Metamaterial analog of a black hole shadow: An exact ray-tracing simulation based on the spacetime index of refraction

M. Nouri-Zonoz,[*] A. Parvizi,[†] and H. Forghani-Ramandy[‡]
Department of Physics, University of Tehran, North Karegar Avenue, Tehran 14395-547, Iran

(Received 2 September 2022; accepted 1 November 2022; published 7 December 2022)

We show that the equation of null geodesics in spherically symmetric spacetimes in isotropic coordinates is identical to the equation of light ray trajectories in isotropic media in flat spacetime. Based on this analogy we introduce an exact simulation of the light ray trajectories both in these spacetimes and in their metamaterial analogs in terms of the spacetime index of refraction. As unstable light trajectories, the photon spheres form in these metamaterial analogs at *exactly* the same radial distances as expected from the corresponding black hole geometries. Using the same ray-tracing simulation we find the analog of a simple black hole shadow formed by the metamaterial analog of a Schwarzschild black hole, eclipsing a line of light sources near its analog horizon. Designing such a metamaterial could provide a laboratory setting to explore this recently observed phenomenon.

DOI: 10.1103/PhysRevD.106.124013

I. INTRODUCTION

Metamaterial analogs of different spacetimes, based on the so-called transformation optics [1,2], have attracted a lot of attention in recent years (see Ref. [3] and references therein). Historically, one could trace this analogy back to the analogy between a spacetime and a material medium with respect to light propagation through which one could assign an index of refraction as well as other optical characteristics to the corresponding spacetime [4–7]. On the other hand, the same analogy appears in the study of Maxwell's equations in a curved spacetime leading to constitutive equations with the geometric analog of the magneto-electric coupling (effect). Through this analogy, one could establish a correspondence between geometric entities of a curved spacetime and the electromagnetic features of a medium such as its electric permittivity and magnetic permeability [6,7].

In this opto-geometric relation one could start from a given spacetime and find its optical characteristics and, based on them, design its optical analog using metamaterials in which the light trajectories mimic the null geodesics of the corresponding spacetime. Unlike the natural materials which have restricted optical features, the metamaterial designs could demonstrate nontrivial optical features (such as a negative index of refraction), and that is why one uses the term metamaterial analogs instead of optical analogs. Obviously, the more exotic a spacetime, the more interesting its optical features; consequently, their metamaterial analogs will enable one to realize these exotic optical features. In other words, these metamaterial analogs could help us examine interesting and perhaps observationally inaccessible optical characteristics of the corresponding spacetime, which could include, for example, interesting optical features associated with black hole spacetimes. To this end, one needs to simulate light rays, as exactly as possible, in the metamaterial analogs of the corresponding black hole spacetimes. In previous studies full-wave simulations were employed to study light propagation in the metamaterial analogs of different spacetimes [8–10]; it was then compared with the light ray trajectories in the geometric optics limit, formulated in the Hamiltonian language [9] or through a ray-tracing mechanism [10]. Here we introduce a new direct and, at the same time, exact simulation of light ray trajectories in the metamaterial analogs of static spherically symmetric black holes, based only on their indices of refraction, which are adapted from the corresponding spacetimes. The simulated trajectories are *exact* duplicates of those in the corresponding spacetime. In particular, we find the analog of black hole photon spheres in their metamaterial analogs at the same exact radial distance as expected from the spacetime geometry. Indeed, by increasing the precision level, rays could be orbiting the analog photon sphere as many times as the simulation cost allows. This means that in the metamaterials designed with isotropic refractive indices borrowed from these black hole geometries, one could obtain photon spheres at specified radii. This process could potentially find diverse applications both in electro-optical devices and in the investigation of the properties of so-called optical black holes [11,12]. One such interesting application is

[*]Corresponding author.
nouri@ut.ac.ir
[†]a.parvizi@ut.ac.ir
[‡]hasan.forghani@ut.ac.ir

provided by showing how a simple metamaterial analog of a black hole shadow could be simulated.

