = \frac{z_{30}}{z_{30} - 2b^2} \left[\left(1 + \frac{2b^2}{z_{30}} \right) g_{\phi\psi} + 2b g_{\phi\phi} + \frac{b}{z_{30}} g_{\psi\psi} \right], \quad (14)$$

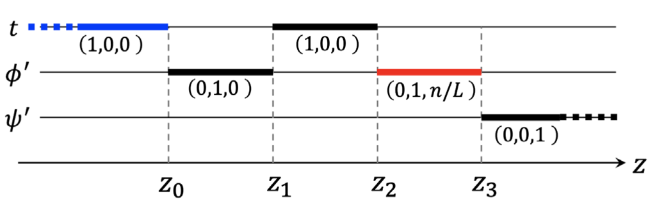

FIG. 2. Rod structure of the black lens inside a bubble of nothing.

where it should be noted that the constraint condition (2) is preserved.

Focusing on the two-dimensional space $\Sigma = \{(\rho, z) | \rho > 0, -\infty < z < \infty\}$, let us study the rod structure of the obtained solution. The rod structure [11,12] allows us to easily obtain stationary and axisymmetric solutions [more precisely, solutions with $(D-2)$ commuting Killing vectors] in a diagrammatic may. The z axis ($\rho = 0$) of the metric, which corresponds to a fixed point set of a certain Killing isometry, is decomposed into five parts: $I_- = \{(\rho, z) | \rho = 0, z < z_0\}$, $I_0 = \{(\rho, z) | \rho = 0, z_0 < z < z_1\}$, $I_1 = \{(\rho, z) | \rho = 0, z_1 < z < z_2\}$, $I_2 = \{(\rho, z) | \rho = 0, z_2 < z < z_3\}$, and $I_+ = \{(\rho, z) | \rho = 0, z_3 < z\}$. Thus, the boundary $\partial\Sigma$ of Σ is composed of $I_\pm, I_i (i = 0, \ldots, 2)$ and the asymptotic region $I_\infty = \{(\rho, z) | \sqrt{\rho^2 + z^2} \to \infty\}$, with $z/\sqrt{\rho^2 + z^2}$ finite.

Now we impose conditions on each rod so that the solution has the same rod structure as in Fig. 2 and has no conical singularities.

(1) $I_3 = \{(\rho, z) | \rho = 0, z > z_3\}$:
The Killing vector $v_3 \coloneqq (0, 0, 1) = \partial/\partial\psi'$ vanishes. The condition for the absence of conical singularities on I_3 is given by

$$\lim_{\rho \to 0} \sqrt{\frac{\rho^2 f_1}{g_{\psi'\psi'}}} = \frac{\Delta\psi'}{2\pi} \Leftrightarrow C_f \frac{z_{30} - 2b^2}{z_{30}} = \left(\frac{\Delta\psi'}{2\pi} \right)^2 \quad (15)$$

for $z \in (z_3, \infty)$. Hence, if we choose the periodicity of ψ' as $\Delta\psi' = 2\pi$, the condition can be satisfied on I_3 by setting

$$C_f = \frac{z_{30}}{z_{30} - 2b^2}. \quad (16)$$

(2) $I_0 = \{(\rho, z) | z_0 < z < z_1, \rho = 0\}$:
The Killing vector $v_{01} \coloneqq (0, 1, 0) = \partial/\partial\phi'$ vanishes. The condition for the absence of conical singularities on I_0 is given by

$$\lim_{\rho \to 0} \sqrt{\frac{\rho^2 f_1}{g_{\phi'\phi'}}} = \frac{\Delta\phi'}{2\pi} \Leftrightarrow \frac{2C_f z_{10}^2 (z_{30} - 2b^2)}{z_{20}^2} = \left(\frac{\Delta\phi'}{2\pi} \right)^2. \quad (17)$$

(3) $I_2 = \{(\rho, z) | z_2 < z < z_3, \rho = 0\}$:
The Killing vector $v_{23} \coloneqq \left(0, 1, \frac{b z_{21}}{z_{30} z_{32} - 2b^2 z_{31}}\right) = \partial/\partial\tilde{\phi}'$ vanishes. The conical-free condition is given by

$$\lim_{\rho\to 0}\sqrt{\frac{\rho^2 f_1}{g_{1ab}v_{23}^a v_{23}^b}} = \frac{\Delta\tilde{\phi}'}{2\pi} \Leftrightarrow \frac{2C_f(z_{30}z_{32}-2b^2z_{31})^2}{z_{32}z_{31}(z_{30}-2b^2)} = \left(\frac{\Delta\tilde{\phi}'}{2\pi}\right)^2. \quad (18)$$

Since ϕ' and $\tilde{\phi}'$ have the dimension of length, it is useful to introduce angular coordinates φ and $\tilde{\varphi}$ with 2π periodicity by $\varphi \coloneqq L\phi'$ and $\tilde{\varphi} \coloneqq L\tilde{\phi}'$. Then, together with Eq. (16), the conditions (17) and (18) can be written as

$$(17) \Leftrightarrow \frac{2z_{10}^2 z_{30}}{z_{20}^2} = L^2\left(\frac{\Delta\varphi}{2\pi}\right)^2, \quad (19)$$

$$(18) \Leftrightarrow \frac{2z_{30}(z_{30}z_{32}-2b^2z_{31})^2}{z_{32}z_{31}(z_{30}-2b^2)^2} = L^2\left(\frac{\Delta\tilde{\varphi}}{2\pi}\right)^2. \quad (20)$$

Moreover, we impose a boundary condition on the parameters (z_i, b) so that the spatial topology of the horizon is the lens space $L(n;1) = S^3/\mathbb{Z}_n$ $(n \in \mathbb{N})$. From the mathematical discussion in Ref. [3], this condition is represented by

$$\det(\bar{v}_{01}, \bar{v}_{23}) = n \Leftrightarrow \frac{Lbz_{21}}{z_{30}z_{32}-2b^2z_{31}} = n, \quad (21)$$

where $(\bar{v}_{01}, \bar{v}_{23}) = L(v_{01}, v_{23})$.

(4) $I_- = \{(\rho,z)|z<z_0, \rho=0\}$:
The Killing vector $v_- \coloneqq (1,0,0) = \partial/\partial t$ vanishes. This semi-infinite rod corresponds to an accelerating horizon.

(5) $I_1 = \{(\rho,z)|z_1<z<z_2, \rho=0\}$:
The Killing vector $v_1 \coloneqq (1,0,0) = \partial/\partial t$ vanishes. This finite rod corresponds to an event horizon.

IV. CTCs

We require the absence of CTCs on $\Sigma\cup\partial\Sigma$. The necessary and sufficient conditions to ensure that CTCs do not exist on $\Sigma\cup\partial\Sigma$ are that $g_{\phi\phi}$ and $g_{\psi\psi}$ (or $g_{\phi'\phi'}$ and $g_{\psi'\psi'}$) become non-negative in the region. The condition for the absence of CTCs is given by

$$\mu_1 W_{03}^2 W_{23}^2 - 4b^2\rho^2\mu_0\mu_2 W_{13}^2 > 0, \quad (22)$$

which imposes an upper bound for b^2 at each point,

$$b^2 < U(\rho,z) \coloneqq \frac{\mu_1 W_{03}^2 W_{23}^2}{4\rho^2\mu_0\mu_2 W_{13}^2}. \quad (23)$$

Therefore, if the minimum $U_{\min}$ of $U(\rho,z)$ exists on $\Sigma\cup\partial\Sigma$ and $b^2 < U_{\min}$ holds, CTCs do not exist in the region. To prove this, we show that the function $U(\rho,z)$ has a minimum at $(\rho,z) = (0,z_3)$ on the rod $I_\pm, I_i (i=0,\ldots,2)$.