II. SPACETIME INDEX OF REFRACTION

Applying Fermat's principle to light rays in stationary spacetime, in the context of (1 + 3) or a threading formulation of spacetime decomposition [6,13], we obtain the following relation [14,15]:

$$\delta \int \left(\frac{1}{\sqrt{g_{00}}} + \mathbf{g}.\hat{\mathbf{k}} \right) dl_c = 0 \tag{1}$$

in which $\mathbf{g}$ is the so-called gravitomagnetic vector potential with components $g_\alpha = -\frac{g_{0\alpha}}{g_{00}}$ ($\alpha = 1, 2, 3$), $\hat{\mathbf{k}}$ is the unit vector along the ray, and dl_c is the spatial line element in the curved spacetime [6,13]. Obviously, from the above equation one could assign the following index of refraction to the 3-space on which dl_c is the spatial line element between any two events in stationary spacetimes,

$$n_s = n_0 + \mathbf{g}.\hat{\mathbf{k}}, \tag{2}$$

in which $n_0 = \frac{1}{\sqrt{g_{00}}}$ is the index of refraction assigned to a static spacetime ($g_\alpha = 0$). Note that for a curved spacetime, the spatial line element is not necessarily flat. Now if we restrict our attention to the static spacetimes and look for their (meta)material analogs in a *flat* spacetime background, we could rewrite Eq. (1) in the following general form:

$$\delta \int \frac{1}{\sqrt{g_{00}}} \left(\frac{dl_c}{dl_f} \right) dl_f = 0 \tag{3}$$

where dl_f is the spatial line element in flat spacetime. The above relation shows that the metamaterial analog of a static spacetime could be assigned with the following index of refraction:

$$n_f = \frac{1}{\sqrt{g_{00}}} \left(\frac{dl_c}{dl_f} \right), \tag{4}$$

in which case the light ray trajectories in the designed metamaterial mimic the null geodesics in the corresponding spacetime. The above argument shows that one could, in principle, assign an appropriate index of refraction to the static spacetime if there is a coordinate system in which the spatial part of the metric is conformally flat.

Specifically, in the case of static spherically symmetric spacetimes with the general form of ($c = 1$),

$$ds^2 = d\tau^2 - dl_c^2 = f(r)dt^2 - \left(\frac{1}{f(r)} dr^2 + r^2 d\Omega^2 \right), \tag{5}$$

the line element could be transformed to the following isotropic form by introducing the radial coordinate $\rho = \text{const}\ \exp\left(\int \frac{dr}{\sqrt{r^2 f(r)}} \right)$ [16],

$$ds^2 = f(r(\rho))dt^2 - F(\rho) dl_f^2. \tag{6}$$

Using the above isotropic form of the spacetime metric and Eq. (3), a static spherically symmetric spacetime, compared to the flat spacetime, is endowed with the following index of refraction [17],

$$n_{\rm sph} = \sqrt{\frac{F(\rho)}{f(\rho)}}. \tag{7}$$

For some spherically symmetric spacetimes, $\rho(r)$ can be obtained analytically. These include the Schwarzschild and Reissner-Nordstrom (RN) black hole geometries for which

$$F_{\rm Sch}(\rho) = \left(1 + \frac{M}{2\rho} \right)^4, \tag{8}$$

$$F_{\rm RN}(\rho) = \left[\left(1 + \frac{M}{2\rho} \right)^2 - \frac{Q^2}{4\rho^2} \right]^2, \tag{9}$$

and we are led to the following indices of refraction,

$$n_{\rm Sch} = \frac{(1 + \frac{M}{2\rho})^3}{(1 - \frac{M}{2\rho})}, \tag{10}$$

$$n_{\rm RN} = \frac{[(M + 2\rho)^2 - Q^2]^2}{4\rho^2 (Q^2 - M^2 + 4\rho^2)}. \tag{11}$$

Since the isotropic coordinates only cover the region outside the black hole horizon (e.g., for Schwarzschild $2M < r < \infty$ or $M/2 < \rho < \infty$) [18], the above relations are also valid only for the same region. Consequently, our simulations are also limited to the same analog region in the corresponding metamaterial.