It is not difficult to show that the function $U(\rho,z)$ has a minimum not on Σ but on $\partial\Sigma$. To see this, one should note that the norm of the gradient can be written as

$$(\partial_\rho U)^2 + (\partial_z U)^2 = \frac{\mu_1^2 W_{03}^4 W_{23}^4}{4\rho^6\mu_0^2\mu_2^2 W_{00}W_{11}W_{22}W_{13}^4}((\mu_0-\mu_1+\mu_2)^2\rho^4 + 2\mu_0\mu_2(2\mu_0\mu_2-(\mu_0+\mu_2)\mu_1+\mu_1^2)\rho^2 + \mu_0^2\mu_1^2\mu_2^2), \quad (24)$$

where the first line is always positive for $\rho > 0$, and the second line is also positive since

$$(\mu_0-\mu_1+\mu_2)^2\rho^4 + 2\mu_0\mu_2(\mu_1^2-(\mu_0+\mu_2)\mu_1+2\mu_0\mu_2)\rho^2 + \mu_0^2\mu_1^2\mu_2^2 = \left((\mu_0-\mu_1+\mu_2)\rho^2 + \frac{\mu_0\mu_2(\mu_1^2-(\mu_0+\mu_2)\mu_1+2\mu_0\mu_2)}{\mu_0-\mu_1+\mu_2}\right)^2 + \frac{4\mu_0^3\mu_2^3(\mu_0-\mu_1)(\mu_1-\mu_2)}{(\mu_0-\mu_1+\mu_2)^2}, \quad (25)$$

where the positivity of the last term can be shown from

$$(\mu_0-\mu_1)(\mu_1-\mu_2) = \frac{z_{10}z_{21}(\mu_1+\mu_0)(\mu_2+\mu_1)}{(\sqrt{\rho^2+(z-z_1)^2}+\sqrt{\rho^2+(z-z_0)^2})(\sqrt{\rho^2+(z-z_2)^2}+\sqrt{\rho^2+(z-z_1)^2})} > 0. \quad (26)$$

Therefore, the gradient of a smooth function $U(\rho,z)$ cannot be zero on Σ, which means that $U(\rho,z)$ must have a minimum not on Σ but on $\partial\Sigma$. Hence, in what follows we consider a minimum of $U(\rho,z)$ on $\partial\Sigma$, which corresponds to $I_i (i=\pm,0,\ldots,2), I_\infty$.

First, let us consider the minimum of $U(0,z)$ on the rod $\rho = 0$, i.e., on $I_i (i = \pm, 0, \ldots, 2)$. On I_+, we have

$$U(0,z) = \frac{(z-z_0)(z-z_2)}{2(z-z_1)},$$
$$U_{,z}(0,z) = \frac{(z-z_1)^2 + z_{10}z_{21}}{2(z-z_1)^2} > 0, \tag{27}$$

and hence the monotonically increasing function $U(0,z)$ on I_+ has a minimum at $z = z_3$, which is written as

$$U(0,z_3) = \frac{z_{30}z_{32}}{2z_{31}}. \tag{28}$$

On the other hand, since on I_2

$$U(0,z) = \frac{(z-z_1)z_{30}^2 z_{32}^2}{2(z-z_0)(z-z_2)z_{31}^2},$$
$$U_{,z}(0,z) = -\frac{[(z-z_1)^2 + z_{10}z_{21}]z_{30}^2 z_{32}^2}{2(z-z_0)^2(z-z_2)^2 z_{31}^2} < 0, \tag{29}$$

the monotonically decreasing function $U(0,z)$ on I_2 has a minimum at $z = z_3$, and hence

$$U(0,z) \geq U(0,z_3). \tag{30}$$

Moreover, observing that on I_0,

$$U(0,z) = \frac{(z-z_2)z_{30}^2}{2(z-z_0)(z-z_1)}, \tag{31}$$

$$U_{,z}(0,z) = -\frac{[(z-z_2)^2 - z_{20}z_{21}]z_{30}^2}{2(z-z_0)^2(z-z_1)^2} \times \begin{cases} <0 & (z_0 < z < z_2 - \sqrt{z_{20}z_{21}}), \\ >0 & (z_2 - \sqrt{z_{20}z_{21}} < z < z_1), \end{cases} \tag{32}$$

we find that the function $U(0,z)$ on I_0 has a local minimum at $z = z_* \coloneqq z_2 - \sqrt{z_{20}z_{21}}$, and hence