III. NULL GEODESICS IN SPHERICALLY SYMMETRIC SPACETIMES IN TERMS OF THE SPACETIME INDEX OF REFRACTION

Starting from the general form of the metric of spherically symmetric spacetimes in isotropic coordinates, namely (6), we introduce the Lagrangian

$$L = f(\rho)\dot{t}^2 - F(\rho)(\dot{\rho}^2 + \rho^2 \dot{\theta}^2 + \rho^2 \sin\theta^2 \dot{\phi}^2) \tag{12}$$

where $\cdot \equiv d/d\lambda$ and λ is an affine parameter along the null geodesics. Due to spherical symmetry, without loss of generality, we take geodesics on the equatorial plane $\theta = \pi/2$ for which L reduces to

$$L = f(\rho)\dot{t}^2 - F(\rho)(\dot{\rho}^2 + \rho^2\dot{\phi}^2). \tag{13}$$

Since the Lagrangian is independent of t and ϕ, we have the following two first integrals from the Euler-Lagrange equations,

$$f(\rho)\dot{t} = E, \tag{14}$$

$$F(\rho)\rho^2\dot{\phi} = D, \tag{15}$$

representing the energy and angular momentum, respectively. Now, since the spatial part of the spherically symmetric spacetimes (6) is conformally flat, the angles are given by the flat space formula; thus, for a small part of the ray trajectory making angle Θ with the radial direction, we have (Fig. 1)

$$\sin\Theta = \frac{\rho d\phi}{\sqrt{\rho^2 d\phi^2 + d\rho^2}} = \frac{\rho\dot{\phi}}{\sqrt{\rho^2\dot{\phi}^2 + \dot{\rho}^2}} = \frac{\rho}{\sqrt{\rho^2 + (\frac{d\rho}{d\phi})^2}}. \tag{16}$$

On the other hand, for null rays $L = 0$, so from (13) we have

$$f(\rho)\dot{t}^2 = F(\rho)(\dot{\rho}^2 + \rho^2\dot{\phi}^2). \tag{17}$$

Now, using the above equation along with Eqs. (14) and (15) to substitute for $\dot{\rho}$ and $\dot{\phi}$ in the second equality in (16), we end up with

$$\sin\Theta = \frac{1}{\rho}\frac{D}{E}\sqrt{\frac{f(\rho)}{F(\rho)}} = \frac{b}{\rho n(\rho)} \tag{18}$$

in which we used (7), and by definition $b = \frac{D}{E}$ is the impact parameter. Substituting $\sin\Theta$ back into the last equality in (16), we find the null geodesic equation in the following form:

$$\frac{d\rho}{d\phi} = \rho\sqrt{\frac{\rho^2 n^2}{b^2} - 1}. \tag{19}$$

This equation gives the null geodesic equation in terms of the spacetime index of refraction. In what follows, we will be primarily concerned with the analog photon spheres, so we need to find the analog of the critical angle the rays should make with the radial direction (i.e., angle of the cone of avoidance) to form the photon sphere. By the above considerations, and from Eq. (18), the critical angle at any given radial coordinate is found to be

$$\sin\Theta_{\mathrm{cr}} = \frac{b_{\mathrm{ph}}}{\rho n(\rho)} \tag{20}$$

where b_{ph} is the impact parameter of rays forming the photon sphere at any radial coordinate ρ. In other words, the critical angle at a given radial coordinate is given in terms of the index of refraction at the same coordinate. Intuitively, this is expected, as the combination $n(\rho)\sin\Theta_{\mathrm{cr}}$ reminds one of Snell's law and the initial refraction needed at each radial coordinate for the rays to be trapped on the unstable photon sphere.

IV. LIGHT RAY TRAJECTORIES IN ISOTROPIC MEDIA

The above-mentioned analogy states that for metamaterial media designed with isotropic refractive indices given by (10) and (11), the light trajectories would be the same as the null geodesics in the corresponding spacetimes. To verify this, one should be able to simulate light rays in isotropic media. Interestingly enough, in media with isotropic indices of refraction, one could obtain the equation of light ray trajectories by considering their geometry in such media. This was already studied by Born and Wolf in their classic text [19], where they showed that due to spherical symmetry, all the rays are plane curves satisfying the relation

$$nr\sin\theta = C \tag{21}$$

where C is a constant and θ is the angle between the radius vector to a point on the light curve and the tangent to the curve at the same point (Fig. 2). To find the equation of light trajectories as plane curves, from the geometry in Fig. 2, it is noted that

$$\sin\theta = \frac{r(\phi)}{\sqrt{r^2(\phi) + (dr/d\phi)^2}}. \tag{22}$$

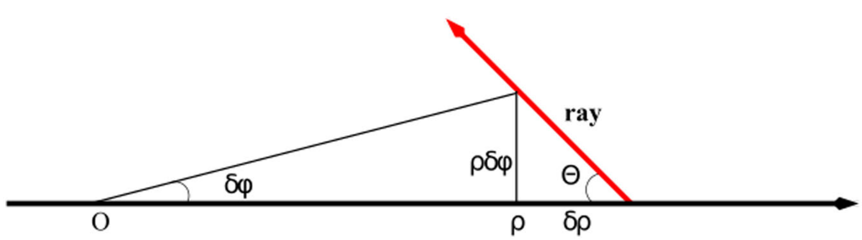

FIG. 1. Spatial geometry of a small portion of a light ray (in red) in spherically symmetric spacetimes in isotropic coordinates.

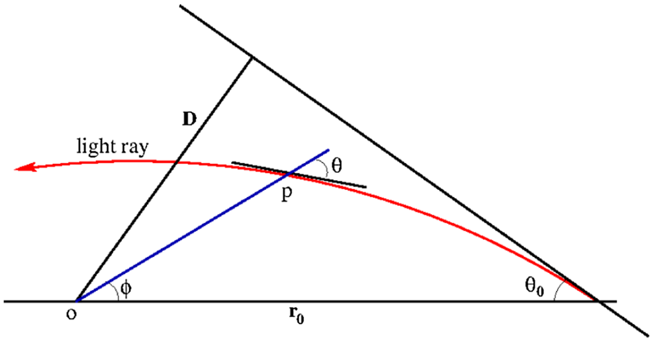

FIG. 2. Geometry of rays in an isotropic media.

Now substituting for $\sin\theta$ in the above equation from (21), we end up with the trajectory equation as [19]

$$\frac{dr}{d\phi} = r\sqrt{\frac{r^2 n^2}{C^2} - 1}. \tag{23}$$

The similarity of Eqs. (21) and (18) is a consequence of the fact that for spherically symmetric spacetimes, in isotropic coordinates, the spatial part of the metric is conformally flat.

V. SIMULATION OF LIGHT RAY TRAJECTORIES

One can employ Eqs. (21) and (22) [or (18) and (16)] to simulate the light ray trajectories in an isotropic medium (or, for that matter, in a spherically symmetric spacetime in isotropic coordinates). For a light ray fired from $r = r_0$ along the θ_0 direction, the impact parameter is $D = r_0 \sin\theta_0$ (Fig. 2) and, from (21), $C = n(r_0)D$. Therefore, the simulation goes as follows: For a given impact parameter and initial distance from the center of the metamaterial, the constant C is fixed. To find the radial coordinate of the next point on the trajectory, we substitute θ_0 and r_0 in (22) to obtain the value of $\frac{dr}{d\phi}$. From that, one can find the radial increment $\Delta r = \frac{dr}{d\phi}(\Delta\phi_0)$, in which $(\Delta\phi_0)$ is the fixed-step increment of the azimuthal angle chosen according to the required precision. Thus, we have $r_1 = r_0 + \Delta r$, and one can repeat the same steps starting from (21), now with $r = r_1$. The results of the simulation for ray trajectories in different (isotropic) metamaterial analogs of spherically symmetric black holes are discussed next.