$$U(0,z) \geq U(0,z_*) = \frac{z_{30}^2}{2(z_{20} + z_{21} - 2\sqrt{z_{20}z_{21}})}, \tag{33}$$

where we note that the ratio of these minima on I_0, I_2, I_+ is computed as

$$\frac{U(0,z_*)}{U(0,z_3)} = \left(1 + \frac{z_{21}}{z_{32}}\right)\left(1 + \frac{z_{32}}{z_{20}}\right)\left(1 - \sqrt{\frac{z_{21}}{z_{20}}}\right)^{-2} > 1. \tag{34}$$

Furthermore, near I_- and I_1, the function $U(\rho,z)$ behaves, respectively, as

$$U(\rho,z) \simeq \frac{2(z_0-z)(z_2-z)(z_3-z)^2}{(z_1-z)\rho^2},$$
$$U(\rho,z) \simeq \frac{2(z-z_1)(z_2-z)(z_3-z)^2 z_{30}^2}{(z-z_0)z_{31}^2\rho^2}, \tag{35}$$

which implies $U(0,z) = \infty$ on I_- and I_1. To summarize, we have shown that the minimum of $U(0,z)$ on $I_i (i = \pm, 0, \ldots, 2)$ is given by Eq. (28).

Next, let us consider the function $U(\rho,z)$ in the asymptotic region, namely, on I_∞. We can show that in the asymptotic region $U(\rho,z)$ behaves as

$$U(\rho,z) \simeq \frac{1}{1 + \frac{z}{\sqrt{\rho^2+z^2}}}\sqrt{\rho^2 + z^2}. \tag{36}$$

Hence, the function $U(\rho,z)$ diverges on I_∞, so it cannot have a minimum on I_∞.

Thus, we conclude that Eq. (28) is a minimum on both $\Sigma \cup \partial\Sigma$ and $\partial\Sigma$, and hence the necessary and sufficient condition for the absence of CTCs is given by

$$b^2 < \frac{z_{30}z_{32}}{2z_{31}}. \tag{37}$$

V. ON THE EXISTENCE OF SOLUTIONS

From the discussion in Sec. III, we have shown that the absence of conical singularities and the black lens condition require

$$\frac{2z_{10}^2 z_{30}}{z_{20}^2} = L^2, \tag{38}$$

$$\frac{2z_{30}(z_{32}z_{30} - 2b^2 z_{31})^2}{z_{32}z_{31}(z_{30} - 2b^2)^2} = L^2, \tag{39}$$

$$\frac{Lbz_{21}}{z_{30}z_{32} - 2b^2 z_{31}} = n. \tag{40}$$

Now, to confirm whether there is really a parameter range such that all of these conditions can be satisfied, we reparametrize the rod interval $z_{i,i-1} \coloneqq z_i - z_{i-1}$ ($i = 1$, 2, 3) and the redefined BZ parameter b as follows:

$$z_{10} = \ell, \qquad z_{21} \coloneqq x\ell, \qquad z_{32} = y\ell, \qquad L = \sqrt{\ell}\hat{L}, \qquad b = \sqrt{\ell}\hat{b}, \tag{41}$$

where ℓ fixes the size of the bubble on I_0, and x and y are the size of the horizon and the distance between the horizon and the center (so-called nut). All of the dimensionless parameters except for $n, x, y, \ell, \hat{L}$, and $\hat{b}$ are assumed to be positive. The condition for avoiding CTCs is now given by

$$b^2 - \frac{y(1+x+y)}{2(x+y)} < 0. \tag{42}$$

From Eq. (40), we have

$$\hat{L} = \frac{n(y^2 - 2\tilde{b}^2(x+y) + y(1+x))}{\hat{b}x}. \tag{43}$$

Eliminating $\hat{L}$ from Eqs. (38) and (39) in terms of Eq. (43), we obtain

$$0 = 2\hat{b}^2(x+y+1)\{2n^2(x+1)^2y^2 + 2n^2(x+1)^2xy + x^2\} - 4\hat{b}^4n^2(x+1)^2(x+y)^2 - n^2(x+1)^2y^2(x+y+1)^2, \tag{44}$$

and

$$0 = \{y(x+y+1) - 2\hat{b}^2(x+y)\}^2 \times [4\hat{b}^4n^2y(x+y) - 2\hat{b}^2(x+y+1) \{2n^2y(x+y) + x^2\} + n^2y(x+y)(x+y+1)^2]. \tag{45}$$

First, we consider the conical singularity-free condition on I_2 in Eq. (45), which admits three branches for $\hat{b}^2$:

$$\hat{b}^2 = \hat{b}^2_\pm := \frac{(x+y+1)(x^2+2n^2(x+y)y \pm x\sqrt{x^2+4n^2xy+4n^2y^2})}{4n^2(x+y)y}, \tag{46}$$

$$\hat{b}^2 = \hat{b}^2_0 := \frac{y(x+y+1)}{2(x+y)}. \tag{47}$$

From Eq. (46), we can show that

$$\hat{b}^2_\pm - \frac{y(1+x+y)}{2(x+y)} = \frac{x(x+y+1)(x+2n^2y \pm \sqrt{x^2+4n^2xy+4n^2y^2})}{4n^2(x+y)y} > 0, \tag{48}$$

where we note that

$$(x+2n^2y)^2 - (x^2+4n^2xy+4n^2y^2) = 4n^2(n^2-1)y^2 > 0. \tag{49}$$

Therefore, this shows that the nonexistence condition of CTCs [Eq. (42)] cannot be satisfied for any $x > 0$ and $y > 0$. On the other hand, substituting Eq. (47) into Eq. (43), we can show that

$$\hat{L} = 0, \tag{50}$$

which cannot satisfy $\hat{L} > 0$. Hence, a solution without a conical singularity on I_2 cannot avoid CTCs.

Next, we consider the conical singularity-free condition on I_0 in Eq. (44), from which we can obtain two branches for $\hat{b}^2$,

$$\hat{b}^2 = \tilde{b}^2_\pm := \frac{(x+y+1)\{x^2+2n^2(x+1)^2(x+y)y \pm x\sqrt{x^2+4n^2(x+1)^2(x+y)y}\}}{4n^2(x+1)^2(x+y)^2}, \tag{51}$$

which lead to

$$\tilde{b}^2_\pm - \frac{y(1+x+y)}{2(x+y)} = \frac{x(x+y+1)(x \pm \sqrt{x^2+4n^2(x+1)^2(x+y)y})}{4n^2(x+1)^2(x+y)^2}. \tag{52}$$

From these, we find that only the branch $\tilde{b}^2_-$ can satisfy the nonexistence condition of CTCs [Eq. (42)].

In summary, if one imposes the absence of conical singularities on the whole axes of symmetry I_0, I_2, I_+, the presence of CTCs cannot be avoided around the center $(\rho, z) = (0, z_3)$. However, if one imposes it only on I_0 and I_+, one can obtain solutions without CTCs, in which the horizon admits the lens space topology $L(n;1)$ for $n \geq 1$.

VI. CONSISTENCY WITH A CHEN-TEO STATIC BLACK LENS

Here we confirm that our solution coincides with the asymptotically flat, static black lens solution of Chen and Teo [25] in a certain scaling limit, for which the following variables are used:

$$z_0 = -\lambda^2, \qquad z_1 = -\lambda c\kappa^2, \qquad z_2 = \lambda c\kappa^2,$$
$$z_3 = \lambda\kappa^2, \qquad L = \sqrt{2}\lambda\bar{L}, \qquad a = \frac{\bar{a}}{\sqrt{2}\kappa^2}. \tag{53}$$