A. Metamaterial analog of a Schwarzschild black hole

The results of the simulation for a congruence of light ray trajectories in a metamaterial analog of the Schwarzschild black hole leading to the formation of a photon sphere are shown in Fig. 3. It is noted that the horizon, $r = 2M$, and the photon sphere, $r = 3M$ are mapped, in isotropic coordinates, to $\rho_{\rm Sch} = M/2$ and $\rho_{\rm ph} = (2+\sqrt{3})M/2$, respectively. The analog photon sphere is formed by

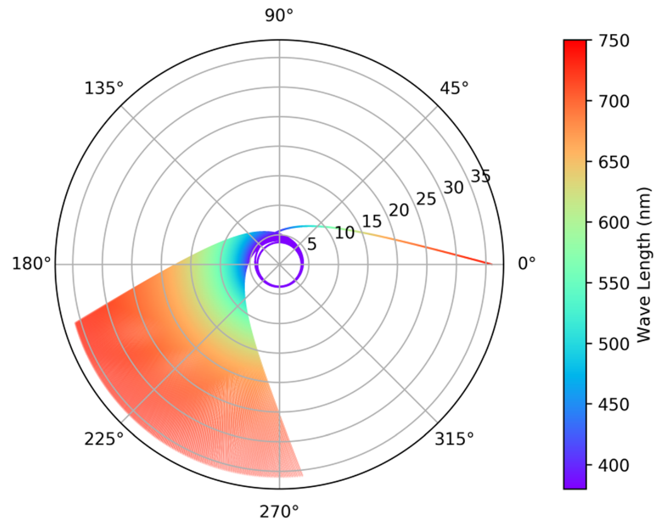

FIG. 3. Light ray trajectories in the metamaterial analog of a Schwarzschild black hole in isotropic coordinates and the formation of a photon sphere (purple circle). Distances are scaled to $M/2$.

267 rays fired from $\rho_o = 36(M/2)$ with the critical angle ranging from 14.98166575° to 14.98166577° ($\delta\Theta_{\rm cr} \sim 10^{-8}$). In our simulations we scanned a whole range of angles reaching the above value which matches exactly with its theoretical value given by (20) for a Schwarzschild black hole, namely,

$$\sin\Theta_{\rm cr} = \frac{3\sqrt{3}M}{\rho}\frac{(1-\frac{M}{2\rho})}{(1+\frac{M}{2\rho})^3} \equiv \frac{3\sqrt{3}M}{\rho n_{\rm Sch}}. \tag{24}$$

In this case the analog photon sphere is formed by rays with the closest distance of approach $\rho_{\rm cda} \approx 3.7321\rho_{\rm Sch}$. All the rays rotate at least three times around the hole before escaping it. To follow different rays we have included their blueshift as they get closer to the analog photon sphere with the help of a color gradient according to the medium's isotropic index of refraction. In the case of the metamaterial medium, both the speed and wavelength of the light ray change as it passes through the medium, but its frequency remains intact. Obviously, this optical feature is different from the *gravitational* blueshift as the light gets closer to the black hole horizon, in which case light frequency and wavelength change inversely but the speed of light is constant.

B. Metamaterial analog of Reissner-Nordstrom black holes

The results of the simulation for the ray trajectories in a metamaterial analog of RN and extreme RN black holes with charges $Q = 0.65M$ and $Q = M$ are shown in Figs. 4 and 5, respectively. The photon sphere is formed by two congruences of 334 and 351 rays fired at critical angles around 13.810519801° and 11.50456352° for $Q = 0.65M$

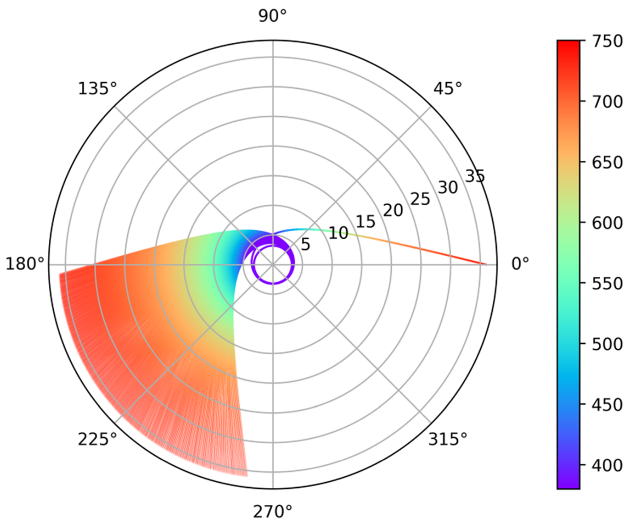

FIG. 4. Light ray trajectories in the metamaterial analog of a Reissner-Nordstrom black hole with $Q/M = 0.65$.

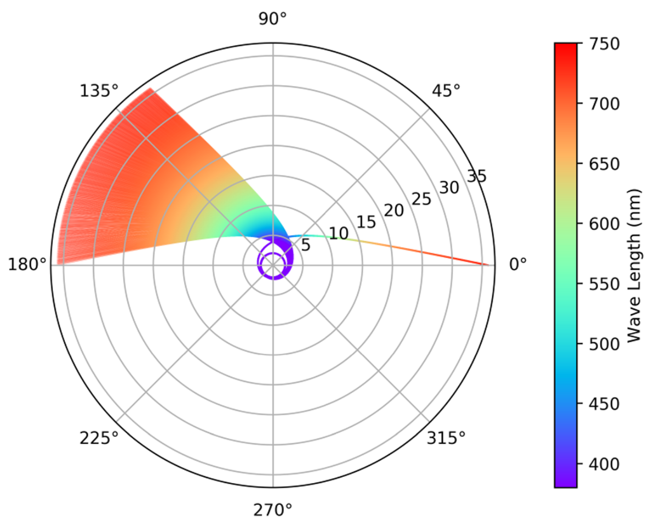

FIG. 5. Light trajectories in the metamaterial analog of an extreme Reissner-Nordstrom black hole, $Q/M = 1$.

and $Q = M$, respectively. Again, this is in complete agreement with the theoretical values given by (20) for the corresponding indices of refraction (11), and the impact parameters given by [20]

$$b_{\rm ph} = \frac{\frac{9}{4}M^2[1+(1-\frac{8Q^2}{9M^2})^{1/2}]^2}{(\frac{3}{2}M^2[1+(1-\frac{8Q^2}{9M^2})^{1/2}]-Q)^{1/2}}. \qquad (25)$$

As expected from the RN black hole geometry, compared to the Schwarzschild black hole with the same mass, the analog photon spheres form at smaller radii in the corresponding metamaterial analogs.

Obviously, in the above simulations M and Q are just parameters in the metamaterial's isotropic index of refraction, Eqs. (10) and (11). Since the metamaterial analog of RN spacetime has two different parameters, one has more control on designing them with the required optical characteristics. In our simulations the maximum winding number (number of rotations on a photon sphere) is 5 for the extreme RN case. Obviously, we can reach higher winding numbers for each case by increasing computational precision and cost, but the present level of precision is high enough for our purpose. Simulation parameters for the above three cases are listed in Table 1.

VI. METAMATERIAL ANALOG OF A BLACK HOLE SHADOW

As another interesting application of the above exact simulation, we consider a metamaterial analog of a simple Schwarzschild black hole shadow. The results of the simulation for a line of light sources placed at $5.07\rho_{\rm Sch} - 8.06\rho_{\rm Sch}$ and eclipsed by the analog of a black hole region are shown in Fig. 6. Light rays emanating from each source (as part of an analog accretion disk), within and at the edge of the corresponding cone of avoidance, are strongly deflected to reach a distant observer, forming the analog of a black hole shadow. Those light rays within the cone of avoidances are lensed to form the outer ring, which is basically the analog of the Einstein ring [21]. The inner ring is produced by those rays emanating at the edge of each cone of avoidance, rotating twice around the photon sphere before escaping to the observer positioned at $72\rho_{\rm Sch}$. This means that, due to the spherical symmetry, an observer/eye in that position will see the pattern shown in Fig. 7. The inner and outer rings are formed by rays coming from 100 point sources placed on a line of length $2.99\rho_{\rm Sch}$. Widths of the outer and inner rings are $1.15\rho_{\rm Sch}$ and $3.75\times10^{-6}\rho_{\rm Sch}$, and from the observer's position they