With the rescaled coordinates

$$\rho \to \lambda\bar{\rho}, \qquad z \to \lambda\bar{z} \tag{54}$$

and the rescaled parameters

$$\bar{z}_0 := z_1/\lambda = -c\kappa^2, \quad \bar{z}_1 := z_2/\lambda = c\kappa^2, \quad \bar{z}_1 := z_3/\lambda = \kappa^2, \tag{55}$$

the limit $\lambda \to \infty$ pushes z_0 to $-\infty$, and one can see that the rod structure in terms of the coordinates $(\bar{\rho}, \bar{z})$ coincides with that of the static black lens in Ref. [25]. In the limit $\lambda \to \infty$, Eq. (38) requires

$$\bar{L} = 1, \tag{56}$$

and then Eqs. (39) and (40) lead, respectively, to the nonexistence condition of conical singularities on $\bar{z} \in (\bar{z}_2, \bar{z}_3)$ and the condition of horizon topology $L(n;1)$ in Ref. [25],

$$\frac{(1-c-\bar{a}^2(1+c))^2}{(1-\bar{a}^2)^2(1-c^2)} = 1, \qquad \frac{2\bar{a}c}{1-c-\bar{a}^2(1+c)} = n, \tag{57}$$

where $\bar{a}$ corresponds to a in Ref. [25]. Moreover, the limit of the nonexistence condition for CTCs [Eq. (37)] can be written as

$$\bar{a}^2 < \frac{c-1}{c+1}, \tag{58}$$

which corresponds to the parameter region "Region I" in Ref. [25]. Here, it should be noted that the solutions in "Region II" in Ref. [25], which admits naked singularities and CTCs, are excluded from our study by the nonexistence condition (37) of CTCs.

VII. SUMMARY AND DISCUSSIONS

In this paper, using the ISM for static and biaxisymmetric Einstein equations, we have constructed a nonrotating black lens inside a bubble of nothing whose horizon is topologically lens space $L(n,1) = S^3/\mathbb{Z}_n$. Our work is entirely the parallel to the work of Chen and Teo [25], where the static black ring as a seed solution was replaced with a static black ring in a bubble of nothing [31]. Using this solution, we have investigated whether a static black lens can be in equilibrium by the force balance between the expansion and gravitational attraction. If we require the absence of CTCs in the domain of outer communication, a nonrotating black lens must have conical singularities between the horizon and the center. It has been shown, however, that for a black lens, the existence of an expanding bubble does not exclude conical singularities, and hence the two forces—the force of the bubble expansion and gravitational attraction—cannot be in static equilibrium, unlike for the black ring.

The authors of Refs. [25,29] studied whether a rotating black lens can be kept in equilibrium by the balance between the gravitational force (attraction) and the centrifugal force (repulsive force), and concluded that it cannot be in equilibrium. The generalization of this rotating black lens solution to one in an expanding bubble may be an interesting issue, since whether such a black lens without conical singularities exists depends on the balance between the gravitational force, the centrifugal force, and the expansion of the bubble. Such a rotating black lens solution is expected to be more than one soliton solution. This deserves our future work.