TABLE I. Details of the simulation for metamaterial analogs of Schwarzschild and RN black holes. Note that $\Theta_{\rm cr}$, $\rho_{\rm cda}$, and λ_m are the critical angle, the closest distance of approach, and the minimum wavelength (maximum blueshift) of the rays, respectively. All the rays have an initial wavelength of 730 nm, and ϕ is the azimuthal angle (in radians) representing the number of rotations around the hole (the winding number).

$q = Q/M$	$\rho_o/\rho_{\rm Sch}$	$\Theta_{\rm cr}$ (in radians)	$\rho_{\rm cda}/\rho_{\rm Sch}$	ϕ	λ_m(nm)
0	36	0.2614793948	3.7321649	$6\pi + 4.82$	292.75
0.65	36	0.2410390419	3.1896082	$8\pi + 4.59$	270.94
1	36	0.2007925125	2.0000398	$10\pi + 3.13$	203.34

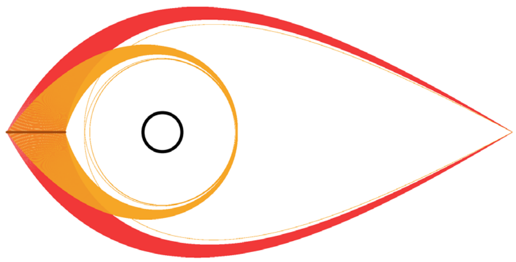

FIG. 6. The side view of the metamaterial analog of a black hole shadow produced by rays (in yellow) from a line of light sources (black line) behind its analog horizon (black circle), reaching a distant observer after rotating twice on its analog photon sphere (yellow circles). Other rays (in red) could reach the observer directly with larger impact parameters.

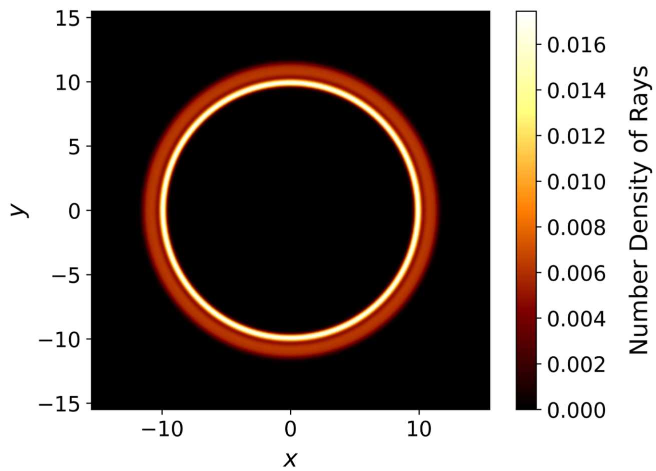

FIG. 7. Metamaterial analog of a black hole shadow as seen by an observer at the radial distance $\rho = 72\rho_{\mathrm{Sch}}$. It is produced by rotating the side view about the optical axis.

are seen at subtended angles 0.898° and $2.932° \times 10^{-6}$, respectively. The color gradient in Fig. 7 shows the density of rays in each ring, and as expected, the relativistic ring (inner ring) is denser and hence brighter than the outer lensed ring (we have to modify the color gradient to make the difference between the two rings visually clear).

Obviously, experimental realization of all the above phenomena in metamaterials needs a very delicate design specifically noting that our simulation is only valid for the region outside the analog horizon in the designed metamaterial. So, for example, in the case of the above analog black hole shadow, one needs sources whose light rays are emanating only inside the cone of avoidance to avoid those falling toward the center.

VII. CONCLUSIONS

In this study we investigated the null ray trajectories in metamaterial (optical) analogs of spherically symmetric black hole spacetimes based only on the metamaterial's index of refraction, which was taken from the spacetime geometry in isotropic coordinates. This was done by showing that the equation of null geodesics in spherically symmetric spacetimes (in isotropic coordinates) is identical to the equation of light ray trajectories in media with isotropic indices of refraction. Using this relation we introduced an exact ray-tracing simulation both for the null trajectories (geodesics) in spherically symmetric spacetimes and for light rays in isotropic metamaterials. Assigning a metamaterial with a refractive index identical to that of a spherically symmetric black hole spacetime in isotropic coordinates [modulated by its spatial conformal factor (4)], it was shown that the structure of light ray trajectories in the metamaterial exactly mimics that of the corresponding spacetime. This was explicitly shown for the case of analog photon spheres in the metamaterial analogs of Schwarzschild and RN black holes. The main advantage of the procedure outlined here is its simplicity and accuracy, allowing for simulation of any spherically symmetric spacetime which could be written in isotropic coordinates. Finally, the same simulation method was employed to find the optical analog of a simple black hole shadow formed by rays escaping the photon sphere of an analog of a Schwarzschild black hole. Designing a metamaterial with an index of refraction borrowed from a spherically symmetric black hole spacetime could provide a laboratory setting to explore this recently observed phenomenon.

ACKNOWLEDGMENTS

The authors would like to thank University of Tehran for supporting this project under the grants provided by the research council. They would also like to thank the Department of Physics for use of its HPC system, and I. Maleki for helping run the simulations. M. N.-Z. thanks Kh. Hasani for useful discussions. This work is based upon research funded by the Iran National Science Foundation (INSF) under Project No. 4005058.

[1] D. R. Smith, J. B. Pendry, and M. C. K. Wiltshire, Science **305**, 788 (2004); J. B. Pendry, D. Schurig, and D. R. Smith, Science **312**, 1780 (2006).
[2] U. Leonhardt, Science **312**, 1777 (2006).
[3] U. Leonhardt and T. G. Philbin, New J. Phys. **8**, 247 (2006).
[4] A. S. Eddington, *Space, Time, and Gravitation* (Cambridge University Press, Cambridge, England, 1920).
[5] W. Gordon, Ann. Phys. (Berlin) **377**, 421 (1923).
[6] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975).
[7] J. Plebanski, Phys. Rev. **118**, 1396 (1960).
[8] H. Chen, R.-X. Miao, and M. Li, Opt. Express **18**, 15183 (2010).
[9] I. Fernández-Núñez and O. Bulashenko, Phys. Lett. A **380**, 1 (2016).
[10] R. A. Tinguely and Andrew P. Turner, Commun. Phys. **3**, 120 (2020).
[11] W. G. Unruh and R. Schützhold, Phys. Rev. D **68**, 024008 (2003).
[12] E. E. Narimanov and A. V. Kildishev, Appl. Phys. Lett. **95**, 041106 (2009).
[13] D. Lynden-Bell and M. Nouri-Zonoz, Rev. Mod. Phys. **70**, 427 (1998).
[14] A. Besharat, M. Miri, and M. Nouri-Zonoz, J. Phys. Commun. **3**, 115019 (2019).
[15] M. Nouri-Zonoz and B. Nazari, Phys. Rev. D **82**, 044047 (2010).
[16] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972).
[17] In the case of spherically symmetric spacetimes in isotropic coordinates, one could see that $ds^2 = 0$ in (6) leads to $\frac{dl_f}{dt} = \sqrt{\frac{f(\rho)}{F\rho)}} = \frac{1}{n}$ [4].
[18] H. A. Buchdal, Int. J. Theor. Phys. **24**, 731 (1985).
[19] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, New York, 1980).
[20] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1986).
[21] R. Narayan, M. D. Johnson, and C. F. Gammie, Astrophys. J. Lett. **885**, L33 (2019).