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[1] A. Strominger and C. Vafa, Microscopic origin of the Bekenstein-Hawking entropy, Phys. Lett. B **379**, 99 (1996).
[2] P. C. Argyres, S. Dimopoulos, and J. March-Russell, Black holes and submillimeter dimensions, Phys. Lett. B **441**, 96 (1998).
[3] S. Hollands and S. Yazadjiev, Uniqueness theorem for 5-dimensional black holes with two axial Killing fields, Commun. Math. Phys. **283**, 749 (2008).
[4] S. Hollands, J. Holland, and A. Ishibashi, Further restrictions on the topology of stationary black holes in five dimensions, Ann. Henri Poincaré **12**, 279 (2011).
[5] M. l. Cai and G. J. Galloway, On the Topology and area of higher dimensional black holes, Classical Quantum Gravity **18**, 2707 (2001).
[6] G. J. Galloway and R. Schoen, A generalization of Hawking's black hole topology theorem to higher dimensions, Commun. Math. Phys. **266**, 571 (2006).
[7] F. R. Tangherlini, Schwarzschild field in n dimensions and the dimensionality of space problem, Nuovo Cimento **27**, 636 (1963).
[8] R. C. Myers and M. J. Perry, Black holes in higher dimensional space-times, Ann. Phys. (N.Y.) **172**, 304 (1986).
[9] R. Emparan and H. S. Reall, A Rotating Black Ring Solution in Five-Dimensions, Phys. Rev. Lett. **88**, 101101 (2002).
[10] A. A. Pomeransky and R. A. Sen'kov, Black ring with two angular momenta, arXiv:hep-th/0612005.
[11] T. Harmark, Stationary and axisymmetric solutions of higher-dimensional general relativity, Phys. Rev. D **70**, 124002 (2004).
[12] R. Emparan and H. S. Reall, Generalized Weyl solutions, Phys. Rev. D **65**, 084025 (2002).
[13] A. A. Pomeransky, Complete integrability of higher-dimensional Einstein equations with additional symmetry, and rotating black holes, Phys. Rev. D **73**, 044004 (2006).
[14] P. Figueras, A black ring with a rotating 2-sphere, J. High Energy Phys. 07 (2005) 039.
[15] T. Mishima and H. Iguchi, New axisymmetric stationary solutions of five-dimensional vacuum Einstein equations with asymptotic flatness, Phys. Rev. D **73**, 044030 (2006).
[16] S. Tomizawa, Y. Morisawa, and Y. Yasui, Vacuum solutions of five dimensional Einstein equations generated by inverse scattering method, Phys. Rev. D **73**, 064009 (2006).
[17] H. Iguchi and T. Mishima, Solitonic generation of five-dimensional black ring solution, Phys. Rev. D **73**, 121501 (2006).
[18] S. Tomizawa and M. Nozawa, Vacuum solutions of five-dimensional Einstein equations generated by inverse scattering method. II. Production of black ring solution, Phys. Rev. D **73**, 124034 (2006).
[19] H. Elvang and P. Figueras, Black saturn, J. High Energy Phys. 05 (2007) 050.
[20] H. Iguchi and T. Mishima, Black di-ring and infinite nonuniqueness, Phys. Rev. D **75**, 064018 (2007); **78**, 069903(E) (2008).
[21] J. Evslin and C. Krishnan, The black di-ring: An inverse scattering construction, Classical Quantum Gravity **26**, 125018 (2009).
[22] H. Elvang and M. J. Rodriguez, Bicycling black rings, J. High Energy Phys. 04 (2008) 045.
[23] K. Izumi, Orthogonal black di-ring solution, Prog. Theor. Phys. **119**, 757 (2008).
[24] J. Evslin, Geometric engineering 5d black holes with rod diagrams, J. High Energy Phys. 09 (2008) 004.
[25] Y. Chen and E. Teo, A rotating black lens solution in five dimensions, Phys. Rev. D **78**, 064062 (2008).
[26] H. K. Kunduri and J. Lucietti, Supersymmetric Black Holes with Lens-Space Topology, Phys. Rev. Lett. **113**, 211101 (2014).
[27] S. Tomizawa and M. Nozawa, Supersymmetric black lenses in five dimensions, Phys. Rev. D **94**, 044037 (2016).
[28] V. Breunhölder and J. Lucietti, Moduli space of supersymmetric solitons and black holes in five dimensions, Commun. Math. Phys. **365**, 471 (2019).
[29] S. Tomizawa and T. Mishima, Stationary and biaxisymmetric four-soliton solution in five dimensions, Phys. Rev. D **99**, 104053 (2019).
[30] J. Lucietti and F. Tomlinson, On the nonexistence of a vacuum black lens, J. High Energy Phys. 02 (2021) 005.
[31] M. Astorino, R. Emparan, and A. Viganò, Bubbles of nothing in binary black holes and black rings, and viceversa, J. High Energy Phys. 07 (2022) 007.
[32] H. Elvang and G. T. Horowitz, When black holes meet Kaluza-Klein bubbles, Phys. Rev. D **67**, 044015 (2003).
[33] S. Tomizawa, H. Iguchi, and T. Mishima, Rotating black holes on Kaluza-Klein bubbles, Phys. Rev. D **78**, 084001 (2008).
[34] H. Iguchi, T. Mishima, and S. Tomizawa, Boosted black holes on Kaluza-Klein bubbles, Phys. Rev. D **76**, 124019 (2007); **78**, 109903(E) (2008).